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MINIMAL VARIANCE HEDGING FOR FRACTIONAL BROWNIAN MOTION *

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Abstract. We discuss the extension to the multi-dimensional case of the Wick-Itô integral with respect to fractional Brownian motion, introduced by [6] in the 1-dimensional case. We prove a multi-dimensional Itô type isometry for such integrals, which is used in the proof of the multi-dimensional Itô formula. The results are applied to study the problem of minimal variance hedging in a market driven by fractional Brownian motions.

1. Introduction. In the following we let $H = (H_1, H_2, \ldots, H_m)$ be an m-dimensional Hurst vector with components $H_i \in (\frac{1}{2}, 1)$ for $i = 1, 2, \ldots, m$, and we let $B^{(H)}(t) = (B_1^{(H)}(t), \ldots, B_m^{(H)}(t))$ be an m-dimensional *fractional Brownian motion* (fBm) with Hurst parameter H. This means that $B^{(H)}(t) = B^{(H)}(t, \omega)$; $t \in \mathbb{R}$, $\omega \in \Omega$ is a continuous Gaussian stochastic process on a filtered probability space $(\Omega, \mathcal{F}_t^{(H)}, \mu)$ with mean

$$\mathbb{E}[B^{(H)}(t)] = 0 = B^{(H)}(0) \qquad \text{for all } t \tag{1.1}$$

and covariance

$$\mathbb{E}[B_i^{(H)}(s) B_j^{(H)}(t)] = \tfrac{1}{2}\{|s|^{2H_i} + |t|^{2H_i} - |s-t|^{2H_i}\}\delta_{ij} \tag{1.2}$$

where

$$\delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j\ ; \qquad i \leq i, j \leq m\ , \end{cases}$$

where $\mathbb{E} = \mathbb{E}_\mu$ denotes the expectation with respect to the probability law μ of $B^{(H)}(\cdot)$.

In other words, $B^{(H)}(t)$ consists of m independent 1-dimensional fractional Brownian motions with Hurst parameters $H_1, \ldots, H_m$, respectively. If $H_i = \frac{1}{2}$ for all i, then $B^{(H)}(t)$ coincides with classical Brownian motion $B(t)$. We refer to [11], [13] and [18] for more information about 1-dimensional fBm. Because of its properties (persistence/antipersistence and self-similarity) fBm has been suggested as a useful mathematical tool in many applications, including finance [10]. For example, these features of fBm seem to appear in the log-returns of stocks [18], in weather derivative models [3] and in electricity prices in a liberated electricity market [20].

In view of this it is of interest to develop a powerful calculus for fBm. Unfortunately, fBm is not a semimartingale nor a Markov process (unless $H_i = \frac{1}{2}$ for all i), so these theories cannot be applied to fBm. However, if $H_i > \frac{1}{2}$ then the paths have zero quadratic variation and it is therefore possible to define a *pathwise integral*, denoted by

$$\int_{\mathbb{R}} f(t, \omega) \delta B^{(H)}(t)\ ,$$

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by a classical result of Young from 1936. See [12] and the references therein. This integral will obey Stratonovich type (i.e. "deterministic") integration rules. Typically the expectation of such integrals is not 0 and it is known ([12], [15], [16], [19]) that the use of these integrals in finance will give markets with *arbitrage*, even in the most basic cases. In fact, this unpleasant situation (from a modelling point of view) occurs whenever we use an integration theory with Stratonovich integration rules in the generation of wealth from a portfolio. See e.g. the simple examples of [4] and [19].

Because of this – and for several other reasons – it is natural to try other types of integration with respect to fBm. Let $\mathcal{L}_\phi^{1,2}$ be the set of (measurable) processes $f(\cdot,\cdot) : \mathbb{R} \times \Omega \to \mathbb{R}$ such that $\|f\|_{\mathcal{L}_\phi^{1,2}} < \infty$, where

$$\|f\|^2_{\mathcal{L}_\phi^{1,2}} := \mathbb{E}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}} f(s)f(t)\phi(s,t)ds\,dt + \Big(\int_{\mathbb{R}} D_t^\phi f(t)dt\Big)^2\Big]. \tag{1.3}$$

In [6] a Wick-Itô type of integral is constructed, denoted by

$$\int_{\mathbb{R}} f(t,\omega)dB^{(H)}(t) \ ,$$

where $B^{(H)}(t)$ is a 1-dimensional fBm with $H \in (\frac{1}{2}, 1)$. This integral exists as an element of $L^2(\mu)$ for all (measurable) processes $f(t,\omega)$ such that $\|f\|_{\mathcal{L}_\phi^{1,2}} < \infty$. Here, and in the following,

$$\phi(s,t) = \phi_H(s,t) = H(2H-1)|s-t|^{2H-2} \ ; \qquad (s,t) \in \mathbb{R}^2, \quad \tfrac{1}{2} < H < 1 \tag{1.4}$$

and

$$D_t^\phi F = \int_{\mathbb{R}} \phi(s,t) D_s F\, ds \tag{1.5}$$

denotes the Malliavin ϕ-derivative of F (see [6, Definition 3.4]). If $f(t,\omega)$ is a step process of the form

$$f(t,\omega) = \sum_{i=1}^n f_i(\omega)\mathcal{X}_{[t_i,t_{i+1})}(t) \ , \qquad \text{where } \ t_1 < t_2 < \cdots < t_{n+1} \ , \tag{1.6}$$

and $\|f\|_{\mathcal{L}_\phi^{1,2}} < \infty$, then the integral is defined by

$$\int_{\mathbb{R}} f(t,\omega)dB^{(H)}(t) = \sum_{i=1}^n f_i(\omega) \diamond (B^{(H)}(t_{i+1}) - B^{(H)}(t_i)) \ , \tag{1.7}$$

where $\diamond$ denotes the Wick product. We have the following basic properties of the Wick-Itô integral:

$$\mathbb{E}\Big[\int_{\mathbb{R}} f(t,\omega)dB^{(H)}(t)\Big] = 0 \qquad \text{for all } \ f \in \mathcal{L}_\phi^{1,2} \tag{1.8}$$

$$\mathbb{E}\Big[\Big(\int_{\mathbb{R}} f(t,\omega)dB^{(H)}(t)\Big)\Big(\int_{\mathbb{R}} g(t,\omega)dB^{(H)}(t)\Big)\Big] = \big(f,g\big)_{\mathcal{L}_\phi^{1,2}} \quad \text{for all } f,g \in \mathcal{L}_\phi^{1,2} \text{ where} \tag{1.9}$$

$$\big(f,g\big)_{\mathcal{L}_\phi^{1,2}} = \mathbb{E}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}} f(s)g(t)\phi(s,t)ds\,dt + \Big(\int_{\mathbb{R}} D_t^\phi f(t)dt\Big)\cdot\Big(\int_{\mathbb{R}} D_t^\phi g(t)dt\Big)\Big]. \tag{1.10}$$

See [6] for details and proofs.

This Wick-Itô fractional calculus was subsequently extended to a white noise setting and applied to finance in [9]. Later this white noise theory was generalized to all $H \in (0,1)$ by [7].

All the above papers [6], [9] and [7] only deal with the 1-dimensional case. In Section 2 of this paper we discuss the extension of this integral to the m-dimensional case, i.e. we discuss the integral

$$\int_{\mathbb{R}} f(t,\omega)dB^{(H)}(t) = \sum_{i=1}^{m} \int_{\mathbb{R}} f_i(t,\omega)dB_i^{(H)}(t) \qquad \text{for } f = (f_1,\dots,f_m) \in \mathcal{L}_\phi^{1,2}(m)$$

where $B^{(H)}(t) = (B_1^{(H)}(t),\dots,B_m^{(H)}(t))$ is m-dimensional fBm, $\phi = (\phi_{H_1},\dots,\phi_{H_m})$ and $\mathcal{L}_\phi^{1,2}(m)$ is the corresponding class of integrands (see (2.5) below). We prove the m-dimensional analogue of the isometry (1.9), which turns out to have some unexpected features (see Theorem 2.1). By combining the multi-dimensional fractional Itô formula (Theorem 2.6) with Theorem 2.1 we obtain another fractional Itô isometry (Theorem 2.7). Finally, we end Section 2 by proving a fractional integration by parts formula (Theorem 2.9 and Theorem 2.10).

In Section 3 we apply the above results to study the problem of minimal variance hedging in a (possibly incomplete) market driven by m-dimensional fBm. Here we use fractional mathematical market model introduced by [9] and by [7]. For classical Brownian motions (and semimartingales) this problem has been studied by many researchers. See for example the survey [17] and the references therein. It turns out that for fBm this problem is even harder than in the classical case and in this paper we concentrate on a special case in order to get more specific results.

2. Multi-dimensional Wick-Itô integration with respect to fBm. Let $B^{(H)}(t) = (B_1^{(H)}(t),\dots,B_m^{(H)}(t))$; $t \in \mathbb{R}$, $\omega \in \Omega$ be m-dimensional fBm with Hurst vector $H = (H_1,\dots,H_m) \in (\frac{1}{2},1)^m$, as in Section 1. Since the $B_k^{(H)}(\cdot)$ are independent, we may regard Ω as a product $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_m$ of identical copies Ω_k of some $\bar{\Omega}$ and write $\omega = (\omega_1,\dots,\omega_m) \in \Omega$.

Let $\mathcal{F} = \mathcal{F}_\infty^{(m,H)}$ be the σ-algebra generated by $\{B_k^{(H)}(s,\cdot); s \in \mathbb{R}, k = 1,2,\dots,m\}$ and let $\mathcal{F}_t = \mathcal{F}_t^{(m,H)}$ be the σ-algebra generated by $\{B_k^{(H)}(s,\cdot); 0 \le s \le t, k = 1,2,\dots,m\}$. If $F : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable, $1 \le k \le m$, we set

$$D_{k,t}^{\phi} F = \int_{\mathbb{R}} \phi_k(s,t) D_{k,t} F \, dt \qquad \text{(if the integral converges)} \tag{2.1}$$

where

$$\phi = (\phi_1,\dots,\phi_m) \tag{2.2}$$

$$\phi_k(s,t) = \phi_{H_k}(s,t) = H_k(2H_k - 1)\big|s-t\big|^{2H_k-2}; \qquad (s,t) \in \mathbb{R}^3, \ \ k = 1,2,\dots,m \tag{2.3}$$

and $D_{k,t} F = \frac{\partial F}{\partial \omega_k}(t,\omega)$ is the Malliavin derivative of F with respect to ω_k, at (t,ω) (if it exists).

Let $\mathcal{B} = \mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}$. Similarly to the 1-dimensional case we can define the multi-dimensional fractional Wick-Itô integral

$$\int_{\mathbb{R}} f(t,\omega)dB^{(H)}(t) = \sum_{k=1}^{m} \int_{\mathbb{R}} f_k(t,\omega)dB_k^{(H)}(t) \in L^2(\mu) \tag{2.4}$$

for all $\mathcal{B}\times\mathcal{F}$-measurable processes $f(t,\omega)=(f_1(t,\omega),\dots,f_m(t,\omega))\in\mathbb{R}^m$ such that

$$\|f_k\|_{\mathcal{L}^{1,2}_{\phi_k}}<\infty \qquad \text{for all } k=1,2,\dots,m, \text{ where}$$

$$\|f_k\|_{\mathcal{L}^{1,2}_{\phi_k}}:=\mathbb{E}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}} f_k(s)f_k(t)\phi_k(s,t)ds\,dt+\Big(\int_{\mathbb{R}} D^{\phi}_{k,t}\, f_k(t)dt\Big)^2\Big]. \tag{2.5}$$

Denote the set of all such m-dimensional processes f by $\mathcal{L}^{1,2}_{\phi}(m)$. As in the 1-dimensional case we obtain the isometries

$$\mathbb{E}\Big[\Big(\int_{\mathbb{R}} f_k dB_k^{(H)}\Big)^2\Big]=\|f_k\|_{\mathcal{L}^{1,2}_{\phi_k}}\,; \qquad k=1,2,\dots,m\,. \tag{2.6}$$

This is intuitively clear, since we (by independence of $B_1^{(H)},\dots,B_m^{(H)}$) can treat the remaining stochastic variables $\omega_1,\dots,\omega_{k-1},\omega_{k+1},\dots,\omega_m$ as parameters and repeat the 1-dimensional approach in the ω_k variable. It is also easy to prove (2.6) rigorously by writing $f_k(t,\omega_1,\omega_2,\dots,\omega_m)$ as a limit of sums of products of functions depending only on (t,ω_k) and only on $(\omega_1,\dots,\omega_{k-1},\omega_{k+1},\dots,\omega_m)$, respectively.

In view of this it is clear that if $f=(f_1,\dots,f_m)\in\mathcal{L}^{1,2}_{\phi}(m)$, then the Wick-Itô integral (2.4) is well-defined as an element of $L^2(\mu)$ and by (2.6) we have

$$\Big\|\int_{\mathbb{R}} f dB^{(H)}\Big\|_{L^2(\mu)}\le\sum_{k=1}^m\|f_k\|_{\mathcal{L}^{1,2}_{\phi_k}}\,. \tag{2.7}$$

It is useful to have an explicit expression for the norm on the left hand side of (2.7). The following formula is our main result of this section:

THEOREM 2.1 (Multi-dimensional fractional Wick-Itô Isometry I). *Let* $f,g\in\mathcal{L}^{1,2}_{\phi}(m)$. *Then*

$$\mathbb{E}\Big[\Big(\int_{\mathbb{R}} f dB^{(H)}\Big)\cdot\Big(\int_{\mathbb{R}} g dB^{(H)}\Big)\Big]=(f,g)_{\mathcal{L}^{1,2}_{\phi}(m)} \tag{2.8}$$

where

$$(f,g)_{\mathcal{L}^{1,2}_{\phi}(m)} =\mathbb{E}\Big[\sum_{k=1}^m\int_{\mathbb{R}}\int_{\mathbb{R}} f_k(s)g_k(t)\phi_k(s,t)ds\,dt+\sum_{k,\ell=1}^m\Big(\int_{\mathbb{R}} D^{\phi}_{\ell,t}\, f_k(t)dt\Big)\cdot\Big(\int_{\mathbb{R}} D^{\phi}_{k,t}\, g_\ell(t)dt\Big)\Big]. \tag{2.9}$$

REMARK. Note the crossing of the indices ℓ,k of the derivatives and the components f_k,g_ℓ in the last terms of the right hand side of (2.9).

To prove Theorem 2.1 we proceed as in [6], but with the appropriate modifications:

In the 1-dimensional case, let $L^2_{\phi_k}$ be the set of deterministic functions $\alpha:\mathbb{R}\to\mathbb{R}$ such that

$$(\alpha,\alpha)_{\phi_k}:=|\alpha|^2_{\phi_k}:=\int_{\mathbb{R}}\int_{\mathbb{R}}\alpha(s)\alpha(t)\phi_k(s,t)ds\,dt<\infty\,. \tag{2.10}$$

If $\alpha \in L^2_{\phi_k}$ then clearly $\alpha \in \mathcal{L}^{1,2}_{\phi_k}$. Hence we can define the *Wick* (or Doleans-Dale) *exponential*

$$\mathcal{E}(\alpha) = \exp^\diamond \Big(\int_{\mathbb{R}} \alpha(t) dB_k^{(H)}(t) \Big) = \exp \Big(\int_{\mathbb{R}} \alpha(t) dB_k^{(H)}(t) - \tfrac{1}{2} \big|\alpha\big|^2_{\phi_k} \Big). \tag{2.11}$$

See e.g. [6, (3.1)] or [9, Example 3.10].

Similarly, in the multidimensional case we put $\phi = (\phi_1, \dots, \phi_m)$ and we let L^2_ϕ be the set of all deterministic functions $\alpha = (\alpha_1, \dots, \alpha_m) : \mathbb{R} \to \mathbb{R}^m$ such that $\alpha_k \in L^2_{\phi_k}$ for $k = 1, \dots, m$. If $\alpha \in L^2_\phi$ we define the corresponding Wick exponential

$$\begin{aligned}\mathcal{E}(\alpha) &= \exp^\diamond \Big(\int_{\mathbb{R}} \alpha(t) dB^{(H)}(t) \Big) = \exp^\diamond \Big(\sum_{k=1}^m \int_{\mathbb{R}} \alpha_k(t) dB_k^{(H)}(t) \Big) \\ &= \exp \Big(\sum_{k=1}^m \int_{\mathbb{R}} \alpha_k(t) dB_k^{(H)}(t) - \tfrac{1}{2} |\alpha|^2_\phi \Big), \end{aligned} \tag{2.12}$$

where

$$|\alpha|^2_\phi = \sum_{k=1}^m \int_{\mathbb{R}} \alpha_k(s)\alpha_k(t)\phi_k(s,t) ds\, dt = \sum_{k=1}^m |\alpha|^2_{\phi_k} \ . \tag{2.13}$$

Let $\mathcal{E}$ be the linear span of all $\mathcal{E}(\alpha)$; $\alpha \in L^2_\phi$. Then we have

THEOREM 2.2. ([6, Theorem 3.1]) *$\mathcal{E}$ is a dense subset of $L^p(\mathcal{F}, \mu)$, for all $p \geq 1$.*

and

THEOREM 2.3. ([6, Theorem 3.2]) *Let $g_i = (g_{i1}, \dots, g_{im}) \in L^2_\phi$ for $i = 1, 2, \dots, n$ such that*

$$|g_{ik} - g_{jk}|_{\phi_k} \neq 0 \qquad \text{if} \quad i \neq j, \quad k = 1, \dots, m \ . \tag{2.14}$$

Then $\mathcal{E}(g_1), \dots, \mathcal{E}(g_n)$ are linearly independent in $L^2(\mathcal{F}, \mu)$.

If $F \in L^2(\mathcal{F}, \mu)$ and $g_k \in L^2_{\phi_k}$ we put, as in [6],

$$D_{k,\Phi(g_k)}\, F = \int_{\mathbb{R}} D^\phi_{k,t}\, F \cdot g_k(t) dt \ . \tag{2.15}$$

We list some useful differentiation and Wick product rules. The proofs are similar to the 1-dimensional case and are omitted.

LEMMA 2.4. *Let $f = (f_1, \dots, f_m) \in L^2_\phi$, $g = (g_1, \dots, g_m) \in L^2_\phi$. Then*

(i) $D_{k,\Phi(g_k)} \Big(\sum_{i=1}^m \int_{\mathbb{R}} f_i dB_i^{(H)} \Big) = (f_k, g_k)_{\phi_k}, \quad k = 1, \dots, m,$

where

$$(f_k, g_k)_{\phi_k} = \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(s) g_k(t) \phi_k(s,t) ds\, dt \ ; \qquad k = 1, \dots, m \ , \tag{2.16}$$

(ii) $D^\phi_{k,s} \Big(\sum_{i=1}^m \int_{\mathbb{R}} f_i dB_i^{(H)} \Big) = \int_{\mathbb{R}} f_k(u)\phi_k(s,u) du \ ; \quad k = 1, \dots, m \ ,$

(iii) $D_{k,\Phi(g_k)}\,\mathcal{E}(f) = \mathcal{E}(f)\cdot(f_k, g_k)_{\phi_k}\ ;\quad k = 1,\dots,m\ ,$
(iv) $D^{\phi}_{k,s}\,\mathcal{E}(f) = \mathcal{E}(f)\cdot\int_{\mathbb{R}} f_k(u)\phi_k(s,u)du\ ;\quad k = 1,\dots,m\ ,$
(v) $\mathcal{E}(f)\diamond\mathcal{E}(g) = \mathcal{E}(f+g)$
(vi) $F\diamond\int_{\mathbb{R}} g_k dB_k^{(H)} = F\cdot\int_{\mathbb{R}} g_k dB_k^{(H)} - D_{k,\Phi(g_k)}\,F\ ,\quad k = 1,\dots,m\ ,$
provided that $F\in L^2(\mathcal{F},\mu)$ *and* $D_{k,\Phi(g_k)}\,F\in L^2(\mathcal{F},\mu)$.
(vii) $\mathbb{E}[\mathcal{E}(f)\cdot\mathcal{E}(g)] = \exp(f,g)_\phi\ .$

We now turn to the multi-dimensional case. We will prove

LEMMA 2.5. *Suppose* $\alpha_k\in L^2_{\phi_k}$, $\beta_\ell\in L^2_{\phi_\ell}$, $D_{\ell,\Phi(\beta_\ell)}\,F\in L^2(\mu)$ *and* $D_{k,\Phi(\alpha_k)}\,G\in L^2(\mu)$. *Then*

$$\mathbb{E}\Big[\Big(F\diamond\int_{\mathbb{R}}\alpha_k dB_k^{(H)}\Big)\cdot\Big(G\diamond\int_{\mathbb{R}}\beta_\ell dB_\ell^{(H)}\Big)\Big] = \mathbb{E}\big[(D_{\ell,\Phi(\beta_\ell)}\,F)\cdot(D_{k,\Phi(\alpha_k)}\,G) + \delta_{k\ell}FG(\alpha_k,\beta_k)_{\phi_k}\big]\ , \tag{2.17}$$

where

$$\delta_{k\ell} = \begin{cases} 1 & \text{if}\quad k=\ell \\ 0 & \text{otherwise}\end{cases}$$

Proof. We adapt the argument in [6] to the multi-dimensional case:
First note that by a density argument we may assume that

$$F = \mathcal{E}(f) = \exp\Big\{\int_{\mathbb{R}} f(t)dB^{(H)}(t) - \tfrac12|f|^2_\phi\Big\}$$

and

$$G = \mathcal{E}(g) = \exp\Big\{\int_{\mathbb{R}} g(t)dB^{(H)}(t) - \tfrac12|g|^2_\phi\Big\}\ ,$$

for some $f\in L^2_\phi$, $g\in L^2_\phi$.

Choose $\delta = (\delta_1,\dots,\delta_m)\in\mathbb{R}^m$, $\gamma = (\gamma_1,\dots,\gamma_m)\in\mathbb{R}^m$ and put $\delta\times f = (\delta_1 f_1,\dots,\delta_m f_m)$ and $\gamma\times g = (\gamma_1 g_1,\dots,\gamma_m g_m)$. Then by Lemma 2.4

$$\mathbb{E}[(\mathcal{E}(f)\diamond\mathcal{E}(\delta\times\alpha))\cdot(\mathcal{E}(g)\diamond\mathcal{E}(\gamma\times\beta))] \tag{2.18}$$
$$= \mathbb{E}[\mathcal{E}(f+\delta\times\alpha)\cdot\mathcal{E}(g+\gamma\times\beta)] = \exp(f+\delta\times\alpha, g+\gamma\times\beta)_\phi$$
$$= \exp\Big\{\sum_{i=1}^m\int_{\mathbb{R}}\int_{\mathbb{R}}(f_i+\delta_i\alpha_i)(s)(g_i+\gamma_i\beta_i)(t)\phi_i(s,t)ds\,dt\Big\}. \tag{2.19}$$

We now compute the double derivatives

$$\frac{\partial^2}{\partial\delta_k\partial\gamma_\ell}$$

of (2.18) and (2.19) at $\delta=\gamma=0$. We distinguish between two cases:

Case 1. $k \neq \ell$

Then if we differentiate (2.18) we get

$$\begin{aligned}
&\frac{\partial^2}{\partial\delta_k\partial\gamma_\ell}\mathbb{E}\big[(\mathcal{E}(f)\diamond\mathcal{E}(\delta\times\alpha))\cdot(\mathcal{E}(g)\diamond\mathcal{E}(\gamma\times\beta))\big]_{\delta=\gamma=0} \\
&\qquad=\frac{\partial}{\partial\gamma_\ell}\mathbb{E}\Big[\Big(\mathcal{E}(f)\diamond\mathcal{E}(\delta\times\alpha)\diamond\int_{\mathbb{R}}\alpha_k dB_k^{(H)}\Big)\cdot(\mathcal{E}(g)\diamond\mathcal{E}(\gamma\times\beta))\Big]_{\delta=\gamma=0} \\
&\qquad=\mathbb{E}\Big[\Big(\mathcal{E}(f)\diamond\int_{\mathbb{R}}\alpha_k dB_k^{(H)}\Big)\cdot\Big(\mathcal{E}(g)\diamond\int_{\mathbb{R}}\beta_\ell dB_\ell^{(H)}\Big)\Big]. \qquad (2.20)
\end{aligned}$$

On the other hand, if we differentiate (2.19) we get

$$\begin{aligned}
&\frac{\partial^2}{\partial\delta_k\partial\gamma_\ell}\big[\exp(f+\delta\times\alpha,g+\gamma\times\beta)_\phi\big]_{\delta=\gamma=0} \\
&=\frac{\partial}{\partial\gamma_\ell}\Big[\exp(f+\delta\times\alpha,g+\gamma\times\beta)_\phi\cdot\int_{\mathbb{R}}\int_{\mathbb{R}}\alpha_k(s)(g_k+\gamma_k\beta_k)(t)\phi_k(s,t)ds\,dt\Big]_{\delta=\gamma=0} \\
&=\exp(f,g)_\phi\cdot\int_{\mathbb{R}}\int_{\mathbb{R}}\alpha_k(s)g_k(t)\phi_k(s,t)ds\,dt\cdot\int_{\mathbb{R}}\int_{\mathbb{R}}\beta_\ell(s)f_\ell(t)\phi_\ell(s,t)ds\,dt \\
&=\exp(f,g)_\phi\cdot(\alpha_k,g_k)_{\phi_k}\cdot(\beta_\ell,f_\ell)_{\phi_\ell} \\
&=\mathbb{E}[\mathcal{E}(f)\cdot(\beta_\ell,f_\ell)_{\phi_\ell}\cdot\mathcal{E}(g)\cdot(\alpha_k,g_k)_{\phi_k}] \\
&=\mathbb{E}\big[D_{\ell,\Phi(\beta_\ell)}\,\mathcal{E}(f)\cdot D_{k,\Phi(\alpha_k)}\,\mathcal{E}(g)\big]\,. \qquad (2.21)
\end{aligned}$$

This proves (2.17) in this case.

Case 2. $k=\ell$.

In this case, if we differentiate (2.18) we get

$$\begin{aligned}
&\frac{\partial^2}{\partial\delta_k\partial\gamma_k}\mathbb{E}\big[(\mathcal{E}(f)\diamond\mathcal{E}(\delta\times\alpha))\cdot(\mathcal{E}(g)\diamond\mathcal{E}(\gamma\times\beta))\big]_{\delta=\gamma=0} \\
&\qquad=\frac{\partial}{\partial\gamma_k}\mathbb{E}\Big[\Big(\mathcal{E}(f)\diamond\mathcal{E}(\delta\times\alpha)\diamond\int_{\mathbb{R}}\alpha_k dB_k^{(H)}\Big)\cdot(\mathcal{E}(g)\diamond\mathcal{E}(\gamma\times\beta))\Big]_{\delta=\gamma=0} \\
&\qquad=\mathbb{E}\Big[\Big(\mathcal{E}(f)\diamond\int_{\mathbb{R}}\alpha_k dB_k^{(H)}\Big)\cdot\Big(\mathcal{E}(g)\diamond\int_{\mathbb{R}}\beta_k dB_k^{(H)}\Big)\Big]. \qquad (2.22)
\end{aligned}$$

On the other hand, if we differentiate (2.19) we get

$$\begin{aligned}
&\frac{\partial^2}{\partial\delta_k\partial\gamma_k}\big[\exp(f+\delta\times\alpha,g+\gamma\times\beta)_\phi\big]_{\delta=\gamma=0} \\
&=\frac{\partial}{\partial\gamma_k}\Big[\exp(f+\delta\times\alpha,g+\gamma\times\beta)_\phi\cdot\int_{\mathbb{R}}\int_{\mathbb{R}}\alpha_k(s)(g_k+\gamma_k\beta_k)(t)\phi_k(s,t)ds\,dt\Big]_{\delta=\gamma=0} \\
&=\exp(f,g)_\phi\cdot\Big[(\alpha_k,g_k)_{\phi_k}\cdot(\beta_k,f_k)_{\phi_k}+\int_{\mathbb{R}}\int_{\mathbb{R}}\alpha_k(s)\beta_k(t)\phi_k(s,t)ds\,dt\Big] \\
&=\mathbb{E}\big[D_{k,\Phi(\beta_k)}\,\mathcal{E}(f)\cdot D_{k,\Phi(\alpha_k)}\,\mathcal{E}(g)+\mathcal{E}(f)\mathcal{E}(g)(\alpha_k,\beta_k)_{\phi_k}\big]\,. \qquad (2.23)
\end{aligned}$$

This proves (2.17) also for Case 2 and the proof of Lemma 2.5 is complete. □

We are now ready to prove Theorem 2.1:

Proof. We may consider $\int_{\mathbb{R}} f_k(t)dB_k^{(H)}(t)$ as the limit of sums of the form

$$\sum_{i=1}^{N} f_k(t_i) \diamond (B_k^{(H)}(t_{i+1}) - B_k^{(H)}(t_i))$$

when $\Delta t_i = t_{i+1} - t_i \to 0,\ t_1 < t_2 < \cdots < t_N,\ N = 2,3,\ldots$ Hence $\mathbb{E}\Big[\big(\int_{\mathbb{R}} f dB^{(H)}\big)^2\Big] = \mathbb{E}\Big[\Big(\sum_{k=1}^{m} \int_{\mathbb{R}} f_k dB_k^{(H)}\Big)^2\Big]$ is the limit of sums of the form

$$\sum_{i,j,k,\ell} \mathbb{E}\Big[(f_k(t_i) \diamond (B_k^{(H)}(t_{i+1}) - B_k^{(H)}(t_i))) \cdot (f_\ell(t_j) \diamond (B_\ell^{(H)}(t_{j+1}) - B_\ell^{(H)}(t_j))\Big],$$

which by Lemma 2.5 is equal to

$$\sum_{i,j,k,\ell} \mathbb{E}\Big[\Big(\int_{t_i}^{t_{i+1}} D_{\ell,t}^{\phi} f_k(t_i)dt\Big) \cdot \Big(\int_{t_j}^{t_{j+1}} D_{k,t}^{\phi} f_\ell(t_j)dt\Big) + \delta_{k\ell} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} f_k(t_i) f_k(t_j) \phi_k(s,t) ds\, dt\Big].$$

When $\Delta t_i \to 0$ this converges to

$$\mathbb{E}\Big[\sum_{k,\ell=1}^{m} \Big(\int_{\mathbb{R}} D_{\ell,t}^{\phi} f_k(t)dt\Big) \cdot \Big(\int_{\mathbb{R}} D_{k,t}^{\phi} f_\ell(t)dt\Big) + \sum_{k=1}^{m} \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(s) f_k(t) \phi_k(s,t) ds\, dt\Big]. \tag{2.24}$$

This proves (2.9) when $f = g$. By polarization the proof of Theorem 2.1 is complete. ∎

Using Theorem 2.1 we can now proceed as in the 1-dimensional case ([6, Theorem 4.3]), with appropriate modifications, and obtain a fractional multi-dimensional Itô formula. We omit the proof.

THEOREM 2.6 (The fractional multi-dimensional Itô formula). *Let $X(t) = (X_1(t), \ldots, X_n(t))$, with*

$$dX_i(t) = \sum_{j=1}^{m} \sigma_{ij}(t,\omega) dB_j^{(H)}(t) ;$$

$$\text{where} \quad \sigma_i = (\sigma_{i1}, \ldots, \sigma_{im}) \in \mathcal{L}_\phi^{1,2}(m) ; \quad 1 \le i \le n . \tag{2.25}$$

Suppose that for all $j = 1, \ldots, m$ there exists $\theta_j > 1 - H_j$ such that

$$\sup_i \mathbb{E}[(\sigma_{ij}(u) - \sigma_{ij}(v))^2] \le C\big|u - v\big|^{\theta_j} \qquad \text{if } |u - v| < \delta \tag{2.26}$$

where $\delta > 0$ is a constant. Moreover, suppose that

$$\lim_{\substack{0 \le u,v \le t \\ |u-v| \to 0}} \big\{ \sup_{i,j,k} \mathbb{E}[(D_{k,u}^{\phi}\{\sigma_{ij}(u) - \sigma_{ij}(v)\})^2\big] = 0 . \tag{2.27}$$

Let $f \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ with bounded second order derivatives with respect to x. Then, *for* $t > 0$,

$$f(t,X(t)) = f(0,X(0)) + \int_0^t \frac{\partial f}{\partial s}(s,X(s))ds + \int_0^t \sum_{i=1}^n \frac{\partial f}{\partial x_i}(s,X(s))dX_i(s)$$

$$+ \int_0^t \Big\{ \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x_i \partial x_j}(s,X(s)) \sum_{k=1}^m \sigma_{ik}(s) D_{k,s}^{\phi}(X_j(s)) \Big\} ds \tag{2.28}$$

$$= f(0,X(0)) + \int_0^t \frac{\partial f}{\partial s}(s,X(s))ds + \sum_{j=1}^m \int_0^t \Big[\sum_{i=1}^n \frac{\partial f}{\partial x_i}(s,X(s))\sigma_{ij}(s,\omega) \Big] dB_j^{(H)}(s)$$

$$+ \int_0^t \mathrm{Tr}\left[\Lambda^T(s) f_{xx}(s,X(s))\right] ds\,. \tag{2.29}$$

Here $\Lambda = [\Lambda_{ij}] \in \mathbb{R}^{n\times m}$ *with*

$$\Lambda_{ij}(s) = \sum_{k=1}^m \sigma_{ik} D_{k,s}^{\phi}(X_j(s))\,; \qquad 1 \le i \le n\,, \quad 1 \le j \le m\,, \tag{2.30}$$

$$f_{xx} = \Big[\frac{\partial^2 f}{\partial x_i \partial x_j}\Big]_{1\le i,j\le n} \in \mathbb{R}^{n\times n} \tag{2.31}$$

and $(\cdot)^T$ *denotes matrix transposed,* $\mathrm{Tr}[\cdot]$ *denotes matrix trace.*

If we combine Theorem 2.6 with Theorem 2.1 we get the following result, which also may be regarded as a fractional Itô isometry:

THEOREM 2.7 (Fractional Itô isometry II). *Suppose* $f = (f_1,\dots,f_m) \in \mathcal{L}_{\phi}^{1,2}(m)$. *Then, for* $T > 0$,

$$\mathbb{E}\Big[\Big(\int_0^T D_{\ell,t}^{\phi} f_k(t)dt\Big)\cdot\Big(\int_0^T D_{k,t}^{\phi} f_\ell(t)dt\Big)\Big]$$

$$= \mathbb{E}\Big[\int_0^T \Big\{ f_k(t)\int_0^t D_{k,t}^{\phi} f_\ell(s)dB_\ell^{(H)}(s) + f_\ell(t)\int_0^T D_{\ell,t}^{\phi} f_k(s)dB_k^{(H)}(s)\Big\}dt\Big] \tag{2.32}$$

Proof. By the Itô formula (Theorem 2.6) we have

$$\mathbb{E}\Big[\Big(\int_0^T f_k dB_k^{(H)}\Big)\cdot\Big(\int_0^T f_\ell dB_\ell^{(H)}\Big)\Big]$$

$$= \mathbb{E}\Big[\int_0^T \Big\{ f_k(t) D_{k,t}^{\phi}\Big(\int_0^t f_\ell(s)dB_\ell^{(H)}(s)\Big) + f_k(t) D_{\ell,t}^{\phi}\Big(\int_0^t f_k(s)dB_k^{(H)}(s)\Big)\Big\}dt\Big]$$

$$= \mathbb{E}\Big[\int_0^T \Big\{ f_k(t)\int_0^t D_{k,t}^{\phi} f_\ell(s)dB_\ell^{(H)}(s) + f_\ell(t)\int_0^t D_{\ell,t}^{\phi} f_k(s)dB_k^{(H)}(s)\Big\}dt\Big]$$

$$+ \delta_{k\ell}\mathbb{E}\Big[\int_0^T\int_0^t \{f_k(t)f_k(s) + f_\ell(t)f_k(s)\}\phi_k(s,t)ds\,dt\Big], \tag{2.33}$$

where we have used that, for $u > 0$,

$$D_{k,t}^{\phi}\Big(\int_0^u f_\ell(s)dB_\ell^{(H)}(s)\Big) = \int_0^u D_{k,t}^{\phi} f_\ell(s)dB_\ell^{(H)}(s) + \delta_{k\ell}\int_0^u f_k(s)\phi_k(t,s)ds\,. \tag{2.34}$$

(See [6, Theorem 4.2].)

On the other hand, the Itô isometry (Theorem 2.1) gives that

$$\mathbb{E}\Big[\Big(\int_0^T f_k dB_k^{(H)}\Big)\cdot\Big(\int_0^T f_\ell dB_\ell^{(H)}\Big)\Big]$$
$$= \mathbb{E}\Big[\Big(\int_0^T D_{\ell,t}^{\phi} f_k(t)dt\Big)\cdot\Big(\int_0^T D_{k,t}^{\phi} f_\ell(t)dt\Big) + \delta_{k\ell}\big|f_k\big|_{\phi_k}^2\Big]. \tag{2.35}$$

Comparing (2.33) and (2.35) we get Theorem 2.7. □

We end this section by proving a fractional integration by parts formula. First we recall

Theorem 2.8 (Fractional Girsanov formula). *Suppose* $\gamma = (\gamma_1, \dots, \gamma_m) \in (L^2(\mathbb{R}))^m$ *and* $\hat\gamma = (\hat\gamma_1, \dots, \hat\gamma_m) \in L_\phi^2$ *are related by*

$$\gamma_k(t) = \int_{\mathbb{R}} \hat\gamma_k(s)\phi_k(s,t)ds\ ; \qquad t \in \mathbb{R}, \quad k = 1, \dots, m\ . \tag{2.36}$$

Let $G \in L^2(\mu)$. *Then*

$$\mathbb{E}[G(\omega+\gamma)] = \mathbb{E}[G(\omega)\exp^{\diamond}(\langle\omega,\hat\gamma\rangle)] = \mathbb{E}\Big[G(\omega)\mathcal{E}\Big(\int_{\mathbb{R}}\hat\gamma dB^{(H)}\Big)\Big]. \tag{2.37}$$

For a proof in the 1-dimensional case see e.g. [9, Theorem 3.16]. The proof in the multi-dimensional case is similar.

If $F \in L^2(\mu)$ and $\gamma = (\gamma_1, \dots, \gamma_m) \in (L^2(\mathbb{R}))^m$ the *directional derivative of* F *in the direction* γ is defined by

$$D_\gamma F(\omega) = \lim_{\varepsilon\to 0} \frac{F(\omega+\varepsilon\gamma) - F(\omega)}{\varepsilon}\ , \tag{2.38}$$

provided the limit exists in $L^2(\mu)$. We say that F is *differentiable* if there exists a process $D_t F(\omega) = (D_{1,t}F(\omega), \dots, D_{m,t}F(\omega))$ such that $D_{k,t}F(\omega) \in L^2(d\mu\otimes dt)$ for all $k = 1, \dots, m$ and

$$D_\gamma F(\omega) = \int_{\mathbb{R}} D_t F(\omega)\cdot\gamma(t)dt \qquad \text{for all } \ \gamma \in (L^2(\mathbb{R}))^m\ . \tag{2.39}$$

Theorem 2.9 (Fractional integration by parts I). *Let* $F, G \in L^2(\mu)$, $\gamma \in (L^2(\mathbb{R}))^m$ *and assume that the directional derivatives* $D_\gamma F$, $D_\gamma G$ *exist. Then*

$$\mathbb{E}[D_\gamma F\cdot G] = \mathbb{E}\Big[F\cdot G\cdot\int_{\mathbb{R}}\hat\gamma dB^{(H)}\Big] - \mathbb{E}[F\cdot D_\gamma G]\ . \tag{2.40}$$

Proof. By Theorem 2.8 we have, for all $\varepsilon > 0$,

$$\mathbb{E}[F(\omega+\varepsilon\gamma)G(\omega)] = \mathbb{E}[F(\omega)G(\omega-\varepsilon\gamma)\exp^{\diamond}(\varepsilon\langle\omega,\hat\gamma\rangle)]\ .$$

Hence

$$\begin{aligned}\mathbb{E}[D_\gamma F \cdot G] &= \mathbb{E}\Big[\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{F(\omega+\varepsilon\gamma)-F(\omega)\}G(\omega)\Big] \\ &= \mathbb{E}\Big[\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{F(\omega)[G(\omega-\varepsilon\gamma)\exp^\diamond(\varepsilon\langle\omega,\hat\gamma\rangle)-G(\omega)]\}\Big] \\ &= \mathbb{E}\Big[F(\omega)\frac{d}{d\varepsilon}\Big\{G(\omega-\varepsilon\gamma)\exp\Big(\varepsilon\int_{\mathbb{R}}\hat\gamma dB^{(H)}-\tfrac{1}{2}\varepsilon^2|\hat\gamma|_\phi^2\Big)\Big\}_{\varepsilon=0}\Big] \\ &= \mathbb{E}\Big[F(\omega)G(\omega)\int_{\mathbb{R}}\hat\gamma dB^{(H)}\Big]-\mathbb{E}[F(\omega)D_\gamma G(\omega)]\end{aligned}$$

□

We now apply the above to the fractional gradient

$$D_t^\phi F = \int_{\mathbb{R}} D_s F\cdot\phi(s,t)ds = \sum_{k=1}^m \int_{\mathbb{R}} D_{k,s}F\cdot\phi_k(s,t)ds = D_\phi F(\omega) \tag{2.41}$$

THEOREM 2.10 (Fractional integration by parts II). *Suppose $F, G \in L^2(\mu)$ are differentiable, with fractional gradients $D_t^\phi F$, $D_t^\phi G$. Then for each $t\in\mathbb{R}$, $k\in\{1,\dots,m\}$ we have*

$$\mathbb{E}[D_{k,t}^\phi F\cdot G] = \mathbb{E}[F\cdot G\cdot B_k^{(H)}(t)] - \mathbb{E}[F\cdot D_{k,t}^\phi G]\ . \tag{2.42}$$

Proof. Choose a sequence $\hat\gamma_k^{(j)}\in L^2_{\phi_k}$; $j=1,2,\dots$, such that $\lim_{j\to\infty}\hat\gamma_k^{(j)}=\delta_t(\cdot)$ (the point mass at t), in the sense that if we define

$$\phi_k^{(j)}(s) = \int_{\mathbb{R}} \hat\gamma_k^{(j)}\phi_k(s,r)dr$$

then $\phi_k^{(j)}(\cdot)\to\phi_k(\cdot,t)$ in $L^2(\mathbb{R})$. Then by Theorem 2.9

$$\begin{aligned}\mathbb{E}[D_{k,t}^\phi F\cdot G] &= \mathbb{E}\Big[\lim_{j\to\infty} D_{\phi_k^{(j)}}F\cdot G\Big] = \lim_{j\to\infty}\mathbb{E}[D_{\phi_k^{(j)}}F\cdot G] \\ &= \lim_{j\to\infty}\mathbb{E}\Big[F\cdot G\cdot\int_{\mathbb{R}}\hat\gamma^{(j)}dB^{(H)}\Big]-\mathbb{E}[F\cdot D_{\phi_k^{(j)}}G] \\ &= \mathbb{E}[F\cdot G\cdot B_k^{(H)}(t)]-\mathbb{E}[F\cdot D_{k,t}G]\ .\end{aligned}$$

□

3. Application to minimal variance hedging. Consider the multidimensional version of the fractional mathematical market model introduced by [9] and by [7], consisting of $n+1$ independent fractional Brownian motions $B_1^{(H)}(t),\dots,B_m^{(H)}(t)$ with Hurst coefficients $H_1,\dots,H_m$ respectively ($\frac{1}{2}<H_i<1$), as follows:

(bond price) $$dS_0(t) = r(t,\omega)dt\ ;\quad S_0(0)=s_0\ ,\quad 0\le t\le T \tag{3.1}$$

(stock prices) $$dS_i(t) = \mu_i(t,\omega)dt + \sum_{j=1}^m \sigma_{ij}(t,\omega)dB_j^{(H)}(t)\ ;\quad S_i(0)=s_i,\quad i=1,\dots,n,\quad 0\le t\le T\ . \tag{3.2}$$

Here $r(t,\omega), \mu_i(t,\omega)$ and $\sigma_{ij}(t,w)$ are $\mathcal{F}_t^{(H)}$-adapted processes satisfying reasonable growth conditions. We refer to [7], [9], [14] and [21] for a general discussion of such markets.

We say that $g = (g_1, \ldots, g_m)$ is an *admissible portfolio* if $g(t)$ is $\mathcal{F}_t^{(H)}$-adapted, $g\sigma \in \mathcal{L}_\phi^{1,2}(m)$ and $\mathbb{E}\Big[\int_0^T \sum_{i=1}^n |g_i(t)\mu_i(t)|dt\Big] < \infty$. Here we denote by σ the volatility matrix $[\sigma]_{i,j}(\cdot) = \sigma_{ij}(\cdot)$. Suppose we are only allowed to trade in some, say k, of the securities $S_0, \ldots, S_n$. Let $\mathcal{K}$ be the set of $i \in \{1, \ldots, n\}$ such that trading in S_i is allowed. Then, according to our model, the *wealth* hedged by an *initial value* $z \in \mathbb{R}$ and an admissible portfolio $g(t) = (g_i(t,\omega))_{i\in\mathcal{K}} \in \mathbb{R}^k$ up to time t is

$$V(t) = V_z^g(t) = z + \sum_{i\in\mathcal{K}} \int_0^t g_i(u)dS_i(u)\ ; \qquad 0 \le t \le T\ . \tag{3.3}$$

Now let $F(\omega)$ be a T-*claim*, i.e. an $\mathcal{F}_T^{(H)}$-measurable random variable in $L^2(\mu)$.

The *minimal variance hedging problem* is to find a $z^* \in \mathbb{R}$ and an admissible portfolio g^* such that

$$\mathbb{E}[(F - V_{z^*}^{g^*}(T))^2] = \inf_{z,g} \mathbb{E}[(F - V_z^g(T))^2]\ . \tag{3.4}$$

This is a difficult problem even in the classical Brownian motion setting. See e.g. [8], [17] and the references therein. For a recent general martingale approach see [5]. For fractional Brownian motion markets a special case is solved in [1] by using optimal control theory.

Here we will discuss the two-dimensional case only, and we will simply assume that

$$dS_0(t) = 0, \quad dS_1(t) = dB_1^{(H)}(t) \qquad \text{and} \quad dS_2(t) = dB_2^{(H)}(t)\ .$$

Assume that only trading in S_0 and S_1 is allowed. Then the problem is to minimize

$$J(z, g_1) = \mathbb{E}\Big[\Big(F - \Big(z + \int_0^T g_1 dS_1\Big)\Big)^2\Big] \tag{3.5}$$

over all $z \in \mathbb{R}$ and all admissible portfolios g_1.

By the fractional Clark-Haussmann-Ocone formula ([9, Theorem 4.15]) we can write

$$F(\omega) = \mathbb{E}[F] + \int_0^T f_1(t)dB_1^{(H)}(t) + \int_0^T f_2(t)dB_2^{(H)}(t) \tag{3.6}$$

where

$$f_i(t) = \widetilde{\mathbb{E}}[D_{i,t}\,F \mid \mathcal{F}_t^{(H)}]\ ; \qquad i = 1,2\ .$$

Substituting this into (3.5) we get, by (1.8),

$$\begin{aligned} J(z, g_1) &= \mathbb{E}\Big[\Big(\mathbb{E}[F] - z + \int_0^T (f_1 - g_1)dB_1^{(H)} + \int_0^T f_2 dB_2^{(H)}\Big)^2\Big] \\ &= (\mathbb{E}[F] - z)^2 + \mathbb{E}\Big[\Big(\int_0^T (f_1 - g_1)dB_1^{(H)} + \int_0^T f_2 dB_2^{(H)}\Big)^2\Big]. \end{aligned} \tag{3.7}$$

Hence it is optimal to choose $z = z^* := \mathbb{E}[F]$. The remaining problem is therefore to minimize

$$J_0(g_1) = \mathbb{E}\Big[\Big(\int_0^T (f_1 - g_1)dB_1^{(H)} + \int_0^T f_2 dB_2^{(H)}\Big)^2\Big]. \tag{3.8}$$

From now on we assume that $f_1 \in \mathcal{L}_{\phi_i}^{1,2}$ for $i = 1, 2$. By a Hilbert space argument on $L^2(\mu)$ we see that g_1^* minimizes (3.8) if and only if

$$\mathbb{E}\Big[\Big(\int_0^T (f_1 - g_1)dB_1^{(H)} + \int_0^T f_2 dB_2^{(H)}\Big) \cdot \Big(\int_0^T \gamma dB_1^{(H)}\Big)\Big] = 0$$
$$\text{for all adapted } \gamma \in \mathcal{L}_{\phi_1}^{1,2}. \tag{3.9}$$

By Theorem 2.1 (3.9) is equivalent to

$$\begin{aligned}\mathbb{E}\Big[\int_0^T\int_0^T &(f_1(t) - g_1(t))\gamma(s)\phi_1(s,t)ds\,dt + \Big(\int_0^T D_{1,t}^{\phi}(f_1(t) - g_1(t))dt\Big)\Big(\int_0^T D_{1,t}^{\phi}\gamma(t)dt\Big)\\ &+ \Big(\int_0^T D_{1,t}^{\phi} f_2(t)dt\Big) \cdot \Big(\int_0^T D_{2,t}^{\phi}\gamma(t)dt\Big)\Big]\\ &= 0 \quad \text{for all adapted } \gamma \in \mathcal{L}_{\phi}^{1,2}.\end{aligned} \tag{3.10}$$

From this we immediately deduce

PROPOSITION 3.1. *The portfolio*

$$g_1(t) = g_1^*(t) := f_1(t)$$

minimizes (3.8) *if and only if*

$$\int_0^T D_{1,t}^{\phi} f_2(t)dt = 0 \quad a.s. \tag{3.11}$$

This result is surprising in view of the corresponding situation for classical Brownian motion, when it is *always* optimal to choose $g_1(t) = g_1^*(t) = f_1(t)$.

We also get

PROPOSITION 3.2. *Suppose* $g_1^*(t)$ *minimizes* (3.8). *Then*

$$\mathbb{E}\Big[\int_0^T (f_1(t) - g_1^*(t))dt\Big] = 0\,. \tag{3.12}$$

Proof. This follows by choosing $\gamma(t)$ deterministic in (3.10). ◻

Now assume that $D^{\phi}_{1,t}(f_1(t))$ and $D^{\phi}_{1,t}(g_1(t))$ are differentiable with respect to $D^{\phi}_{1,s}$ and that $D^{\phi}_{1,t}\, f_2(t)$ is differentiable with respect to $D^{\phi}_{2,s}$ for all $s \in [0,T]$. Then we can use integration by parts (Theorem 2.10) to rewrite equation (3.10) as follows:

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T\int_0^T \{(f_1(t)-g_1(t))\gamma(s)\phi_1(s,t) + D^{\phi}_{1,t}(f_1(t)-g_1(t))\cdot D^{\phi}_{1,s}\,\gamma(s) \\
&\qquad + D^{\phi}_{1,t}\, f_2(t)\cdot D^{\phi}_{2,s}\,\gamma(s)\}ds\,dt\Big] \\
&= \int_0^T\int_0^T \mathbb{E}[(f_1(t)-g_1(t))\phi_1(s,t)\gamma(s) + D^{\phi}_{1,t}(f_1(t)-g_1(t))\gamma(s)B_1^{(H)}(s) \\
&\qquad - D^{\phi}_{1,s}\, D^{\phi}_{1,t}(f_1(t)-g_1(t))\gamma(s) + D^{\phi}_{1,t}\, f_2(t)\gamma(s)B_2^{(H)}(s) \\
&\qquad - D^{\phi}_{2,s}\, D^{\phi}_{1,t}\, f_2(t)\gamma(s)\Big]ds\,dt \\
&= \mathbb{E}\Big[\int_0^T K(s)\gamma(s)ds\Big] = 0\ ,
\end{aligned} \tag{3.13}$$

where

$$K(s) = \int_0^T G(s,t)dt\ , \tag{3.14}$$

with

$$\begin{aligned}
G(s,t) &= (f_1(t)-g_1(t))\phi_1(s,t) + D^{\phi}_{1,t}(f_1(t)-g_1(t))B_1^{(H)}(s) \\
&\quad - D^{\phi}_{1,s}\, D^{\phi}_{1,t}(f_1(t)-g_1(t)) + D^{\phi}_{1,t}\, f_2(t)B_2^{(H)}(s) - D^{\phi}_{2,s}\, D^{\phi}_{1,t}\, f_2(t)\ .
\end{aligned} \tag{3.15}$$

Since $\gamma(s)$ is $\mathcal{F}_s^{(H)}$-measurable we get from (3.13) that

$$\begin{aligned}
0 &= \int_0^T \mathbb{E}[K(s)\gamma(s)]ds = \int_0^T \mathbb{E}\big[\mathbb{E}[K(s)\gamma(s) \mid \mathcal{F}_s^{(H)}]\big]ds \\
&= \int_0^T \mathbb{E}\big[\gamma(s)\mathbb{E}[K(s) \mid \mathcal{F}_s^{(H)}]\big]ds = \mathbb{E}\Big[\int_0^T \mathbb{E}\big[K(s) \mid \mathcal{F}_s^{(H)}\big]\gamma(s)ds\Big]\ .
\end{aligned} \tag{3.16}$$

Since this holds for all adapted $\gamma \in \mathcal{L}_{\phi}^{1,2}$ we conclude that

$$\mathbb{E}[K(s) \mid \mathcal{F}_s^{(H)}] = 0 \qquad \text{for a.a. } (s,\omega)\ . \tag{3.17}$$

or, using (3.14),

$$\begin{aligned}
&\int_0^T \{\mathbb{E}_s[f_1(t)-g_1(t)]\phi_1(s,t) + \mathbb{E}_s[D^{\phi}_{1,t}(f_1(t-g_1(t))]B_1^{(H)}(s) \\
&- \mathbb{E}_s[D^{\phi}_{1,s}\, D^{\phi}_{1,t}(f_1(t)-g_1(t))] + \mathbb{E}_s[D^{\phi}_{1,t}\, f_2(t)]B_2^{(H)}(s) - \mathbb{E}_s[D^{\phi}_{2,s}\, D^{\phi}_{1,t}\, f_2(t)]\}dt = 0\ ,
\end{aligned} \tag{3.18}$$

where we have used the shorthand notation

$$\mathbb{E}_s[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_s^{(H)}]\ .$$

We have proved:

THEOREM 3.3. *Suppose the claim F represented by* (3.6) *is such that* $D^{\phi}_{1,s} D^{\phi}_{1,t} f_1(t)$ *and* $D^{\phi}_{2,s} D^{\phi}_{1,t} f_2(t)$ *exist for all* $s,t \in [0,T]$. *Suppose* $\hat{g}_1(t)$ *is an adapted process in* $\mathcal{L}^{1,2}_{\phi}$ *such that* $D^{\phi}_{1,t}\hat{g}_1(t)$ *and* $D^{\phi}_{1,s} D^{\phi}_{1,t}\hat{g}_1(t)$ *exist for all* $s,t \in [0,T]$. *Then the following are equivalent:*

(i) $\hat{g}_1(t)$ *is a minimal variance hedging portfolio for* F, *i.e.* $\hat{g}_1(t)$ *minimizes* (3.8) *over all adapted* $g_1(t) \in \mathcal{L}^{1,2}_{\phi}$

(ii) $g_1(t) = \hat{g}_1(t)$ *satisfies equation* (3.18).

Note that the same method also applies if we assume a fractional exponential dynamics for the asset prices, which represents a more realistic financial model.
To illustrate this result we consider the following special case:

EXAMPLE 3.4. Suppose $f_1(t) = 0$ and

$$D^{\phi}_{1,t} f_2(t) = h(t)\ , \quad \text{a deterministic function}\ . \tag{3.19}$$

We seek a minimal variance hedging portfolio $g_1^*(t)$ for the claim

$$F(\omega) = \int_0^T f_2(t)dB_2^{(H)}(t)\ . \tag{3.20}$$

In this case (3.18) gets the form

$$\int_0^T \{-\mathbb{E}_s[g_1(t)]\phi_1(s,t) - \mathbb{E}_s[D^{\phi}_{1,t}\, g_1(t)]B_1^{(H)}(s) + \mathbb{E}_s[D^{\phi}_{1,s}\, D^{\phi}_{1,t}\, g_1(t)] + h(t)B_2^{(H)}(s)\}dt = 0 \qquad \text{for a.a.}\ \ (s,\omega)\ . \tag{3.21}$$

Let us try to choose $g_1(t)$ such that

$$D^{\phi}_{1,t}\, g_1(t) = 0\ . \tag{3.22}$$

Then (3.19) reduces to

$$\int_0^T \mathbb{E}_s[g_1(t)]\phi_1(s,t)dt = B_2^{(H)}(s)\int_0^T h(t)dt \tag{3.23}$$

or, since g_1 is adapted,

$$\int_0^s g_1(t)\phi_1(s,t)dt + \int_s^T \mathbb{E}_s[g_1(t)]\phi_1(s,t)dt = B_2^{(H)}(s)\int_0^T h(t)dt, \quad s \in [0,T]\ . \tag{3.24}$$

In particular, if we choose $s = T$ we get the equation

$$\int_0^T g_1(t)\phi_1(T,t)dt = B_2^{(H)}(T)\int_0^T h(t)dt\ , \tag{3.25}$$

which clearly has no adapted solution $g_1(t)$. (However, it obviously has a *non-adapted* solution.) Therefore an optimal portfolio $g_1(t) = g_1^*(t)$ for the claim (3.20), if it exists, cannot satisfy (3.22).

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