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BYPASS ATTACHMENTS AND HOMOTOPY CLASSES OF 2-PLANE FIELDS IN CONTACT TOPOLOGY

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We use the generalized Pontryagin–Thom construction to analyze the effect of attaching a bypass on the homotopy class of the contact structure. In particular, given a three-dimensional contact manifold with convex boundary, we show that the bypass triangle attachment changes the homotopy class of the contact structure relative to the boundary, and the difference is measured by the homotopy group $\pi_3(S^2)$.

The goal of this paper is to study how bypass attachments affect the homotopy type of the contact structure on a given contact manifold with convex boundary. Although the notion of a *bypass* was defined by K. Honda in [**5**] and has been used in various classification problems in three-dimensional contact geometry, it has not been clear until now how this operation changes the homotopy class of the underlying 2-plane field distribution. In particular, we will see in this paper how a special sequence of bypass attachments, namely, a *bypass triangle attachment*, affects the homotopy type of the contact structure.

Let M be an compact oriented 3-manifold with boundary. Let ξ and ξ' be two co-oriented contact structures on M such that $\xi = \xi'$ in the complement of an open ball $B^3 \subset \mathrm{int}(M)$. Using a slight generalization of the Pontryagin–Thom construction, also known as the *framed cobordism construction*, to the case of compact manifolds with boundary, we define a three-dimensional obstruction class $o_3(\xi, \xi') \in \mathbb{Z}/d(\xi)$, where $d(\xi)$ is the divisibility of the Euler class $e(\xi) = e(\xi') \in H^2(M, \mathbb{Z})$, and use it to distinguish the homotopy classes of ξ and ξ'. See [**9**] for the original, and more general, discussion of such constructions.

The main result of this paper is as follows. The definition of a bypass triangle attachment will be given in Definition 2, and its construction will be discussed in more details in later sections.

Theorem 1. *If (M, ξ) is a contact manifold with convex boundary, and ξ' is the contact structure obtained from ξ by attaching a bypass triangle on ∂M, then $o_3(\xi, \xi') = -1$. In particular, ξ' is not homotopic to ξ relative to the boundary as 2-plane field distributions.*

Theorem 1 is an important ingredient in the analysis of the universal cover of a contact category $\mathscr{C}(\Sigma)$ defined in [**4**], which we denote by $\widetilde{\mathscr{C}}(\Sigma)$. Here Σ is a closed, oriented surface. Note that the objects of $\mathscr{C}(\Sigma)$ are dividing sets on Σ, and the morphisms are isotopy classes of tight contact structures on $\Sigma \times [0,1]$. Loosely speaking $\widetilde{\mathscr{C}}(\Sigma)$ in addition keeps track of the homotopy class of the contact structure as a 2-plane field, which defines a grading on $\widetilde{\mathscr{C}}(\Sigma)$. It is observed in [**4**] that $\widetilde{\mathscr{C}}(\Sigma)$ is a category equipped with distinguished triangles given by the bypass triangle attachment defined above. Theorem 1 asserts that the shift functor in $\widetilde{\mathscr{C}}(\Sigma)$ actually decreases the grading by 1. In fact, this observation has been further developed from an algebraic point of view in the very recent work of Y. Tian [**10, 11**].

Another application of Theorem 1 is that it plays an important role in the author's reproof of Eliashberg's theorem [**1**] on the classification of overtwisted contact structures. See [**7**] for more details.

Now let us clarify some technical terms involved in the statement of Theorem 1. First we recall the definition of a bypass from [**5**]. Let Σ be a convex surface, and α be a Legendrian arc on Σ which intersects the dividing set Γ_Σ in three points. A *bypass* along α on Σ is half of an overtwisted disk whose boundary is the union of two Legendrian arcs $\alpha \cup \beta$, where the *Thurston–Bennequin invariants*[1] of α and β are -1 and 0, respectively. A bypass attachment can be viewed in the following two different ways: (1) Inside a contact manifold, a bypass attachment isotopes a convex surface across the bypass to obtain a new convex surface, where the dividing set changes as depicted in Figure 1. (2) For a contact manifold with convex boundary, a bypass attachment on the boundary changes the isotopy class of the contact structure. In particular, the dividing set on the boundary changes in the same way as depicted in Figure 1. Throughout this paper we adopt the second point of view unless otherwise specified. We denote by σ_α the bypass attachment along a Legendrian arc α. See Section 1 for a detailed construction of the bypass attachment.

In this paper, we study the effect of a bypass attachment on the homotopy class of the contact structure using the generalized Pontryagin–Thom construction discussed in Section 2.2. More precisely, we consider a contact 3-manifold (M, ξ) with convex boundary. Let $\alpha \subset \partial M$ be a Legendrian arc along which a bypass can be attached, and let $\xi * \sigma_\alpha$ denote the new contact structure on M obtained by attaching a bypass to (M, ξ) along α. Fix a

[1]By fixing framing at endpoints, the Thurston–Bennequin invariant is well-defined for Legendrian arcs.

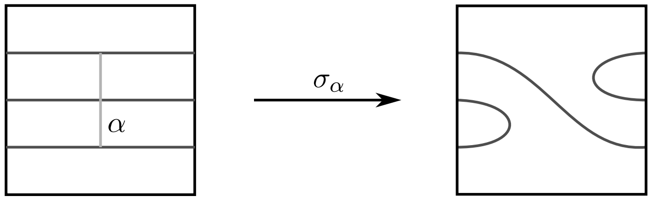

Figure 1. The effect of a bypass attachment.

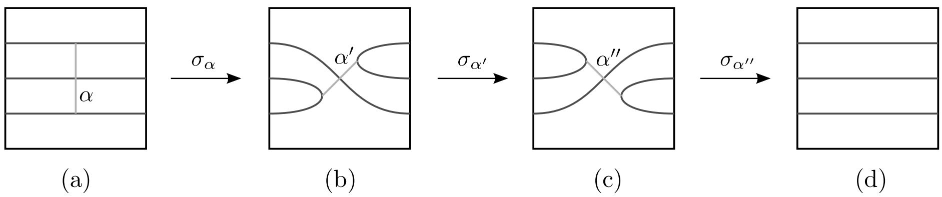

Figure 2. A bypass triangle attachment.

trivialization of the tangent bundle TM. We compute the difference of the *relative Pontryagin submanifolds* associated with ξ and $\xi * \sigma_\alpha$. See Section 2.2 for a discussion of the relative Pontryagin submanifold. However, since the setup for application of the generalized Pontryagin–Thom construction, and therefore the computation of the relative Pontryagin submanifolds, involves several choices which are quite technical, we refer the reader to Section 3 (cf. Theorem 20) for a precise statement of the main result.

As an application, we study the effect of a bypass triangle attachment on the homotopy class of the contact structure. We first define a bypass triangle attachment as follows.

Definition 2. Let (M, ξ) be a contact 3-manifold with convex boundary and $\alpha \subset \partial M$ be a Legendrian arc. A *bypass triangle attachment* along α is the composition of three bypass attachments along Legendrian arcs α, α' and α'' as depicted in Figure 2. We denote the bypass triangle attachment along α by $\triangle_\alpha = \sigma_\alpha * \sigma_{\alpha'} * \sigma_{\alpha''}$, where the composition $*$ is from left to right, i.e., we attach σ_α first, followed by $\sigma_{\alpha'}$ and then $\sigma_{\alpha''}$.

It follows from Giroux's Flexibility Theorem (cf. Theorem 3) that a bypass triangle attachment does not change the contact structure in a neighborhood of ∂M up to isotopy. In fact, it only affects the contact structure within a ball embedded in the interior of M, which can be measured by a three-dimensional obstruction class o_3 defined in Section 2. Finally, the exact measure of this obstruction class for a bypass triangle attachment is the content of Theorem 1.

This paper is organized as follows. In Section 1, we review some basic material in contact geometry including convex surface theory and bypasses. In Section 2, we recall the classical Pontryagin–Thom construction for closed manifold M, and generalize it to the case when ∂M is nonempty. As an application, we define the three-dimensional obstruction class $o_3(\xi, \xi')$ for homotopy classes of 2-plane fields ξ and ξ' with the same Euler class. Finally, we give the proof of Theorem 20 and Theorem 1 in Section 3.

1. Contact geometry preliminaries

1.1. Convex surfaces. Let (M, ξ) be a contact 3-manifold. A vector field v in M is a *contact vector field* if the flow of v preserves ξ. We say a closed, oriented surface $\Sigma \subset M$ is *convex* if there exists a contact vector field v transverse to Σ. For the rest of the paper, we always assume that ∂M is convex if nonempty.

Given a convex surface Σ, we define the *dividing set* $\Gamma_\Sigma := \{x \in \Sigma \mid v(x) \in \xi_x\}$, where v is a contact vector field transverse to Σ. The *characteristic foliation* Σ_ξ is a singular foliation on Σ obtained by integrating the singular line field $T\Sigma \cap \xi$. We summarize the basic properties of dividing set as follows.

(1) Γ_Σ is a nonempty smooth one-dimensional submanifold of Σ.
(2) Γ_Σ is transverse to Σ_ξ.
(3) The isotopy class of Γ_Σ does not depend on the choice of the transverse contact vector field v.

It is well-known that if two contact structures induce the same characteristic foliation on Σ, then they are isotopic in a neighborhood of Σ. In fact, E. Giroux [**2**] showed that one needs much less information — only the dividing set — to determine the isotopy class of contact structures in a neighborhood of convex surface. More precisely, we have the following *Giroux's Flexibility Theorem.*

Theorem 3 (Giroux). *Let Σ be a convex surface with characteristic foliation Σ_ξ, v be a contact vector field transverse to Σ, and Γ_Σ be the dividing set. If $\mathscr{F}$ is another singular foliation on Σ adapted to Γ_Σ (i.e., there is a contact structure ξ' in a neighborhood of Σ such that $\Sigma_{\xi'} = \mathscr{F}$ and Γ_Σ is also a dividing set for ξ'), then there exists an isotopy $\phi_t, t \in [0, 1]$ such that*

(1) *$\phi_0 = id$ and $\phi_t|_{\Gamma_\Sigma} = id$ for all t.*
(2) *v is transverse to $\phi_t(\Sigma)$ for all t.*
(3) *The characteristic foliation on $\phi_1(\Sigma)$ is $\mathscr{F}$.*

Remark 4. In the light of Theorem 3, any adapted characteristic foliation can be realized by a C^0-small isotopy of the convex surface, or equivalently,

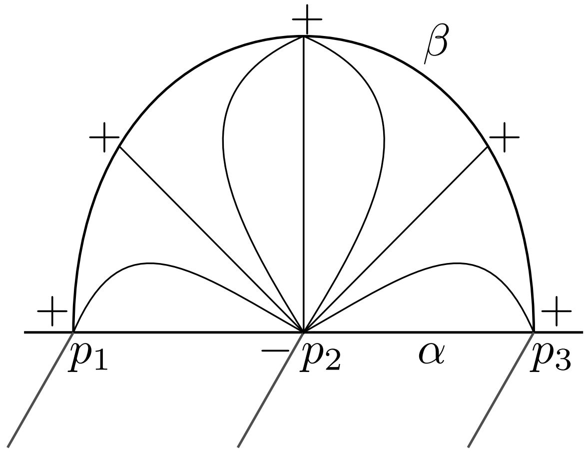

Figure 3. A bypass.

it can be realized by a C^0-small isotopy of the contact structure near a fixed convex surface. Therefore we will only specify the dividing set to determine the isotopy class of a contact structure near a convex surface for the rest of this paper.

1.2. Bypasses. A properly embedded arc γ in (M, ξ) is *Legendrian* if $T_x\gamma \subset \xi_x$ for any $x \in \gamma$. Following [**6**], let Σ be a convex surface. A *bypass* D for Σ is a convex disk with Legendrian boundary $\partial D = \alpha \cup \beta$ such that the following conditions hold:

(1) $\alpha = \Sigma \cap D$.
(2) $\Gamma_\Sigma \cap \alpha = \{p_1, p_2, p_3\}$, where p_1, p_2, p_3 are distinct points.
(3) $\alpha \cap \beta = \{p_1, p_3\}$.
(4) for an appropriate orientation of D, p_1 and p_3 are both positive elliptic singular points of D, p_2 is a negative elliptic singular point of D, and all the singular points along β are positive and alternate between elliptic and hyperbolic.

Remark 5. Given a Legendrian arc $\alpha \subset \Sigma$ as above, a bypass along α does not always exist in a contact manifold. However in this paper, we do not worry about the existence of bypasses because we will attach bypasses from outside of the contact manifold. (Compare with the discussion of bypasses in the third paragraph on Page 1.)

Given a convex surface and a bypass as above, we now describe a bypass attachment.

Lemma 6 (Honda). *Assume D is a bypass for a convex surface Σ. Then there exists a (closed) neighborhood of $\Sigma \cup D \subset M$ diffeomorphic to $\Sigma \times [0, 1]$,* *such that $\Sigma_i = \Sigma \times \{i\}, i = 0, 1,$ are convex, and Γ_{Σ_1} is obtained from Γ_{Σ_0} by performing the bypass attachment operation depicted in Figure* 1 *in a neighborhood of the attaching Legendrian arc α.*

In practice, we construct a neighborhood of $\Sigma \cup D$ with a contact structure given by the bypass attachment as follows. Let $D \times [-\epsilon, \epsilon]$ be a thickening of D with an invariant contact structure in the $[-\epsilon, \epsilon]$-direction, and similarly $\Sigma \times [-\epsilon, 0]$ be a thickening of Σ, where $\epsilon > 0$ is small. Here we assume $D \times [-\epsilon, \epsilon]$ is attached to $\Sigma \times \{0\}$. Then a neighborhood of $\Sigma \cup D$ can be obtained by rounding the corners of $(\Sigma \times [-\epsilon, 0]) \cup (D \times [-\epsilon, \epsilon])$. A more precise construction will be given in Section 3.

2. The Pontryagin–Thom construction

2.1. The Pontryagin–Thom construction for closed manifolds. The Pontryagin–Thom construction is designed to study homotopy types of smooth maps $f : M \to S^n$, where M is a closed manifold. The idea is that instead of working with maps between manifolds, we study framed submanifolds of M associated with these maps and framed cobordism between them. Throughout this paper, we always assume M is three-dimensional and $n = 2$.

Fix a Riemannian metric on M. Let $L \subset M$ be a link. A *framing* of L is the homotopy class of a smooth function σ which assigns to each point $x \in L$ a basis $\{v_1(x), v_2(x)\}$ of the orthogonal complement of T_xL in T_xM. We call the pair (L, σ) a *framed link*. Two framed links (L, σ) and (L', σ') are *framed cobordant* if there exists a framed surface (Σ, δ) in the 4-manifold $M \times [0, 1]$ such that $(\Sigma, \delta)|_{M\times 0} = (L, \sigma)$ and $(\Sigma, \delta)|_{M\times 1} = (L', \sigma')$, where the framing δ is the homotopy class of a smooth function which assigns to each point $y \in \Sigma$ a basis of the orthogonal complement of $T_y\Sigma$ in $T_y(M \times [0, 1])$.

Remark 7. The above construction is independent of the choice of a Riemannian metric on M since the space of Riemannian metrics on M is path-connected.

The main result of the Pontryagin–Thom construction is the following theorem. See Chapter 7 of [**8**] for more details.

Theorem 8. *Let M be a closed 3-manifold. Then there exists a one-to-one correspondence*

$$\{\textit{smooth maps } f : M \to S^2 \textit{up to homotopy}\} \xleftrightarrow{1-1} \{\textit{framed links in } M \textit{up to framed cobordism}\}.$$

Sketch of proof. To construct a framed link in M from a smooth map $f : M \to S^2$, let $p \in S^2$ be a regular value of f. By choosing a basis $\{v_1, v_2\}$

of T_pS^2, we obtain a framed link $(L_{f,p}, \sigma_{f,p})$ in M, where $L_{f,p} = f^{-1}(p)$ and $\sigma_{f,p}(x)$ is the pull-back of $\{v_1, v_2\}$ via the isomorphism $f_* : T_xL^\perp \to T_pS^2, \forall x \in L$.

Conversely, let (L, σ) be a framed link in M. Identify an open tubular neighborhood $N(L)$ of L with $L \times \mathbb{R}^2$ via σ. Choose a smooth map $\phi : \mathbb{R}^2 \to S^2$ which maps every x with $||x|| \geq 1$ to a base point $y \in S^2$, and maps the open unit disk $||x|| < 1$ diffeomorphically onto $S^2 \backslash \{y\}$. We define a smooth map $f : M \to S^2$ in two steps. First we define $f|_{N(L)} : N(L) \simeq L \times \mathbb{R}^2 \xrightarrow{\pi_2} \mathbb{R}^2 \xrightarrow{\phi} S^2$, where $\pi_2 : L \times \mathbb{R}^2 \to \mathbb{R}^2$ is the projection onto the second factor. Then we extend $f|_{N(L)}$ to $f : M \to S^2$ by the constant map $f|_{M \backslash N(L)} \equiv y \in S^2$.

One can show that the above construction in both directions establishes the desired one-to-one correspondence. □

Definition 9. Given a smooth map $f : M \to S^2$, we call the framed link $(L_{f,p}, \sigma_{f,p})$ constructed above the *Pontryagin submanifold associated with* f.

Remark 10. Although the construction of $(L_{f,p}, \sigma_{f,p})$ depends on the choice of p, its framed cobordism class does not. Compare with the relative Pontryagin–Thom construction discussed in Section 2.2.

However, Theorem 8 is still not satisfactory for our purposes because we will be working with contact manifolds with boundary. Before we generalize the Pontryagin–Thom construction to manifolds with boundary, we look at a simple application of Theorem 8 which computes the homotopy group $\pi_3(S^2)$. We will use a similar computation in a relative case in Section 2.3.

Lemma 11. *There exists an isomorphism* $h : \pi_3(S^2) \xrightarrow{\sim} \mathbb{Z}$

Sketch of proof. Since any continuous map $f : S^3 \to S^2$ can be approximated by a smooth map, we can assume that the elements in $\pi_3(S^2)$ are represented by smooth maps. Now it follows immediately from Theorem 8 that $\pi_3(S^2) = \{(L, \sigma)\}/\sim$, where $(L, \sigma) \sim (L', \sigma')$ if and only if they are framed cobordant. The group structure on $\{(L, \sigma)\}/\sim$ is defined by $[(L_1, \sigma_1)] + [(L_2, \sigma_2)] = [(L_1 \sqcup L_2, \sigma_1 \sqcup \sigma_2)]$ and $-[(L, \sigma)] = [(L, -\sigma)]$. If $\widetilde{L}$ is a parallel copy of L given by the framing σ, then we define $n(L, \sigma)$ to be the self-linking number $lk(L, \widetilde{L})$. Now we define the group homomorphism $h : \pi_3(S^2) \to \mathbb{Z}$ by sending $[(L, \sigma)]$ to $n(L, \sigma)$. It is not hard to verify that h is well-defined and is an isomorphism. See for example [**3**]. □

2.2. The Pontryagin–Thom construction for manifolds with boundary. Let M be a compact 3-manifold with boundary. Let $f : M \to S^2$ be a smooth map and $p \in S^2$ be a regular value of f. The *Pontryagin* *submanifold* $(f^{-1}(p), \sigma_{f,p})$ associated with the pair (f, p) is a framed one-dimensional submanifold of M, i.e., it is the disjoint union of a framed link and a finite collection of framed arcs with endpoints contained in ∂M. Two framed one-dimensional submanifolds (L, σ) and (L', σ') of M are *relatively framed cobordant* if there exists a framed surface (Σ, δ) in $M \times [0,1]$ such that (i) $(\Sigma, \delta)|_{M\times\{0\}} = (L, \sigma)$, (ii) $(\Sigma, \delta)|_{M\times\{1\}} = (L', \sigma')$, and (iii) $(\Sigma, \delta)|_{\partial M\times\{t\}} = (L, \sigma)|_{\partial M\times\{0\}} = (L', \sigma')|_{\partial M\times\{1\}}$ for any $t \in [0,1]$. We have the following theorem which can be viewed as the relative analog of Theorem 8.

Theorem 12. *Let M be a compact 3-manifold with boundary. If $f, f' : M \to S^2$ are smooth maps such that $f|_{\partial M} = f'|_{\partial M}$, then f is homotopic to f' relative to the boundary if and only if for any $p \in S^2$ which is a common regular value of f and f', $(f^{-1}(p), \sigma_{f,p})$ is relatively framed cobordant to $(f'^{-1}(p), \sigma_{f',p})$.*

Proof. Let $H : M \times [0,1] \to S^2$ be a homotopy between f and f' relative to the boundary. Generically we can assume $p \in S^2$ is also a regular value of H. Hence the Pontryagin submanifold $(H^{-1}(p), \delta_H)$ defines a relative framed cobordism between $(f^{-1}(p), \sigma_{f,p})$ and $(f'^{-1}(p), \sigma_{f',p})$.

Conversely, let $(\Sigma, \delta) \subset M \times [0,1]$ be a relative framed cobordism between $(f^{-1}(p), \sigma_{f,p})$ and $(f'^{-1}(p), \sigma_{f',p})$. Let $\partial M \times [-1, 0] \subset M$ be a collar neighborhood of ∂M where $\partial M \times \{0\}$ is identified with ∂M, and $\widetilde{M}$ be the metric closure of $M \backslash (\partial M \times [-1, 0])$. Abusing notation, we shall write Σ for $\Sigma \cap (\widetilde{M} \times [0,1])$. As in the proof of Theorem 8, we identify an open tubular neighborhood $N(\Sigma)$ of Σ in $\widetilde{M} \times [0,1]$ with $\Sigma \times \mathbb{R}^2$ via δ, and define a smooth map $H_1 : \widetilde{M} \times [0,1] \to S^2$ by (i) $H_1|_{N(\Sigma)} : N(\Sigma) \simeq \Sigma \times \mathbb{R}^2 \xrightarrow{\pi_2} \mathbb{R}^2 \xrightarrow{\phi} S^2$ where $\pi_2 : \Sigma \times \mathbb{R}^2 \to \mathbb{R}^2$ is the projection onto the second factor, and (ii) $H_1|_{\widetilde{M}\backslash N(\Sigma)} \equiv y \in S^2$. Observe that $H_1|_{\partial\widetilde{M}\times\{t\}} : \partial\widetilde{M} \times \{t\} \to S^2$ is homotopic to $f : \partial M \times \{t\} \to S^2$ for any $t \in [0,1]$, and let $H_2^t : \partial M \times [-1,0] \times \{t\} \to S^2$ be the homotopy, i.e., $H_2^t|_{s=-1} = H_1|_{\partial\widetilde{M}\times\{t\}}$ and $H_2^t|_{s=0} = f$, where $s \in [-1, 0]$. Define $H_2 : \partial M \times [-1,0] \times [0,1]$ by $H_2(x, s, t) = H_2^t(x, s)$ for $x \in \partial M$, $s \in [-1, 0]$ and $t \in [0,1]$. We construct a map $H : M \times [0,1] \to S^2$ by gluing H_1 and H_2 along $\partial M \times \{-1\} \times [0,1]$ which satisfies $H|_{\partial M\times\{t\}} = f|_{\partial M} = f'|_{\partial M}$ for any $t \in [0,1]$. One can verify that $H|_{M\times\{0\}}$ and $H|_{M\times\{1\}}$ are homotopic to f and f' relative to the boundary, respectively, as in the closed case. Hence the conclusion follows. $\square$

Remark 13. The proof of Theorem 12 above is slightly more involved than the proof in the closed case, namely, we used the submanifold $\widetilde{M} \subset M$ onto which M deformation retracts. This is because the restriction of the

Pontryagin submanifold to ∂M does not determine $f|_{\partial M} : \partial M \to S^2$ (resp. $f'|_{\partial M}$). It only determines the homotopy type of $f|_{\partial M}$ (resp. $f'|_{\partial M}$).

Corollary 14. *Let $f, f' : M \to S^2$ be smooth maps such that $f|_{\partial M} = f'|_{\partial M}$. If $(f^{-1}(p), \sigma_{f,p})$ is relatively framed cobordant to $(f'^{-1}(p), \sigma_{f',p})$ for some common regular value p of f and f', then the same holds for all common regular values of f and f'.*

Proof. This follows immediately from the proof of Theorem 12. □

Hence in practice, in order to verify that f is homotopic to f' relative to the boundary, it suffices to check the framed cobordant condition for a preferred common regular value.

Remark 15. One can easily generalize Theorem 12 to arbitrary dimension using the same proof.

2.3. The three-dimensional obstruction class $o_3(\xi, \xi')$ of 2-plane field distributions. Let M be a compact oriented 3-manifold, and ξ and ξ' be two oriented 2-plane field distributions on M such that $\xi = \xi'$ on $M \backslash B^3$ for a 3-ball $B^3 \subset \mathrm{int}(M)$. Fix a trivialization of TM. Let $G_\xi : M \to S^2$ and $G_{\xi'} : M \to S^2$ be the Gauss maps associated with ξ and ξ', respectively. Take a common regular value $p \in S^2$ of G_ξ and $G_{\xi'}$, and let (L, σ) and (L', σ') be the Pontryagin submanifolds associated with (G_ξ, p) and $(G_{\xi'}, p)$, respectively, i.e., $L = G_\xi^{-1}(p)$ and $L' = G_{\xi'}^{-1}(p)$. By assumption, $(L, \sigma) = (L', \sigma')$ on $M \backslash B^3$. Hence, we may focus on the relative framed cobordism classes of $(L, \sigma)|_{B^3}$ and $(L', \sigma')|_{B^3}$. Since B^3 is contractible, L is always relatively cobordant to L' but the framing may not extend to the cobordism. To fix this issue, let $C \subset \mathrm{int}(B^3)$ be a trivial loop which does not link with L'. Observe that (L, σ) is relatively framed cobordant to $(L' \sqcup C, \sigma' \sqcup \delta)$ in B^3 for some framing δ of C. If C' is a parallel copy of C given by δ, then we define $n(C, \delta)$ to be the self-linking number $lk(C, C')$ with respect to the orientation of B^3 inherited from the orientation of M.

Definition 16. Let ξ and ξ' be oriented 2-plane field distributions on M such that $\xi = \xi'$ on $M \backslash B^3$ for a 3-ball $B^3 \subset M$. We define the three-dimensional obstruction class $o_3(\xi, \xi') \in \mathbb{Z}/d(\xi)$ to be $n(C, \delta)$ as constructed above modulo $d(\xi)$, where $d(\xi)$ is the divisibility of the Euler class $e(\xi) \in H^2(M, \mathbb{Z})$.

Remark 17. One can think of $o_3(\xi, \xi')$ as a relative version of $\pi_3(S^2)$ discussed in Lemma 11.

It is easy to see that the definition of $o_3(\xi, \xi')$ is independent of various choices involved, namely, the trivialization of TM, the 3-ball $B^3 \subset M$, the trivial loop C and the common regular value $p \in S^2$. The independence of the choice of common regular values is slightly nontrivial, so we prove this in the following lemma.

Lemma 18. *The obstruction class $o_3(\xi,\xi')\in\mathbb{Z}/d(\xi)$ is independent of the choice of $p\in S^2$.*

Proof. Let $\widehat{M}=M\cup_{\partial M}(-M)$ be a closed oriented 3-manifold, where $-M$ is M with the opposite orientation. Glue G_ξ and $G_{\xi'}$ along ∂M to obtain a smooth map $\widehat{G}:\widehat{M}\to S^2$ given by:

$$\widehat{G}(x)=\begin{cases} G_\xi(x) & \text{if } x\in M,\\ G_{\xi'}(x) & \text{if } x\in -M.\end{cases}$$

If $q\in S^2$ is another common regular value of G_ξ and $G_{\xi'}$, then p and q are both regular values of $\widehat{G}$. We write $o_3^p(\xi,\xi')$ (resp. $o_3^q(\xi,\xi')$) for the obstruction class to indicate the potential dependence on the choice of p (resp. q). According to Proposition 4.1 in [**3**], we have $o_3^p(\xi,\xi')-o_3^q(\xi,\xi')=0\in\mathbb{Z}/d(\xi)$. Hence $o_3(\xi,\xi')$ is independent of the choice of p modulo $d(\xi)$. $\square$

Using the same argument as in proof of Proposition 4.1 in [**3**], we also obtain the following result.

Proposition 19. *If ξ and ξ' are two contact structures on M such that $\xi|_{M\setminus B^3}=\xi'|_{M\setminus B^3}$ for some 3-ball $B^3\subset\operatorname{int}(M)$, then ξ is homotopic to ξ' relative to the boundary if and only if $o_3(\xi,\xi')=0\in\mathbb{Z}/d(\xi)$.*

3. Proof of the main results

3.1. Computation of the homotopy class of a bypass attachment. We first compute the homotopy class of a bypass attachment in a local model which we describe now. Let $V=[-3/4,3/4]\times[-1,1]\times[0,1]\subset\mathbb{R}^3$ be a 3-manifold with boundary equipped with the standard coordinates, and ξ_V be a contact structure on V defined by $\xi_V=\ker\lambda$, where $\lambda=\cos(2\pi x)dy-\sin(2\pi x)dz$. Let $\alpha=[-1/2,1/2]\times\{0\}\times\{1\}\subset V$ be a Legendrian arc. Let $\xi_V*\sigma_\alpha$ denote the contact structure given by attaching a bypass to (V,ξ_V) along α (cf. Lemma 6). Trivialize TV by the standard embedding $V\subset\mathbb{R}^3$ and look at the associated Gauss map $G_{\xi_V*\sigma_\alpha}:V\to S^2$. Observe that $p=(1,0,0)\in S^2$ is a regular value of $G_{\xi_V*\sigma_\alpha}$ by construction. We state the main result of this paper as follows.

Theorem 20. *Let $(V,\xi_V*\sigma_\alpha)$ be the contact manifold described above. Then the Pontryagin submanifold $G^{-1}_{\xi_V*\sigma_\alpha}(p)\subset V$ is a properly embedded framed arc with framing as depicted in Figure* 4.

Remark 21. The Pontryagin submanifold $G^{-1}_{\xi_V*\sigma_\alpha}(p)$ in Theorem 20 depends on various choices including the trivialization of TV and the regular value p. For example, it will be clear from the proof of Theorem 20 that

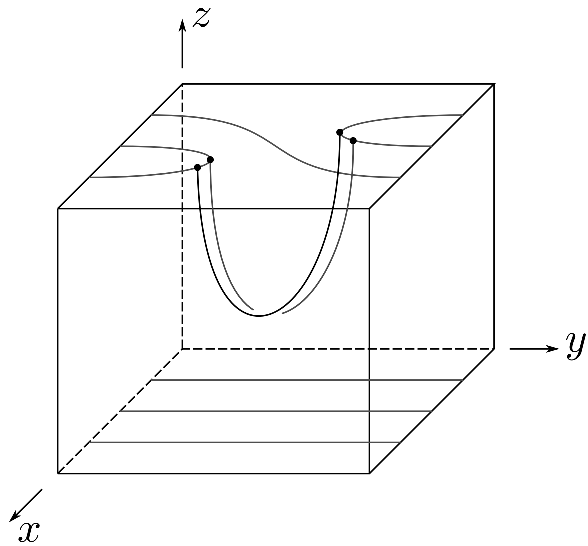

Figure 4. The Pontryagin submanifold $G^{-1}_{\xi_V * \sigma_\alpha}(p)$ in V. The blue arc is a parallel copy of $G^{-1}_{\xi_V * \sigma_\alpha}(p)$ which defines the framing.

$q = (-1, 0, 0) \in S^2$ is also a regular value of $G_{\xi_V * \sigma_\alpha}$, but $G^{-1}_{\xi_V * \sigma_\alpha}(q)$ is the empty set.

The following corollary follows immediately from Theorem 20 by the local nature of the bypass attachment.

Corollary 22. *Let (M, ξ) be a contact 3-manifold with convex boundary and $\alpha \subset \partial M$ be a Legendrian arc along which a bypass can be attached. Then there exists a trivialization of TM and a common regular value $p \in S^2$ of G_ξ and $G_{\xi * \sigma_\alpha}$ such that the Pontryagin submanifold $G^{-1}_{\xi * \sigma_\alpha}(p) = G^{-1}_\xi(p) \cup \gamma$, where $G^{-1}_\xi(p)$ is the Pontryagin submanifold associated with ξ and $\gamma \subset M$ is a properly embedded framed arc as depicted in Figure 4 which does not link $G^{-1}_\xi(p)$.*

Proof. Let $N(\alpha) \subset \partial M$ be a small neighborhood of α. Consider a thickening $N(\alpha) \times [-\epsilon, 0] \subset M$ of $N(\alpha)$ for small $\epsilon > 0$, such that $N(\alpha) \times \{0\} \subset \partial M$. Since ∂M is convex by assumption, we may assume that $N(\alpha) \times [-\epsilon, 0]$ with the restricted contact structure is contactomorphic to (V, ξ_V) as constructed above. Moreover we can choose a trivialization of TM such that its restriction to $N(\alpha) \times [-\epsilon, 0]$ is pushed forward to the standard trivialization of TV under the contactomorphism (which is, in particular, a diffeomorphism). With this setup, the conclusion is an immediate consequence of Theorem 20. $\square$

Now we are ready to compute the relative Pontryagin submanifold associated with the contact 3-manifold $(V, \xi * \sigma_\alpha)$ as constructed in Theorem 20.

Proof of Theorem 20. Let $\Sigma_t = [-3/4, 3/4] \times [-1, 1] \times \{t\} \subset V$ be a foliation by convex surfaces with respect to the contact vector field $\partial/\partial z$ for $t \in [0, 1]$. The dividing set Γ_t on Σ_t, $t \in [0, 1]$, is the disjoint union of three parallel intervals $(\{1/2\} \times [-1, 1] \times \{t\}) \cup (\{0\} \times [-1, 1] \times \{t\}) \cup (\{-1/2\} \times [-1, 1] \times \{t\})$ which divide Σ_t into positive and negative regions. Let $\alpha = [-1/2, 1/2] \times \{0\} \times \{1\} \subset \Sigma_1$ be the Legendrian arc along which an I-invariant neighborhood of the bypass $D_\alpha = \{(x, y, z) \mid 1 \leq z \leq 1 + \sqrt{1/4 - x^2}, y = 0\}$ is attached. We choose the characteristic foliation on D_α so that it is half of an overtwisted disk with one negative elliptic singular point at the center and alternating positive elliptic and hyperbolic singular points on the boundary, and the dividing set Γ_{D_α} is a semi-circle centered at $(0, 0, 1)$ with radius $1/4$. By gluing a $\partial/\partial y$-invariant neighborhood $D_\alpha \times [-\epsilon, \epsilon]$ of D_α for small $\epsilon > 0$ to (V, ξ_V), we obtain a contact manifold $(V_\alpha, \xi_{V_\alpha})$ with corners where $V_\alpha = V \cup (D_\alpha \times [-\epsilon, \epsilon])$. Abusing notation, we also denote the contact manifold obtained by rounding corners on $D_\alpha \times [-\epsilon, \epsilon] \subset V_\alpha$ by $(V_\alpha, \xi_{V_\alpha})$. By slightly tilting $D_\alpha \times \{-\epsilon\}$ and $D_\alpha \times \{\epsilon\}$, we can further assume that the $\partial/\partial z$-direction is transverse to $\partial_+ V_\alpha$, the top boundary of V_α. Observe that, up to isotopy, $\Gamma_{\partial_+ V_\alpha}$ is as depicted in the right-hand side of Figure 1.

Choose a non-positive smooth function $g : V_\alpha \to \mathbb{R}_{\leq 0}$ supported in a neighborhood of $D_\alpha \times [-\epsilon, \epsilon]$ such that the time-1 map ϕ^1_X of the flow of $X = g\partial/\partial z$ sends V_α diffeomorphically onto V. We identify V_α with V via ϕ^1_X, and we denote the contact structure $(\phi^1_X)_*(\xi_{V_\alpha})$ by $\xi_V * \sigma_\alpha$, where $\xi_V * \sigma_\alpha$ is known as the contact structure obtain by attaching a bypass along α to ξ_V.

Now we study the homotopy type of $(V, \xi_V * \sigma_\alpha)$ using the Pontryagin–Thom construction. We first consider the homotopy class of the contact structure ξ_{V_α} constructed above. Observe that by construction $p = (1, 0, 0) \in S^2$ is a regular value of the Gauss map $G_{\xi_{V_\alpha}} : V_\alpha \to S^2$ associated to ξ_{V_α}. Here TV_α is trivialized by the standard embedding into $\mathbb{R}^3$ as constructed above. Then the (unframed) Pontryagin submanifold $G^{-1}_{\xi_{V_\alpha}}(p) = \{(x, 0, 5/4) \mid -\epsilon \leq x \leq \epsilon\}$ is just a horizontal arc with endpoints on the dividing set. This is depicted as the black arc in Figure 5(a). In order to keep track of the framing of $G^{-1}_{\xi_{V_\alpha}}(p)$, we fix another regular value $p' = (1 - \delta, \sqrt{2\delta - \delta^2}, 0) \in S^2$ near p for small $\delta > 0$. Then $G^{-1}_{\xi_{V_\alpha}}(p')$ must be an arc obtained by isotoping $G^{-1}_{\xi_{V_\alpha}}(p)$ inside V_α with the endpoints on the dividing set. This isotopy can be explicitly visualized by looking at the Pontryagin submanifolds associated to the path of regular values

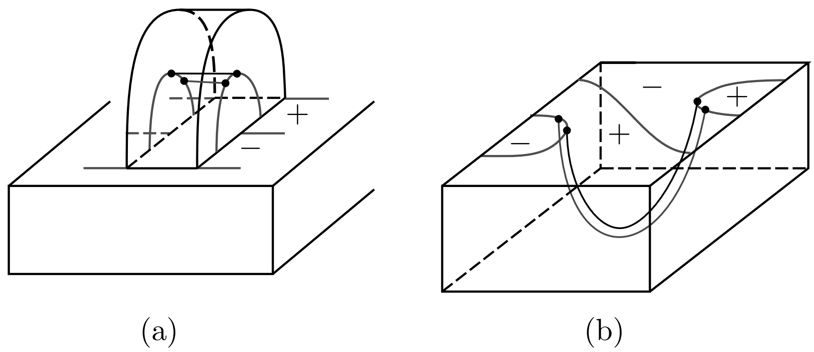

Figure 5. (a) The Pontryagin submanifold $G^{-1}_{\xi_{V_\alpha}}(p)$ contained in V_α. (b) The Pontryagin submanifold $G^{-1}_{\xi_V * \sigma_\alpha}(p)$ contained in V. The blue arc is a parallel copy of $G^{-1}_{\xi_V * \sigma_\alpha}(p)$ which defines the framing.

$(1-t, \sqrt{2t-t^2}, 0)$, where $0 \leq t \leq \delta$. It turns out that the endpoints of $G^{-1}_{\xi_{V_\alpha}}(p')$ are located at different sides of the endpoints of $G^{-1}_{\xi_{V_\alpha}}(p)$, according to the way the contact planes rotate. This is depicted as the blue arc in Figure 5(a). It is important to notice that our determination of the framing of $G^{-1}_{\xi_{V_\alpha}}(p)$ only uses a perturbation in the xy-plane, but not the z-direction. Therefore we can safely apply the diffeomorphism ϕ^1_X constructed above, which only deforms in the z-direction, to flatten the bump to obtain the Pontryagin submanifold $G^{-1}_{\xi_V * \sigma_\alpha}(p)$ associated with $\xi_V * \sigma_\alpha$ with framing given by $G^{-1}_{\xi_V * \sigma_\alpha}(p')$ as depicted in Figure 5(b). This finishes the proof of Theorem 20. □

3.2. Proof of Theorem 1. We proceed to the proof of Theorem 1 which involves three bypass attachments. Our strategy is to first construct a local model for the bypass triangle attachment in $\mathbb{R}^3$, and compute the associated Pontryagin submanifold based on essentially the same methods used in the proof of Theorem 20. Next we identify a neighborhood of the arc of attachment α in M where the bypass triangle is attached with our previously constructed local model (cf. proof of Corollary 22), and conclude that the bypass triangle attachment drops o_3 by 1.

We first establish a technical lemma which enables us to isotop characteristic foliations on a disk adapted to a fixed dividing set without affecting the Pontryagin submanifold.

Lemma 23. *Let $(D^2 \times [0,1], \xi)$ be a contact 3-manifold with $T(D^2 \times [0,1])$ trivialized by the standard embedding $D^2 \times [0,1] \subset \mathbb{R}^3$, i.e., D^2 is contained in* *the xy-plane and* $[0,1]$ *is in the direction of the z-axis. Suppose the following conditions hold:*

(1) *There exists a contact vector field on* $D^2 \times [0,1]$*, with respect to which* $D^2 \times \{t\}$ *are convex and the dividing sets* $\Gamma_{D^2\times\{t\}}$ *agree for all* $t \in [0,1]$*.*
(2) *The characteristic foliations* $\mathscr{F}_{D^2\times\{t\}}$ *agree in a neighborhood of* $\Gamma_{D^2\times\{t\}}$ *for all* $t \in [0,1]$*.*
(3) *The Gauss map* G_ξ *satisfies: (i)* $G_\xi(\Gamma_{D^2\times\{t\}}) \subset \{z=0\} \subset S^2$*, (ii)* $G_\xi(R_+(D^2 \times \{t\})) \subset \{z>0\} \subset S^2$*, and (iii)* $G_\xi(R_-(D^2\times\{t\})) \subset \{z<0\} \subset S^2$ *for any* $t \in [0,1]$*.*
(4) $p=(1,0,0) \in S^2$ *is a regular value of* G_ξ*, and* $G_\xi^{-1}(p)$ *is disjoint from* $\partial D^2 \times [0,1]$*.*

Then $G_\xi^{-1}(p)$ *is framed cobordant to* $G_{\xi_0}^{-1}(p)$ *relative to the boundary, where* ξ_0 *is the I-invariant contact structure on* $D^2 \times [0,1]$ *with* $\xi_0|_{D^2\times\{0\}} = \xi|_{D^2\times\{0\}}$*.*

Proof. The conclusion follows from the observation that $G_\xi^{-1}(p) \cap (D^2 \times \{t\}) \subset \Gamma_{D^2\times\{t\}}$ for all $t \in [0,1]$, and ξ is I-invariant in a neighborhood of $\Gamma_{D^2\times\{0\}} \times [0,1]$ in $D^2 \times [0,1]$. □

The following proposition constructs a local model for the bypass triangle attachment explicitly and computes its Pontryagin submanifold.

Proposition 24. *Let* $T = [-3/4,3/4]\times[-1,1]\times[0,3] \subset \mathbb{R}^3$ *be a 3-manifold,* $\eta = \ker(\cos(2\pi x)dy - \sin(2\pi x)dz)$ *be a contact structure on* T*, and* $\alpha = \{-1/2 \le x \le 1/2, y=z=0\}$ *be a Legendrian arc. Then there exists a contact 3-manifold* $(T, \eta * \triangle_\alpha)$ *where* $\eta * \triangle_\alpha$ *is the contact structure obtained from* η *by attaching a bypass triangle along* α*, such that the Pontryagin submanifold* $G_{\eta*\triangle_\alpha}^{-1}(p)$ *is the unknot with framing* -1 *with respect to the standard orientation. Here* $p=(1,0,0)\in S^2$ *is a regular value of* $G_{\eta*\triangle_\alpha}^{-1}$*.*

Proof. We construct $(T, \eta*\triangle_\alpha)$ and compute its Pontryagin submanifold in three steps corresponding to three bypass attachments σ_α, $\sigma_{\alpha'}$ and $\sigma_{\alpha''}$ respectively.

Step 1. We simply use the construction of $(V, \eta*\sigma_\alpha)$ [2] in the proof of Theorem 20. Recall that the Pontryagin submanifold $G_{\eta*\sigma_\alpha}^{-1}(p)$ is a framed arc in V as depicted in Figure 5(b).

Step 2. We compute the Pontryagin submanifold associated with the second bypass attachment $\sigma_{\alpha'}$ in two substeps.

[2]The contact structure η here is the same as ξ_V in the notation of Theorem 20.

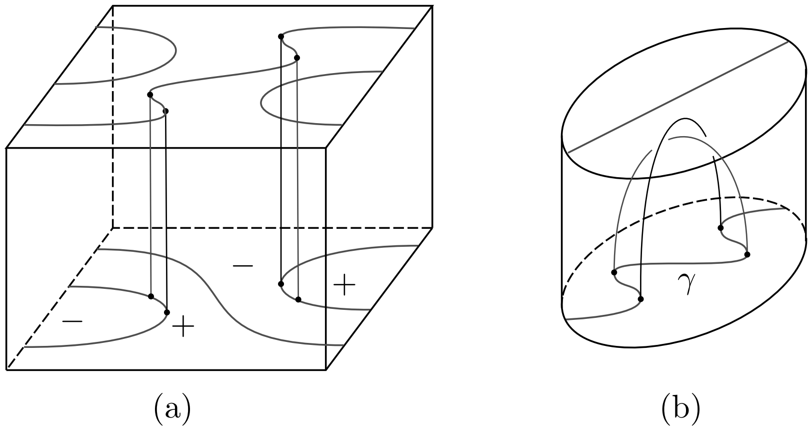

Figure 6. (a) The Pontryagin submanifold $G^{-1}_{(\eta*\sigma_\alpha|_{\Sigma_1})*\sigma_{\alpha'}}(p)$ contained in U. (b) The Pontryagin submanifold $G^{-1}_\tau(p)$ contained in $N(\gamma)\times[2,3]$.

Substep 2.1. We attach the second bypass in a similar manner. Let $U=[-3/4,3/4]\times[-1,1]\times[1,2]\subset\mathbb{R}^3$ be a contact 3-manifold with contact structure obtained by a $\partial/\partial z$-invariant extension of $\eta*\sigma_\alpha|_{\Sigma_1}$, where $\Sigma_1=[-3/4,3/4]\times[-1,1]\times\{1\}$. Recall that the second bypass is attached along the Legendrian arc α' as depicted in Figure 2(b). Let $D_{\alpha'}$ be the bypass along α', and $(U_{\alpha'},\eta_{\alpha,\alpha'})$ be the contact 3-manifold obtained by rounding the corners of $U\cup(D_{\alpha'}\times[-\epsilon,\epsilon])$ with the glued contact structure for small $\epsilon>0$. By Lemma 23, we can choose a Legendrian representative α' within its isotopy class such that $p\notin G_{\eta_{\alpha,\alpha'}}(D_{\alpha'}\times[-\epsilon,\epsilon])$, the image of $D_{\alpha'}\times[-\epsilon,\epsilon]$ under the associated Gauss map $G_{\eta_{\alpha,\alpha'}}$. Since the contact structure remains I-invariant away from a neighborhood of α', by pushing $D_{\alpha'}\times[-\epsilon,\epsilon]$ into U, we obtain the contact 3-manifold $(U,(\eta*\sigma_\alpha|_{\Sigma_1})*\sigma_{\alpha'})$ whose Pontryagin submanifold $G^{-1}_{(\eta*\sigma_\alpha|_{\Sigma_1})*\sigma_{\alpha'}}(p)$ is as depicted in Figure 6(a).

Substep 2.2. Let $\gamma\subset\Gamma_{\Sigma_2}$ be the arc containing the endpoints of $G^{-1}_{(\eta*\sigma_\alpha|_{\Sigma_1})*\sigma_{\alpha'}}(p)$ on $\Sigma_2=[-3/4,3/4]\times[-1,1]\times\{2\}$, and $N(\gamma)$ be a neighborhood of γ on Σ_2. Here $N(\gamma)$ is chosen suitably large so that it contains also the endpoints of a parallel copy of $G^{-1}_{(\eta*\sigma_\alpha|_{\Sigma_1})*\sigma_{\alpha'}}(p)$ which determines the framing. See for example the bottom of Figure 6(b). Observe that the Gauss map $G_{(\eta*\sigma_\alpha|_{\Sigma_1})*\sigma_{\alpha'}}|_{N(\gamma)}$, restricted to $N(\gamma)$, hits p twice only because the dividing set is "curly" by the straightforward bypass attachments we performed in earlier steps. By Giroux's Flexibility Theorem (Theorem 3), there is an isotopy relative to the boundary $\phi_t:N(\gamma)\to N(\gamma)$, $t\in[0,1]$, $\phi_0=id$, which "straightens" the dividing set contained in $N(\gamma)$ in the sense

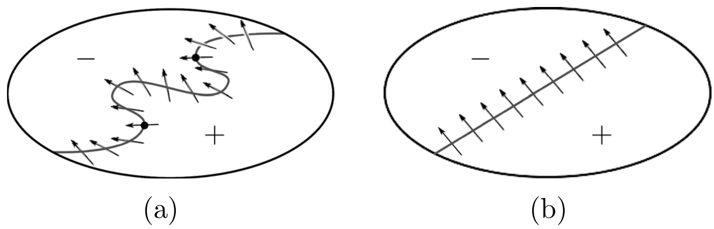

Figure 7. The characteristic foliation on $N(\gamma)$. (a) The characteristic foliation after the second bypass attachment. The two black dots denote the preimage of p under the Gauss map. (b) The characteristic foliation after the isotopy ϕ_1.

that $p \notin G_{\phi_{1*}((\eta*\sigma_\alpha|_{\Sigma_1})*\sigma_{\alpha'})}(N(\gamma))$. More precisely, the dividing set on $N(\gamma)$ with respect to the contact structure $\phi_{1*}((\eta * \sigma_\alpha|_{\Sigma_1}) * \sigma_{\alpha'})$ looks like the one depicted on the top of Figure 6(b). By construction the Gauss map $G_{\phi_{t*}((\eta*\sigma_\alpha|_{\Sigma_1})*\sigma_{\alpha'})}$ hits p exactly at points on the dividing set where the characteristic foliation is parallel to the y-axis, and the positive region is (locally) on the right side of the negative region. So the actual effect of isotoping the contact structure by ϕ_t is to modify the characteristic foliation on $N(\gamma)$ such that p is not contained in the image of $G_{\phi_{1*}((\eta*\sigma_\alpha|_{\Sigma_1})*\sigma_{\alpha'})}(N(\gamma))$. See Figure 7 for a pictorial illustration of the effect of the isotopy of the characteristic foliation.

Now we define $\Phi : N(\gamma) \times [2,3] \to N(\gamma) \times [2,3]$ by $\Phi(x,t) = (\phi_t(x), t)$ for $x \in N(\gamma)$, $t \in [2,3]$, then we can push-forward a $\partial/\partial z$-invariant contact structure $\eta * \sigma_\alpha * \sigma_{\alpha'}|_{N(\gamma)}$ on $N(\gamma) \times [2,3]$ via Φ to obtain a new contact structure on $N(\gamma)\times[2,3]$, which we denote by τ. The Pontryagin submanifold $G_\tau^{-1}(p)$ in $N(\gamma) \times [2,3]$ is a framed arc as depicted in Figure 6(b). To see how $G_\tau^{-1}(p)$ is linked with the blue arc in Figure 6(b) which determines the framing, we note that the endpoints of the blue arc are in between of the endpoints of $G_\tau^{-1}(p)$. If we suppose the interval $[2,3]$ is parameterized by time, then the endpoints of the blue arc will merge and disappear before the endpoints of $G_\tau^{-1}(p)$ do as time goes from 2 to 3. Hence we obtain a contact manifold $(U \cup (N(\gamma)\times[2,3]), ((\eta*\sigma_\alpha)|_{\Sigma_1}*\sigma_{\alpha'})\cup\tau)$. By rounding the corners of $N(\gamma)\times[2,3]$ and pushing it into U as before, we obtain the contact 3-manifold which we still denote by $(U, (\eta * \sigma_\alpha)|_{\Sigma_1} * \sigma_{\alpha'})$ whose associated Pontryagin submanifold $G^{-1}_{(\eta*\sigma_\alpha|_{\Sigma_1})*\sigma_{\alpha'}}(p)$ is a framed arc as depicted in Figure 8.

Step 3. We finish the bypass triangle by attaching the third bypass $D_{\alpha''}$ along α'' as depicted in Figure 2(c). As in previous steps, let $W = [-3/4, 3/4] \times [-1,1] \times [2,3] \subset \mathbb{R}^3$ be a contact 3-manifold with contact structure obtained by a $\partial/\partial z$-invariant extension of $\eta * \sigma_\alpha * \sigma_{\alpha'}|_{\Sigma_2}$. Again by Lemma 23, we can choose α'' so that p is not contained in the image

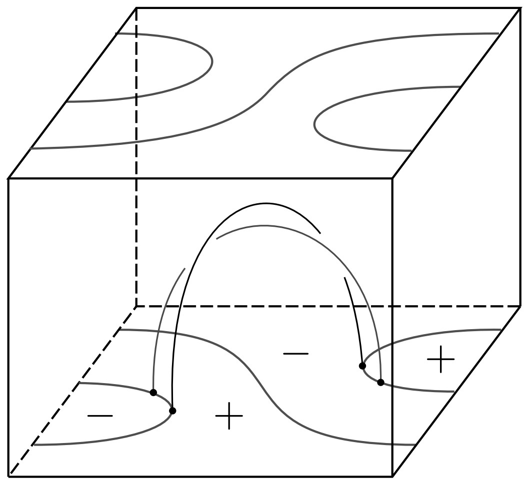

Figure 8. The Pontryagin submanifold $G^{-1}_{(\eta*\sigma_\alpha|_{\Sigma_1})*\sigma_{\alpha'}}(p)$ contained in U after an isotopy.

of $D_{\alpha''} \times [-\epsilon, \epsilon]$ under the Gauss map. Hence the same argument as before produces the third contact 3-manifold $(W, (\eta * \sigma_\alpha * \sigma_{\alpha'}|_{\Sigma_2}) * \sigma_{\alpha''})$ whose associated Pontryagin submanifold $G^{-1}_{(\eta*\sigma_\alpha*\sigma_{\alpha'}|_{\Sigma_2})*\sigma_{\alpha''}}(p)$ is the empty set.

Finally, in order to construct $(T, \eta * \triangle_\alpha)$ with the desired properties, we simply let $(T, \eta * \triangle_\alpha) = (V, \eta * \sigma_\alpha) \cup (U, (\eta * \sigma_\alpha|_{\Sigma_1}) * \sigma_{\alpha'}) \cup (W, (\eta * \sigma_\alpha * \sigma_{\alpha'}|_{\Sigma_2}) * \sigma_{\alpha''})$ glued along adjacent faces. It is easy to see that the associated Pontryagin submanifold $G^{-1}_{\eta*\triangle_\alpha}(p)$ obtained by gluing the framed arcs from Steps 1, 2, and 3 is the unknot with framing -1. See Figure 9. □

Proof of Theorem 1. Let $\alpha \subset \partial M$ be the Legendrian arc such that $\xi' \simeq \xi * \triangle_\alpha$ relative to the boundary, and $N(\alpha)$ be a neighborhood of α on ∂M. Let $\partial M \times [-1, 0] \subset M$ be a collar neighborhood of ∂M with an I-invariant contact structure such that ∂M is identified with $\partial M \times \{0\}$. Assume up to a boundary relative isotopy that $\triangle_\alpha$ is supported in $N(\alpha) \times [-2/3, -1/3] \subset \mathrm{int}(M)$, i.e., $\xi = \xi'$ on $M \backslash (N(\alpha) \times [-2/3, -1/3])$, and that there exists a contactomorphism $\psi : (N(\alpha) \times [-2/3, -1/3], \xi') \to (T, \eta * \triangle_\alpha)$ where $(T, \eta * \triangle_\alpha)$ is the local model for a bypass triangle attachment constructed in Proposition 24. Without loss of generality, we also choose the trivialization of TM so that its restriction to $N(\alpha) \times [-2/3, -1/3]$ coincides with the pull-back of $T\mathbb{R}^3$ via ψ. Let $p = (1, 0, 0) \in S^2$ be a common regular value of G_ξ and $G_{\xi'}$. Observe that the Pontryagin submanifold $G^{-1}_{\xi'}(p)$ restricted to $N(\alpha) \times [-2/3, -1/3]$ is the unknot with framing -1. Since $G^{-1}_\xi(p)$ restricted to $N(\alpha) \times [-2/3, -1/3]$ is the empty set, it follows from Definition 16 that $o_3(\xi, \xi') = -1$ as desired.

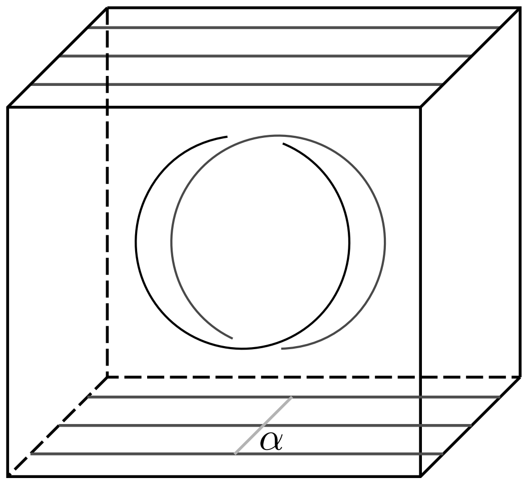

Figure 9. The Pontryagin submanifold $G^{-1}_{\eta * \triangle_\alpha}(p)$. The blue circle is a parallel copy of $G^{-1}_{\eta * \triangle_\alpha}(p)$ which defines the framing.

In particular, ξ is not homotopic to ξ' relative to the boundary by Proposition 19 since $d(\xi)$ is always even. $\square$

References

[1] Y. Eliashberg, *Classification of overtwisted contact structures on* 3*-manifolds*, Invent. Math. **98**(3) (1989), 623–637.

[2] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helv. **66**(4) (1991), 637–677.

[3] R.E. Gompf, *Handlebody construction of Stein surfaces*, Ann. Math. (2), **148**(2) (1998), 619–693.

[4] K. Honda, Contact structures, Heegaard floer homology and triangulated categories. In preparation.

[5] K. Honda, *On the classification of tight contact structures. I*, Geom. Topol. **4** (2000), 309–368.

[6] K. Honda, *Gluing tight contact structures*, Duke Math. J. **115**(3) (2002), 435–478.

[7] Y. Huang, *A proof of the classification theorem of overtwisted contact structures via convex surface theory*, J. Symplecitc Geom. **11**(4) (2013), 1–39.

[8] J.W. Milnor, *Topology from the differentiable viewpoint*, Based on notes by David W. Weaver. The University Press of Virginia, Charlottesville, VA. 1965.

[9] L. Pontryagin, *A classification of mappings of the three-dimensional complex into the two-dimensional sphere*, Rec. Math. [Mat. Sbornik] N. S. **9**(51) (1941), 331–363.

[10] Y. Tian, *A categorification of $U_q sl(1,1)$ as an algebra*, preprint. `arXiv:1210.5680`.

[11] Y. Tian, *A categorification of $U_T sl(1,1)$ and its tensor product representations*, preprint. `arXiv:1301.3986`.

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