GKM-sheaves and nonorientable surface group representations

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Let T be a compact torus and X a nice compact T-space (say a manifold or variety). We introduce a functor assigning to X a *GKM-sheaf* $\mathcal{F}_X$ over a *GKM-hypergraph* Γ_X. Under the condition that X is *equivariantly formal*, the ring of global sections of $\mathcal{F}_X$ are identified with the equivariant cohomology, $H_T^*(X;\mathbb{C}) \cong H^0(\mathcal{F}_X)$. We show that GKM-sheaves provide a general framework able to incorporate numerous constructions in the GKM-theory literature.

In the second half of the paper we apply these ideas to study the equivariant topology of the representation variety $\mathcal{R}_K := \mathrm{Hom}(\pi_1(\Sigma), K)$ under conjugation by K, where Σ is a nonorientable surface and K is a compact connected Lie group. We prove that $\mathcal{R}_{SU(3)}$ is equivariantly formal for all Σ and compute its equivariant cohomology ring. We also produce conjectural betti number formulas for some other Lie groups.

1. Introduction

1.1. GKM-sheaves

The first goal of this paper is to develop a general formalism for calculating the equivariant cohomology of torus actions. Let $T \cong U(1)^r$ be a compact torus of rank r and X a T-space. We call X *nice* if it admits the structure of a finite T-CW complex[1].

The equivariant cohomology[2] $H_T^*(X) = H_T^*(X;\mathbb{C})$ is a graded algebra over the equivariant cohomology of a point $H_T^*(pt) = H^*(BT)$. A T-space X is called *equivariantly formal* if $H_T^*(X)$ is a free $H^*(BT)$-module. In this

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[1] This class of T-spaces include compact, smooth T-manifolds and compact, real algebraic subsets of linear T representations [PS98].

[2] We use complex coefficients throughout.

case there is an isomorphism of graded $H^*(BT)$-modules,

$$H_T^*(X) \cong H^*(X) \otimes H^*(BT).$$

Many interesting examples of T-spaces are known to be equivariantly formal, including compact Hamiltonian T-manifolds and spaces whose ordinary cohomology is zero in odd degrees.

Consider the equivariant topological filtration

$$X^T = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$$

where X_j is the union of T-orbits of dimension less than or equal to j. The pair (X_1, X_0) is called the *one-skeleton* of X. If X is a nice, equivariantly formal T-space, the *Chang–Skjelbred Lemma* [CS74][3] (see also [FP07]) identifies $H_T^*(X)$ with the kernel of the coboundary map of the pair

(1) $$\delta : H_T^*(X_0) \to H_T^{*+1}(X_1, X_0).$$

An important advantage of the Chang–Skjelbred lemma over other methods of calculating equivariant cohomology (such as Morse theory) is that the isomorphism $H_T^*(X) \cong \ker(\delta)$ is an isomorphism of rings, not simply of graded vector spaces. It is thus desirable to have methods to calculate $\ker(\delta)$.

A class of T-spaces called *GKM manifolds*, introduced by Goresky et al. [GKM97] (see also [GZ01]), are particularly amenable to this calculation. A GKM-manifold is a compact Hamiltonian T-manifold X for which X_0 is a finite set, and X_1 is two dimensional. This forces X_1 to be a union of 2-spheres upon which T acts by rotation through characters $\alpha : T \to U(1)$. These spheres intersect at the their fixed points or "poles". The topology of (X_1, X_0) is neatly encoded in a GKM-graph or moment graph; this is the finite graph with vertices corresponding to points in X_0, edges corresponding to spheres in X_1, and edges labelled by characters of T. The equivariant cohomology $H_T^*(X)$ can be calculated entirely in terms of this GKM-graph using linear algebra.

In the current paper, we generalize this GKM procedure to all nice T-spaces X. In Section 2.2 we define a functor $X \mapsto \Gamma_X$ from the category of nice T-spaces to the category of *GKM-hypergraphs*, so that Γ_X encodes the combinatorics of the one-skeleton (X_1, X_0). In Section 2.3 we define the notion of a *GKM-sheaf* over a GKM-hypergraph, and associate to any nice

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[3]The conclusion of the Chang–Skjelbred Lemma holds under more general hypotheses, see Allday et al. [AFP11]

T-space X a GKM-sheaf $\mathcal{F}_X$ over Γ_X such that the algebra of global sections $H^0(\mathcal{F}_X)$ is naturally isomorphic to $\ker(\delta)$ from (1). If X is equivariantly formal, then

$$(2) \qquad\qquad H^0(\mathcal{F}_X) \cong H_T^*(X)$$

by the Chang–Skjelbred Lemma.

Using GKM-sheaves opens access to sheaf theoretic operations that have proven useful in calculations. We introduce several such operations in Section 2.4, including push-fowards, external tensor products, induction and convolution.

GKM-sheaves are related to and were inspired by the Γ-sheaves of Braden and MacPherson [BM01], which are used in the study of equivariant intersection cohomology. In Section 2.6 we show that every *pure* Γ-sheaf determines a GKM-sheaf in such a way that their modules of global sections are isomorphic. GKM-sheaves are also a useful framework for understanding constructions due to Guillemin and Zara [GZ03] and Guillemin and Holm [GH04], related to the topology of Hamiltonian actions, a point we address in Section 2.6.

1.2. Representation varieties

Let π be the fundamental group of a manifold M and let K be a compact, connected Lie group of rank r. The space of homomorphisms $\mathrm{Hom}(\pi, K)$ is called a *representation variety*. The group K acts by conjugation on $\mathrm{Hom}(\pi, K)$ producing a topological quotient stack which is naturally isomorphic to the moduli stack of flat K-bundles over M. The second half of this paper studies the equivariant cohomology ring $H_K^*(\mathrm{Hom}(\pi, K))$ when $M = \Sigma$ is a nonorientable 2-manifold (In fact, we work with a wider class of varieties associated to a punctured surface that includes to $\mathrm{Hom}(\pi, K)$ as a special case).

In [Bai10], the author derived formulas for the cohomology rings $H_{SU(2)}^*(\mathrm{Hom}(\pi, SU(2)))$, describing Poincaré polynomials and cup product structure (unless otherwise indicated, cohomology is singular with complex coefficients). This calculation was greatly facilitated by the following fact:

Theorem 1.1. *For $K = SU(2)$, the K-space $\mathrm{Hom}(\pi, K)$ is equivariantly formal.*

Recall that a nice K-space X is called equivariantly formal if $H_K^*(X)$ is free as a module over $H^*(BK)$ or equivalently, if $H_K^*(X) \cong H^*(X)$

$\otimes_{\mathbb{C}} H^*(BK)$ as graded vector spaces. The second objective of the current paper is to test whether this equivariant formality property generalizes to other Lie groups. We get a positive result for $K = SU(3)$.

Theorem 1.2. *Let Σ be the connected sum of $g + 1$ copies of $\mathbb{R}P^2$ and let $\pi = \pi_1(\Sigma)$. The $SU(3)$-space $\mathrm{Hom}(\pi, SU(3))$ is equivariantly formal, with equivariant Poincaré series*

$$(3) \qquad \frac{(1 + t^3 + t^5 + t^8)^g + (1 + t^2 + t^4)(t^3 + 2t^4 + t^5)^g}{(1 - t^2)(1 - t^4)(1 - t^6)}.$$

Moreover, we obtain an explicit description of the cup product structure on $H^*_{SU(3)}(\mathrm{Hom}(\pi, SU(3)))$. The formula (3) for the Poincaré series was conjectured by Ho and Liu [HL08] and later proven by the author [Bai09] using the Morse theory of the Yang–Mills functional, but this approach concealed the cup product structure. Because GKM-theory respects the cup product structure we are able to prove Theorem 1.2 by combining GKM methods with the previously obtained formula (3).

In an earlier draft of this paper we claimed to show that the conjecture fails for $K = SU(5)$ when Σ is a Klein bottle. This claim relied on computer calculations that are now in doubt, so the claim has been withdrawn. We are currently only able to prove that equivariant formality does not generally hold for the wider class of representation varieties $\mathcal{R}^g_K(c)$ defined in Section 3 (see Sections 4.6 and 4.7).

We briefly outline the strategy. Recall that if $T \subset K$ is a maximal torus and $W = N(T)/T$ the Weyl group, then for any T-space X, there is a canonical isomorphism

$$H_K(X) = H^*_T(X)^W,$$

so to a great extent the study of connected compact group actions reduces to torus actions.

Let $R := \mathrm{Hom}(\pi, K)$. Because T is maximal abelian in K it follows easily that

$$R_0 := \mathrm{Hom}(\pi, K)^T = \mathrm{Hom}(\pi, T).$$

The codimension one tori in T that occur as stabilizer groups of the torus action are precisely the root hypertori of T in K. Thus if we let Δ_+ denote the set of positive roots of K, we have

$$(4) \qquad R_1 = \bigcup_{\alpha \in \Delta_+} \mathrm{Hom}(\pi, K_\alpha),$$

where K_α is the centralizer of the kernel of α. The groups K_α have semisimple rank one, so admit a covering $U(1)^{r-1} \times SU(2) \to K_\alpha$, where r is the rank of K. The representation variety $\text{Hom}(\pi, K_\alpha)$ can then be described using the known properties of $\text{Hom}(\pi, SU(2))$ from [Bai10]. Since we have a good understanding of the one-skeleton (R_1, R_0), we are able to construct the GKM-sheaf $\mathcal{F}_R$. If $H_T^*(R)$ is equivariantly formal, then necessarily $H^0(\mathcal{F}_R)$ must be a free module, so computing $H^0(\mathcal{F}_R)$ provides a test of the equivariant formality of $H_T^*(R)$. In the case of $K = SU(3)$, this is combined with Morse theoretic information to compute $H_T^*(R)$ as a ring.

2. GKM theory

2.1. Equivariant cohomology

Fix a compact torus T of rank r with complexified Lie algebra $\mathfrak{t} = \text{Lie}(T) \otimes \mathbb{C}$ and dual $\mathfrak{t}^* = \text{Hom}(\mathfrak{t}, \mathbb{C})$. Let $A := \mathbb{C}[\mathfrak{t}] \cong S(\mathfrak{t}^*)$ denote the symmetric algebra of polynomial functions on $\mathfrak{t}$, graded so that $\mathfrak{t}^*$ has degree 2. As a graded ring, A is naturally isomorphic to $H^*(BT)$. We say that a T-space X is *nice* if it admits the structure of a finite $T - CW$ complex.

Let X be a nice T-space and let Y be a nice T-subspace. Let $\mathfrak{t}_x$ denote the complexified infinitesimal stabilizer of a point $x \in X$. The *Borel localization theorem* says that as a module over A, the support of $H_T^*(X, Y)$ satisfies

$$(5) \qquad \text{Supp}(H_T^*(X, Y)) \subseteq \bigcup_{x \in X \setminus Y} \mathfrak{t}_x.$$

Notice that by the finiteness condition on X, the right-hand side of (5) is a finite union of linear spaces so it is an algebraic set.

An important special case is $Y = X_0 = X^T$. Combining the localization theorem with the long exact sequence of the pair $i : X_0 \hookrightarrow X$, we conclude that the kernel and cokernel of the homomorphism

$$(6) \qquad i^* : H_T^*(X) \to H_T^*(X_0)$$

are torsion A-modules. If X is equivariantly formal, then i^* restricts to an isomorphism between $H_T^*(X)$ and the kernel of (1)

$$\delta : H_T^*(X_0) \to H_T^{*+1}(X_1, X_0).$$

Recall the filtration $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$ described in Section 1.1. This filtration is preserved by T, so it determines a spectral sequence

$(E_r^{p,q}, d_r)$ converging to $H_T^*(X)$ (see [FP07]). The first page of the spectral sequence satisfies

$$E_1^{p,q} \cong H_T^{p+q}(X_p, X_{p-1})$$

and $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ equals the coboundary operator of the triple (X_{p+1}, X_p, X_{p-1}). In particular, we identify $\ker(\delta)$ with the initial column $E_2^{0,*}$.

Proposition 2.1. *Let X be a nice T-space. There is a natural short exact sequence of graded A-algebras,*

$$0 \to \mathrm{Tor}_A(H_T^*(X)) \to H_T^*(X) \xrightarrow{\psi} \ker(\delta) \to \mathrm{cok}(\psi) \to 0,$$

where $\mathrm{cok}(\psi)$ is an A-module with support in codimension ≥ 2. If X is equivariantly formal, then ψ is an isomorphism.

Proof. First, note that $\ker(\delta) \subset H^*(X_0) = H^*(X_0) \otimes A$ is torsion free, so $\mathrm{Tor}_A(H_T^*(X)) \subseteq \ker(\psi)$. Next, observe that because the spectral sequence $E_r^{p,q}$ converges to $H_T^*(X)$, $\ker(\psi)$ admits a finite filtration whose associated graded object is isomorphic to a subquotient of the A-module $\oplus_{p=1}^r H_T^*(X_p, X_{p-1})$, which is a torsion module by the localization theorem (5). We deduce that $\ker(\psi)$ is torsion and thus $\ker(\psi) = \mathrm{Tor}_A(H_T^*(X))$.

Similarly, an associated graded version of $\mathrm{cok}(\psi)$ is equal to

$$\bigoplus_{p=2}^r \mathrm{im}(d_p : E_p^{0,*} \to E_p^{p,(*-p+1)})$$

which is a subquotient of $\bigoplus_{p=2}^r H_T^*(X_p, X_{p-1})$, so it has support in codimension two by the localization theorem.

If X is equivariantly formal, then the Chang–Skjelbred Lemma says that ψ is an isomorphism. $\qquad\square$

2.2. GKM-hypergraphs

Let

$$\Lambda := \mathrm{Hom}(T, U(1))$$

be the weight lattice of T, which we think of as embedded in $\mathfrak{t}^*$ in the usual way. Let $\mathbb{P}(\Lambda)$ denote the set of nonzero weights modulo scalar multiplication

or *projective weights*. The elements of $\mathbb{P}(\Lambda)$ are in one-to-one correspondence with the codimension one subtori of T by the rule

$$\alpha \in \mathbb{P}(\Lambda) \leftrightarrow \ker(\tilde{\alpha}) \subset T,$$

where $\tilde{\alpha} \in \Lambda$ is a primitive representative of α. We will generally be sloppy and write α in place of $\tilde{\alpha}$.

Definition 1. A **GKM-hypergraph** $\Gamma = (\mathcal{V}_\Gamma, \sim)$ consists of

(i) a finite set $\mathcal{V}_\Gamma$ called the *vertices* of Γ;

(ii) an equivalence relation $\sim_\alpha$ on $\mathcal{V}_\Gamma$ for each projective weight $\alpha \in \mathbb{P}(\Lambda)$.

We denote by $\mathrm{Part}_\Gamma(\alpha)$ the set of nonempty equivalence classes of $\sim_\alpha$. We frequently drop the subscript Γ and write $\mathcal{V}$ and $\mathrm{Part}(\alpha)$ when there is little risk of confusion.

A *morphism* of GKM-hypergraphs $\phi \in \mathrm{Hom}_{GKM}(\Gamma, \Gamma')$ is a map of sets

$$\phi : \mathcal{V}_\Gamma \to \mathcal{V}_{\Gamma'},$$

such that for $v, w \in \mathcal{V}_\Gamma$ and $\alpha \in \mathbb{P}(\Lambda)$, $v \sim_\alpha w$ implies $\phi(v) \sim_\alpha \phi(w)$.

The main motivating example of GKM-hypergraphs is the following.

Definition 2. To any nice T-space X, we associate the GKM-hypergraph Γ_X by

(i) $\mathcal{V}_X = \mathcal{V}_{\Gamma_X} := \pi_0(X^T)$, is the set of path components of the fixed point set X^T,

(ii) If $v_1, v_2 \in \mathcal{V}_X$ then $v_1 \sim_\alpha v_2$ if and only if corresponding path components lie in the same path component of the fixed point set $X^{\ker(\alpha)}$.

The function $X \mapsto \Gamma_X$ extends to a functor from the category of nice T-spaces and equivariant maps to the category of GKM-hypergraphs.

Let $\wp(\mathcal{V})$ denote the power set of $\mathcal{V}$. We consider $\mathrm{Part}(\alpha)$ to be a subset of $\wp(\mathcal{V})$ (by convention, the empty set $\emptyset \notin \mathrm{Part}(\alpha)$). To a GKM-hypergraph Γ define the set of *hyperedges* $\mathcal{E}_\Gamma$ by

$$\mathcal{E} = \mathcal{E}_\Gamma := \{(S, \alpha) \in \wp(\mathcal{V}) \times \mathbb{P}(\Lambda) \mid S \in \mathrm{Part}(\alpha)\}.$$

Projection defines maps

- $\alpha : \mathcal{E} \to \mathbb{P}(\Lambda)$, the *axial function*, and
- $I : \mathcal{E} \to \wp(\mathcal{V})$ the *incidence function*.

We say a hyperedge $e \in \mathcal{E}$ is *incident* to a vertex $v \in \mathcal{V}$ if $v \in I(e)$. Every hyperedge is incident to a least one vertex. A hyperedge e is called *degenerate* if it is incident to only one vertex. We say that a GKM-hypergraph is *discrete* if all edges are degenerate. We use the term *GKM-graph* to mean a GKM-hypergraph for which no hyperedge is incident to more than two vertices.

Remark 1. Observe that a GKM-hypergraph can be reconstructed from $\mathcal{V}$, $\mathcal{E}$ and the incidence and axial functions. We will sometimes construct a GKM-hypergraph by specifying this data.

Given $\alpha \in \mathbb{P}(\Lambda)$, denote $\mathcal{E}^{\alpha} := \{e \in \mathcal{E} | \alpha(e) = \alpha\}$. This forms a partition

$$(7) \qquad \mathcal{E} = \coprod_{\alpha \in \mathbb{P}(\Lambda)} \mathcal{E}^{\alpha}$$

and the elements in $\mathcal{E}^{\alpha}$ correspond to equivalence classes of $\sim_{\alpha}$.

Proposition 2.2. *Let X be a nice T-space. There is a natural injection*

$$\mathcal{E}_X^{\alpha} \hookrightarrow \pi_0(X^{\ker(\alpha)}),$$

whose image is those path components of $X^{\ker(\alpha)}$ containing T-fixed points.

Proof. This is clear from the definitions. □

Remark 2. When considering the GKM-hypergraph Γ_X associated to a nice T-space X, we use lower case letters v and e to denote vertices and hyperedges, and upper case letters V and E to denote the corresponding path components of X^T and $X^{\ker(\alpha(e))}$ respectively.

2.2.1. The topology of a GKM-hypergraph. Given a GKM-hypergraph Γ, we define $\mathrm{Top}(\Gamma)$ to be the a topological space with underlying set $\mathcal{V}_{\Gamma} \cup \mathcal{E}_{\Gamma}$ and with *basic open sets*

- $U_v := \{v\}$ for $v \in \mathcal{V}_{\Gamma}$ and
- $U_e := \{e\} \cup I(e)$ for $e \in \mathcal{E}_{\Gamma}$.

Somewhat counterintuitively, vertices are open points and hyperedges are closed points in this topology. We now prove that Top extends to a functor from the category of GKM-hypergraphs to the category of topological spaces.

Proposition 2.3. *Any GKM-morphism, $\phi : \Gamma \to \Gamma'$ induces a map of sets $\varphi : \mathcal{E}_\Gamma \to \mathcal{E}_{\Gamma'}$ preserving the axial function and commuting with the incidence map.*

Proof. Let $e = (I(e), \alpha(e)) \in \mathcal{E}_\Gamma$ be a hyperedge of Γ. By the definition of GKM-morphism, there exists a unique $e' \in \mathcal{E}_{\Gamma'}$ such that $\alpha(e) = \alpha(e')$ and $I(e) \subseteq I(e')$. Defining $\varphi(e) = e'$ completes the proof. $\square$

Abusing notation, we denote by

$$\phi : \mathcal{V}_\Gamma \cup \mathcal{E}_\Gamma \to \mathcal{V}_{\Gamma'} \cup \mathcal{E}_{\Gamma'}$$

the map of sets restricting to $\phi : \mathcal{V}_\Gamma \to \mathcal{V}_{\Gamma'}$ and $\varphi : \mathcal{E}_\Gamma \to \mathcal{E}_{\Gamma'}$.

Proposition 2.4. *Let $\phi : \Gamma \to \Gamma'$ be a morphism of GKM-hypergraphs. The preimage of any basic open set $U_x \subseteq \mathrm{Top}(\Gamma')$ is a disconnected union of basic open sets:*

$$(8) \qquad\qquad \phi^{-1}(U_x) = \coprod_{y \in \phi^{-1}(x)} U_y.$$

In particular, $\phi : \mathrm{Top}(\Gamma) \to \mathrm{Top}(\Gamma')$ is a continuous map and Top defines a functor from the category of GKM-hypergraphs to the category of topological spaces.

Proof. Clearly (8) holds for singleton sets $U_v = \{v\}$ centered on vertices. For a hyperedge $e' \in \mathcal{E}_{\Gamma'}$, we have $U_{e'} = e' \cup I(e')$, so we must show that if $\phi(v) \in I(e')$ then there exists $e \in \mathcal{E}_\Gamma$ such that $v \in I(e)$ and $\phi(e) = e'$.

Suppose that $\phi(v) \in I(e')$. By the partition condition on GKM-hypergraphs, there is a unique $e \in \mathcal{E}$ such that $\alpha(e) = \alpha(e')$ and $v \in I(e)$. Thus $\phi(v) \in \phi(I(e)) \subseteq I(\phi(e))$ and we deduce that $\phi(e) = e'$. Functoriality is clear. $\square$

2.3. GKM-sheaves

We begin with an abstract definition.

Definition 3. Let Γ be a GKM-hypergraph. A **GKM-sheaf** over Γ is a sheaf $\mathcal{F}$ of finitely generated, $\mathbb{Z}$-graded A-modules over $\mathrm{Top}(\Gamma)$ satisfying the following three conditions.

1) $\mathcal{F}$ is locally free (that is, for every basic open set $U_x \in \mathrm{Top}(\Gamma)$, the stalk $\mathcal{F}(U_x)$ is a free A-module).

2) For all $e \in \mathcal{E}_\Gamma$, the restriction map $\mathrm{res}_e : \mathcal{F}(U_e) \to \mathcal{F}(I(e))$ becomes an isomorphism upon inverting $\alpha(e)$:

$$(9) \qquad \mathcal{F}(U_e) \otimes_A A[\alpha(e)^{-1}] \cong \mathcal{F}(I(e)) \otimes_A A[\alpha(e)^{-1}].$$

3) $\mathrm{res}_e : \mathcal{F}(U_e) \to \mathcal{F}(I(e))$ is an isomorphism for all but a finite number of $e \in \mathcal{E}_\Gamma$.

We denote by $GKM(\Gamma)$ the full subcategory of the category of sheaves of graded A-modules on $\mathrm{Top}(\Gamma)$, whose objects are GKM-sheaves.

A nice T-space X determines a sheaf of graded A-algebras on $\mathrm{Top}(\Gamma_X)$, denoted $\mathcal{F}_X$, with stalks

$$\mathcal{F}_X(U_V) = \mathcal{F}_X(V) = H_T^*(V)$$

at vertices $V \subseteq X^T$ and

$$\mathcal{F}_X(U_E) = \mathcal{F}_X(E \cup I(E)) = H_T^*(E)/\mathrm{Tor}_A(H_T^*(E))$$

at edges $E \subseteq X^{\ker(\alpha(E))}$, where $\mathrm{Tor}_A(M) := \{m \in M | am = 0 \text{ for some } a \in A \setminus \{0\}\}$ is the torsion submodule of the A-module M. If $i : V \hookrightarrow E$ is a subset inclusion then the restriction map $\mathcal{F}_X(U_E) \to \mathcal{F}_X(U_V)$ is identified with the cohomology morphism $i^* : H_T^*(E) \to H_T^*(V)$. This is well defined because $H_T^*(V)$ is torsion-free so $\mathrm{Tor}_A(H_T^*(E)) \subseteq \ker(i^*)$.

Proposition 2.5. *If X is a nice T-space then $\mathcal{F}_X$ is a GKM-sheaf.*

Proof. For any $E \in \mathcal{E}$, the restriction map $\mathrm{res}_E : \mathcal{F}_X(U_E) \to \mathcal{F}(I(E))$ is identified with the map $H_T^*(E)/\mathrm{Tor}_A(H_T^*(E)) \to H_T^*(E^T)$ induced by the inclusion $E^T \subset E$. Since the only isotropy algebra for $E \setminus E^T$ is $(\alpha_E)^\perp \subset \mathfrak{t}$, the Borel localization theorem (5) tells us that the kernel and cokernel of res_E are $\alpha(E)$-torsion. The finiteness condition follows easily from compactness of X. Local freeness of $\mathcal{F}_X$ is an immediate consequence of Lemma 2.6. $\square$

Lemma 2.6. *If X is a nice T-space and $H \subset T$ is a codimension one subtorus, then $H_T^*(X^H)$ is the direct sum of a free and a torsion A-module. If $H_T^*(X)$ is torsion free, then $H_T^*(X^H)$ is free.*

Proof. Let $\alpha \in \Lambda \subset \mathfrak{t}^*$ be the character for which H is the kernel. Because H acts trivially on X^H, $H_T^*(X^H) \cong H_{T/H}^*(X^H) \otimes_{\mathbb{C}[\alpha]} A$. Since $\mathbb{C}[\alpha]$ is a PID, the fundamental theorem of finitely generated modules over a PID implies that $H_{T/H}^*(X^H)$ (hence also $H_T^*(X^H)$) is isomorphic to the sum of a free module and a torsion module, proving the first statement.

Furthermore, because $H_{T/H}^*(X^H)$ is a graded $\mathbb{C}[\alpha]$-module, its torsion submodule decomposes into a direct sum of modules of the form

$$\mathbb{C}[\alpha]/\alpha^n \mathbb{C}[\alpha]$$

for some positive integer n (up to degree shifts). Thus $H_T^*(X^H)$ is free if and only if it does not contain a summand isomorphic to $A/\alpha^n A$.

Assume that $H_T^*(X)$ is torsion free over A. Localizing at the prime ideal $(\alpha) \subset A$ we have

$$H_T^*(X)_{(\alpha)} \cong H_T^*(X^H)_{(\alpha)}$$

by the localization theorem. Localization is an exact functor so it preserves torsion-freeness and we infer that $H_T^*(X^H)_{(\alpha)}$ is torsion free over $A_{(\alpha)}$. It follows that $H_T^*(X^H)$ cannot contain a summand of the form $A/\alpha^n A$, so we conclude that $H_T^*(X^H)$ is free over A. $\square$

Notice that $\mathcal{F}_X$ is in fact a sheaf of A-algebras, so the set of global sections $H^0(\mathcal{F}_X)$ is an A-algebra. Recall from Section 1.1 the one-skeleton (X_1, X_0) of a T-space X.

Proposition 2.7. *Let X be a nice T-space. The space of global sections $H^0(\mathcal{F}_X)$ fits into an exact sequence of graded A-modules*

$$0 \to H^0(\mathcal{F}_X) \xrightarrow{r} H_T^*(X_0) \xrightarrow{\delta} H_T^*(X_1, X_0)[+1]$$

for which r is a homomomorphism of A-algebras.

Proof. Let $\Gamma_X = (\mathcal{V}, \mathcal{E}, I, \alpha)$. The map r is identified with the sheaf restriction map $H^0(\mathcal{F}_X) \to \mathcal{F}_X(\mathcal{V}) \cong H_T^*(X_0)$, so r is certainly a homomorphism of algebras.

Since $\mathcal{F}_X$ is locally free and the restriction maps $\text{res}_E : \mathcal{F}_X(U_E) \to \mathcal{F}_X(I(E))$ are isomorphisms modulo torsion, they must be injective. Thus any element of $\mathcal{F}_X(\mathcal{V})$ extends in at most one way to each hyperedge and we deduce that r is injective.

It remains to show that $\operatorname{im}(r) = \ker(\delta)$. Decomposing cohomology into connected components we have an isomorphism

$$H_T^*(X_0) \cong \bigoplus_{V \in \mathcal{V}} H_T^*(V)$$

and

$$H_T^*(X_1, X_0) \cong \Big(\bigoplus_{E \in \mathcal{E}} H_T^*(E, E^T) \Big) \oplus H_T^*(X_1'),$$

where X_1' is the union of components of X_1 that do not intersect X_0. Clearly the projection of δ onto $H_T^*(X_1')$ is zero, so $\ker(\delta) \cong \ker(d)$, where d is the block decomposition

$$(10) \qquad d : \bigoplus_{V \in \mathcal{V}} H_T^*(V) \to \bigoplus_{E \in \mathcal{E}} H_T^{*+1}(E, E^T)$$

with matrix blocks $d_{E,V}$, such that $d_{E,V} = 0$ when $V \not\subseteq E$, and when $V \subseteq E$ the diagram

$$(11)$$

$$H_T^*(V) \xrightarrow{\quad d_{E,V} \quad} H_T^{*+1}(E, E^T)$$

$$\iota \searrow \qquad \nearrow \delta_E$$

$$H_T^*(E^T)$$

commutes, where δ_E is the boundary map of the pair (E, E^T) and ι is inclusion as a summand. Thus, $(a_V) \in \bigoplus_{V \in \mathcal{V}} H_T^*(V)$ lies in $\ker(d)$ if and only if for all $E \in \mathcal{E}$,

$$(12) \qquad 0 = \sum_{V \in \mathcal{V}} d_{E,V}(a_V) = \sum_{V \subseteq E} d_{E,V}(a_V) = \delta_E \Big(\sum_{V \subseteq E} \iota(a_V) \Big).$$

By the long exact sequence of the pair (E, E^T), (12) holds if and only if $\sum_{V \subseteq E^T} \iota(a_V) \in H_T^*(E^T)$ lies in the image of $H_T^*(E) \to H_T^*(E^T)$ and this is equal to the image of $\operatorname{res}_E : \mathcal{F}_X(U_E) \to \mathcal{F}(I(E))$. Thus $\ker(d)$ corresponds to those sections in $\mathcal{F}(\mathcal{V})$ that can extend to every hyperedge and we conclude that $\operatorname{im}(r) = \ker(d) = \ker(\delta)$. $\qquad\qquad \square$

Theorem 2.8. *Let X be a nice T-space. There is a natural short exact sequence of graded A-algebras,*

$$0 \to \operatorname{Tor}_A(H_T^*(X)) \to H_T^*(X) \xrightarrow{\psi} H^0(\mathcal{F}_X) \to \operatorname{cok}(\psi) \to 0,$$

where $\operatorname{cok}(\psi)$ is an A-module with support in codimension ≥ 2. If X is equivariantly formal, then ψ is an isomorphism

$$H_T^*(X) \overset{\psi}{\cong} H^0(\mathcal{F}_X).$$

Proof. This follows immediately from Propositions 2.1 and 2.7. □

The following examples are equivariantly formal.

Example 1. Let S be a finite discrete set on which T acts trivially. Then $\mathcal{F}_S$ is simply the constant sheaf $A_{\operatorname{Top}(\Gamma_S)}$. We abuse notation and denote this special case A_S.

Example 2. The GKM-sheaf of a GKM-manifold has stalks

$$\mathcal{F}_X(U_x) = H_T^*(pt) = A$$

when x is a vertex or degenerate edge, and $\mathcal{F}_X(U_e) \cong H_T^*(S^2) \cong A \oplus A[2]$ as an A-module when e is a nondegenerate edge (we use the grading shift convention $M[d]^* \cong M^{*-d}$). The restriction map at a nondegenerate edge is the map from $\mathcal{F}_X(U_e) \cong A \oplus A[2]$ to $\mathcal{F}_X(I(e)) \cong A \oplus A$ defined by the matrix

$$\begin{pmatrix} 1 & \alpha(e) \\ 1 & -\alpha(e) \end{pmatrix}.$$

Example 3. Consider a compact, connected Lie group K with maximal torus T acting on K by conjugation, and let $\Delta \subset \mathbb{P}(\Lambda)$ be the set of roots of K, modulo scalar multiplication. The GKM-hypergraph Γ_K is the unique GKM-hypergraph with one vertex v. The GKM-sheaf $\mathcal{F}_K$ has stalk

$$\mathcal{F}_K(U_v) = H_T^*(T) \cong \wedge \mathfrak{t}^* \otimes_{\mathbb{C}} A$$

at v. For every edge e with $\alpha(e) \notin \Delta$, we have $\mathcal{F}_K(U_e) = \mathcal{F}_K(U_v) = \wedge \mathfrak{t}^* \otimes A$ with restriction map the identity. For every edge e with $\alpha(e) \in \Delta$, let $H_{\alpha(e)} \subset \mathfrak{t}^*$ be the hyperplane fixed by the Weyl reflection $S_{\alpha(e)}$. Then $\mathcal{F}_K(U_e)$ is the A-submodule of $\mathfrak{t}^* \otimes_{\mathbb{C}} A$ generated by $(\wedge^* H_{\alpha(e)}) \otimes 1$ and $\alpha(e) \wedge (\wedge^* H_{\alpha(e)}) \otimes \alpha(e)$ with sheaf restriction map the inclusion.

There is another useful description of the global sections of a GKM-sheaf. For each $\alpha \in \mathbb{P}(\Lambda)$, denote be i_α^* the restriction map from $\mathcal{F}(\mathcal{V} \cup \mathcal{E}^\alpha)$ to $\mathcal{F}(\mathcal{V})$. Note that by the finiteness condition in Definition 3, i_α^* is an isomorphism for all but a finite set of $\alpha \in \mathbb{P}(\Lambda)$.

Proposition 2.9. *The restriction map* $i^* : H^0(\mathcal{F}) \to \mathcal{F}(\mathcal{V})$ *is injective, with image*

$$(13) \qquad \operatorname{im}(i^*) = \bigcap_{\alpha \in \mathbb{P}(\Lambda)} \operatorname{im}(i_\alpha^*).$$

Proof. For each hyperedge e the restriction morphism $\mathcal{F}(U_e) \to \mathcal{F}(I(e))$ is injective, because $\mathcal{F}(U_e)$ is a free A-module and the kernel must be torsion free. This implies that sections in $\mathcal{F}(\mathcal{V})$ can extend in at most one way to $\mathcal{F}(\mathcal{V} \cup \mathcal{E}) = H^0(\mathcal{F})$, which is equivalent to i^* being injective. By the proof of Proposition 2.7, this means both sides of (13) are equal the set of sections in $\mathcal{F}(\mathcal{V})$ that extend to global sections. $\qquad \square$

2.4. Operations on GKM sheaves

2.4.1. Pushforwards. Given a continuous map between topological spaces $f : X \to Y$ and a sheaf $\mathcal{F}$ on X, that the *pushforward sheaf* $f_*(\mathcal{F})$ is defined by the rule $f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ for open sets $U \subseteq Y$.

Proposition 2.10. *Let* $f : \Gamma \to \tilde{\Gamma}$ *be a morphism of GKM-hypergraphs and* $\mathcal{F}$ *a GKM-sheaf over* Γ. *The pushforward sheaf,* $f_*(\mathcal{F})$, *is a GKM-sheaf over* $\tilde{\Gamma}$ *satisfying*

$$(14) \qquad H^0(\mathcal{F}) = H^0(f_*(\mathcal{F})).$$

Proof. For any $y \in \operatorname{Top}(\tilde{\Gamma})$ we have by Proposition 2.4 that

$$f_*(\mathcal{F})(U_y) = \bigoplus_{x \in f^{-1}(y)} \mathcal{F}(U_x),$$

which is a direct sum of free modules, hence free. By the same proposition, for a hyperedge $\tilde{e} \in \mathcal{E}_{\tilde{\Gamma}}$, the restriction map $\operatorname{res}_{\tilde{e}} : f_*(\mathcal{F})(U_{\tilde{e}}) \to f_*(\mathcal{F})(I(\tilde{e}))$

is identified with the direct sum of maps

$$\bigoplus_{e \in f^{-1}(\tilde{e})} \Big(\mathrm{res}_e : \mathcal{F}(U_e) \to \mathcal{F}(I(e)) \Big),$$

which is an isomorphism modulo $\alpha(e)$ and is an isomorphism for all but finitely many $\tilde{e} \in \mathcal{E}_{\tilde{\Gamma}}$. Equation (14) holds simply by the definition of the pushforward sheaf. $\square$

Proposition 2.11. *For any GKM-hypergraph morphism $\phi : \Gamma_1 \to \Gamma_2$ the pushforward ϕ_* of sheaves defines a functor $\phi_* : GKM(\Gamma_1) \mapsto GKM(\Gamma_2)$.*

Proof. This follows from the fact that push-forward is a functor for sheaves of A-modules. $\square$

Proposition 2.12. *Let $\Phi : X \to Y$ be a T-equivariant map between nice T-spaces and let $\phi : \Gamma_X \to \Gamma_Y$ be the induced morphism of GKM-hypergraphs. There is a natural morphism $h : \mathcal{F}_Y \to \phi_*(\mathcal{F}_X)$ for which the following diagram commutes:*

(15)
$$\begin{array}{ccc} H_T^*(Y) & \xrightarrow{\quad \Phi^* \quad} & H_T^*(X) \\ \downarrow & & \downarrow \\ H^0(\mathcal{F}_Y) & \xrightarrow{\quad H^0(h) \quad} & H^0(\phi_*(\mathcal{F}_X)) = H^0(\mathcal{F}_X) \end{array}$$

where the vertical arrows come from Theorem 2.8.

Proof. A T-equivariant map $\Phi : X \to Y$ restricts to a map of the pairs $(X_1, X_0) \to (Y_1, Y_0)$. By Proposition 2.7, there is an induced map $l : H^0(\mathcal{F}_Y) \to H^0(\mathcal{F}_X)$ which fits into the commutative diagram in place of $H^0(h)$. It remains to find a GKM-sheaf morphism $h : \mathcal{F}_Y \to \phi_*(\mathcal{F}_X)$ such that $H^0(h) = l$.

If $\tilde{V}$ is a connected component of $Y_0 = Y^T$ representing a vertex of Γ_Y, then at the level of stalks we have

$$\mathcal{F}_Y(U_{\tilde{V}}) = H_T^*(\tilde{V}) \to \bigoplus_{V \in \phi^{-1}(\tilde{V})} H_T^*(V) = \phi_*(\mathcal{F}_X)(U_{\tilde{V}})$$

is the direct sum of maps $H_T^*(\tilde{V}) \to H_T^*(V)$ induced by the restriction of Φ. The morphism is defined at the stalk of hyperedges similarly, but using components of $Y^{\ker(\alpha)}$ and $X^{\ker(\alpha)}$. This sheaf morphism is natural with

respect to the block decomposition from the proof of Proposition 2.7, so $H^0(h) = l$. $\qquad\square$

Remark 3. It is worth pointing out that the pull-back of a GKM-sheaf under GKM-morphism is *not* necessarily a GKM-sheaf. Consider for example, the case $T = S^1$, Γ is the GKM-graph with two vertices v_1 and v_2 and one edge e, Γ' is the GKM-graph with one vertex, and $\phi : \Gamma \to \Gamma'$ is the unique morphism. Then the constant sheaf $A_{\Gamma'}$ is GKM over Γ', but the pull-back $\mathcal{F} := \phi^*(A_{\Gamma'})$ is not because $\mathcal{F}(U_e) = \mathcal{F}(\Gamma) = A$ and $\mathcal{F}(I(e)) = \mathcal{F}(v_1) \oplus \mathcal{F}(v_2) = A \oplus A$ have different ranks.

2.4.2. Group actions. Let G be a finite group acting by automorphisms on a GKM-hypergraph Γ. Define the quotient GKM-hypergraph Γ/G with vertex set $\mathcal{V}_{\Gamma/G} = \mathcal{V}_\Gamma/G$ and equivalence relations $[v] \sim_\alpha [w]$ if and only if $gv \sim_\alpha w$ for some $g \in G$. The quotient map $\mathcal{V}_\Gamma \mapsto \mathcal{V}_\Gamma/G$ determines a GKM-morphism

$$\pi : \Gamma \mapsto \Gamma/G.$$

Denote by $GKM_G(\Gamma)$ the category of G-equivariant GKM-sheaves. For any $\mathcal{F} \in GKM_G(\Gamma)$ the pushforward $\pi_*(\mathcal{F})$ is a sheaf of modules over the group ring AG, so we may decompose into $\mathbb{C}G$-isotypical components

$$\pi_*(\mathcal{F}) \cong \bigoplus_{\chi \in \hat{G}} \pi_*(\mathcal{F})^\chi,$$

indexed by the set $\hat{G}$ of irreducible, complex G-representations, where for an open set $U \subset \mathrm{Top}(\Gamma/G)$

$$\pi_*(\mathcal{F})^\chi(U) := (\pi_*(\mathcal{F})(U))^\chi \cong \chi \otimes_\mathbb{C} \mathrm{Hom}_{\mathbb{C}G}(\chi, \pi_*(\mathcal{F})(U)).$$

Lemma 2.13. *For each $\chi \in \hat{G}$, the isotypical component $\pi_*(\mathcal{F})^\chi$ is a GKM-sheaf over Γ/G and there is a natural isomorphism*

$$H^0(\mathcal{F})^\chi \cong H^0(\pi_*(\mathcal{F})^\chi).$$

Proof. Everything follows easily from the fact that summands of free A-modules are free A-modules and that G-equivariant maps respect isotypical components. $\qquad\square$

An important special case of Lemma 2.13 is the G-invariant subsheaf which we denote $\pi_*(\mathcal{F})^G$.

Lemma 2.14. *Let G be a finite group and let X be a nice $G \times T$-space so that X/G is a nice T-space. Then Γ_X inherits a G-action and there is a natural isomorphism of GKM-sheaves*

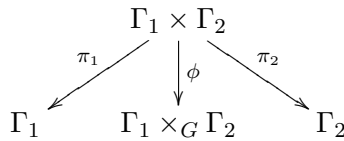$$\mathcal{F}_{X/G} \cong \pi_*(\mathcal{F}_X)^G.$$

Proof. This follows pretty directly from the natural isomorphisms $H_T^*((X/G)^T) \cong H_T^*((X^T)/G) \cong H_T^*(X^T)^G$ and similarly for codimension one subtori $H \subset T$. $\square$

2.4.3. Tensor products.

Definition 4. For $i = 1, 2$, let $\Gamma_i = (\mathcal{V}_i, \sim)$ be GKM-hypergraphs. Define the **product GKM-hypergraph** $\Gamma_1 \times \Gamma_2$ with vertices $\mathcal{V}_1 \times \mathcal{V}_2$ and relations $(v_1, v_2) \sim_\alpha (w_1, w_2)$ if and only if $v_1 \sim_\alpha w_1$ and $v_2 \sim_\alpha w_2$.

The projection maps $\pi_i : \Gamma_1 \times \Gamma_2 \to \Gamma_i$ are GKM-morphisms making $\Gamma_1 \times \Gamma_2$ a product object in the category of GKM-hypergraphs.

Suppose now that Γ_1 and Γ_2 admit a G-actions. Then we have a diagram of GKM-hypergraphs

$$
\begin{array}{ccc}
 & \Gamma_1 \times \Gamma_2 & \\
\overset{\pi_1}{\swarrow} \quad \downarrow{\scriptstyle\phi} & & \overset{\pi_2}{\searrow} \\
\Gamma_1 \qquad \Gamma_1 \times_G \Gamma_2 & & \Gamma_2
\end{array}
$$

where ϕ is the quotient map for the diagonal G-action on $\Gamma_1 \times \Gamma_2$.

Proposition 2.15. *Let $\mathcal{F}_i$ be a GKM-sheaf on Γ_i for $i = 1, 2$. The **external tensor product***

$$\mathcal{F}_1 \boxtimes_G \mathcal{F}_2 := \phi_*(\pi_1^*(\mathcal{F}_1) \otimes_A \pi_2^*(\mathcal{F}_2))^G$$

is a GKM sheaf on $\Gamma_1 \times_G \Gamma_2$. This construction determines a bifunctor

$$GKM_G(\Gamma_1) \times GKM_G(\Gamma_2) \to GKM(\Gamma_1 \times_G \Gamma_2).$$

Proof. This bifunctor factors into $GKM_G(\Gamma_1) \times GKM_G(\Gamma_2) \mapsto GKM_G(\Gamma_1 \times \Gamma_2)$ and the G-invariant pushforward $GKM_G(\Gamma_1 \times \Gamma_2) \mapsto GKM(\Gamma_1 \times_G \Gamma_2)$, so by Lemma 2.13 it suffices to consider the case that G is trivial.

The finiteness and locally free conditions clearly hold. It remains to prove that for all hyperedges e,

$$\mathrm{res}_e : (\mathcal{F}_1 \boxtimes \mathcal{F}_2)(U_e) \to (\mathcal{F}_1 \boxtimes \mathcal{F}_2)(I(e))$$

is an isomorphism modulo $\alpha(e)$. It is an easy check that $e = (I(e_1) \times I(e_2), \alpha(e))$ for some hyperedges $e_1 \in \mathcal{E}_{\Gamma_1}$ and $e_2 \in \mathcal{E}_{\Gamma_2}$ with $\alpha(e) = \alpha(e_1) = \alpha(e_2)$. By definition, res_e is identified with

$$\mathrm{res}_{e_1} \otimes \mathrm{res}_{e_2} : \mathcal{F}_1(U_{e_1}) \otimes \mathcal{F}_2(U_{e_2}) \to \mathcal{F}_1(I(e_1)) \otimes \mathcal{F}_2(I(e_2))$$

and res_{e_i} is an isomorphism after inverting $\alpha(e) = \alpha(e_i)$ for $i = 1, 2$, so res_e is also an isomorphism after inverting $\alpha(e)$. $\qquad\square$

Proposition 2.16. *The external tensor product is associative. That is, if* $\mathcal{F}_i \in GKM_{G_i \times G_{i+1}}(\Gamma_i)$ *for* $i = 1, 2, 3$ *then there is a natural isomorphism*

$$(\mathcal{F}_1 \boxtimes_{G_2} \mathcal{F}_2) \boxtimes_{G_3} \mathcal{F}_3 \cong \mathcal{F}_1 \boxtimes_{G_2} (\mathcal{F}_2 \boxtimes_{G_3} \mathcal{F}_3)$$

in $GKM_{G_1 \times G_4}(\Gamma_1 \times_{G_2} \Gamma_2 \times_{G_3} \Gamma_3)$.

Proof. Both sides are identified with the $G_2 \times G_3$-invariant pushforward of the sheaf

$$\pi_1^* \mathcal{F}_1 \otimes \pi_2^* \mathcal{F}_2 \otimes \pi_3^* \mathcal{F}_3,$$

where $\pi_i : \Gamma_1 \times \Gamma_2 \times \Gamma_3 \to \Gamma_i$ are the projection maps. $\qquad\square$

Proposition 2.17. *Suppose that X and Y are nice $G \times T$-spaces for G a finite group, and suppose that both $H_T^*(X^{\ker(\alpha)})$ and $H_T^*(Y^{\ker(\alpha)})$ are free A-modules for all $\alpha \in \mathbb{P}(\Lambda)$ (this holds if $H_T^*(X)$ are $H_T^*(Y)$ are torsion free by Lemma 2.6). Then*

$$\mathcal{F}_X \boxtimes_G \mathcal{F}_Y \cong \mathcal{F}_{X \times_G Y}$$

for the diagonal T-action on $X \times_G Y$.

Proof. By Lemma 2.14, it suffices to consider the case that G is trivial.

It is evident that $(X \times Y)^T = X^T \times Y^T$ and that $X^{\ker(\alpha)} \times Y^{\ker(\alpha)} = (X \times Y)^{\ker(\alpha)}$ for all $\alpha \in \mathbb{P}(\Lambda)$, so the corresponding equalities on connected components produces an isomorphism of GKM graphs $\Gamma_X \times \Gamma_Y \cong \Gamma_{X \times Y}$.

The isomorphism of sheaves is defined through the Kunneth morphisms

$$H_T^*(X^T) \otimes_A H_T^*(Y^T) \to H_T^*(X^T \times Y^T),$$
$$H_T^*(X^{\ker(\alpha)}) \otimes_A H_T^*(Y^{\ker(\alpha)}) \to H_T^*((X \times Y)^{\ker(\alpha)}),$$

which are isomorphisms because the modules are all free over A. $\qquad\square$

Proposition 2.18. *Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are G-equivariant GKM-sheaves. Then the kernel and cokernel of the natural map*

(16) $$\phi : H^0(\mathcal{F}_1) \otimes_{AG} H^0(\mathcal{F}_2) \to H^0(\mathcal{F}_1 \boxtimes_G \mathcal{F}_2)$$

are torsion with support in codimension greater than one. If additionally $H^0(\mathcal{F}_k)$ is free over A for $k = 1, 2$, then ϕ is an isomorphism.

Proof. Because $H^0(\mathcal{F}_1) \otimes_{AG} H^0(\mathcal{F}_2) = (H^0(\mathcal{F}_1) \otimes_A H^0(\mathcal{F}_2))^G$ and $H^0(\mathcal{F}_1 \boxtimes_G \mathcal{F}_2) = H^0(\mathcal{F}_1 \boxtimes \mathcal{F}_2)^G$, it suffices to consider the case that G is trivial.

We use Proposition 2.9 to describe the global sections functor. Let

$$i_{k,\alpha}^* : \mathcal{F}_k(\mathcal{V}_k \cup \mathcal{E}_k^\alpha) \to \mathcal{F}_k(\mathcal{V}_k)$$

denote the restriction map and let $\Delta \subset \mathbb{P}(\Lambda)$ be the finite set of α for which $i_{k,\alpha}^*$ is not an isomorphism for either $k = 1$ or 2. The map (16) is equivalent to the functorial map

$$\psi : \left(\bigcap_{\alpha \in \Delta} \mathrm{im}(i_{1,\alpha}^*) \right) \otimes \left(\bigcap_{\alpha \in \Delta} \mathrm{im}(i_{2,\alpha}^*) \right) \to \bigcap_{\alpha \in \Delta} \mathrm{im}(i_{1,\alpha}^* \otimes i_{2,\alpha}^*).$$

Our hypotheses imply that i_α^* is an injective map between free A-modules and that i_α^* becomes an isomorphism after localizing to the hyperplane $(\alpha)^\perp \subset \mathfrak{t}$. If Δ has cardinality one, then ψ is an isomorphism because everything is free. Since localization commutes with tensor products and finite intersections ([Eis95], Section 2.2), we deduce that if $x \in \mathfrak{t}$ is annihilated by no more than one element of Δ, then the localization of ψ at x is an isomorphism. Consequently, the support of both $\ker(\psi)$ and $\mathrm{cok}(\psi)$ must lie in the union of codimension two planes $(\alpha_0)^\perp \cap (\alpha_1)^\perp$ where α_0 and α_1 vary over distinct elements of Δ.

Now suppose that $H^0(\mathcal{F}_k)$ is a free A-module for $k = 1, 2$ and let M and N denote the source and target of ψ respectively. Then M is free, so $M \cong A^d$ for some non-negative intger d (ignoring the grading). Because

$\ker(\psi)$ is torsion submodule of M, it must be zero and we have a short exact sequence

$$(17) \qquad 0 \to M \xrightarrow{\psi} N \to \operatorname{cok}(\psi) \to 0.$$

Now suppose for the sake of contradiction that $\operatorname{cok}(\psi)$ is nonzero. By the theory of associated primes ([Eis95], Section 3.1) there exists some nonzero $y \in \operatorname{cok}(\psi)$ for which the annihilator $\operatorname{ann}(y) = \{a \in A | ay = 0\}$ is a prime ideal. By the result on the support of $\operatorname{cok}(\psi)$, there must exist distinct elements $\alpha_0, \alpha_1 \in \Delta \cap \operatorname{ann}(y)$. We apply the functor $\operatorname{Tor}_*(A/\alpha_0 A, -)$ to (17). We know N is torsion free because it injects into $(A^n)^{\otimes m}$, so we obtain an exact sequence

$$0 \to \operatorname{Tor}_1(A/\alpha_0 A, \operatorname{cok}(\psi)) \to (A/\alpha_0 A)^d,$$

where $y \in \operatorname{Tor}_1(A/\alpha_0 A, \operatorname{cok}(\psi)) = \{z \in \operatorname{cok}(\psi) | \alpha_0 z = 0\}$. This implies that $(A/\alpha_0 A)^d$ contains a nonzero element annihilated by α_1 which is a contradiction. Thus $\operatorname{cok}(\psi) = 0$ and ψ is an isomorphism. $\qquad\square$

2.4.4. Induction. Let G be a finite group and consider the discrete GKM-hypergraph Γ_G with vertex set G. Then $G \times G$ acts on Γ_G by left and right multiplication, and the constant sheaf $A_G := A_{\operatorname{Top}(\Gamma_G)}$ is a $G \times G$-equivariant GKM-sheaf.

Given a homomorphism $H \to G$, we make A_G a $G \times H$-equivariant GKM-sheaf. If a GKM-hypergraph Γ has left H-action, define the induction functor

$$\operatorname{Ind}_H^G : GKM_H(\Gamma) \to GKM_G(G \times_H \Gamma)$$

by $\operatorname{Ind}_H^G(\mathcal{F}) = A_G \boxtimes_H \mathcal{F}$. This is a well-defined functor because it is obtained from the external tensor product. Note that $\operatorname{Ind}_H^G(\mathcal{F})$ depends on the homomorphism $\phi : H \to G$ even though it has been suppressed in the notation. For the identity morphism on G we have natural isomorphism $G \times_G \Gamma = \Gamma = \Gamma \times_G G$ and

$$(18) \qquad \operatorname{Ind}_G^G(\mathcal{F}) = A_G \boxtimes_G \mathcal{F} \cong \mathcal{F} \cong \mathcal{F} \boxtimes_G A_G.$$

2.4.5. Convolution. Suppose now that Γ is a GKM-hypergraph equipped with the action of an *abelian* group G which is free and transitive on the set of vertices.

Lemma 2.19. *There is an isomorphism of GKM-hypergraphs*

$$\phi : \Gamma \cong \Gamma \times_G \Gamma,$$

which is canonically defined up to a choice of base vertex in Γ. Furthermore, ϕ is equivariant with respect to a residual G-action on $\Gamma \times_G \Gamma$.

Proof. Let $\Gamma = (\mathcal{V}, \sim)$ and choose a base vertex $v_* \in \mathcal{V}$. Define a morphism $\phi : \Gamma \to \Gamma \times_G \Gamma$ as the composition $\pi \circ i$

$$\Gamma \xrightarrow{\ i\ } \Gamma \times \Gamma$$
$$\downarrow \pi$$
$$\Gamma \times_G \Gamma$$

where $i(v) = (v, v_*)$ and π is the quotient map. It is easy to see that ϕ is a bijective on vertices, because every G orbit in $\mathcal{V} \times \mathcal{V}$ passes once through $\mathcal{V} \times \{v_*\}$. To see that ϕ is a GKM-isomorphism, it is enough to count hyperedges $\mathcal{E}^\alpha$ and $\mathcal{E}^\alpha \times_G \mathcal{E}^\alpha$ for each $\alpha \in \mathbb{P}(\Lambda)$ and show they have the same cardinality.

Because G acts transitively on $\mathcal{V}$, it also acts transitively on $\mathcal{E}^\alpha$. Because G is abelian the stabilizer G_e is the common stabilizer of all hyperedges $e \in \mathcal{E}^\alpha$ and thus also the common stabilizer of all elements in $\mathcal{E}^\alpha \times \mathcal{E}^\alpha$ under the anti-diagonal action. Consequently by the orbit-stabilizer theorem, we have

$$|(\mathcal{E}^\alpha \times \mathcal{E}^\alpha)/G| = |\mathcal{E}^\alpha \times \mathcal{E}^\alpha|/|\mathcal{E}^\alpha| = |\mathcal{E}^\alpha|$$

proving that $\pi \circ i$ is an isomorphism.

Because the diagonal subgroup $\Delta G \subseteq G \times G$ is normal, the $G \times G$-action on $\Gamma \times \Gamma$ descends to a $(G \times G/\Delta G)$-action on $\Gamma \times_G \Gamma$. Under the isomorphism

$$G \cong (G \times G)/\Delta G, \ \ g \mapsto [(g, id_G)],$$

ϕ becomes G-equivariant. $\square$

We call the composition of functors

$$GKM_G(\Gamma) \times GKM_G(\Gamma) \mapsto GKM_G(\Gamma \times_G \Gamma) \cong GKM_G(\Gamma)$$

the *convolution product*, denoted $*$.

Remark 4. The convolution product as defined is not symmetric in general. This can be rectified by working instead with $\Gamma^{\mathrm{op}} \times_G \Gamma$, where $\Gamma^{\mathrm{op}} = \Gamma$

with G acting through the inverse map $g \mapsto g^{-1}$. In our applications, G is a 2-torsion group, so the inverse map is the identity and this issue is irrelevant.

Lemma 2.20. *For $i = 1, 2$ let Γ_i be a GKM-hypergraph with an action by an abelian group G_i which is free and transitive on vertices. The external tensor product determines a bifunctor*

$$\boxtimes : GKM_{G_1}(\Gamma_1) \times GKM_{G_2}(\Gamma_2) \to GKM_{G_1 \times G_2}(\Gamma_1 \times \Gamma_2)$$

for which there are natural isomorphisms

(19) $$(\mathcal{F}_1 * \mathcal{G}_1) \boxtimes (\mathcal{F}_2 * \mathcal{G}_2) \cong (\mathcal{F}_1 \boxtimes \mathcal{F}_2) * (\mathcal{G}_1 \boxtimes \mathcal{G}_2)$$

for $\mathcal{F}_i, \mathcal{G}_i \in GKM_{G_i}(\Gamma_i)$.

Proof. The product operation of GKM-hypergraphs is associative and commutative, so $(\Gamma_1 \times \Gamma_1) \times (\Gamma_2 \times \Gamma_2)$ is naturally isomorphic to $(\Gamma_1 \times \Gamma_2) \times (\Gamma_1 \times \Gamma_2)$. Both sides of (19) are invariant pushforwards of the naturally isomorphic GKM-sheaves $(\mathcal{F}_1 \boxtimes \mathcal{G}_1) \boxtimes (\mathcal{F}_2 \boxtimes \mathcal{G}_2) \cong (\mathcal{F}_1 \boxtimes \mathcal{F}_2) \boxtimes (\mathcal{G}_1 \boxtimes \mathcal{G}_2)$. $\square$

Proposition 2.21. *Let Γ be GKM-graph equipped with a transitive and vertex-free action by a finite abelian group H, and let $\phi : H \to G$ be a homomorphism of finite abelian groups. Then the induced G-action on $G \times_H \Gamma$ is transitive and vertex-free, and the induction functor $\mathrm{Ind}_H^G : GKM(\Gamma) \mapsto GKM(G \times_H \Gamma)$ respects the convolution product,*

$$\mathrm{Ind}_H^G(\mathcal{F} * \mathcal{G}) \cong \mathrm{Ind}_H^G(\mathcal{F}) * \mathrm{Ind}_H^G(\mathcal{G}).$$

Proof. Applying (18) and associativity we have

$$\begin{aligned}
Ind_H^G(\mathcal{F} * \mathcal{G}) &= A_G \boxtimes_H (\mathcal{F} \boxtimes_H \mathcal{G}) \\
&= A_G \boxtimes_H (\mathcal{F} \boxtimes_G A_G) \boxtimes_H \mathcal{G} \\
&= (A_G \boxtimes_H \mathcal{F}) \boxtimes_G (A_G \boxtimes_H \mathcal{G}) = \mathrm{Ind}_H^G(\mathcal{F}) * \mathrm{Ind}_H^G(\mathcal{G}).
\end{aligned}$$

$\square$

2.5. Twisted actions

Let W be a finite group. Given a torus T a *twist* τ is a homomorphism

$$\tau : W \to \mathrm{Aut}(T), \; w \mapsto \tau_w.$$

In all examples we consider, W will be a Weyl group acting in the standard way on T. A twist induces actions of W on $\mathbb{P}(\Lambda)$ and on $A = S(\mathfrak{t}^*)$ that we also denote by τ.

A τ-*twisted action on a GKM-hypergraph* $\Gamma = (\mathcal{V}, \mathcal{E}, I, \alpha)$ consists of W-actions on the sets $\mathcal{V}$ and $\mathcal{E}$ that are equivariant with respect to I and α : $\mathcal{E} \to \mathbb{P}(\Lambda)$. This differs from an ordinary GKM-action, because W is allowed to act nontrivially on $\mathbb{P}(\Lambda)$.

A τ-*twisted W-action on a GKM sheaf* $\mathcal{F}$ over Γ is a lift of a τ-twisted W-action ρ on Γ to an W-action $\tilde{\rho}$ on $\mathcal{F}$ as a sheaf of $\mathbb{Z}$-graded abelian groups, satisfying the identity $\tilde{\rho}_w(fs) = \tau_w(f)\tilde{\rho}_w(s)$ for all $w \in W$, $f \in A = S(\mathfrak{t}^*)$ and sections s of $\mathcal{F}$. Given such an action, W-invariant global sections $H^0(\mathcal{F})^W$ form a graded A^W-module.

Let K be a compact Lie group with maximal torus T such that the normalizer $N_K(T)$ intersects every path component of K, and let X be a nice K-space. Restricting to the action of T, we may associate a GKM-hypergraph Γ_X and GKM-sheaf $\mathcal{F}_X$. Since the one-skeleton (X_1, X_0) is preserved by $N_K(T)$, we gain a twisted action of $W = N_K(T)/T$ on Γ_X lifting to $\mathcal{F}_X$. The following proposition is a straightforward consequence of Theorem 2.8 and the isomorphism $H_K^*(X) \cong H_T^*(X)^W$.

Proposition 2.22. *If X is equivariantly formal, then there is a natural isomorphism of graded A^W-algebras*

$$\phi : H_K^*(X) \cong H^0(\mathcal{F}_X)^W.$$

2.6. Examples

2.6.1. Monodromy sheaves.   So far our examples of a GKM-sheaf have come from T-spaces. Now we consider a construction of GKM-sheaves from combinatorial and algebraic data. Let Γ be a GKM-*graph* with a finite number of nondegenerate edges. Let $\mathcal{E}^{\mathrm{nd}} \subset \mathcal{E}$ denote the set of nondegenerate edges. Choose an orientation for each $e \in \mathcal{E}^{\mathrm{nd}}$, by defining source and target maps $s, t : \mathcal{E}^{\mathrm{nd}} \to \mathcal{V}$, such that $I(e) = \{s(e), t(e)\}$. A *local system* on Γ consists of a (finitely generated, $\mathbb{Z}$-graded) free A-module M called the *fibre* and a map $\rho : \mathcal{E}^{\mathrm{nd}} \to \mathrm{Aut}(M)$.

Definition 5. The *monodromy GKM-sheaf* associated to the local system (Γ, M, ρ) is the GKM-sheaf $\mathcal{F} = \mathcal{F}_\rho$ with stalks $\mathcal{F}(U_x) = M$ at vertices and degenerate edges, and $\mathcal{F}(U_e) = M \oplus M[2]$ at non-degenerate edges with restriction maps $\mathcal{F}(U_e) = M \oplus M[2] \overset{\mathrm{res}_e}{\to} M \oplus M = \mathcal{F}(U_{s(e)}) \oplus \mathcal{F}(U_{t(e)})$

defined by the matrix

$$(20) \qquad \text{res}_e := \begin{pmatrix} 1 & \alpha(e) \\ \rho(e) & -\alpha(e)\rho(e) \end{pmatrix},$$

where we have abusively used $\alpha(e)$ to denote a generator of the projective weight $\alpha(e) \in \mathbb{P}(\Lambda)$.

This construction produces a GKM-sheaf because the matrix (20) becomes invertible after inverting $\alpha(e)$.

Example 4. The traditional GKM-manifolds of [GZ00] provide the simplest example of a monodromy sheaf. In this case the fibre $M = A$ and all the automorphisms $\rho(e) \in \text{Aut}(A)$ are the identity.

Monodromy sheaves were inspired by work of Guillemin and Holm [GH04] in the context of Hamiltonian actions on symplectic manifolds. A *GKM manifold with nonisolated fixed points* is a closed T-manifold X, all of whose fixed point components are homeomorphic to a fixed reference space F, with GKM-graph $\Gamma_X = (\mathcal{V}, \mathcal{E}, I, \alpha)$ such that for each nondegenerate edge E with (necessarily distinct) vertices V_s, V_t, there exists a commutative diagram of T-spaces:

$$V_s \underset{i_s}{\overset{\pi_s}{\rightleftarrows}} E \underset{i_t}{\overset{\pi_t}{\rightleftarrows}} V_t$$

where π_s and π_t are T-equivariant S^2-fibre bundles for which the inclusions i_s and i_t are sections.

Proposition 2.23. *If $\mathcal{F}_\rho$ is the monodromy GKM-sheaf on Γ with fibre $H_T^*(F) \cong H_T^*(V)$ and $\rho(E) = (\pi_s \circ i_t)^*$ then $\mathcal{F}_\rho \cong \mathcal{F}_X$.*

Proof. The isomorphisms $\mathcal{F}_\rho(U_V) \cong H_T^*(F) \cong \mathcal{F}_X(U_V)$ at vertices is clear from the definition. The isomorphism

$$\mathcal{F}_\rho(U_E) = H_T^*(F) \oplus H_T^*(F)[2] \cong H_T^*(E) = \mathcal{F}_X(U_E)$$

follows from the Thom isomorphism for the sphere bundle $\pi_s : E \to V_s \cong F$. That the restriction maps match up is an easy exercise. $\square$

In Section 3, we show that for regular value $c \in K$, the representation variety $\mathcal{R}_K^1(c)$ is a GKM-manifold with nonisolated fixed points, and that $\mathcal{F}_{\mathcal{R}_K^1(c)}$ is a monodromy sheaf with nontrivial monodromy in general.

2.6.2. Pure Γ-sheaves.   In [BM01] Braden and MacPherson introduce the notion of a pure Γ-sheaf $\mathcal{M}$ over a moment graph. They also show that in many interesting cases, the equivariant intersection cohomology $IH^*_T(X)$ of a complex projective variety X is equal to the global sections $H^0(\mathcal{M})$ of a pure Γ-sheaf $\mathcal{M}$ associated with X.

Proposition 2.24.   *To any pure Γ-sheaf $\mathcal{M}$ there is a canonically associated GKM-sheaf $\mathcal{M}'$ such that $H^0(\mathcal{M}) \cong H^0(\mathcal{M}')$.*

In our framework, a moment graph is essentially the same thing as a GKM-graph $(\mathcal{V}, \mathcal{E}, I, \alpha)$ with only a finite number of nondegenerate edges $\mathcal{E}^{\mathrm{nd}} \subset \mathcal{E}$, and an ordering $I(e) = \{v_s, v_t\}$ for $e \in \mathcal{E}^{\mathrm{nd}}$ subject to some further conditions. A Γ-*sheaf* $\mathcal{M}$ consists of A-modules $\mathcal{M}(v)$ for each vertex and $\mathcal{M}(e)$ for each $e \in \mathcal{E}^{\mathrm{nd}}$ and a homomorphism $\rho_{v,e} : \mathcal{M}(v) \to \mathcal{M}(e)$ for every pair $\{(v,e) \in \mathcal{V} \times \mathcal{E}^{\mathrm{nd}} | v \in I(e)\}$. The module of global sections is

$$(21) \qquad H^0(\mathcal{M}) = \{(m_x) \in \bigoplus_{x \in \mathcal{V} \cup \mathcal{E}^{\mathrm{nd}}} \mathcal{M}(x) |\ \rho_{v,e}(m_v) = m_e\ \forall \rho_{v,e}\}.$$

Remark 5.   It is explained in [BM01] that the data defining a Γ-sheaf is equivalent to a sheaf of A-modules over $\mathcal{V} \cup \mathcal{E}^{\mathrm{nd}}$ in the topology dual to the one we use, in the sense that their closed sets are equal to our open sets and vice versa.

A Γ-sheaf is called *pure* if $\mathcal{M}(v)$ is free for all $v \in \mathcal{V}$ and if for every edge $e \in \mathcal{E}^{\mathrm{nd}}$ with $I(e) = \{v_s, v_t\}$ satisfies, $\mathcal{M}(e) \cong \mathcal{M}(v_s)/\alpha(e)\mathcal{M}(v_s)$ with

$$\rho_{v_s,e} : \mathcal{M}(v_s) \to \mathcal{M}(e) = \mathcal{M}(v_s)/\alpha(e)\mathcal{M}(v_s)$$

the projection, plus some additional conditions.

Proof of Proposition 2.24. Let $\mathcal{M}$ be a pure Γ-sheaf. Define a new sheaf of A-modules $\mathcal{M}'$ by

- $\mathcal{M}'(\{v\}) = \mathcal{M}'(U_v) = \mathcal{M}(v)$ for all $v \in \mathcal{V}$;

- for all $e \in \mathcal{E}^{\mathrm{nd}}$ the stalk $\mathcal{M}'(U_e)$ and restriction map res_e are defined by the short exact sequence

$$(22) \qquad 0 \to \mathcal{M}'(U_e) \overset{\mathrm{res}_e}{\to} \mathcal{M}(v_s) \oplus \mathcal{M}(v_t) \overset{\rho_{v_s,e} - \rho_{v_t,e}}{\longrightarrow} \mathcal{M}(e) \to 0;$$

- for all degenerate edges $e \in \mathcal{E} \setminus \mathcal{E}^{\mathrm{nd}}$, $\mathcal{M}'(e) = \mathcal{M}'(I(e))$ with res_e the identity map.

Clearly $\mathcal{M}'$ is a sheaf and res_e is an isomorphism modulo $\alpha(e)$ for all $e \in \mathcal{E}$. Both $H^0(\mathcal{M})$ and $H^0(\mathcal{M}')$ inject into $\bigoplus_{v \in \mathcal{V}} \mathcal{M}(v)$ by projection, and the short exact sequence (22) ensures that they have the same image, so $H^0(\mathcal{M}) \cong H^0(\mathcal{M}')$.

It remains to show that $\mathcal{M}'(U_e)$ is a free A-module for every $e \in \mathcal{E}^{\text{nd}}$. Because $\mathcal{M}(v_s)$ and $\mathcal{M}(v_t)$ are projective, $\rho_{v_s,e} - \rho_{v_t,e}$ lifts through (21) to a surjective map $f : \mathcal{M}(v_s) \oplus \mathcal{M}(v_t) \to \mathcal{M}(v_s)$. Because f admits a section, $\ker(f) \cong \mathcal{M}(v_t)$ and $\mathcal{M}'(U_e)$ fits into an exact sequence $0 \to \ker(f) \to \mathcal{M}'(U_e) \to \alpha(e)\mathcal{M}(v_s) \to 0$ which must also split so

$$\mathcal{M}'(U_e) \cong \mathcal{M}(v_t) \oplus \alpha(e)\mathcal{M}(v_s)$$

is free. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.6.3. Mutants of compactified representations. These are examples of closed T-manifolds introduced by Franz and Puppe [FP08] whose equivariant cohomology is torsion-free but not free over A. There are three examples: Z_r, $r = 2, 4, 8$ with action by $T = (U(1))^{r+1}$, whose construction makes use of the Hopf fibration $S^{r-1} \to S^{2r-1} \to S^r$. Franz and Puppe prove that as graded A-modules

$$H_T^*(Z_r) \cong A \oplus m[r-1] \oplus A[2r+2] \oplus A[3r+1],$$

where m is the augmentation ideal of A.

The GKM-sheaf $\mathcal{F}_{Z_r}$ is easily determined from the description of Z_r in [FP08]. The GKM-hypergraph Γ_{Z_r} consists of two vertices and $r+1$ nondegenerate edges labelled by distinct weights $\alpha_0, \ldots, \alpha_r$ which form a basis of $\mathfrak{t}^*$. The GKM-sheaf $\mathcal{F}_{Z_r}$ is the trivial monodromy sheaf with fibres $M := H^*(S^{r-1}) \otimes A \cong A \oplus A[r-1]$. It follows that the global sections of $\mathcal{F}_{Z_r}$ may be identified with the image of the matrix

$$\begin{pmatrix} 1 & f \\ 1 & -f \end{pmatrix}$$

in $M \oplus M$, where $f = \prod_{i=0}^r \alpha_i$ has degree $2(r+1)$. Thus as graded A-modules

$$H^0(\mathcal{F}_{Z_r}) \cong A \oplus A[r-1] \oplus A[2r+2] \oplus A[3r+1].$$

This example shows that $H^0(\mathcal{F}_X)$ may be a free A-module even when $H_T^*(X)$ is not.

2.6.4. Equivariant de Rham theory and graphs. Given a GKM-graph Γ, consider the trivial monodromy sheaf $\mathcal{F}_{id}$ over Γ with fibre A. The *graph cohomology* $H_T(\Gamma)$ is defined to equal $H^0(\mathcal{F}_{id})$. It is interesting to ask under what circumstances $H_T(\Gamma)$ is a free A-module and what its Betti numbers are.

In a series of papers, [GZ00, GZ01, GZ03] Guillemin and Zara translate concepts from Hamiltonian actions on symplectic manifolds to the GKM theory, motivated in part by these questions. They a define the notion of moment map on a GKM-graph Γ, define the reduction $\Gamma//S^1$ with respect to such a moment map and prove a version of Kirwan surjectivity $\kappa : H_T(\Gamma) \rightarrow H_{T/S^1}(\Gamma//S^1)$.

In certain circumstances, the reduction $\Gamma//S^1$ is not a graph but a GKM-*hyper*graph and in this situation, addressed in [GZ03], the arguments become rather technical. We believe the reduction process could be more clearly understood in our framework (we do not pursue this in this paper). For example, the definition of $H_T(\Gamma//S^1)$ "by duality" is a strong hint that our topology, which is dual to the more obvious one in Braden–MacPherson [BM01] may be important (see Remark 5). Also, the locality of $H_T(\Gamma//S^1)$ would be manifest if it were defined as the global sections of a GKM-sheaf.

3. Representation varieties for a punctured surface

It will be useful to introduce a larger class of representation varieties than those defined in the introduction. Gauge theoretically, these varieties correspond to moduli spaces of flat bundles over a nonorientable surface with a single puncture and prescribed holonomy around the puncture.

Let Σ_g denote the connected sum of $g + 1$ copies[4] of $\mathbb{R}P^2$. By the classification of compact surfaces, every nonorientable compact surface without boundary is isomorphic to Σ_g for some $g = 0, 1, 2, \ldots$. The fundamental group $\pi_1(\Sigma_g)$ has presentation

(23)
$$\pi_1(\Sigma_g) \cong \left\{ a_0, \ldots, a_g | \prod_{i=0}^{g} a_i^2 = \mathbb{1} \right\}.$$

Let Γ_g denote the free group on $g + 1$ generators $\{a_0, \ldots, a_g\}$ so that the presentation (23) determines a surjection $\Gamma_g \rightarrow \pi_1(\Sigma_g)$. For K a compact,

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[4]The index is chosen so that g is the genus of the orientable double cover.

connected Lie group of rank r and $c \in K$, define

$$\mathcal{R}_K^g(c) := \left\{ \phi \in \mathrm{Hom}(\Gamma_g, K) \big| \phi \left(\prod_{i=0}^{g} a_i^2 \right) = c \right\}.$$

The embedding $\mathcal{R}_K^g(c) \hookrightarrow K^{g+1}$ sending ϕ to $(\phi(a_0), \ldots, \phi(a_g))$ identifies $\mathcal{R}_K^g(c)$ with the compact real algebraic set

$$(24) \qquad \mathcal{R}_K^g(c) \cong \left\{ (k_0, \ldots, k_g) \in K^{g+1} \big| \prod_{i=0}^{g} k_i^2 = c \right\}$$

on which the centralizer $Z_K(c) \subseteq K$ acts by conjugation. Notice that

$$\mathcal{R}_K^g(\mathbb{1}) \cong \mathrm{Hom}(\pi_1(\Sigma_g), K),$$

where $\mathbb{1} \in K$ is the identity, so we recover the representation varieties described in the introduction.

We will always choose c to lie in a fixed maximal torus $T \subset K$, with complexified lie algebra $\mathfrak{t}$. We call c *regular* if $Z_K(c) = T$. We use notation $W := N_K(T)/T$ for the Weyl group, and $W_c := N_{Z_K(c)}(T)/T$ for the portion of the Weyl group that stabilizes c. The isomorphism

$$H_{Z_K(c)}^*(\mathcal{R}_K^g(c)) \cong H_T^*(\mathcal{R}_K^g(c))^{W_c}$$

allows us to divide the study of general compact group actions into torus actions and Weyl group actions. As before, we use notation $A := \mathbb{C}[\mathfrak{t}] = S(\mathfrak{t}^*) \cong H^*(BT)$.

3.1. Main results

In this section we construct the GKM-sheaf of the representation variety $\mathcal{R}_K^g(c)$ using a monodromy sheaf, convolution products and induction.

Let T_2 denote the subgroup of 2-torsion elements of T. That is,

$$T_2 := \{ t \in T | t^2 = \mathbb{1} \}.$$

Then T_2 is isomorphic to the finite group $(\mathbb{Z}/2)^r$. Let $\mathcal{V}$ be a torsor for T_2, meaning that $\mathcal{V}$ is a finite set of order 2^r equipped with a free and transitive action $T_2 \times \mathcal{V} \to \mathcal{V}$. Let $\Delta \subset \mathbb{P}(\Lambda)$ denote the set of roots of $T \subset K$ modulo nonzero scalar multiplication. For each $\alpha \in \Delta$, there exists a pair of co-roots

$\pm h_\alpha \in \mathfrak{t}$ and an element $\exp(\pm i\pi h_\alpha) \in T_2$ (observe that the sign ambiguity disappears after exponentiating). For each $\alpha \in \mathbb{P}(\Lambda)$, define the equivalence relation $\sim_\alpha$ on $\mathcal{V}$ to be trivial if $\alpha \notin \Delta$ and to be generated by the relation

$$v \sim_\alpha \exp(i\pi h_\alpha)v$$

for all $v \in \mathcal{V}$, if $\alpha \in \Delta$. Together $(\mathcal{V}, \sim)$ form a GKM-graph which we denote $\Gamma(K)$. Our first result is

Theorem 3.1. *For $g \geq 1$, the isomorphism type of the GKM-hypergraph $\Gamma_{\mathcal{R}_K^g(c)}$ is independent of both g and c and is equal to $\Gamma(K)$.*

Observe next that the T_2-action on $\mathcal{V}$ respects the equivalence relations $v \sim_\alpha \exp(i\pi h_\alpha)v$ because T_2 is abelian. Thus we may consider the category $GKM_{T_2}(\Gamma(K))$ of T_2-equivariant GKM-sheaves on $\Gamma(K)$ and this category acquires a convolution product $*$ by Section 2.4.5.

Theorem 3.2. *Let $c \in T \subseteq K$ be a regular element. Then for any $g \geq 1$, the GKM-sheaf $\mathcal{F}_{\mathcal{R}_K^g(c)}$ is T_2-equivariant and there is an isomorphism*

$$\mathcal{F}_{\mathcal{R}_K^g(c)} \cong \mathcal{F}_{\mathcal{R}_K^1(c)} * \cdots * \mathcal{F}_{\mathcal{R}_K^1(c)}$$

with the g-fold convolution product of $\mathcal{F}_{\mathcal{R}_K^1(c)}$.

Suppose now that $c \in T \subseteq K$ is not necessarily regular.

Theorem 3.3. *Let $c, c' \in T \subseteq K$ and suppose that c' is regular. There exist twisted actions of W_c on both $\mathcal{F}_{\mathcal{R}_K^g(c)}$ and $\mathcal{F}_{\mathcal{R}_K^g(c')}$ such that*

$$H^0(\mathcal{F}_{\mathcal{R}_K^g(c)})^{W_c} \cong H^0(\mathcal{F}_{\mathcal{R}_K^g(c')})^{W_c}.$$

In Theorem 3.3, the twisted action of W_c on $\mathcal{F}_{\mathcal{R}_K^g(c)}$ is induced by the topological action of $Z_K(c)$ on the representation variety $\mathcal{R}_K^g(c)$ in the standard way (see Proposition 2.22). The twisted action of W_c on $\mathcal{F}_{\mathcal{R}_K^g(c')}$ is more subtle, and is not induced from an explicit action on $\mathcal{R}_K^g(c')$.

Theorems 3.2 and 3.3 reduce the problem of constructing the GKM-sheaf $\mathcal{F}_{\mathcal{R}_K^g(c)}$ to the special case when $g = 1$ and c is regular. In this case, $\mathcal{F}_{\mathcal{R}_K^1(c)}$ is isomorphic to a monodromy GKM-sheaf over $\Gamma(K)$, supposing a

mild additional condition on K, as we now explain. Let

$$M = \wedge(\mathfrak{t}^*) \otimes_{\mathbb{C}} A$$

with $\wedge(\mathfrak{t}^*)$ graded so that $\mathfrak{t}^*$ has degree one. Each $\alpha \in \Delta$ determines a Weyl group reflection on $\mathfrak{t}$ and we denote by S_α the induced action on $\wedge(\mathfrak{t}^*)$. For a given nondegenerate edge e of $\Gamma(K)$ with $\alpha(e) \in \Delta$, we define

$$\rho(e) = S_{\alpha(e)} \otimes_{\mathbb{C}} Id_A \in \mathrm{Aut}(M).$$

Observe that $\rho(e) \circ \rho(e)$ is the identity on M, so it is unnecessary to specify the orientation of the edges.

Theorem 3.4. *Let $c \in T \subseteq K$ be regular and suppose that $\pi_1(K)$ is 2-torsion free (i.e., only the identity element squares to the identity). Then there is an isomorphism between the GKM-sheaf $\mathcal{F}_{\mathcal{R}_K^1(c)}$ and the monodromy sheaf $\mathcal{F}_\rho$ with data M and ρ described above.*

Finally, suppose that $\pi_1(K)$ contains 2-torsion. Choose a finite, connected covering group $\phi : \tilde{K} \to K$ such that $\pi_1(\tilde{K})$ is 2-torsion free. Let $\tilde{T} \subseteq \tilde{K}$ and $T \subset K$ be maximal tori with $\phi(\tilde{T}) = T$. Let $\tilde{c}$ and c be respective regular elements and let $\tilde{T}_2$ and T_2 be respective 2-torsion subgroups. Restricting ϕ to the homomorphism $\tilde{T}_2 \to T_2$ induces an induction functor

$$\mathrm{Ind}_{\tilde{T}_2}^{T_2} : GKM_{\tilde{T}_2}(\Gamma(\tilde{K})) \mapsto GKM_{T_2}(\Gamma(K)).$$

Theorem 3.5. *If c is regular, then there is an isomorphism of T_2-equivariant GKM-sheaves*

$$\mathcal{F}_{\mathcal{R}_K^g(c)} \cong \mathrm{Ind}_{\tilde{T}_2}^{T_2}(\mathcal{F}_{\mathcal{R}_{\tilde{K}}^g(\tilde{c})}).$$

3.2. Fixed points

Our first task is to describe the fixed points of $\mathcal{R}_K^g(c)$ under the conjugation action by T. Maximal tori are maximal abelian, so it follows that a homomorphism $\phi \in \mathcal{R}_K^g(c)$ is fixed by T if and only if $\mathrm{im}(\phi) \subset T$, thus

$$\mathcal{R}_K^g(c)^T = \mathcal{R}_T^g(c).$$

For this reason, it is useful to describe with some care the case that $K = T = U(1)^r$ is a torus.

Any homomorphism from $\pi := \pi_1(\Sigma_g)$ to an abelian group must factor through the abelianization $\pi/[\pi,\pi] \cong H_1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^g \oplus \mathbb{Z}_2$ so we obtain

$$(25) \qquad \mathcal{R}_T^g(\mathbb{1}) = \mathrm{Hom}(\pi, T) \cong \mathrm{Hom}(H_1(\Sigma_g, \mathbb{Z}), T) \cong T^g \times T_2,$$

where $T_2 = \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2, T) \cong (\mathbb{Z}/2)^r$ is the 2-torsion subgroup of T. More canonically, regarding $\mathcal{R}_T^g(\mathbb{1})$ as a group under pointwise multiplication, we have a short exact sequence

$$(26) \qquad 0 \longrightarrow T^g \overset{\rho}{\underset{}{\longleftrightarrow}} \mathcal{R}_T^g(\mathbb{1}) \longrightarrow T_2 \longrightarrow 0$$

and we choose a splitting $\rho : \mathcal{R}_T^g(\mathbb{1}) \to T^g$, $\rho(\phi) = (\phi(a_1), \ldots, \phi(a_g))$, using the presentation (23) of the fundamental group.

For general $c \in T$, pointwise multiplication determines a free and transitive action of $\mathcal{R}_T^g(\mathbb{1})$ on $\mathcal{R}_T^g(c)$, i.e.,

Lemma 3.6. *For $g \geq 0$, T a rank r compact torus and $c \in T$, the representation variety $\mathcal{R}_T^g(c)$ is a torsor for $\mathcal{R}_T^g(\mathbb{1})$, thus is diffeomorphic to $T^g \times T_2$.*

Remark 6. The action of T_2 on $\mathcal{R}_T^g(c)$ is canonically defined up to isotopy and acts freely and transitively on the set of components. Thus we have a canonical isomorphism $H_T^*(F_i) \cong H_T^*(F_j)$ for every pair of connected components F_i, F_j of $\mathcal{R}_T^g(c)$. Less canonically, these cohomology rings may all be identified with $H_T(T^g) \cong (\wedge \mathfrak{t}^*)^{\otimes g} \otimes_{\mathbb{C}} A$, where $A := \mathbb{C}[\mathfrak{t}]$ as before.

Corollary 3.7. *For any compact, connected Lie group K of rank r, the GKM-hypergraph of $\mathcal{R}_K^g(c)$ has a vertex set $\mathcal{V}$ consisting of 2^r vertices which are naturally identified with the set of square roots of c,*

$$\mathcal{V} \cong \{t \in T | t^2 = c\}$$

upon which T_2 acts freely and transitively by multiplication. The stalks of the GKM-sheaf $\mathcal{F}_{\mathcal{R}_K^g(c)}$ at vertices may be identified with $\wedge(\mathfrak{t}^*)^{\otimes g} \otimes A$ and the T_2-action preserves these identifications.

Observe that Corollary 3.7 implies that the vertex set of $\Gamma_{\mathcal{R}_K^g(c)}$ is independent of both c and g, in keeping with Theorem 3.1.

Because T_2 acts freely on the vertex set $\mathcal{V}$, we gain a convolution product $*$ on T_2-equivariant GKM-sheaves (defined in Section 2.4.5).

Lemma 3.8. *For $g \geq 0$ and $c \in T$ let $\mathcal{F}_g := \mathcal{F}_{\mathcal{R}_T^g(c)}$. Then*

$$\mathcal{F}_g \cong \mathcal{F}_1 * \cdots * \mathcal{F}_1 = (\mathcal{F}_1)^{*g}.$$

Proof. We begin with the case $c = \mathbb{1}$. Use the characterization (24)

$$\mathcal{R}_T^g(\mathbb{1}) = \{(t_0, \ldots, t_g) \in T^{g+1} | t_0^2 \ldots t_g^2 = \mathbb{1}\}$$

with T_2 acting by multiplying t_0. There is a T_2-equivariant homeomorphism

$$\psi : \mathcal{R}_T^1(\mathbb{1}) \times_{T_2} \mathcal{R}_T^g(\mathbb{1}) \cong \mathcal{R}_T^{g+1}(\mathbb{1}),$$

where $\psi((s_0, s_1) \times (t_0, \ldots, t_g)) = (s_0 t_0, s_1, t_1, t_2, \ldots, t_g)$. Applying Proposition 2.17 and induction completes the proof. The case of general $c \in T$ follows by the fact that $\mathcal{R}_T^g(c)$ is a torsor for $\mathcal{R}_T^g(\mathbb{1})$, thus is T_2-equivariantly diffeomorphic. $\square$

Remark 7. In the proof of Lemma 3.8, T_2 acts on $\mathcal{R}_T^g(\mathbb{1})$ by multiplying the 0th entry t_0. We could have chosen instead to multiply one of the other entries and the action would be the same up to isotopy.

For $c \neq \mathbb{1}$, the isomorphism $\mathcal{F}_g \cong \mathcal{F}_1 * \cdots * \mathcal{F}_1$ of Lemma 3.8 has only been defined up to an automorphism of $\mathcal{F}_g$ induced by T_2, or equivalently up to a choice of base vertex. The following Lemma is meant to address this ambiguity.

Lemma 3.9. *A continuous path $\gamma : [0, 1] \to T$ induces an isotopy class of T_2-equivariant homeomorphisms $\mathcal{R}_T^g(\gamma(0)) \to \mathcal{R}_T^g(\gamma(1))$ (where T_2 acts as in Lemma 3.6). This class contains a map sending $(t_0, \ldots, t_g)$ to $(st_0, t_1, \ldots, t_g)$, where $s \in T$ satisfies $s^2 = \gamma(0)^{-1}\gamma(1)$.*

Proof. The variety $\mathcal{R}_T^g(c)$ is the fibre at $c \in T$ of the submersion $T^{g+1} \to T$ sending $(t_0, \ldots, t_g)$ to $\prod_{i=0}^g t_i^2$. Pulling back by γ determines a fibre bundle $\gamma^* T^{g+1}$ over $[0, 1]$ inducing a homeomorphism up to isotopy between fibres $\mathcal{R}_T^g(\gamma(0))$ and $\mathcal{R}_T^g(\gamma(1))$.

The T_2-action is induced by a canonical $\mathcal{R}_T^g(\mathbb{1})$-action. That the homeomorphism can be made equivariant follows from the classical result of Palais and Stewart [PS60] on the rigidity of compact group actions on compact manifolds.

Define the path $\delta : [0, 1] \to T$ by $\delta(x) = \gamma(0)^{-1}\gamma(x)$. Because the squaring map is a covering, there is a unique path $\sqrt{\delta} : [0, 1] \to T$ such that

$(\sqrt{\delta}(x))^2 = \delta(x)$ and $\sqrt{\delta}(0) = \mathbb{1}$. Define a T_2-equivariant bundle trivialization $\phi : [0,1] \times \mathcal{R}_T^g(\gamma(0)) \to \gamma^* T^{g+1}$ by $\phi(x, (t_0, \ldots, t_g)) = (x, (\sqrt{\delta}(x) t_0, \ldots, t_g))$. Setting $s = \sqrt{\delta}(1)$ completes the proof. $\square$

Corollary 3.10. *Let $\gamma : [0,1] \to T$ be a continuous path and denote $\mathcal{R}_i = \mathcal{R}_K^g(\gamma(i))$ for $i = 0, 1$. Then γ determines a T_2-equivariant bijection between the vertex sets $\mathcal{V}_0 \cong \mathcal{V}_1$ that lifts to a T_2-equivariant isomorphism of restricted sheaves*

$$\mathcal{F}_{\mathcal{R}_0}|_{\mathcal{V}_0} \cong \mathcal{F}_{\mathcal{R}_1}|_{\mathcal{V}_1}.$$

3.3. The case K=SU(2)

We review here the main results from [Bai10], using the language of GKM-sheaves. Throughout Section 3.3, set $K = SU(2)$, and $T \subset K$ is a maximal torus with $c \in T$. The centre $Z(K)$ consists of $\pm \mathbb{1} = T_2$, and all other values of c are regular.

Theorem 3.11 ([Bai10] Theorem 1.2). *The representation varieties $\mathcal{R}_K^g(c)$ are equivariantly formal under conjugation by T for all $c \in K$ and $g \in \{0, 1, 2, \ldots\}$. This means in particular that $H_T^*(\mathcal{R}_K^g(c)) \cong H^0(\mathcal{F}_{\mathcal{R}_K^g(c)})$ is free over A.*

Since T has rank one, $\mathbb{P}(\Lambda)$ is a single point.

Proposition 3.12. *The GKM-hypergraph $\Gamma_{\mathcal{R}_K^g(c)}$ for any $c \in T$ and $g \geq 1$ consists of two vertices and a single edge e connecting them, with $\alpha(e)$ equal to the sole element of $\mathbb{P}(\Lambda)$.*

Proof. This follows from Corollary 3.7 and the fact that $\mathcal{R}_K^g(c)$ is connected. $\square$

Because $\Gamma_{\mathcal{R}_K^g(c)}$ has only a single edge e, the GKM-sheaf $\mathcal{F}_{\mathcal{R}_K^g(c)}$ is completely determined by the localization map

$$H_T^*(\mathcal{R}_K^g(c)) = \mathcal{F}_{\mathcal{R}_K^g(c)}(U_e) \to H_T^*(\mathcal{R}_T^g(c)) = \mathcal{F}_{\mathcal{R}_K^g(c)}(I(e)).$$

The case when $g = 1$ and c is regular can be described quite explicitly. In this case, $\mathcal{R}_K^1(c)$ is diffeomorphic to $S^1 \times S^2$ and T acts via rotation on S^2 with weight 2. In particular (recall the definitions of GKM-manifold with nonisolated coefficients and monodromy sheaf from Section 2.6.1):

Proposition 3.13. *For regular c, $\mathcal{R}^1_K(c)$ is a GKM-manifold with non-isolated fixed points (Section 2.6.1). The GKM-sheaf $\mathcal{F}_{\mathcal{R}^1_K(c)}$ is isomorphic to a monodromy sheaf with fibre $H^*_T(T) \cong \wedge \mathfrak{t}^* \otimes A$, with holonomy map $\rho(e) = S_{\alpha(e)} \otimes Id_A$, where $S_{\alpha(e)}$ is the automorphism of $\wedge(\mathfrak{t}^*)$ induced by multiplication by -1 on $\mathfrak{t}$, which we think of as reflection in the root hyper-plane $\alpha(e)^\perp = \{0\}$.*

Proof. That $\mathcal{R}^1_K(c)$ is a GKM-manifold with nonisolated fixed points is clear from the preceding description. The fibres were described in Remark 6. The holonomy map can be inferred from Propositions 5.3 and 5.4 of [Bai10]. $\square$

Remark 8. Notice that because the graph $\Gamma_{\mathcal{R}^1_K(c)}$ is "simply connected", the local system above can be trivialized. However, the convention adopted in Remark 6 of using the T_2-action to identify fixed point components forces the local system to be nontrivial. Furthermore, when we consider higher rank groups in 3.5, the analogous local system will not be trivializable in general.

Because $T_2 = \{\pm\mathbb{1}\}$ lies in the centre of K, the T_2-action on $\mathcal{R}^g_T(c)$ described in Remark 6 extends to $\mathcal{R}^g_K(c)$ in the obvious way. This makes $\mathcal{F}_{\mathcal{R}^g_K(c)}$ into a T_2-equivariant GKM-sheaf.

Proposition 3.14. *Let $c \in T$ be a regular element and let $\mathcal{F}_g := \mathcal{F}_{\mathcal{R}^g_K(c)}$. For $g \geq 1$ we have an isomorphism between the GKM-sheaves $\mathcal{F}_g \cong \mathcal{F}_1 * \cdots * \mathcal{F}_1 = (\mathcal{F}_1)^{*g}$.*

Proof. Follows from Propositions 5.3 and 5.4 of [Bai10]. $\square$

Remark 9. In view of Proposition 2.17, one might suspect that Proposition 3.14 is the consequence of a homeomorphism or at least a T-map between $\mathcal{F}_{\mathcal{R}^g_K(c)}$ and $\mathcal{F}_{\mathcal{R}^g_K(c)} \times_{T_2} \ldots \times_{T_2} \mathcal{F}_{\mathcal{R}^g_K(c)}$. As explained in [Bai10], the real story is trickier than this.

Now we turn to the nonregular cases $\epsilon \in \{\pm\mathbb{1}\}$. The full group K central-izes ϵ, so K acts by conjugation on $\mathcal{R}^g_K(\epsilon)$, and $\mathcal{F}_{\mathcal{R}^g_K(\epsilon)}$ acquires a twisted $W_\epsilon = W$-action as described in Section 2.5. Choose a path $\gamma : [0,1] \to K$ connecting ϵ with some regular element c. Using Lemma 3.9 we obtain a twisted action of W on the restriction of $\mathcal{F}_{\mathcal{R}^g_K(c)}$ to the vertex set.

Proposition 3.15. *The action described above extends to a twisted action of the Weyl group W on $\mathcal{F}_{\mathcal{R}_K^g(c)}$, twisted by the standard action of W on $\mathfrak{t}$. Taking W-invariants produces an isomorphism*

$$H_K^*\big(\mathcal{R}_K^g(\epsilon)\big) \cong H^0(\mathcal{F}_{\mathcal{R}_K^g(c)})^W.$$

Proof. Use (26) to identify $\mathcal{R}_T^g(\mathbb{1}) \cong T^g \times T_2$. By Lemma 3.9, any path connecting $\mathbb{1}$ to $-\mathbb{1}$ also determines an isomorphism $\mathcal{R}_T^g(-\mathbb{1}) \cong T^g \times T_2$. In terms of these isomorphisms, the action of the nontrivial element $w \in W$, sends $(t_1, \ldots, t_g, z)$ to $(wt_1w^{-1}, \ldots, wt_gw^{-1}, \epsilon z)$. The result now follows from the explicit description of the image of the localization map $H_T^*(\mathcal{R}_K^g(\epsilon)) \to H_T^*(\mathcal{R}_T^g(\epsilon))$ found in Propositions 5.3, 5.4 and 5.5 from [Bai10]. □

Remark 10. This result is stranger than it might first appear. The W-action on $\mathcal{F}_{\mathcal{R}_K^g(c)}$ is *not* in general induced by one on $\mathcal{R}_K^g(c)$. Moreover, while the result implies that $H_T^*\big(\mathcal{R}_K^g(\epsilon)\big)^W \cong H^0(\mathcal{F}_{\mathcal{R}_K^g(c)})^W$, it is *not* true in general that $H_T^*\big(\mathcal{R}_K^g(\epsilon)\big) \cong H^0(\mathcal{F}_{\mathcal{R}_K^g(c)})$.

Remark 11. It will be important later to observe that for both $\epsilon \in \{\pm\mathbb{1}\}$, the action of W on $\mathcal{F}_{\mathcal{R}_K^g(c)}$ described in Proposition 3.15 commutes with the T_2-action, and these actions differ by the nontrivial element of T_2.

3.4. K has semisimple rank one

Let K be a compact connected Lie group of rank r with complexified Lie algebra $\mathfrak{k}$ and let $T \subset K$ be a maximal torus with complexified Lie algebra $\mathfrak{t}$. We may decompose $\mathfrak{k}$ into its central and semisimple parts:

$$\mathfrak{k} = Z(\mathfrak{k}) \oplus \mathfrak{k}_{ss}.$$

In this section, we consider the case where $\mathfrak{k}_{ss}$ has rank 1 or equivalently, $\mathfrak{k}_{ss} \cong \mathfrak{su}(2) \otimes \mathbb{C}$. The following lemma is elementary.

Lemma 3.16. *If K has semisimple rank one, then it is isomorphic to one of the following:*
(i) $U(1)^{r-1} \times SU(2)$,
(ii) $U(1)^{r-2} \times U(2)$,
(iii) $U(1)^{r-1} \times SO(3)$.

If $K = SU(2) \times U(1)^{r-1}$ and $c = (c_1, c_2)$, then the representation variety factors as a product of spaces already described

$$(27) \qquad \mathcal{R}_K^g(c) = \mathcal{R}_{SU(2)}^g(c_1) \times \mathcal{R}_{U(1)^{r-1}}^g(c_2).$$

If K has form (ii) or (iii), then it fits into a short exact sequence

$$(28) \qquad 0 \to C_2 \to \tilde{K} \xrightarrow{\phi} K \to 1,$$

where $\tilde{K} = U(1)^{r-1} \times SU(2)$ and $C_2 \cong \mathbb{Z}/2$ is central in $\tilde{K}$. Thus

$$(29) \qquad \mathcal{R}_K^g(c) \cong \coprod_{\phi(\tilde{c})=c} \mathcal{R}_{\tilde{K}}^g(\tilde{c})/C_2^{g+1},$$

where C_2^{g+1} acts by multiplying $g+1$-tuples entry-wise.

Proposition 3.17. *For all $g \geq 0$ and $c \in T$ the representation variety $\mathcal{R} := \mathcal{R}_K^g(c)$ is equivariantly formal, so $H^0(\mathcal{F}_\mathcal{R}) \cong H_T^*(\mathcal{R})$ by Theorem 2.8.*

Proof. If K is type (i) then the result follows from (27) because the product of equivariantly formal spaces is equivariantly formal under the product group action.

In the remaining cases, apply equivariant cohomology to (29)

$$H_T^*(\mathcal{R}_K^g(c)) \cong H_{\tilde{T}}^*(\mathcal{R}_K^g(c)) \cong \bigoplus_{\phi(\tilde{c})=c} H_{\tilde{T}}^*(\mathcal{R}_{\tilde{K}}^g(\tilde{c}))^{C_2^{g+1}},$$

where $\tilde{T}$ is the maximal torus in $\tilde{K}$ mapping to T. Thus $H_T^*(\mathcal{R}_K^g(c))$ is a summand of a free A-module, hence free. $\square$

Remark 12. There is a small subtlety in the above proof that should be explained. We are working with complex coefficients so the homomorphism $\tilde{T} \to T$ induces an isomorphism $H_{\tilde{T}}^*(\mathcal{R}_{\tilde{K}}^g(\tilde{c})) \cong H_T^*(\mathcal{R}_{\tilde{K}}^g(\tilde{c}))$. Thus formality with respect to T is equivalent to formality with respect to $\tilde{T}$.

Remark 13. Since C_2 is a subset of the two torsion group $\tilde{T}_2 \subset \tilde{T} \subseteq \tilde{K}$, it follows from Remark 7 each factor of C_2^{g+1} acts the same way on $\mathcal{R}_{\tilde{T}}^g(\tilde{c}))$ up to homotopy. Thus the GKM-sheaf of $\mathcal{R}_K^g(c)$ coincides with the GKM-sheaf of $\mathcal{R}_{\tilde{K}}^g(\tilde{c})/C_2$ where C_2 acts by multiplying (say) the 0th entry.

We turn to studying the GKM-sheaf $\mathcal{F}_{\mathcal{R}_K^g(c)}$. We begin with determining the GKM-hypergraph. Observe that since K has semisimple part of rank

one, the pair (K, T) has a unique pair of roots, which we denote $\pm\alpha$. Every point of $\mathcal{R}_K^g(c)$ is fixed by $\ker(\alpha)$, so the one-skeleton is the whole space $(\mathcal{R}_K^g(c), \mathcal{R}_T^g(c))$.

Define the co-root $h_\alpha \in \mathfrak{t}$ to be the unique element in $\mathfrak{t} \cap \mathfrak{k}_{ss}$ satisfying $\alpha(h_\alpha) = 2$. The exponential $\exp(2\pi i h_\alpha) = \mathbb{1} \in K$, so $\exp(\pi i h_\alpha) \in T_2$.

Proposition 3.18. *For any $c \in T$ and $g \geq 1$, denote the GKM-hypergraph $\Gamma_{\mathcal{R}_K^g(c)} = (\mathcal{V}, \sim)$. The vertex set $\mathcal{V}$ is naturally identified with the T_2-torsor*

$$\mathcal{V} \cong \{t \in T | t^2 = c\}.$$

The equivalence relation $\sim_\alpha$ is discrete unless α is the root, in which case

$$(30) \qquad\qquad\qquad v \sim_\alpha \exp(\pi i h_\alpha) \cdot v$$

for all $v \in \mathcal{V}$. In particular $\Gamma_{\mathcal{R}_K^g(c)} = \Gamma(K)$.

Proof. The vertex set was explained in Corollary 3.7. If $K \cong \tilde{K} \cong SU(2) \times U(1)^{r-1}$, then $\mathcal{R}_K^g(c) = \mathcal{R}_{SU(2)}^g(c_1) \times \mathcal{R}_{U(1)^{r-1}}^g(c_2)$ and the result follows easily. For the remaining cases we use Remark 13 to identify $\Gamma_{\mathcal{R}_K^g(c)}$ with the quotient of the C_2 action on $\coprod_{\phi(\tilde{c})=c} \Gamma_{\mathcal{R}_{\tilde{K}}^g(\tilde{c})}$ and the result follows.   □

Remark 14. The element $\exp(\pi i h_\alpha) \in T_2$ is nontrivial if K has type (i) or (ii) and is the identity when K has type (iii). Thus $\Gamma_{\mathcal{R}_K^g(c)}$ is a bipartite graph in the first two cases and is discrete in the third.

3.4.1. If c is regular. As explained in Section 3.1, the GKM-graph $\Gamma(K) = \Gamma_{\mathcal{R}_K^g(c)}$ admits a T_2-action that is free and transitive on vertices. Denote $\mathcal{F}_g := \mathcal{F}_{\mathcal{R}_K^g(c)}$ for c regular. The homomorphism $\tilde{K} \to K$ of (28) restricts to a homomorphism $\tilde{T}_2 \to T_2$ of 2-torsion groups.

Lemma 3.19. *If $c \in T$ is regular, then the GKM-sheaf $\mathcal{F}_g := \mathcal{F}_{\mathcal{R}_K^g(c)}$ is T_2-equivariant, and the induction functor*

$$\mathrm{Ind}_{\tilde{T}_2}^{T_2} : GKM_{\tilde{T}_2}(\Gamma(\tilde{K})) \mapsto GKM_{T_2}(\Gamma(K))$$

satisfies $\mathrm{Ind}_{\tilde{T}_2}^{T_2}(\tilde{\mathcal{F}}_g) \cong \mathcal{F}_g$, where $\tilde{\mathcal{F}}_g = \mathcal{F}_{\mathcal{R}_{\tilde{K}}^g(c)}$.

Proof. If $K \cong SU(2) \times U(1)^{r-1}$, then T_2 lies in the centre of K so the action of T_2 on $\mathcal{R}_K^g(c)$ is simply by multiplying the zeroth entry, and this induces the action on $\mathcal{F}_g$.

For the remaining cases we use of Remark 13 to identify

$$\mathcal{F}_g \cong (\tilde{\mathcal{F}}_g \coprod \tilde{\mathcal{F}}_g)/C_2 = \mathrm{Ind}_{C_2}^{\mathbb{Z}_2}(\mathcal{F}_g),$$

where the induction is with respect to the zero morphism $C_2 \to \mathbb{Z}_2$. Because $C_2 \subset \tilde{T}_2$ is the kernel of the morphism $\tilde{T}_2 \to T_2$, we can also say

$$\mathcal{F}_g \cong \mathrm{Ind}_{\tilde{T}_2}^{T_2}(\tilde{\mathcal{F}}_g)$$

thereby making $\mathcal{F}_g$ equivariant. □

Proposition 3.20. *Denote* $\mathcal{F}_g = \mathcal{F}_{\mathcal{R}_K^g(c)}$ *where* $c \in K$ *is regular. Then* $\mathcal{F}_g$ *is isomorphic as a GKM-sheaf to the g-fold convolution product* $\mathcal{F}_1 * \cdots * \mathcal{F}_1$.

Proof. For K of type (i) we have (27) so the result follows from Lemmas 3.8 and 2.20 and Proposition 3.14.

The result for general K follows from the isomorphisms

$$\mathcal{F}_1 * \cdots * \mathcal{F}_1 \cong \mathrm{Ind}_{\tilde{T}_2}^{T_2}(\tilde{\mathcal{F}}_1 * \cdots * \tilde{\mathcal{F}}_1) \cong \mathrm{Ind}_{\tilde{T}_2}^{T_2}(\tilde{\mathcal{F}}_g) \cong \mathcal{F}_g$$

from Proposition 2.21 and Lemma 3.19. □

Proposition 3.21. *If K is of type (i) or (ii), then $\mathcal{F}_1$ is monodromy sheaf, with vertex stalk $\wedge(\mathfrak{t}^*) \otimes A$ and holonomy $S_{\alpha(e)} \otimes Id_A$ for all nondegenerate edges e, where $S_{\alpha(e)} \in \mathrm{Aut}(\wedge(\mathfrak{t}^*))$ is induced by the Weyl reflection in the hyperplane $\alpha(e)^\perp \subset \mathfrak{t}$.*

Proof. For type (i) the result follows from the $SU(2)$ case, Proposition 3.13. In type (ii) we have $\mathcal{F}_1 = \mathrm{Ind}_{\tilde{T}_2}^{T_2}(\tilde{\mathcal{F}}_1)$ where $\tilde{\mathcal{F}}_1$ where is a monodromy sheaf as described above. Because the identification of the vertex stalks $\tilde{\mathcal{F}}_1(v) \cong \wedge(\mathfrak{t}^*) \otimes A$ were specifically chosen to be invariant under the $\tilde{T}_2$-action, it is clear that $\mathcal{F}_1$ is also a monodromy sheaf with the required monodromy. □

3.4.2. If c is not regular. Let $c \in T$ be a not necessarily regular element with centralizer $Z(c) \subseteq K$ and define the Weyl group at c by the formula $W_c := N_{Z(c)}(T)/T$. Because K has semi-simple rank one, $W_c = W = \mathbb{Z}/2\mathbb{Z}$ if c is not regular and W_c is trivial if c is regular. The T-action on $\mathcal{R}_K^g(c)$ extends to a $Z(c)$-action and there is a well known formula:

$$H_{Z(c)}^*(\mathcal{R}_K^g(c)) \cong H_T^*(\mathcal{R}_K^g(c))^{W_c}.$$

Proposition 3.22. *Let $\mathcal{R} := \mathcal{R}_K^g(c)$ for c not necessarily regular and let $\mathcal{F}_g$ be as defined in Section 3.4.1. Then W_c acts on both $\mathcal{F}_\mathcal{R}$ and $\mathcal{F}_g$ giving rise to an isomorphism of W_c-invariants,*

$$H_T^*(\mathcal{R})^{W_c} \cong H^0(\mathcal{F}_\mathcal{R})^{W_c} \cong H^0(\mathcal{F}_g)^{W_c}.$$

Proof. If W_c is trivial then c is regular and the statement is vacuous. So assume c is not regular so $W_c = W \cong \mathbb{Z}/2\mathbb{Z}$.

The twisted W-actions on $H_T^*(\mathcal{R})$ and on $\mathcal{F}_\mathcal{R}$ are the standard ones described in Proposition 2.22 determining the isomorphism

$$H_T^*(\mathcal{R})^W \cong H^0(\mathcal{F}_\mathcal{R})^W.$$

To construct the twisted W-action on $\mathcal{F}_g$, consider first the case of $K = \tilde{K} \cong SU(2) \times U(1)^{r-1}$. In this case the representation variety splits as $\mathcal{R} \cong \mathcal{R}_{SU(2)}^g(c_1) \times \mathcal{R}_{U(1)^{r-1}}^g(c_2)$. If $\gamma : [0,1] \to T$ is a path connecting c to a regular value, then Lemma 3.9 and Proposition 3.15 imply that $\mathcal{F}_g$ acquires a twisted W-action and $H^0(\mathcal{F}_g)^W \cong H_T^*(\mathcal{R})^W$.

Now consider K of type (ii) or (iii), fitting into the short exact sequence (28). Denote $\mathcal{R} = \coprod_{\phi(\tilde{c})=c} \mathcal{R}_{\tilde{K}}^g(\tilde{c})$ which by (29) forms a C_2^{g+1}-Galois covering $\tilde{\mathcal{R}} \to \mathcal{R}$. The elements $\tilde{c} \in \phi^{-1}(c)$ are either both regular or both nonregular. If they are both regular, then $\mathcal{R}$ is isomorphic as a T-space to $\mathcal{R}_K^g(c')$ for regular $c' \in K$ and consequently $\mathcal{F}_\mathcal{R} \cong \mathcal{F}_g$ even before taking W-invariants. Otherwise, both $\tilde{c} \in \phi^{-1}(c)$ are nonregular and W acts on both cofactors, so $H_T^*(\tilde{\mathcal{R}})^W \cong H^0(\tilde{\mathcal{F}}_g)^W \oplus H^0(\tilde{\mathcal{F}}_g)^W$ since we have already confirmed this isomorphism for $K = \tilde{K}$.

By Remark 11 the actions by C_2^{g+1} and W commute, on both $H_T^*(\mathcal{R})$ and on $\mathcal{F}_g$. Thus

$$\begin{aligned}
H_T^*(\mathcal{R})^W &\cong (H_T^*(\tilde{\mathcal{R}})^{C_2^{g+1}})^W \cong (H_T^*(\tilde{\mathcal{R}})^W)^{C_2^{g+1}} \\
&\cong (H^0(\tilde{\mathcal{F}}_g)^W \oplus H^0(\tilde{\mathcal{F}}_g)^W)^{C_2^{g+1}} \\
&\cong ((H^0(\tilde{\mathcal{F}}_g) \oplus H^0(\tilde{\mathcal{F}}_g))^{C_2^{g+1}})^W \cong H^0(\mathcal{F}_g)^W.
\end{aligned}$$

$\square$

Remark 15. It is useful to describe more explicitly the twisted W_c-action on $\mathcal{F}_g$. This action is completely determined by its restriction to the vertex set $\mathcal{V}$. We may identify the vertex set $\mathcal{V} \cong \{t \in T | t^2 \in c\}$ and W_c acts on $\mathcal{V}$ by restricting the standard action on T. The stalks over every $v \in \mathcal{V}$ are identified with $\mathcal{F}_g(v) = \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$ (see Corollary 3.7) on which W_c-acts

by $S_\alpha \otimes S_\alpha$, where S_α denotes the extension of the Weyl reflection on $\mathfrak{t}^*$ to algebra automorphisms of $\wedge(\mathfrak{t}^*)$ and $S(\mathfrak{t}^*)$.

3.5. Proving the theorems of Section 3.1

In this section, let K denote an arbitrary compact connected Lie group with maximal torus T. Denote by Δ the set of roots of $T \subset K$ and let $\Delta \subset \mathbb{P}(\Lambda)$ be the set of roots modulo scalars.

For $\alpha \in \mathbb{P}(\Lambda)$, denote by K_α the centralizer of $\ker(\alpha)$ in K. It is easy to see that

$$(31) \qquad \mathcal{R}_K^g(c)^{\ker(\alpha)} = \mathcal{R}_{K_\alpha}^g(c) \subseteq \mathcal{R}_K^g(c).$$

Lemma 3.23. *The centralizer $K_\alpha \subseteq K$ is strictly larger than T if and only if $\alpha \in \Delta$.*

Proof. Centralizers of tori in K are connected (see [BtD85] IV Theorem 2.3) so K_α is connected for any $\alpha \in \mathbb{P}(\Lambda)$. Thus K_α is strictly larger than T if and only the adjoint action of $\ker(\alpha) \subset T$ on $\mathfrak{k}$ has fixed point set larger than $\mathfrak{t}$. Since the roots $\Delta \subset \Lambda$ record the weights of the adjoint action, the result follows. $\qquad \square$

Lemma 3.23 and (31) imply that the one-skeleton of $\mathcal{R}_K^g(c)$ is the pair

$$(32) \qquad \Big(\bigcup_{\alpha \in \Delta} \mathcal{R}_{K_\alpha}^g(c), \ \mathcal{R}_T^g(c) \Big).$$

Consequently, the construction of the GKM-sheaf $\mathcal{F}_{\mathcal{R}_K^g(c)}$ reduces to the torus and the semi-simple rank one cases.

Proof of Theorem 3.1. The description of the vertex set is from Corollary 3.7. Lemma 3.23 reduces the description of the edge set to the semisimple rank one case, which is described in Proposition 3.18. $\Gamma_{\mathcal{R}_K^g(c)}$ is T_2-equivariant because $\Gamma_{\mathcal{R}_{K_\alpha}^g(c)}$ is for all $\alpha \in \mathbb{P}(\Lambda)$. $\qquad \square$

Proof of Theorem 3.2. $\mathcal{F}_g := \mathcal{F}_{\mathcal{R}_K^g(c)}$. Restricting to vertices, there is a natural isomorphism

$$(33) \qquad \mathcal{F}_g|_\mathcal{V} \cong (\mathcal{F}_1)^{*g}|_\mathcal{V}$$

as defined in Corollary 3.7. For any α, Proposition 3.20 combined with (31) provides an isomorphism $\mathcal{F}_g|_{\mathcal{V} \cup \mathcal{E}^\alpha} \cong (\mathcal{F}_1)^{*g}|_{\mathcal{V} \cup \mathcal{E}^\alpha}$ respecting (33). Gluing together completes the result. $\qquad \square$

Proof of Theorem 3.3. Let $\mathcal{R} = \mathcal{R}_K^g(c)$ where c is not necessarily regular and let $\mathcal{F}_g$ be as defined in Section 3.4.1.

The centralizer $Z(c)$ acts on $\mathcal{R}$ by conjugation, and this determines a twisted W_c-action on $\mathcal{F}_\mathcal{R}$ as described in Proposition 2.22.

The action of W_c on the restriction $\mathcal{F}_g|_\mathcal{V}$ is the one described in Remark 15. Namely W_c acts on $\mathcal{V} \cong \{t \in T | t^2 = c\}$ by the restriction of the standard action on T, and W_c acts on stalks $\mathcal{F}_g(v) \cong \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$ by the tensor product of standard representations of W_c on $\wedge(\mathfrak{t}^*)$ and on $S(\mathfrak{t}^*)$. To see that this action extends to all of $\mathcal{F}_g \cong (\mathcal{F}_1)^{*g}$, observe that W_c respects the local system of the monodromy sheaf $\mathcal{F}_1$ because for any $w \in W_c$ and $e \in \mathcal{E}$

$$(w \otimes w)(S_{\alpha(e)} \otimes Id_A)(w^{-1} \otimes w^{-1}) = S_{w\alpha(e)w^{-1}} \otimes Id_A = S_{\alpha(w \cdot e)} \otimes A$$

as an automorphism of $\wedge(\mathfrak{t}^*) \otimes Id_A$.

Now let $i_\alpha^* : \mathcal{F}_\mathcal{R}(\mathcal{V} \cup \mathcal{E}^\alpha) \to \mathcal{F}_\mathcal{R}(\mathcal{V})$ and $j_\alpha^* : \mathcal{F}_g(\mathcal{V} \cup \mathcal{E}^\alpha) \to \mathcal{F}_g(\mathcal{V})$ denote the restriction maps and identify

$$\mathcal{F}_g(\mathcal{V}) = \mathcal{F}_\mathcal{R}(\mathcal{V}) = \mathbb{C}\mathcal{V} \otimes \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*).$$

By Proposition 3.22 we know that $\operatorname{im}(i_\alpha^*)^{S_\alpha \otimes S_\alpha} = \operatorname{im}(j_\alpha^*)^{S_\alpha \otimes S_\alpha}$, so

$$H^0(\mathcal{F}_\mathcal{R})^{W_c} \cong \left(\bigcap_{\alpha \in \Delta} \operatorname{im}(i_\alpha^*) \right)^{W_c} = \left(\bigcap_{\alpha \in \Delta} \operatorname{im}(i_\alpha^*)^{S_\alpha} \right)^{W_c}$$

$$= \left(\bigcap_{\alpha \in \Delta} \operatorname{im}(j_\alpha^*)^{S_\alpha} \right)^{W_c} \cong H^0(\mathcal{F}_g)^{W_c}.$$

$\square$

Every connected compact Lie group K possesses a finite covering group $\tilde{K} \to K$ for which $\pi_1(\tilde{K})$ is torsion-free. We can exploit this fact to simplify arguments, much as we did in the previous section.

Lemma 3.24. *If K is a compact, connected Lie group for which $\pi_1(K)$ is 2-torsion free, then for all roots $\alpha \in \Delta_+$, the fundamental group $\pi_1(K_\alpha)$ is also 2-torsion-free. This implies that the exponential of the coroot $\exp(\pi i h_\alpha) \in K$ does not equal the identity.*

Proof. Consider the exact sequence of homotopy groups

$$\pi_2(K) \to \pi_2(K/K_\alpha) \to \pi_1(K_\alpha) \to \pi_1(K).$$

Here $\pi_2(K) = 0$ by Whitehead's theorem, and $\pi_2(K/K_\alpha) \cong H_2(K/K_\alpha; \mathbb{Z})$ is torsion free because K/K_α admits a Bruhat decomposition into even dimensional cells. Since $\pi_1(K)$ contains no 2-torsion, we deduce that $\pi_1(K_\alpha)$ does not either.

Since K_α has semisimple rank one, it is isomorphic one of the groups in Lemma 3.16 and is not equal to $U(1)^{(r-1)} \times SO(3)$, so it is easy to verify that $\exp(\pi i h_\alpha) \neq \mathbb{1}$. $\square$

Proof of Theorem 3.4. The restriction of $\mathcal{F}_{\mathcal{R}_K^1(c)}$ to $\mathcal{V} \cup \mathcal{E}^\alpha$ is naturally isomorphic to $\mathcal{F}_{\mathcal{R}_{K_\alpha}^1(c)}$ where K_α is of type (i) or (ii) so this follows directly from Proposition 3.21. $\square$

Proof of Theorem 3.5. The restriction of $\mathcal{F}_{\mathcal{R}_K^1(c)}$ to $\mathcal{V} \cup \mathcal{E}^\alpha$ is naturally isomorphic to $\mathcal{F}_{\mathcal{R}_{K_\alpha}^1(c)}$ so this follows directly from Lemma 3.19. $\square$

4. Cohomology calculations

For simplicity, we focus on simple groups K for which $\pi_1(K)$ has finite and odd cardinality. Let $\Phi = \mathrm{Span}_\mathbb{Z}\{\pm h_\alpha|\ \alpha \in \Delta\}$ denote the coroot lattice of K in $\mathfrak{t}$. Lemma 3.24 implies that the map $\Phi \to T_2$, $\xi \mapsto \exp(\pi i \xi)$ defines an isomorphism

$$(34) \qquad\qquad \Phi/2\Phi \cong T_2.$$

4.1. Regular c

Let $\mathcal{F}_1 = \mathcal{F}_\rho \cong \mathcal{F}_{\mathcal{R}_K^1(c)}$ is the monodromy GKM-sheaf of Theorem 3.4. Because $\mathcal{F}_1$ is T_2-equivariant, we may decompose $H^0(\mathcal{F}_1)$ into isotypical components

$$H^0(\mathcal{F}_1) \cong \bigoplus_{\chi \in \hat{T}_2} H^0(\mathcal{F}_1)^\chi.$$

The localization map $i^* : H^0(\mathcal{F}_1) = \mathcal{F}_1(\mathcal{V} \cup \mathcal{E}) \to \mathcal{F}_1(\mathcal{V})$ is injective and T_2-equivariant, so we may identify $H^0(\mathcal{F}_1)^\chi$ with the image of

$$i_\chi^* : H^0(\mathcal{F}_1)^\chi \hookrightarrow \mathcal{F}_1(\mathcal{V})^\chi \cong \wedge(\mathfrak{t}^*) \otimes A.$$

Employing Proposition 2.9, we have

$$(35) \qquad\qquad \mathrm{im}(i_\chi^*) = \bigcap_{\alpha \in \Delta} \mathrm{im}(i_{\alpha,\chi}^*),$$

where $i_{\alpha,\chi}^* : \mathcal{F}_1(\mathcal{V} \cup \mathcal{E}^\alpha)^\chi \hookrightarrow \mathcal{F}_1(\mathcal{V})^\chi$ is the sheaf restriction.

Given $\alpha \in \Delta$ a root, choose a basis $\{\alpha, \beta_1, \ldots, \beta_{r-1}\}$ of $\mathfrak{t}^*$ where the β_i are orthogonal to α (equivalently, the β_i are $+1$ eigenvectors for the reflection defined by α). If $m \in \wedge(\mathfrak{t}^*)$ is a monomial in this basis define $\deg_\alpha(m) = 1$ if α is a factor of m and $\deg_\alpha(m) = 0$ if not.

Proposition 4.1. *The image of $i^*_{\alpha,\chi}$ in $\wedge(\mathfrak{t}^*) \otimes A$ is generated as an A-submodule by elements*

$$m \otimes 1 \ if \ (-1)^{\deg_\alpha(m)}\chi(\exp(\pi i h_\alpha)) = 1$$

and

$$m \otimes \alpha \ if \ (-1)^{\deg_\alpha(m)}\chi(\exp(\pi i h_\alpha)) = -1$$

as m varies over a monomial basis of $\wedge(\mathfrak{t}^*)$.

Proof. The basis is chosen so that for a monomial $m \in \wedge(\mathfrak{t}^*)$,

$$S_\alpha(m) = (-1)^{\deg_\alpha(m)}m.$$

T_2 factors as $T_2' \times T_2''$, where T_2'' acts freely on $\mathcal{E}^\alpha$ and $T_2' \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $\exp(\pi i h_\alpha)$, which fixes the edges in $\mathcal{E}^\alpha$ and transposes their incident vertices. Thus we may restrict attention to a single edge, and identify i^*_α with the intersection of the image of the matrix

$$\begin{pmatrix} 1 \otimes 1 & 1 \otimes \alpha \\ S_\alpha \otimes 1 & -S_\alpha \otimes \alpha \end{pmatrix}$$

in $(\wedge(\mathfrak{t}^*) \otimes A)^{\oplus 2}$ with either the diagonal if $\chi(\exp(\pi i h_\alpha)) = 1$ or the anti-diagonal if $\chi(\exp(\pi i h_\alpha)) = -1$, from which the result follows. $\square$

In [Bai08] Chapter 8, it was shown that the projection map $\mathcal{R}_K^g(c) \to K^g$ sending $(k_0, \ldots, k_g)$ to $(k_1, \ldots, k_g)$ is a "cohomological covering map". In particular, this means that the induced map on cohomology is an injection $H_T^*(K^g) \hookrightarrow H_T^*(\mathcal{R}_K^g(c))$. The analogue of this for $\mathcal{F}_{\mathcal{R}_K^g(c)}$ is the following.

Corollary 4.2. *Let $\mathcal{F}_g$ denote the g-fold convolution product $\mathcal{F}_1 * \cdots * \mathcal{F}_1$. Then the T_2 invariant part of cohomology $H^0(\mathcal{F}_g)^{T_2}$ is isomorphic to $H_T^*(K^g)$.*

Proof. For $g = 1$, just compare Proposition 4.1 with Example 3. For $g > 1$ apply Proposition 2.18. $\square$

Remark 16. Using Theorem 35, we identify $H^*_T(K) \cong H^0(\mathcal{F}_1)^{T_2}$ as a sub-module of $\wedge(\mathfrak{t}^*) \otimes S(\mathfrak{t}^*)$. The equivariant fundamental class of K spans the one-dimensional subspace $\wedge^r(\mathfrak{t}^*) \otimes (\prod_{\alpha \in \Delta} \alpha)$. Compare this with Example 3.

Let proj_r denote projection from $\wedge(\mathfrak{t}^*) \otimes A$ onto $\wedge^r(\mathfrak{t}^*) \otimes A$.

Lemma 4.3. *Use Theorem 35 to identify $H^0(\mathcal{F}_1)^\chi$ with a submodule of $\wedge(\mathfrak{t}^*) \otimes A$. Then*

$$\mathrm{proj}_r(H^0(\mathcal{F}_1)^\chi H^0(\mathcal{F}_1)^\chi) \subseteq \wedge^r(\mathfrak{t}^*) \otimes (\prod_{\alpha \in \Delta} \alpha)A.$$

Proof. The fact that $H^0(\mathcal{F}_1)$ is a T_2-equivariant algebra implies that $H^0(\mathcal{F}_1)^\chi H^0(\mathcal{F}_1)^\chi \subseteq H^0(\mathcal{F}_1)^{T_2} \subseteq \wedge(\mathfrak{t}^*) \otimes A$. By Remark 16, we know

$$\mathrm{proj}_r(H^0(\mathcal{F}_1)^{T_2}) = H^0(\mathcal{F}_1)^{T_2} \cap (\wedge^r(\mathfrak{t}^*) \otimes A) = \wedge^r(\mathfrak{t}^*) \otimes \left(\prod_{\alpha \in \Delta} \alpha\right) A.$$

$\square$

We will use this Lemma 4.3 in combination with

Lemma 4.4. *Let $d \in wedge^r(\mathfrak{t}^*) \otimes A$ be a homogeneous element and let M be a free A-submodule of the A-algebra $\wedge(\mathfrak{t}^*) \otimes A$ with homogeneous free generators $\{x_1, \ldots, x_{2^r}\}$ satisfying*

$$\mathrm{proj}_r(x_i x_{2^r - j}) = a_{i,j}d,$$

where $[a_{i,j}]$ is a nonsingular matrix of complex numbers. Then M is maximal among submodules satisfying $\mathrm{proj}_r(MM) \subseteq d \otimes A$.

Proof. Suppose there exists $y \in (\wedge(\mathfrak{t}^*) \otimes A) \setminus M$ for which $\mathrm{proj}_r(My) \subseteq d \otimes A$. Then by subtracting an A-linear combination of x_i from y, we may assume that $\mathrm{proj}_r(My) = 0$. But this implies that $y = 0$ because M is of maximal rank and the proj_r is a nondegenerate pairing for $\wedge(\mathfrak{t}^*) \otimes A$. $\square$

4.2. Nonregular c

For $c \in T \subseteq K$, the W_c-action on $\mathcal{F}_g$ defined in Remark 15 does not generally commute with the T_2-action. Instead W_c acts by conjugation on T_2, restricting the standard action of W_c on T. Consequently, $H^0(\mathcal{F}_g)^{W_c}$ decomposes

into a sum over the orbit space $\hat{T}_2/W_c$,

$$(36) \qquad H^0(\mathcal{F}_g)^{W_c} \cong \bigoplus_{[\chi] \in \hat{T}_2/W_c} (H^0(\mathcal{F}_g)^{W_c})[\chi].$$

Corollary 4.5. *Let $(W_c)_\chi$ be the stabilizer of $\chi \in T_2$. If $H^0(\mathcal{F}_1)^\chi$ is free over A, then*

$$(H^0(\mathcal{F}_g)^{W_c})[\chi] \cong (H^0(\mathcal{F}_1)^\chi \otimes_A \cdots \otimes_A H^0(\mathcal{F}_1)^\chi)^{(W_c)_\chi}.$$

Proof. The W_c-action permutes the summands of $\bigoplus_{\chi \in \hat{T}_2} H^0(\mathcal{F}_g)^\chi$ according to its action on $\hat{T}_2$, so

$$(H^0(\mathcal{F}_g)^{W_c})[\chi] \cong (H^0(\mathcal{F}_g)^\chi)^{(W_c)_\chi}$$

and the result then follows from the isomorphism $H^0(\mathcal{F}_g)^\chi \cong H^0(\mathcal{F}_1)^\chi \otimes_A \cdots \otimes_A H^0(\mathcal{F}_1)^\chi$ of Theorem 3.2 and Proposition 2.18. $\qquad\square$

A choice of base vertex v_* determines an isomorphism

$$(37) \qquad \mathcal{F}_g(\mathcal{V})^\chi \cong \mathcal{F}_g(v_*) = \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$$

and we want to describe the twisted $(W_c)_\chi$-action on $\mathcal{F}_g(\mathcal{V})^\chi$ in terms of this identification.

Lemma 4.6. *Given a base vertex v_*, let*

$$\tau : (W_c)_\chi \to \mathrm{Aut}(\wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*))$$

denote the twisted action induced from (37). Let τ^{st} denote the standard twisted W-action on $\wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$, induced by the standard Weyl group action on $\mathfrak{t}$. Then $\tau_w = \chi(t)\tau_w^{\mathrm{st}}$ where $t \in T_2$ satisfies $t \cdot v_* = w \cdot v_*$. In particular, the restriction of τ to $(W_c)_\chi$ is the standard action if either $(W_c)_\chi$ fixes the base vertex v_* or if χ is trivial.

Proof. It was explained in the proof of Theorem 3.3 that W acts on $\mathcal{F}_g(\mathcal{V})^\chi \cong \mathbb{C}\mathcal{V} \otimes \wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$ by the tensor product of τ^{st} with the action on $\mathbb{C}\mathcal{V}$ induced by the action on $\mathcal{V}$. If v_* is fixed by $(W_c)_\chi$ then the identification (37) is undisturbed and $\tau = \tau^{\mathrm{st}}$. Otherwise it is necessary to correct by multiplying by t, which introduces the factor $\chi(t)$. $\qquad\square$

4.3. Type A_2

In this section we let $K = SU(3)$ or $PSU(3)$. Then because $\pi_1(K)$ has odd order, (34) holds. Let $\{e_1, e_2, e_3\}$ denote the standard basis in $\mathbb{C}^3$ with the standard pairing. Identify $\mathfrak{t} = (e_1 + e_2 + e_3)^\perp \subset \mathbb{C}^3$. We choose positive roots $\Delta_+ = \{\alpha_{i,j} = e_i - e_j | i < j\}$ and use the pairing to identify $\alpha_{i,j} = h_{\alpha_{i,j}}$. The group $\Phi/2\Phi \cong T_2 \cong (\mathbb{Z}/2)^2$ has coset representatives consisting of 0 and the three positive roots Δ_+. There are three nontrivial characters χ_1, χ_2, χ_3 of $T_2 \cong \Phi/2\Phi$, defined by $\chi_k([\alpha_{i,j}]) = 1$ if and only if i, j, k are pairwise distinct. The Weyl group S_3 acts on the three nontrivial characters χ_1, χ_2, χ_3 in the standard fashion.

Proposition 4.7. *The summands $H^0(\mathcal{F}_1)^{\chi_k}$ are pairwise isomorphic free modules for $k = 1, 2, 3$. Under the injection*

$$i^* : H^0(\mathcal{F}_1)^{\chi_3} \hookrightarrow \wedge\mathfrak{t}^* \otimes S(\mathfrak{t}^*)$$

described in Theorem 35 and using coordinates $x_1 = \alpha_{1,3}$ and $x_2 = \alpha_{2,3}$, the free basis is

$$(38) \qquad\qquad 1 \otimes x_1 x_2, \qquad x_1 \wedge x_2 \otimes (x_1 - x_2),$$
$$(39) \qquad x_1 \otimes x_2 + x_2 \otimes x_1, \qquad x_1 \otimes x_2(x_1 - x_2) - x_2 \otimes x_1(x_1 - x_2).$$

Proof. That the $H^0(\mathcal{F}_1)^{\chi_k}$ are pairwise isomorphic follows from the fact that the χ_k are permuted by W.

Let M denote the submodule of $\wedge(\mathfrak{t}^*) \otimes A$ generated by the purported free basis. It is straightforward linear algebra to check that M is contained in the intersection $\bigcap_{\alpha \in \Delta} \mathrm{im}(i_\alpha^*)$ of Theorem 35, so we conclude that $M \subseteq H^0(\mathcal{F}_1)^{\chi_3}$.

To prove $M \supseteq H^0(\mathcal{F}_1)^{\chi_3}$ observe that M satisfies the conditions of Lemma 4.4 with d an element of $\wedge^2(\mathfrak{t}^*) \otimes x_1 x_2(x_1 - x_2)$. Combining this with Lemma 4.3 completes the proof. $\qquad\qquad\square$

Corollary 4.8. *For $g \geq 1$, the A-module $H^0(\mathcal{F}_g)$ is free with Poincaré series*

$$P_t(H^0(\mathcal{F}_g)) = ((1 + t^3 + t^5 + t^8)^g + 3(t^3 + 2t^4 + t^5)^g)P_t(BT).$$

Proof. By Corollary 4.2 we know $H^0(\mathcal{F}_g)^{T_2} = H_T^*(K^g)$ and this explains the term $(1 + t^3 + t^5 + t^8)^g P_t(BT)$. The remaining three isotypical components

$H^0(\mathcal{F}_g)^\chi \cong (H^0(\mathcal{F}_1)^\chi)^{\otimes g}$ are isomorphic and have Poincaré series $(t^3 + 2t^4 + t^5)^g P_t(BT)$ by Proposition 4.7. $\qquad\square$

Corollary 4.9. *Let $\epsilon \in Z(K)$ and $\mathcal{R} = \mathcal{R}_K^g(\epsilon)$. Then $H^0(\mathcal{F}_R)^W$ is a free A^W-module with Poincaré series*

$$((1 + t^3 + t^5 + t^8)^g + (1 + t^2 + t^4)(t^3 + 2t^4 + t^5)^g)P_t(BK).$$

Proof. Observe that every $\epsilon \in Z(K)$ has a square root in $Z(K)$, so the $W_\epsilon = W$ action fixes one vertex and permutes the remaining three. By Lemma 4.6 the induced W-action on $\wedge(\mathfrak{t}^*)^{\otimes g} \otimes S(\mathfrak{t}^*)$ is the standard one.

Because W permutes the three nontrivial characters $\{\chi_1, \chi_2, \chi_3\}$, $H^0(\mathcal{F}_\mathcal{R})^W$ decomposes into a sum of two A^W-modules as explained in Corollary 4.5. The trivial character contributes the first summand $H_T^*(K^g)^W = H_K^*(K^g)$ with Poincaré series

$$(1 + t^3 + t^5 + t^8)^g P_t(BK).$$

The remaining summand is

$$\left(\bigoplus_{i=1}^{3} H^0(\mathcal{F}_\mathcal{R}^{\chi_i}) \right)^W \cong \left((H^0(\mathcal{F}_1)^{\chi_3})^{\otimes g} \right)^{W_{\chi_3}},$$

where $W_{\chi_3} = <(1,2)>$ is the stabilizer of χ_3. The free basis of Proposition 4.7 of $H^0(\mathcal{F}_1)^{\chi_3}$ is invariant under $(1,2)$ so the basis of $(H^0(\mathcal{F}_1)^{\chi_3})^{\otimes g}$ is also invariant and we deduce that

$$P_t(((H^0(\mathcal{F}_1)^{\chi_3})^{\otimes g})^{W_{\chi_3}}) = (t^3 + 2t^4 + t^5)^g P_t(A^{W_{\chi_3}}) = \frac{(t^3 + 2t^4 + t^5)^g}{(1 - t^2)(1 - t^4)}$$
$$= (t^3 + 2t^4 + t^5)^g (1 + t^2 + t^4) P_t(BK).$$

$\qquad\square$

Proof of Theorem 1.2. In [Bai09] the Betti numbers of $H_{U(3)}^*(\mathcal{R}_{U(3)}^g(\mathbb{1}))$ were computed and it was shown to be torsion-free as an $H^*(BU(3))$-module. It is easily deduced that $H_{SU(3)}^*(\mathcal{R}_{SU(3)}^g(\mathbb{1}))$ is torsion-free and that

$$P_t^{SU(3)}(\mathcal{R}_{SU(3)}^g(\mathbb{1})) = ((1 + t^3 + t^5 + t^8)^g + (1 + t^2 + t^4)(t^3 + 2t^4 + t^5)^g)$$
$$\times oP_t(BSU(3)).$$

By Theorem 2.8 this means that there is an injective morphism of graded algebras

$$H^*_{SU(3)}(\mathcal{R}^g_{SU(3)}(\mathbb{1})) \hookrightarrow H^0(\mathcal{F}_{\mathcal{R}^g_{SU(3)}(\mathbb{1})})$$

which by comparing Betti numbers must be an isomorphism, so $H^0(\mathcal{F}_{\mathcal{R}^g_{SU(3)}(\mathbb{1})})$ is free by Corollary 4.9. $\qquad\square$

4.4. Type B_2

The root system of $Spin(5)$ has positive roots $\{2e_1, 2e_2, -e_1 + e_2, e_1 + e_2\} \subset \mathbb{C}^2$, with corresponding coroots $\{e_1, e_2, e_1 + e_2, e_1 - e_2\}$ under the standard pairing. Thus $T_2 \cong \Phi/2\Phi$ is generated by $[e_1]$ and $[e_2]$, so $[e_1 + e_2] = [e_1 - e_2] \in \Phi/2\Phi$. The character group $\hat{T}_2$ has four elements defined in the basis $\{[e_1], [e_2]\}$ by matrices $[1, 1], [-1, -1], [1, -1], [-1, 1]$, and the Weyl group interchanges only $[1, -1]$ with $[-1, 1]$.

Proposition 4.10. *For $K = \mathrm{Spin}(5)$ and all $\chi \in \hat{T}_2$, the A-submodule $H^0(\mathcal{F}_1)^\chi \subset \wedge(\mathfrak{t}^*) \otimes A$ is free. The generators of $H^0(\mathcal{F}_1^{[-1,-1]})$ are*

$$1 \otimes e_1 e_2, e_1 \wedge e_2 \otimes (e_1 + e_2)(e_1 - e_2),$$
$$e_1 \otimes e_2 + e_2 \otimes e_1 e_1 \otimes e_2(e_1^2 - e_2^2) + e_2 \otimes e_1(e_2^2 - e_1^2).$$

Changing basis to $x = e_1 + e_2$ and $y = e_1 - e_2$, the generators of $H^0(\mathcal{F}_1^{[-1,1]})$ are

$$1 \otimes xy(x + y), x \wedge y \otimes (x - y),$$
$$x \otimes y(x + y) + y \otimes x(x + y), x \otimes y(x - y) - y \otimes x(x - y).$$

Proof. Analogous to Proposition 4.7. $\qquad\square$

Corollary 4.11. *For $K = \mathrm{Spin}(5)$ and $g \geq 1$, the ring $H^0(\mathcal{F}_g)$ is a free module over A with Poincaré series*

$$P_t(H^0(\mathcal{F}_g)) = (P_t(K)^g + (t^3 + t^4 + t^6 + t^7)^g + 2(t^4 + 2t^5 + t^6)^g)P_t(BT).$$

Furthermore, under the augmentation morphism $A \to \mathbb{C}$, the extension of scalars $H^0(\mathcal{F}_g) \otimes_A \mathbb{C}$ is a dimension $10g$ Poincaré duality ring over $\mathbb{C}$.

Proof. The calculation of the Poincaré series is analogous to Corollary 4.8. Since $H^0(\mathcal{F}_g)$ is free over A, we have

$$P_t(H^0(\mathcal{F}_g) \otimes_A \mathbb{C}) = P_t(H^0(\mathcal{F}_g))(1 - t^2)^2$$
$$= (1 + t^3 + t^7 + t^{10})^g + (t^3 + t^4 + t^6 + t^7)^g$$
$$+ 2(t^4 + 2t^5 + t^6)^g$$

The fundamental class for the Poincaré duality pairing is a nonzero element of

$$\left((\wedge^r(\mathfrak{t}^*) \otimes \prod_{\alpha_0 \in \Delta_+} \alpha_0) \right)^{\otimes g} \in H^0(\mathcal{F}_1^{T_2})^{\otimes g} = H^0(\mathcal{F}_g^{T_2})$$

and the pairing is just the extension of scalars of $(\mathrm{proj}_r)^{\otimes g}$. The generators listed in Proposition 4.10 provide the nondegenerate pairs.   □

Remark 17. For any regular $c \in \mathrm{Spin}(5)$, if $\mathcal{R}^g_{\mathrm{Spin}(5)}(c)$ is equivariantly formal then its ordinary cohomology ring is isomorphic to $H^0(\mathcal{F}_g) \otimes_A \mathbb{C}$. Since we know that $\mathcal{R}^g_{\mathrm{Spin}(5)}(c)$ is an orientable manifold of dimension $10g$, Corollary 4.11 is consistent with the conjecture that $\mathcal{R}^g_{\mathrm{Spin}(5)}(c)$ is equivariantly formal.

In terms of the basis e_1, e_2, the Weyl group for type B_2 is generated by (any two of) the reflections:

$$s_1 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Corollary 4.12. *Let $\epsilon \in \mathrm{Spin}(5) = K$ be the nontrivial central element and let $\mathcal{R} = \mathcal{R}^g_K(\epsilon^i)$ for $i = 0$ or 1. The ring of Weyl invariants $H^0(\mathcal{F}_\mathcal{R})^{W_{\epsilon^i}}$ is a free module over $A^W = H^*(BK)$ with Poincaré series*

$$(P_t(K)^g + (t^3 + t^4 + t^6 + t^7)^g + t^2(1 + t^4)(t^4 + 2t^5 + t^6)^g)P_t(BK)$$

if $g + i$ is even and

$$(P_t(K)^g + t^4(t^3 + t^4 + t^6 + t^7)^g + t^2(1 + t^4)(t^4 + 2t^5 + t^6)^g)P_t(BK)$$

if $g + i$ is odd.

Proof. The generators of $H^0(\mathcal{F}_1)^{[-1,-1]}$ are not invariant under the standard $W_{\epsilon^i} = W$ action, but instead are weight vectors, of weight -1 for s_1 and s_2

and weight $+1$ for s_3. Taking tensor powers, it follows that for even g, the free A-generators of $H^0(\mathcal{F}_g)^{[-1,-1]}$ equal the free $A^W = H^*(BK)$ generators of $(H^0(\mathcal{F}_g)^{[-1,-1]})^{W_1}$, while for odd g the generators of $(H^0(\mathcal{F}_g)^{[-1,-1]})^{W_1}$ are $e_1 e_2$ times the generators of $H^0(\mathcal{F}_g)^{[-1,-1]}$. On the other hand, one checks that under the twisted action of W_ϵ on $H^0(\mathcal{F}_1)^{[-1,-1]}$ the generators are invariant, so for odd g the generators of $(H^0(\mathcal{F}_g)^{[-1,-1]})^{W_\epsilon}$ coincide with those of $H^0(\mathcal{F}_g)^{[-1,-1]}$ while for even g we must shift by $e_1 e_2$.

The remaining pair of characters are permuted by the Weyl group, so we must consider $(H^0(\mathcal{F}_g^{[1,-1]}) \oplus H^0(\mathcal{F}_g^{[-1,1]}))^{W_\epsilon} \cong H^0(\mathcal{F}_g^{[1,-1]})^{W_{[1,-1]}}$, which will be a free module over $A^{W_{[1,-1]}}$. Similar considerations now apply, but in this case generators must be shifted by a degree two element (either e_1 or e_2 depending on the parity of $g + i$). $\qquad\square$

4.5. Type G_2

The compact group G_2 has root system spanning the orthogonal complement of $e_1 + e_2 + e_3$ in $\mathbb{C}^3$, with simple roots $\alpha = e_1 - e_2$ and $\beta = -2e_1 + e_2 + e_3$ and remaining positive roots $\alpha + \beta$, $2\alpha + \beta$, $3\alpha + \beta$ and $3\alpha + 2\beta$. The Weyl group is isomorphic to the dihedral group D_6 and acts transitively on the nontrivial characters of $T_2 \cong \Phi/2\Phi$, with stabilizer isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. If we let χ denote the nontrivial character sending $[h_\alpha]$ and $[h_{3\alpha+2\beta}]$ to 1 then W_χ is generated by matrices

$$s_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the basis $x = \alpha + \beta$ and $y = 2\alpha + \beta$.

Proposition 4.13. *For $K = G_2$, the module $H^0(\mathcal{F}_1)^\chi$ is a free A-module generated by*

$$1 \otimes xy(2x - y)(2y - x), (x \otimes y - y \otimes x)(x^2 - y^2),$$
$$x \wedge y \otimes (x^2 - y^2), (x \otimes y + y \otimes x)(2x - y)(2y - x).$$

Proof. Analogous to Proposition 4.7. $\qquad\square$

Corollary 4.14. *For $K = G_2$, the graded ring $H^0(\mathcal{F}_g)$ is a free A-module with Poincaré series $(P_t(K)^g + 3(t^6 + 2t^7 + t^8)^g)P_t(BT)$. Moreover the extension of scalars by the augmentation map $H^0(\mathcal{F}_g) \otimes_A \mathbb{C}$ is a Poincaré duality ring of dimension $14g$.*

Proof. Analogous to Corollary 4.11.   □

The simply connected group G_2 has trivial centre.

Corollary 4.15. *For $K = G_2$, the ring $H^0(\mathcal{F}_{\mathcal{R}^g_K(\mathbb{1})})^W \cong (H^0(\mathcal{F}_g))^W$ is a free A^W-module with Poincaré series*

$$(P_t(K)^g + \frac{1}{2}(1 + t^4 + t^8)(t^6 + t^7)^g((t^4 + t^2)(t + 1)^g + (t^2 - 1)(t - 1)^g))P_t(BK).$$

Proof. We have $P_t(H^0(\mathcal{F}_g)^W) = P_t(K^g)P_t(BK) + P_t((H^0(\mathcal{F}_1)^\chi)^{\otimes g})^{W_\chi})$. The formula follows by computation from Proposition 4.13.   □

4.6. Type A_3

Let $\{e_1, e_2, e_3, e_4\}$ denote the standard basis in $\mathbb{C}^4$ with the standard pairing. Identify $\mathfrak{t} = (e_1 + e_2 + e_3 + e_4)^\perp \subset \mathbb{C}^3$. We choose positive roots $\Delta_+ = \{\alpha_{i,j} = e_i - e_j | i < j\}$ and use the pairing to identify $\alpha_{i,j} = h_{\alpha_{i,j}}$. The Weyl group $W = S_4$ acts on the group of characters of $T_2 = \Phi/2\Phi$ with two nontrivial orbits, distinguished by where they send $-\mathbb{1}$.

Proposition 4.16. *Let $\chi : \Phi/2\Phi \to \mathbb{Z}/2$ be the nontrivial character sending $[h_{\alpha_{1,2}}], [h_{\alpha_{1,3}}], [h_{\alpha_{2,3}}]$ to 1 (χ lies in orbit of size 4). Then $H^0(\mathcal{F}_1)^\chi$ is a free A-module. Choosing coordinates $x_i = \alpha_{i,4}$, $i = 1, 2, 3$, we have free basis*

$$1 \otimes x_1 x_2 x_3, (x_1 \wedge x_2 \wedge x_3) \otimes (x_1 - x_2)(x_2 - x_3)(x_3 - x_1),$$
$$\sum_{c.p} x_1 \otimes x_2 x_3, \sum_{c.p.} (x_1 \wedge x_2) \otimes (x_1 - x_2) x_1 x_2 x_3,$$
$$\sum_{c.p.} x_1 \otimes x_1 x_2 x_3, \sum_{c.p.} (x_1 \wedge x_2) \otimes (x_1 - x_2) x_2 x_3,$$
$$\sum_{c.p.} x_1 \otimes (x_2 x_3)^2, \sum_{c.p.} (x_1 \wedge x_2) \otimes (x_1 - x_2) x_3.$$

where the sums are over cyclic permutations by the 3-cycle $(1, 2, 3)$.

Proof. Analogous to Proposition 4.7.   □

On the other hand, the remaining Weyl orbit of characters gives rise to nonfree modules:

Proposition 4.17. *Let* $\chi : \Phi/2\Phi \to \mathbb{Z}/2$ *be the nontrivial character satisfying* $\chi(h_{\alpha_{1,2}}) = \chi(h_{\alpha_{3,4}}) = 1$. *Then* $H^0(\mathcal{F}_1)^\chi$ *is not free over* A. *The Hilbert series of* $H^0(\mathcal{F}_1)^\chi$ *is*

$$P_t(H^0(\mathcal{F}_1)^\chi) \cong \frac{(2t^7 + 4t^8 + 2t^9 + t^{10} + t^{11} - t^{12} - t^{13})}{(1 - t^2)^3}.$$

Proof. The calculation was done using MAGMA [BCP97] (explained in Appendix A). Because the numerator of the Hilbert series is has negative coefficients, the module cannot be free. □

This example disproves the general conjecture that $\mathcal{R}^g_K(c)$ is equivariantly formal for all c and K, because it fails when $K = SU(4)$ and c is regular.

4.7. Data tables

The following tables list the Hilbert series of $H^0(\mathcal{F}_1)$ and whether or not it is a free A-module, for several simply connected compact Lie groups K. All results not already described in Section 4 were obtained using MAGMA [BCP97].

The terms correspond to into Weyl orbits of T_2-isotypical components of $H^0(\mathcal{F}_1)$. Every term containing no negative coefficients corresponds to a free A-module summand.

Lie type	Free?	Hilbert series of $H^0(\mathcal{F}_{\mathcal{R}^1_K(c)})$ times $(1-t)^{\mathrm{rk}(K)}$ for regular c.
A_2	Yes	$P_t(A_2) + 3(t^3 + 2t^4 + t^5)$
B_2	Yes	$P_t(B_2) + 2(t^4 + 2t^5 + t^6) + (t^3 + t^4 + t^6 + t^7)$
G_2	Yes	$P_t(G_2) + 3(t^6 + 2t^7 + t^8)$
A_3	No	$P_t(A_3) + 4(1+t)^2(t^8 + t^5) + 3(1+t)(-t^{12} + t^{10} + 2t^8 + 2t^7)$
B_3	No	$P_t(B_3) + 3t^7(t+1)(1 + 2t^3 + t^6) + 4t^9(t+1)^2(t^3 + 1)(-t^3 + t^2 + 1)$
C_3	No	$P_t(C_3) + 3t^8(t+1)^2(t^3 + 1) + 3t^9(t+1)(-t^9 + 2t^5 + t^2 + t + 1) + t^5(t+1)(t^3 + 1)(t^7 + 1)$
A_4	No	$P_t(A_4) + 5t^7(t+1)^2(t^3 + 1)(t^5 + 1) + 10t^{11}(t+1)^2(t^3 + 1)(-t^4 + t + 2)$
B_4	No	$P_t(B_4) + 4t^{11}(t+1)(t^3 + 1)^2(t^7 + 1) + 3t^{15}(t+1)(t^3 + 1)(-t^{11} + t^7 + 2t^3 + t^2 + 1) + 8t^{16}(t+1)^2(t^3 + 1)(-t^8 + t^5 + t^2 + 1)$

C_4	No	$P_t(C_4) + 4t^{12}(t+1)^2(t^3+1)(t^7+1) + t^7(t+1)(t^3+1)(t^7+1)(t^{11}+1) + 6t^{16}(t+1)^2(-t^{13}+3t^5+t^2+1) + 4t^{15}(t+1)^2(t^3+1)^2(-t^9+t^8-t^7+2t^6-t^5+t^4-2t^3+t^2+1)$
D_4	No	$P_t(D_4) + 12t^{11}(t+1)^2(t^3+1)(-t^8+t^7+t^2+1) + 3t^{12}(t+1)^2(-3t^9+6t^5+t^2-t+1)$
F_4	No	$P_t(F_4) + 3t^{15}(t+1)(t^3+1)(t^7+1)(t^{11}+1) + 11t^{24}(t+1)^2(t^3+1)(-t^{14}+t^{11}-t^8+t^6+t^5-t^3+t^2+1)$

In the following table we collect Hilbert series for $H^0(\mathcal{F}_{\mathcal{R}^1_K(\mathbb{1})})^W$ divided by $P_t(BK)$, where $\mathbb{1} \in K$ is the identity.

Lie type	Free?	Hilbert series for $H^0(\mathcal{F}_{\mathcal{R}^1_K(\mathbb{1})})^W$ divided by $P_t(BK)$
A_2	Yes	$P_t(A_2) + (t^4+t^2+1)(t^5+2t^4+t^3)$
B_2	Yes	$P_t(B_2) + t^4(t^7+t^6+t^4+t^3) + t^2(t^4+1)(t^6+2t^5+t^4)$
G_2	Yes	$P_t(G_2) + \frac{1}{2}t^6(t+1)(t^8+t^4+1)(t^5+t^4+2t^3-t+1)$

Appendix A. MAGMA Calculation

To compute the Hilbert series of $H^0(\mathcal{F}_1)$ and to check whether it is free over A, we use a Magma program. The program uses Magma functions for Coxeter groups and root systems. The program begins by defining an A-module M isomorphic to $\wedge(\mathfrak{t}^*) \otimes A$. Then it constructs the submodules $\mathrm{im}(i^*_{\chi,\alpha})$ using the generators described in Proposition 4.1. Finally, it intersects these submodules to get $\mathrm{im}(i^*_\chi) = H^0(\mathcal{F}_1)^\chi$ and applies the functions IsFree and HilbertSeries, to test freeness and obtain the Hilbert series.

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