CONVERGENCE OF KÄHLER TO REAL POLARIZATIONS ON FLAG MANIFOLDS VIA TORIC DEGENERATIONS

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In this paper, we construct a family of complex structures on a complex flag manifold that converge to the real polarization coming from the Gelfand–Cetlin integrable system, in the sense that holomorphic sections of a prequantum line bundle converge to delta-function sections supported on the Bohr–Sommerfeld fibers. Our construction is based on a toric degeneration of flag varieties and a deformation of Kähler structure on toric varieties by symplectic potentials.

1. Introduction

Let (M, ω) be a $2n$-dimensional symplectic manifold. A prequantum line bundle (L, h, ∇) is a complex line bundle L on M with a Hermitian metric h and a Hermitian connection ∇, whose curvature equals $-2\pi\sqrt{-1}\,\omega$. Geometric quantization is a procedure to assign a certain vector space, which is called a quantum Hilbert space, to (M, ω). To perform a geometric quantization procedure, we must choose a polarization, which is an integrable Lagrangian subbundle of the (complexification of the) tangent bundle TM of M. Then, the quantum Hilbert space $\mathcal{H}(P)$ for a polarization P is naively a subspace of (a certain completion of) the space of sections of L, consisting of covariantly constant sections along the polarization P. See [W] for general properties of geometric quantization.

The most common example of a polarization comes from an integrable complex structure J on M such that (M, ω, J) is a Kähler manifold. In this case, the anti-holomorphic tangent bundle $P_J = T^{0,1}M$ is a polarization, which we call a Kähler polarization. The quantum Hilbert space $\mathcal{H}(P_J)$ is the space of holomorphic sections $H^0(L, \overline{\partial}^J)$ with respect to the natural holomorphic structure $\overline{\partial}^J$ on L induced by J.

Another type of polarization, called a real polarization, is given by a foliation of M into Lagrangian submanifolds. A completely integrable system $\mu\colon M \to \mathbb{R}^n$ (which is assumed to be proper) defines a singular real polarization P_μ, where $(P_\mu)_x$ is the tangent space of the fiber of μ at each point $x \in M$. We set $BS(\mu) = \{p \in \mu(M) | H^0((L, h, \nabla)|_{\mu^{-1}(p)}) \neq 0\}$, where $H^0((L, h, \nabla)|_{\mu^{-1}(p)}) = \{s \in \Gamma((L, h, \nabla)|_{\mu^{-1}(p)}) | \nabla s = 0\}$. Namely, $p \in BS(\mu)$ if and only if $\mu^{-1}(p)$ is a Bohr–Sommerfeld fiber. Then, the quantum Hilbert space $\mathcal{H}(P_\mu)$ is defined to be $\bigoplus_{p \in BS(\mu)} H^0((L, h, \nabla)|_{\mu^{-1}(p)})$ [**S**].

From the point of view of physics, the quantum Hilbert space should be independent of the choice of polarization. In particular, although Kähler and real polarizations seem to be quite different, the quantum Hilbert space for a Kähler polarization should be isomorphic to the one for a real polarization. There are several examples where this principle is observed to be true. A non-singular projective toric variety has a natural Kähler structure, and its moment map for the torus action induces a (singular) real polarization. It is well known that the dimension of the space of holomorphic sections of the prequantum line bundle is the number of lattice points in the image of the moment map, which is also the number of Bohr–Sommerfeld fibers in the variety. This implies that the principle holds in this case. In [**JW**], Jeffrey–Weitsman showed that the principle also holds in the case of the moduli space of flat connections over a compact Riemann surface.

A flag manifold with an integral symplectic structure has a singular real polarization defined by the Gelfand–Cetlin system, which was introduced by Guillemin–Sternberg in [**GS**], as well as a natural Kähler polarization since it is a complex manifold. In [**GS**], the authors studied the quantization of flag manifolds, and showed that the two polarizations give rise to quantizations with the same dimensions. However, their proof did not give any sort of direct relationship between the quantizations; rather, they computed the dimensions of the quantizations by other means (representation-theoretical and combinatoric) and showed they are equal.

One way of approaching the principle of independence of polarization is the following, considered by Baier, Florentino, Mourão and Nunes in [**BFMN**]. Fix a Kähler polarization P_J and a real polarization P_μ on (M, ω). Then, the principle can be understood naturally if there is a family $\{P_{J_s}\}_{s \in [0, \infty)}$ of Kähler polarizations on M with $P_{J_0} = P_J$ which converges to P_μ in the sense that there exists a basis $\{\sigma_s^m\}_{m \in BS(P_\mu)}$ of $\mathcal{H}(P_{J_s})$ for each $s \in [0, \infty)$ such that, for each $m \in BS(P_\mu)$, σ_s^m converges to a delta-function section supported on the Bohr–Sommerfeld fiber $\mu^{-1}(m)$ as s goes to ∞. In [**BFMN**], the authors carried out such a construction in the case of a non-singular projective toric variety by changing symplectic potentials, an important notion in the deformation theory of toric Kähler structures due to Guillemin [**Gu1, Gu2**] and Abreu [**Ab1, Ab2**].

In this paper, we construct a family of Kähler polarizations on a flag manifold that converge to the real polarization coming from the Gelfand–Cetlin system. See Theorem 2.1 for details. In doing so, we provide a direct relationship between the two quantizations. Our construction is based on the construction due to [**BFMN**] and the toric degeneration of a flag variety due to Kogan and Miller [**KM**]. Originally, a toric degeneration of a flag variety was constructed in terms of representation theory [**GL, C**]. Later Kogan and Miller introduced deformed actions of a Borel subgroup on the space of matrices and described a toric degeneration of a flag variety explicitly. Moreover, they constructed a "degeneration in stages" of a flag variety to study the geometric meaning of the Gelfand–Cetlin basis of the irreducible representation of the unitary group. In [**NNU**], Nishinou, Nohara and Ueda pointed out that through the degeneration in stages one can identify the Gelfand–Cetlin system on the flag manifold with the integrable system on the limiting toric variety.

Our construction of a family of Kähler polarizations on a flag manifold proceeds as follows. We start from a flag manifold Fl_n embedded in the product of projective spaces $\mathbb{P} = \prod_{l=1,\ldots,n-1} \mathbb{P}(\bigwedge^l \mathbb{C}^n)$. For each $(a_1,\ldots,a_{n-1}) \in (\mathbb{Z}_{>0})^{n-1}$, we fix a prequantum line bundle on $\mathbb{P}$ inducing a natural symplectic structure on Fl_n. The toric degeneration of the flag variety Fl_n due to [**KM**] is a family of complex subvarieties $\{V_t\}_{t\in\mathbb{C}}$ in $\mathbb{P}$, where $V_1 = Fl_n$ and V_0 is a toric variety. Since all V_t are diffeomorphic to each other for $t \neq 0$, the family $\{V_t\}_{t\neq 0}$ can be considered as a family of Kähler structures on the flag manifold Fl_n. On the other hand, there is a family of toric Kähler structures $\{V_{0,s}\}_{s\in[0,\infty)}$ on V_0 with $V_{0,0} = V_0$, as considered in [**BFMN**] (explained above). If we could identify Fl_n with $V_{0,s}$ as a symplectic manifold, we could pull back the complex structures on $V_{0,s}$ to Fl_n. However, the toric variety V_0 is not diffeomorphic to the flag manifold Fl_n.

Instead, we consider a space V_t, which is still diffeomorphic to the flag manifold Fl_n but also is an approximation to V_0. We show that the deformation $\{V_{0,s}\}_{s\in[0,\infty)}$ can be realized as the restriction of a deformation of the ambient toric variety $\mathbb{P}$. The deformation of the ambient space induces a family of Kähler structures $\{V_{t,s}\}_{s\in[0,\infty)}$ on V_t with $V_{t,0} = V_t$ for each $t \in \mathbb{C}$. We develop a method to identify $V_{t,s}$ with $V_{t,0} = V_t$ as a symplectic manifold. Moreover, we identify Fl_n with V_t as a symplectic manifold by using the gradient-Hamiltonian flow (a notion that is due to Ruan [**R**]) along a path that is an approximation of the path for degeneration in stages. Hence, we can pull the complex structure of $V_{t,s}$ back to Fl_n. We also lift this identification to the prequantum line bundle in order to pull back holomorphic sections. Thus, we have a family of complex structures on the flag manifold with a fixed symplectic structure and a family of sections of the

prequantum line bundle on the flag manifold, which are holomorphic with respect to the corresponding complex structure. Moreover, we give a precise estimate of these holomorphic sections, which allows us to prove that the holomorphic sections converge to delta-function sections supported on the Bohr–Sommerfeld fibers if we perform these two types of deformations simultaneously in an appropriate way.

The content of this paper is organized as follows. In Section 2, we state our main result. We review the results on a toric degeneration of a flag variety in Section 3. Then, we recall the gradient-Hamiltonian flow and construct its lift to the line bundle in Section 4. In Section 5, we review toric Kähler structures of toric manifolds, in particular, their deformation due to [**BFMN**]. In Section 6, we develop a method to identify submanifolds under the deformation of toric Kähler structures of the ambient toric manifolds. We also give an estimate of the change of holomorphic sections under this deformation. In Section 7, we prove the main result, constructing a family of complex structures on the flag manifold, and proving that holomorphic sections converge to delta-function sections supported on Bohr–Sommerfeld fibers.

2. Main results

Let GL_n and B be the general linear group and its Borel subgroup consisting of upper triangular matrices with $\mathbb{C}$-coefficients, respectively. The flag manifold is defined to be the complex manifold $Fl_n = GL_n/B$. Let Λ_n be the set of increasing indexes $I = (i_1 < \cdots < i_l)$ with $1 \leq i_1, i_l \leq n$. For $I = (i_1 < \cdots < i_l) \in \Lambda_n$ and $V = (v_{ij}) \in GL_n$, we set $|I| = l$ and

$$p_I(V) = \det \begin{pmatrix} v_{i_1 1} & \ldots & v_{i_1 l} \\ \vdots & \ddots & \vdots \\ v_{i_l 1} & \ldots & v_{i_l l} \end{pmatrix}.$$

Then, the Plücker embedding

$$\rho \colon Fl_n \to \mathbb{P} = \prod_{l=1}^{n-1} \mathbb{P}(\bigwedge^l \mathbb{C}^n)$$

is defined by $[V] \mapsto ([p_I(V); |I| = 1], \ldots, [p_I(V); |I| = n-1])$, where $[x_I; |I| = l]$ is the homogeneous coordinate of $\mathbb{P}(\bigwedge^l \mathbb{C}^n)$. Since the left $U(n)$-action on $M_n(\mathbb{C})$ commutes with the right B-action on $M_n(\mathbb{C})$, $U(n)$ acts on Fl_n from the left.

Next, we define a holomorphic line bundle on Fl_n and a Hermitian connection on it. Let H_l be the hyperplane bundle on $\mathbb{P}(\bigwedge^l \mathbb{C}^n)$. It has a natural Hermitian metric h_l such that $\frac{\sqrt{-1}}{2\pi} R^{\nabla^l} = \omega_l$, where R^{∇^l} is the curvature of the Chern connection ∇_l for the Hermitian metric h_l and $\omega_l \in \Omega^2(\mathbb{P}(\bigwedge^l \mathbb{C}^n))$ is the Fubini–Study form. Let $\pi_l \colon \mathbb{P} \to \mathbb{P}(\bigwedge^l \mathbb{C}^n)$ be the projection. Fix

$\mathbf{a} = (a_1, \ldots, a_{n-1}) \in (\mathbb{Z}_{>0})^{n-1}$. Then, we define a Kähler form $\omega_{\mathbb{P}}$ and a prequantum line bundle $(L^{\mathbb{P}}, h^{\mathbb{P}}, \nabla^{\mathbb{P}})$ on $\mathbb{P}$ by

$$\omega_{\mathbb{P}} = \sum_{l=1}^{n-1} a_l \pi_l^* \omega_l \in \Omega^2(\mathbb{P}), (L^{\mathbb{P}}, h^{\mathbb{P}}, \nabla^{\mathbb{P}}) = \bigotimes_{l=1}^{n-1} \pi_l^*(H_l, h_l, \nabla^l)^{\otimes a_l}.$$

Then, $\nabla^{\mathbb{P}}$ is the Chern connection of $(L^{\mathbb{P}}, h^{\mathbb{P}})$ and satisfies $\frac{\sqrt{-1}}{2\pi} R^{\nabla^{\mathbb{P}}} = \omega_{\mathbb{P}}$. We set $(L^{Fl_n}, h^{Fl_n}, \nabla^{Fl_n}) = \rho^*(L^{\mathbb{P}}, h^{\mathbb{P}}, \nabla^{\mathbb{P}})$, i.e., L^{Fl_n} is a holomorphic line bundle on Fl_n with a Hermitian metric h^{Fl_n} and the Chern connection ∇^{Fl_n} whose first Chern form is $\rho^* \omega_{\mathbb{P}}$. The $U(n)$-action on Fl_n preserves $\rho^* \omega_{\mathbb{P}}$ with a moment map $\mu_{U(n)} \colon Fl_n \to \mathfrak{u}(n)^*$.

Next, we recall a certain completely integrable system on Fl_n. Consider $U(l)$ for $l = 1, \ldots, n-1$ as a subgroup of $U(n)$ defined by

$$(2.1) \qquad U(l) = \left\{ \begin{pmatrix} A & O_{l,n-l} \\ O_{n-l,l} & E_{n-l} \end{pmatrix} \in U(n) \right\},$$

where $O_{l,n-l} \in M_{l,n-l}(\mathbb{C})$ and $O_{n-l,l} \in M_{n-l,l}(\mathbb{C})$ are the zero matrices, $E_{n-l} \in M_{n-l}(\mathbb{C})$ is the unit element, and $A \in M_l(\mathbb{C})$. Let $\iota_l^* \colon \mathfrak{u}(n)^* \to \mathfrak{u}(l)^*$ be the dual map of the inclusion $\iota_l \colon \mathfrak{u}(l) \to \mathfrak{u}(n)$. Define a map $\lambda_l^j \colon \mathfrak{u}(l) \to \mathbb{R}$ such that $\lambda_l^1(A) \geq \cdots \geq \lambda_l^l(A)$ are eigenvalues of $-\sqrt{-1}A$ for $A \in \mathfrak{u}(l)$. We identify $\mathfrak{u}(l)$ with $\mathfrak{u}(l)^*$ by the invariant inner product. In [**GS**], Guillemin and Sternberg proved that

$$\mu_{\mathrm{GC}} = (\lambda_l^j \circ \iota_l^* \circ \mu_{U(n)}; 1 \leq l \leq n-1, 1 \leq j \leq l) \colon Fl_n \to \mathbb{R}^d$$

is a completely integrable system, where $d = \frac{1}{2} \dim_{\mathbb{R}} Fl_n = \frac{n(n-1)}{2}$. The completely integrable system $\mu_{\mathrm{GC}} \colon Fl_n \to \mathbb{R}^d$ and its image $\Delta_{\mathrm{GC}} = \mu_{\mathrm{GC}}(Fl_n) \subset \mathbb{R}^d$ are called the Gelfand–Cetlin system and the Gelfand–Cetlin polytope, respectively. Note that $\mu_{\mathrm{GC}} \colon Fl_n \to \mathbb{R}^d$ is a continuous map and that it is smooth on $\mu_{\mathrm{GC}}^{-1}(\mathrm{Int}\Delta_{\mathrm{GC}})$, where $\mathrm{Int}\Delta_{\mathrm{GC}}$ is the interior of Δ_{GC}. Moreover, $\mu_{\mathrm{GC}}^{-1}(m)$ is a d-dimensional real torus for each $m \in \mathrm{Int}\Delta_{\mathrm{GC}}$. In [**GS**], Guillemin and Sternberg also proved that, for $m \in \mathrm{Int}\Delta_{\mathrm{GC}} \subset \mathbb{R}^d$, the fiber $\mu_{\mathrm{GC}}^{-1}(m)$ is a Bohr–Sommerfeld fiber if and only if $m \in \mathrm{Int}\Delta_{\mathrm{GC}} \cap \mathbb{Z}^d$ and that the number of the points $\Delta_{\mathrm{GC}} \cap \mathbb{Z}^d$ coincides with the dimension of the space of holomorphic sections $H^0(L^{Fl_n}, \overline{\partial}^{Fl_n})$, where $\overline{\partial}^{Fl_n}$ is the holomorphic structure on L^{Fl_n}. Namely, the quantum Hilbert space for the Kähler polarization on Fl_n is isomorphic to the one for the real polarization $P_{\mu_{\mathrm{GC}}}$ coming from the Gelfand–Cetlin system μ_{GC}.

In this paper, we construct a family of complex structures $\{J_s\}_{s \in [0,\infty)}$ on Fl_n such that the family of Kähler polarizations $\{P_{J_s}\}_{s \in [0,\infty)}$ converges to the real polarization $P_{\mu_{\mathrm{GC}}}$ in the following sense.

Theorem 2.1. *Let $\mathbb{F}$ and $J_{\mathbb{F}}$ be the underlying C^{∞}-manifold and the complex structure of Fl_n, respectively. Set $\omega_{\mathbb{F}} = \rho^*\omega_{\mathbb{P}} \in \Omega^2(\mathbb{F})$ and $d = \dim_{\mathbb{R}} \mathbb{F} = \frac{n(n-1)}{2}$. Let $(L^{\mathbb{F}}, h^{\mathbb{F}}, \nabla^{\mathbb{F}})$ be the underlying C^{∞} line bundle of $(L^{Fl_n}, h^{Fl_n}, \nabla^{Fl_n})$. Then, there exists a one-parameter family of $\{J_s\}_{s \in [0,\infty)}$ of complex structures on $\mathbb{F}$ which satisfies the following:*

(1) *J_s is continuous with respect to the parameter $s \in [0,\infty)$.*

(2) *$J_0 = J_{\mathbb{F}}$.*

(3) *$(\mathbb{F}, \omega_{\mathbb{F}}, J_s)$ is a Kähler manifold for each $s \in [0,\infty)$. So, for each $s \in [0,\infty)$, the Hermitian line bundle $(L^{\mathbb{F}}, h^{\mathbb{F}}, \nabla^{\mathbb{F}})$ induces the holomorphic structure $\overline{\partial}^s$ on $L^{\mathbb{F}}$.*

(4) *For each $s \in [0,\infty)$, there exists a basis $\{\sigma_s^m \mid m \in \Delta_{\mathrm{GC}} \cap \mathbb{Z}^d\}$ of the space of holomorphic sections $H^0(L^{\mathbb{F}}, \overline{\partial}^s)$ such that, for each $m \in \mathrm{Int}\Delta_{\mathrm{GC}} \cap \mathbb{Z}^d$, the section $\frac{\sigma_s^m}{\|\sigma_s^m\|_{L^1(\mathbb{F})}}$ converges to a delta-function section supported on the Bohr–Sommerfeld fiber $\mu_{\mathrm{GC}}^{-1}(m)$ in the following sense: there exists a covariantly constant section $\delta_m^{\mathbb{F}}$ of $(L^{\mathbb{F}}, h^{\mathbb{F}}, \nabla^{\mathbb{F}})|_{\mu_{\mathrm{GC}}^{-1}(m)}$ and a measure $d\theta_m$ on $\mu_{\mathrm{GC}}^{-1}(m)$ such that, for any smooth section ϕ of the dual line bundle $(L^{\mathbb{F}})^*$, the following holds:*

$$\lim_{s \to \infty} \int_{\mathbb{F}} \left\langle \phi, \frac{\sigma_s^m}{\|\sigma_s^m\|_{L^1(\mathbb{F})}} \right\rangle \frac{\omega_{\mathbb{F}}^d}{d!} = \int_{\mu_{\mathrm{GC}}^{-1}(m)} \langle \phi, \delta_m^{\mathbb{F}} \rangle d\theta_m.$$

Remark 2.2. By a similar argument as in the proof of Theorem 2.1, we can also prove that the support of the section σ_s^m converges to $\mu_{\mathrm{CG}}^{-1}(m)$ as $s \to \infty$ for any $m \in (\Delta_{\mathrm{GC}} \setminus \mathrm{Int}\Delta_{\mathrm{GC}}) \cap \mathbb{Z}^d$. However, we cannot prove that σ_s^m converges to a delta-function section for $m \in (\Delta_{\mathrm{GC}} \setminus \mathrm{Int}\Delta_{\mathrm{GC}}) \cap \mathbb{Z}^d$, because we do not yet have a sufficient description of $\mu_{\mathrm{CG}}^{-1}(m)$.

3. Toric degeneration of flag varieties

In [**KM**], Kogan and Miller constructed a toric degeneration of a flag variety based on a deformed Borel action. They also introduced degeneration in stages of a flag variety. In this section, we review their construction and recall its symplectic geometric aspects due to Nishinou, Nohara and Ueda [**NNU**].

3.1. Deformed Borel action and toric degeneration. First, we define the right action $\bullet$ of the product group $(GL_n)^n$ on $M_n(\mathbb{C})$ by

$$V \bullet g = \begin{pmatrix} \mathbf{v}_1 g_1 \\ \vdots \\ \mathbf{v}_n g_n \end{pmatrix} \quad \text{for} \quad V = \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} \in M_n(\mathbb{C}) \text{ and } g = (g_1, \ldots, g_n) \in (GL_n)^n.$$

Set $M_n(\mathbb{C}^\times) = \{(a_{ij}) \in M_n(\mathbb{C}) | a_{ij} \neq 0 \text{ for } i, j = 1, \ldots, n\}$. Define a map $\iota \colon M_n(\mathbb{C}^\times) \to (GL_n)^n$ by

$$\iota((a_{ij})) = \left(\begin{pmatrix} a_{11} & & O \\ & \ddots & \\ O & & a_{1n} \end{pmatrix}, \ldots, \begin{pmatrix} a_{n1} & & O \\ & \ddots & \\ O & & a_{nn} \end{pmatrix} \right).$$

Note that $\iota(M_n(\mathbb{C}^\times))$ is the maximal torus of $(GL_n)^n$. We also set

$$T_{\mathrm{GC}} = \left\{ \iota \left(\begin{pmatrix} 1 & & & 1 \\ a_{21} & & \ddots & \\ \vdots & \ddots & & \\ a_{n1} & \cdots & a_{nn-1} & 1 \end{pmatrix} \right) \middle| \begin{pmatrix} 1 & & & 1 \\ a_{21} & & \ddots & \\ \vdots & \ddots & & \\ a_{n1} & \cdots & a_{nn-1} & 1 \end{pmatrix} \in M_n(\mathbb{C}^\times) \right\}.$$

We also define a k-dimensional algebraic subtorus $T_{\mathrm{GC}}^{(k)}$ of T_{GC} by

$$T_{\mathrm{GC}}^{(k)} = \{\iota((a_{ij})) | (a_{ij}) \in M_n(\mathbb{C}^\times), \quad i = k+1 \text{ and } j \leq k \text{ if } a_{ij} \neq 1\}.$$

Then, we have

$$T_{\mathrm{GC}} = \{1\} \times T_{\mathrm{GC}}^{(1)} \times \cdots \times T_{\mathrm{GC}}^{(n-2)} \times T_{\mathrm{GC}}^{(n-1)}.$$

Next, we define the deformed Borel action as follows. For $t \in \mathbb{C}^\times$, we define $t^\omega \in M_n(\mathbb{C}^\times)$ by

$$(3.1) \qquad (t^\omega)_{ij} = t^{\omega_{ij}}, \text{ where } \omega_{ij} = \begin{cases} 3^{i-j-1} & \text{if } i > j, \\ 0 & \text{if } i \leq j. \end{cases}$$

In the above $(t^\omega)_{ij}$ is the (i, j)-component of $t^\omega \in M_n(\mathbb{C}^\times)$. Then, we define the deformed action $\bullet_t$ of B on $M_n(\mathbb{C})$ by

$$V \bullet_t b = V \bullet \{\iota(t^\omega)(b, \ldots, b)(\iota(t^\omega))^{-1}\},$$

where $\iota(t^\omega), (b, \ldots, b), \iota(t^\omega)^{-1} \in (GL_n)^n$.

Let $\mathbb{C}[v_{ij} | 1 \leq i, j \leq n]$ be the coordinate ring of $M_n(\mathbb{C})$. Let $U \subset B$ the subgroup consisting of the matrices with 1's on the diagonal. Then, the ring of U-invariant functions $\mathbb{C}[v_{ij} | 1 \leq i, j \leq n]^U$ for the deformed action $\bullet_t$ of U is generated by the deformed Plücker coordinates

$$\{q_I(V, t) = d_I(t^\omega)^{-1} p_I(V \bullet \iota(t^\omega)) | I \in \Lambda_n\}, \text{ where } d_I(t^\omega) = \prod_{k=1}^{|I|} (t^\omega)_{i_k k}.$$

From the definition of $\omega \in M_n(\mathbb{Z})$, we see that $q_I(V, t)$ is a polynomial of v_{ij} $(1 \leq i, j \leq n)$ and t. Moreover, the deformed action $\bullet_t$ can be naturally extended to the case $t = 0$. Thus, we have a quotient $Fl_n(t) = M_n(\mathbb{C})//_t B$ for all $t \in \mathbb{C}$, where the right-hand side is a GIT quotient by the deformed action $\bullet_t$. We also have a family $f \colon (M_n(\mathbb{C}) \times \mathbb{C})//B \to \mathbb{C}$ with $f^{-1}(t) = Fl_n(t)$. $Fl_n(1)$ is nothing but the flag variety Fl_n. Note that each $Fl_n(t)$ is embedded in $\mathbb{P}$ by the deformed Plücker embedding $\rho_t \colon Fl_n(t) \to \mathbb{P}$, which is defined by $[V] \mapsto ([q_I(V, t); |I| = 1], \ldots, [q_I(V, t); |I| = n - 1])$ as in the

case of the usual Plücker embedding. In [**KM**], Kogan and Miller proved the following, based on the argument in [**GL**].

Proposition 3.1. (1) *The family* $f\colon (M_n(\mathbb{C}) \times \mathbb{C})//B \to \mathbb{C}$ *is flat.*
(2) $Fl_n(t)$ *is biholomorphic to* Fl_n *for any* $t \in \mathbb{C}^\times$*. Moreover,* $Fl_n(0)$ *is a toric variety on which the torus* T_{GC} *acts with an open dense orbit.*

Let us give a few remarks about Proposition 3.1. Note that, if we set

$$GL_n(t) = \{V \in M_n(\mathbb{C}) | V \bullet \iota(t^\omega) \in GL_n\},$$

then we have $Fl_n(t) = GL_n(t)/_t B$ for $t \in \mathbb{C}^\times$, where the right-hand side is a geometric quotient of $GL_n(t)$ by the deformed action $\bullet_t$ of the Borel subgroup B. So, we see that $Fl_n(t)$ is biholomorphic to Fl_n for any $t \in \mathbb{C}^\times$. Moreover, since the action $\bullet g$ on $M_n(\mathbb{C})$ for $g \in T_{\mathrm{GC}}$ commutes with the action $\bullet_0 b$ on $M_n(\mathbb{C})$ for $b \in B$, the torus T_{GC} acts on $Fl_n(0) = M_n(\mathbb{C})//_0 B$. Thus, the family $f\colon (M_n(\mathbb{C}) \times \mathbb{C})//B \to \mathbb{C}$ can be viewed as a toric degeneration of a flag variety. The existence of a toric degeneration of a flag variety is originally proved in [**GL, C**] in terms of representation theory. In [**AB**], a toric degeneration of a more general variety is constructed systematically.

3.2. Degeneration in stages. To relate the $U(n)$-action on $Fl_n = Fl_n(1)$ with the T_{GC}-action on $Fl_n(0)$, Kogan and Miller introduced degeneration in stages as follows. For $\tau = (t_2, \ldots, t_n) \in (\mathbb{C}^\times)^{n-1}$, we define $\tau^\omega \in M_n(\mathbb{C}^\times)$ by

$$(\tau^\omega)_{ij} = t_i^{\omega_{ij}}, \text{ where } t_1 = 1 \text{ and } \omega_{ij} \text{ is given in (3.1).}$$

In the above $(\tau^\omega)_{ij}$ is the (i, j)-component of $\tau^\omega \in M_n(\mathbb{C}^\times)$. Then, we define the deformed action $\bullet_\tau$ of B on $M_n(\mathbb{C})$ by

$$V \bullet_\tau b = V \bullet \{\iota(\tau^\omega)(b, \ldots, b)(\iota(\tau^\omega))^{-1}\}.$$

Thus, we have $Fl_n(\tau) = M_n(\mathbb{C})//_\tau B$ for $\tau \in (\mathbb{C}^\times)^{n-1}$ in the same way as in Section 3.1. We note that $Fl_n(\tau)$ is also embedded in $\mathbb{P}$ by the deformed Plücker relations as $Fl_n(t)$. Set

$$\tau_k^t = (\underbrace{1, \ldots, 1}_{n-1-k}, t, \underbrace{0, \ldots, 0}_{k-1}) \in \mathbb{C}^{n-1} \text{ for } t \in [0, 1] \text{ and } k = 1, \ldots, n-1.$$

It is easy to see that $Fl_n(\tau_k^t) = M_n(\mathbb{C})//_{\tau_k^t} B$ is well defined. Note that $Fl_n(\tau_k^t)$ has singularities if $\tau_k^t = \tau_1^0$ or $k \geq 2$. We call the family $\{Fl_n(\tau_k^t)\}_{t \in [0,1]}$ the kth stage of the degeneration. Note that

$$U(n - k + 1) \times T_{\mathrm{GC}}^{(n-1)} \times \cdots \times T_{\mathrm{GC}}^{(n-k+1)} \text{ acts on } Fl_n(\tau_k^1),$$
$$U(n - k) \times T_{\mathrm{GC}}^{(n-1)} \times \cdots \times T_{\mathrm{GC}}^{(n-k+1)} \text{ acts on } Fl_n(\tau_k^t) \text{ for } t \in (0, 1),$$
$$U(n - k) \times T_{\mathrm{GC}}^{(n-1)} \times \cdots \times T_{\mathrm{GC}}^{(n-k)} \text{ acts on } Fl_n(\tau_k^0),$$

where $U(l)$ for $l = 1, \ldots, n-1$ are the subgroups of $U(n)$ defined by (2.1). Kogan and Miller considered the following degeneration in stages:

$$Fl_n = Fl_n(\tau_1^1) \xrightarrow{1st} Fl_n(\tau_1^0) = Fl_n(\tau_2^1) \longrightarrow \cdots \longrightarrow Fl_n(\tau_k^1) \xrightarrow{kth} Fl_n(\tau_k^0)$$
$$\longrightarrow \cdots \longrightarrow Fl_n(\tau_{n-1}^0) = Fl_n(0).$$

In [**NNU**], Nishinou, Nohara and Ueda clarified the relation between the Gelfand–Cetlin system on the flag variety Fl_n and the completely integrable system on $Fl_n(0)$ coming from its toric structure as follows. The smooth part $Fl_n(\tau_k^t)^{\mathrm{reg}}$ of $Fl_n(\tau_k^t)$ has a symplectic structure $\iota_{\tau_k^t}^* \omega_{\mathbb{P}}$, where $\iota_{\tau_k^t} \colon Fl_n(\tau_k^t)^{\mathrm{reg}} \to \mathbb{P}$ is the deformed Plücker embedding. Let $\mu_{U(n-k)} \colon Fl_n(\tau_k^t)^{\mathrm{reg}} \to \mathfrak{u}(n-k)$ be the moment map for $U(n-k)$-action on $Fl_n(\tau_k^t)^{\mathrm{reg}}$ for $t \in [0,1]$, where $\mathfrak{u}(n-k)$ is identified with $\mathfrak{u}(n-k)^*$ by the invariant inner product. Define a map $\lambda_{n-k}^j \colon \mathfrak{u}(n-k) \to \mathbb{R}$ such that $\lambda_{n-k}^1(A) \geq \cdots \geq \lambda_{n-k}^{n-k}(A)$ are eigenvalues of $-\sqrt{-1}A$ for $A \in \mathfrak{u}(n-k)$ as in Section 2. Then, in [**NNU**], the authors proved the following.

Proposition 3.2. *There exists an open dense subset $Fl_n(\tau_k^t)^\circ \subset Fl_n(\tau_k^t)^{\mathrm{reg}}$ and a symplectic diffeomorphism $\varphi_k^{t_2,t_1} \colon Fl_n(\tau_k^{t_1})^\circ \to Fl_n(\tau_k^{t_2})^\circ$ for each $k = 1, \ldots, n-1$, $t \in [0,1]$ and $0 \leq t_2 \leq t_1 \leq 1$ which satisfy the following:*

(1) *$Fl_n(\tau_1^1)^\circ = \mu_{\mathrm{GC}}^{-1}(\mathrm{Int}\Delta_{\mathrm{GC}}) \subset Fl_n$ holds.*
(2) *$\varphi_k^{t,t}$ is the identity map for any $t \in [0,1]$. Moreover, $\varphi_k^{t_3,t_2} \circ \varphi_k^{t_2,t_1} = \varphi_k^{t_3,t_1}$ holds for $0 \leq t_3 \leq t_2 \leq t_1 \leq 1$.*
(3) *Under the identification of $Fl_n(\tau_k^t)^\circ$ for all $t \in [0,1]$ by the maps $\varphi_k^{t_2,t_1}$, the moment map for $U(n-k) \times T_{\mathrm{GC}}^{(n-1)} \times \cdots \times T_{\mathrm{GC}}^{(n-k+1)}$-action on $Fl_n(\tau_k^t)^\circ$ is independent of $t \in (0,1]$.*
(4) *$(\lambda_{n-k}^j \circ \mu_{U(n-k)} | 1 \leq j \leq n-k) \colon Fl_n(\tau_k^0)^\circ \to \mathbb{R}^{n-k}$ coincides with the moment map for the $T_{\mathrm{GC}}^{(n-k)}$-action on $Fl_n(\tau_k^0)^\circ$.*

The diffeomorphism $\varphi_k^{t_2,t_1} \colon Fl_n(\tau_k^{t_1})^\circ \to Fl_n(\tau_k^{t_2})^\circ$ is constructed by using the gradient-Hamiltonian flow due to Ruan [**R**], which is explained in the next section. The moment map for $U(n-k) \times T_{\mathrm{GC}}^{(n-1)} \times \cdots \times T_{\mathrm{GC}}^{(n-k+1)}$-action on $Fl_n(\tau_k^t)^\circ$ induces the completely integrable system on $Fl_n(\tau_k^t)^\circ$ in the same way as in the case of the Gelfand–Cetlin system. Proposition 3.2 implies the completely integrable system on $Fl_n(\tau_k^t)^\circ$ for $t \in [0,1]$ and $1 \leq k \leq n-1$ remains the same during the degeneration in stages.

Due to Proposition 3.2, we have a diffeomorphism

(3.2) $$\Psi_0 = \varphi_{n-1}^{0,1} \circ \varphi_{n-2}^{0,1} \circ \cdots \circ \varphi_1^{0,1} \colon Fl_n^\circ \to Fl_n(0)^\circ,$$

where $Fl_n^\circ = Fl_n(\tau_1^1)^\circ$ and $Fl_n(0)^\circ = Fl_n(\tau_{n-1}^0)^\circ$. Then, Nishinou, Nohara and Ueda proved the following.

Corollary 3.3. *Let* $\mu_{\mathrm{GC}}\colon Fl_n \to \mathbb{R}^{\frac{n(n-1)}{2}}$ *be the Gelfand–Cetlin system. Let* $\mu_{T_{\mathrm{GC}}}\colon Fl_n(0) \to (\mathfrak{t}_{\mathrm{GC}})^*$ *be the moment map for the action of* T_{GC} *on* $Fl_n(0)$. *Then, there is a linear isomorphism* $i\colon \mathbb{R}^{\frac{n(n-1)}{2}} \to (\mathfrak{t}_{\mathrm{GC}})^*$ *such that* $i \circ \mu_{\mathrm{GC}} = \mu_{T_{\mathrm{GC}}} \circ \Psi_0 \colon Fl_n^\circ \to (\mathfrak{t}_{\mathrm{GC}})^*$. *In particular,* $Fl_n(0)^\circ = \mu_{T_{\mathrm{GC}}}^{-1}(\mathrm{Int}\Delta_{\mathrm{GC}}) \subset Fl_n(0)$ *holds.*

Therefore, the authors concluded that $Fl_n(0)$ is a toric variety constructed from the Gelfand–Cetlin polytope Δ_{GC}. This fact is originally proved in [**KM**] in a different way. So, $Fl_n(0)$ is called a Gelfand–Cetlin toric variety. Moreover, the Gelfand–Cetlin polytope Δ_{GC} can be considered naturally as a subset of $(\mathfrak{t}_{\mathrm{GC}})^*$. From now on, we consider the Gelfand–Cetlin system to be the map $\mu_{\mathrm{GC}}\colon Fl_n \to (\mathfrak{t}_{\mathrm{GC}})^*$.

4. Gradient-Hamiltonian flow

Let (M, ω, J) be a Kähler manifold. Let $f\colon M \to \mathbb{C}$ be a holomorphic function. Set $B = f(M)$ and $V_c = f^{-1}(c)$ for $c \in B$. Denote the inclusion map of V_c by $\rho_c\colon V_c \to M$. Then, we have a family of symplectic manifolds $\{(V_c, \rho_c^*\omega)\}_{c \in B_{\mathrm{reg}}}$ where B_{reg} is the set of regular values of f. To identify these symplectic manifolds, Ruan introduced the gradient-Hamiltonian flow in [**R**]. In this section, we recall the gradient-Hamiltonian flow and its basic properties. We also discuss the lift of the gradient-Hamiltonian flow to the prequantum line bundle.

By simple computations, we see that the following holds.

Lemma 4.1. *Let* (M, ω, J) *be a Kähler manifold. Let* $\Re f$ *and* $\Im f$ *be the real and imaginary part of the holomorphic function* $f\colon M \to \mathbb{C}$, *respectively. Let* $X_{\Im f} \in \mathcal{X}(M)$ *be the Hamiltonian vector field of the function* $\Im f$. *Then, the following holds:*

$$X_{\Im f} = -\mathrm{grad}(\Re f), \text{ that is, } i(-\mathrm{grad}(\Re f))\omega = -d(\Im f),$$

where $i(-\mathrm{grad}(\Re f))\omega$ *is the contraction. In particular,* $X_{\Im f} = -\mathrm{grad}(\Re f)$ *is non-zero at a regular point of* f.

Suppose that f is proper and that each point in M is a regular point of f. Then, we have the following vector field:

$$Z = -\frac{\mathrm{grad}(\Re f)}{|\mathrm{grad}(\Re f)|^2} = \frac{X_{\Im f}}{|X_{\Im f}|^2} \in \mathcal{X}(M).$$

It is easy to see that

$$Z(\Re f) = -1, \quad Z(\Im f) = 0 \text{ on } M.$$

Since $f: M \to B$ is proper, for any $c \in B$ there exists $\epsilon_c > 0$ such that the flow $\{\varphi_t\}_t$ generated by the vector field $Z \in \mathcal{X}(M)$ induces a diffeomorphism $\varphi_t|_{V_c}: V_c \to V_{c-t}$ for $t \in (-\epsilon_c, \epsilon_c)$. In [**R**], Ruan found the following remarkable property.

Proposition 4.2. $(\varphi_t|_{V_c})^*(\rho^*_{c-t}\omega) = \rho^*_c\omega$ *for* $t \in (-\epsilon_c, \epsilon_c)$.

We call $Z \in \mathcal{X}(M)$ the gradient-Hamiltonian vector field and $\{\varphi_t\}_t$ the gradient-Hamiltonian flow, respectively.

Next, we discuss the lift of the gradient-Hamiltonian flow to the prequantum line bundle. Let us assume that there exists a prequantum line bundle (L, h, ∇) on M in addition to the above setting. For any $c \in B$, we denote the restriction of (L, h, ∇) to the fiber V_c by $(L^{V_c}, h^{V_c}, \nabla^{V_c})$.

The horizontal lift $\tilde{Z} \in \mathcal{X}(L)$ of $Z \in \mathcal{X}(M)$ induces the flow $\{\tilde{\varphi}_t\}_t$, which is a lift of the gradient-Hamiltonian flow $\{\varphi_t\}_t$. Similarly, for any $c \in B$, there exists $\epsilon_c > 0$ such that the flow $\{\tilde{\varphi}_t\}_t$ induces a bundle isomorphism $\tilde{\varphi}_t|_{L^{V_c}}: L^{V_c} \to L^{V_{c-t}}$ for $t \in (-\epsilon_c, \epsilon_c)$.

Then, we have the following proposition. Since its proof does not seem to be found in the literature, we give a proof here.

Proposition 4.3. $(\tilde{\varphi}_t|_{L^{V_c}})^*\nabla^{V_{c-t}} = \nabla^{V_c}$ *and* $(\tilde{\varphi}_t|_{L^{V_c}})^*h^{V_{c-t}} = h^{V_c}$ *for* $t \in (-\epsilon_c, \epsilon_c)$.

Proof. Since the connection ∇ preserves the Hermitian metric h, the second assertion is obvious. So, we prove the first assertion.

Since $Z(\Im f) = 0$ on M, the gradient-Hamiltonian flow $\{\varphi_t\}_t$ preserves $M_{\Im f = \Im c} = \{p \in M | \Im f(p) = \Im c\}$. First, we show that $i(Z)\omega = 0$ on $M_{\Im f = \Im c}$. In fact, we have

$$i(Z)\omega = i\left(\frac{X_{\Im f}}{|X_{\Im f}|^2}\right)\omega = \frac{-d(\Im f)}{|X_{\Im f}|^2} = 0 \text{ on } M_{\Im f = \Im c}.$$

Let $S \subset L$ be the unit sphere bundle and $p: S \to M$ the projection. If we denote the connection form of ∇ by $\alpha \in \Omega^1(S)$, then we have $d\alpha = p^*\omega$. Since the restriction of the horizontal lift $\tilde{Z} \in \mathcal{X}(L)$ to S can be considered as $\tilde{Z} \in \mathcal{X}(S)$, we have $i(\tilde{Z})\alpha = 0$ and $p_*\tilde{Z} = Z$. So, on $p^{-1}(M_{\Im f = \Im c})$, we have

$$L_{\tilde{Z}}\alpha = i(\tilde{Z})(p^*\omega) = p^*\{i(p_*\tilde{Z})\omega\} = p^*\{i(Z)\omega\} = 0.$$

Thus, the flow induced by the vector field $\tilde{Z} \in \mathcal{X}(S)$ preserves the connection ∇ on $p^{-1}(M_{\Im f = \Im c})$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. Toric Kähler structures of toric manifolds

In this section, we review toric Kähler structures of toric manifolds. Starting from a Delzant polytope, we construct a symplectic toric manifold in Section 5.1 (due to [**D**], also described in [**Gu1**]) and a complex toric manifold in Section 5.2 (described in [**Au**] and in Appendix 1 of [**Gu2**]). See [**F**] for general properties of a complex toric manifold. We identify them according to a choice of symplectic potentials due to [**Ab1, Ab2, Gu1, Gu2**] in Section 5.3. We also review a certain deformation of toric Kähler structures by changing symplectic potentials, which was introduced in [**BFMN**].

Let T^n be a real torus with the Lie algebra $\mathfrak{t}^n$. Let

$$(5.1) \qquad \Delta = \{p \in (\mathfrak{t}^n)^* | \langle p, r_j \rangle + \lambda_j \geq 0 \text{ for } j = 1, \ldots, d\}$$

be a bounded Delzant polytope, where $\langle,\rangle \colon (\mathfrak{t}^n)^* \times \mathfrak{t}^n \to \mathbb{R}$ is the natural pairing and r_j is a primitive vector in the lattice $\mathfrak{t}^n_{\mathbb{Z}}$ for $j = 1, \ldots, d$. We assume $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$. We set

$$(5.2) \qquad l_j(p) = \langle p, r_j \rangle + \lambda_j, F_j = \{p \in (\mathfrak{t}^n)^* | l_j(p) = 0\} \text{ for } j = 1, \ldots, d.$$

Let T^d be a real torus with the Lie algebra $\mathfrak{t}^d$ and $X_1, \ldots, X_d \in \mathfrak{t}^d_{\mathbb{Z}}$ be the standard basis of $\mathfrak{t}^d$. Let $\pi \colon \mathfrak{t}^d \to \mathfrak{t}^n$ be the surjective Lie algebra homomorphism defined by $\pi(X_j) = r_j$ for $j = 1, \ldots, d$. Then, the kernel of the corresponding Lie group homomorphism $\tilde{\pi} \colon T^d \to T^n$ is a connected subtorus K of T^d with the Lie algebra $\mathfrak{k}$. Let $u_1, \ldots, u_d \in (\mathfrak{t}^d)^*$ be the dual basis of $X_1, \ldots, X_d \in \mathfrak{t}^d_{\mathbb{Z}}$. We set $\lambda_\Delta = \lambda_1 u_1 + \cdots + \lambda_d u_d \in (\mathfrak{t}^d)^*_{\mathbb{Z}}$.

5.1. A symplectic toric manifold M_{symp}.
Let $\tilde{\omega}$ be the standard symplectic form on $\mathbb{C}^d$. The natural action of T^d on $(\mathbb{C}^d, \tilde{\omega})$ admits a moment map $\mu_{T^d} \colon \mathbb{C}^d \to (\mathfrak{t}^d)^*$, given by $\mu_{T^d}(z) = \pi \sum_{j=1}^d |z_j|^2 u_j$, where $z = (z_1, \ldots, z_d)$. If we denote the dual map of the inclusion $\iota \colon \mathfrak{k} \to \mathfrak{t}^d$ by $\iota^* \colon (\mathfrak{t}^d)^* \to \mathfrak{k}^*$, then the moment map $\mu_K \colon \mathbb{C}^d \to \mathfrak{k}^*$ for the action of the subtorus K on $(\mathbb{C}^d, \tilde{\omega})$ is given by $\mu_K(z) = \pi \sum_{j=1}^d |z_j|^2 \iota^* u_j$. The compact symplectic toric manifold M_{symp} is defined to be the symplectic quotient $M_{\text{symp}} = \mu_K^{-1}(\iota^* \lambda_\Delta)/K$ with the natural symplectic structure $\omega \in \Omega^2(M_{\text{symp}})$. The quotient torus $T^n = T^d/K$ acts on $(M_{\text{symp}}, \omega)$ with the moment map $\mu_{T^n} \colon M_{\text{symp}} \to (\mathfrak{t}^n)^*$. Since $\mu_{T^d}(z) - \lambda_\Delta \in \ker\{\iota^* \colon (\mathfrak{t}^d)^* \to \mathfrak{k}^*\} = \text{image}\{\pi^* \colon (\mathfrak{t}^n)^* \to (\mathfrak{t}^d)^*\}$, it is given by $\mu_{T^n}([z]) = (\pi^*)^{-1}(\mu_{T^d}(z) - \lambda_\Delta) \in (\mathfrak{t}^n)^*$. It is well known that $\mu_{T^n}(M_{\text{symp}}) = \Delta$.

Next, we define a prequantum line bundle on M_{symp}. Let $\tilde{L}_{\text{symp}} = \mathbb{C}^d \times \mathbb{C}$ be the trivial line bundle with the standard fiber metric $\tilde{h}$. Let $\tilde{\nabla}$ be a Hermitian connection on $\tilde{L}_{\text{symp}}$ defined by $\tilde{\nabla} = d - \sqrt{-1}\pi \sum_{i=j}^d (x_j dy_j - y_j dx_j)$, where x_j, y_j are the real and imaginary part of z_j, respectively. The action of T^d on $\tilde{L}_{\text{symp}}$ defined by $(z, v)\text{Exp}_{T^d}\xi = (z\text{Exp}_{T^d}\xi, ve^{2\pi\sqrt{-1}\langle\lambda_\Delta, \xi\rangle})$ preserves the Hermitian metric $\tilde{h}$ and the connection $\tilde{\nabla}$, where $\text{Exp}_{T^d}\colon \mathfrak{t}^d \to T^d$ is the exponential map. Then, the prequantum line bundle $(L_{\text{symp}}, h, \nabla)$ on M_{symp} is defined to be the quotient of the restriction of $\tilde{L}_{\text{symp}}$ to $\mu_K^{-1}(\iota^*\lambda_\Delta)$ by the action of the subtorus K. Moreover, the quotient torus $T^n = T^d/K$ acts on L_{symp}, preserving h and ∇. Let $[z]_K \in M_{\text{symp}}$ denote a point represented by $z \in \mu_K^{-1}(\iota^*\lambda_\Delta)$. Similarly, $[z, v]_K$ denotes a point in L_{symp} represented by $(z, v) \in \mu_K^{-1}(\iota^*\lambda_\Delta) \times \mathbb{C}$.

Set $M_{\text{symp}}^0 = \mu_{T^n}^{-1}(\text{Int}\Delta)$, where $\text{Int}\Delta$ is the interior of the Delzant polytope Δ. Then, it is easy to see that $\left(\sqrt{\frac{l_1(p)}{\pi}}, \ldots, \sqrt{\frac{l_d(p)}{\pi}}\right) \in \mu_K^{-1}(\iota^*\lambda_\Delta)$ for any $p \in \text{Int}\Delta$. Therefore, the map $\psi_{\text{symp}}^0 \colon \text{Int}\Delta \times \mathfrak{t}^n/\mathfrak{t}_\mathbb{Z}^n \to M_{\text{symp}}^0$ defined by

$$(5.3) \quad \psi_{\text{symp}}^0(p, [q]) = \left[\left(\sqrt{\frac{l_1(p)}{\pi}}, \ldots, \sqrt{\frac{l_d(p)}{\pi}}\right)\right]_K \text{Exp}_{T^n}(q)$$

$$= \left[\left(\sqrt{\frac{l_1(p)}{\pi}}e^{2\pi\sqrt{-1}\langle u_1, \tilde{q}\rangle}, \ldots, \sqrt{\frac{l_d(p)}{\pi}}e^{2\pi\sqrt{-1}\langle u_d, \tilde{q}\rangle}\right)\right]_K$$

is a diffeomorphism, where $\tilde{q} \in \mathfrak{t}^d$ is taken so that $\pi(\tilde{q}) = q$. Note that we have

$$(5.4) \qquad \mu_{T^n} \circ \psi_{\text{symp}}^0(p, [q]) = p \quad \text{for } (p, [q]) \in \text{Int}\Delta \times \mathfrak{t}^n/\mathfrak{t}_\mathbb{Z}^n.$$

Next, we define a section s_{symp}^0 of L_{symp} restricted to M_{symp}^0 by

$$s_{\text{symp}}^0(p, [q]) = \left[\left(\sqrt{\frac{l_1(p)}{\pi}}, \ldots, \sqrt{\frac{l_d(p)}{\pi}}\right), 1\right]_K \text{Exp}_{T^n}(q) \in L_{\text{symp}}.$$

This section induces a unitary trivialization of the prequantum line bundle L_{symp} on M_{symp}^0.

Fix a $\mathbb{Z}$-basis $p_1, \ldots, p_n \in (\mathfrak{t}^n)_\mathbb{Z}^*$ and its dual basis $q_1, \ldots, q_n \in \mathfrak{t}_\mathbb{Z}^n$. Set $\Delta^0 = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n | \sum_{i=1}^n x_i p_i \in \text{Int}\Delta\}$. Then, we have a coordinate $(x, [\theta]) \in \Delta^0 \times \mathbb{R}^n/\mathbb{Z}^n$ on $\text{Int}\Delta \times \mathfrak{t}^n/\mathfrak{t}_\mathbb{Z}^n$. So, $(x, [\theta]) \in \Delta^0 \times \mathbb{R}^n/\mathbb{Z}^n$ can be considered as a coordinate on M_{symp}^0. It is easy to see the following by simple computations.

Lemma 5.1. *Let* $(x, [\theta]) \in \Delta^0 \times \mathbb{R}^n/\mathbb{Z}^n$ *be the coordinate on* M_{symp}^0 *induced by the fixed basis* $p_1, \ldots, p_n \in (\mathfrak{t}^n)_\mathbb{Z}^*$. *Then, the symplectic form* ω *on* M_{symp}^0

and the connection ∇ on $L_{\mathrm{symp}}|_{M^0_{\mathrm{symp}}}$ are described as follows.

(1) $\omega|_{M^0_{\mathrm{symp}}} = \sum_{i=1}^n dx_i \wedge d\theta_i$.

(2) $\nabla|_{M^0_{\mathrm{symp}}} = d - 2\pi\sqrt{-1}\sum_{i=1}^n x_i d\theta_i$ *with respect to the unitary trivialization defined by the section s^0_{symp} on M^0_{symp}.*

(3) *For $m \in \mathrm{Int}\Delta$, $\mu_{T^n}^{-1}(m)$ is a Bohr–Sommerfeld fiber for the prequantum line bundle $(L_{\mathrm{symp}}, h, \nabla)$ if and only if $m \in \mathrm{Int}\Delta \cap (\mathfrak{t}^n)^*_{\mathbb{Z}}$. Moreover, $\delta_m([\theta]) = e^{2\pi\sqrt{-1}\sum_{i=1}^n m_i\theta_i} s^0_{\mathrm{symp}}|_{\mu_{T^n}^{-1}(m)}$ is a covariantly constant section of $(L_{\mathrm{symp}}, h, \nabla)|_{\mu_{T^n}^{-1}(m)}$ for $m = \sum_{i=1}^n m_i p_i \in \mathrm{Int}\Delta \cap (\mathfrak{t}^n)^*_{\mathbb{Z}}$, where $[\theta] \in \mathbb{R}^n/\mathbb{Z}^n$ is a coordinate on $\mu_{T^n}^{-1}(m)$.*

5.2. A complex toric manifold M_{comp}. Let Δ be a Delzant polytope defined by (5.1), and denote the set of its vertices by $\Delta(0)$. Let $F_j \subset (\mathfrak{t}^n)^*$ be the hyperplane defined in (5.2) for $j = 1,\ldots,d$. For each $v \in \Delta(0)$, we set $\Lambda_v = \{j | v \in F_j\}$, $\mathbb{C}^d_v = \{z \in \mathbb{C}^d | z_j \neq 0 \text{ if } j \in \{1,\ldots,d\} \setminus \Lambda_v\}$ and $\mathbb{C}^d_\Delta = \bigcup_{v \in \Delta(0)} \mathbb{C}^d_v$. Then, the compact complex toric manifold M_{comp} is defined to be the quotient space $M_{\mathrm{comp}} = \mathbb{C}^d_\Delta/K_\mathbb{C}$, where $K_\mathbb{C}$ is the complexification of the subtorus K. Similarly, the complexification of the torus T^d is denoted by $T^d_\mathbb{C}$. The quotient torus $T^n_\mathbb{C} = T^d_\mathbb{C}/K_\mathbb{C}$ acts on M_{comp}, preserving its complex structure J.

Next, we define a holomorphic line bundle on M_{comp}. Let $\tilde{L}_{\mathrm{comp}} = \mathbb{C}^d \times \mathbb{C}$ be a trivial holomorphic line bundle on $\mathbb{C}^d$. Define the action of $T^d_\mathbb{C}$ on $\tilde{L}_{\mathrm{comp}}$ by $(z,v)\mathrm{Exp}_{T^d_\mathbb{C}}\xi = (z\mathrm{Exp}_{T^d_\mathbb{C}}\xi, ve^{2\pi\sqrt{-1}\langle\lambda_\Delta,\xi\rangle})$. The holomorphic line bundle L_{comp} is defined to be the quotient of the restriction of $\tilde{L}_{\mathrm{comp}}$ to $\mathbb{C}^d_\Delta$ by the action of $K_\mathbb{C}$. Then, the quotient torus $T^n_\mathbb{C} = T^d_\mathbb{C}/K_\mathbb{C}$ acts on L_{comp}, preserving its holomorphic structure $\bar{\partial}$. Let $[z]_{K_\mathbb{C}} \in M_{\mathrm{comp}}$ denote a point represented by $z \in \mathbb{C}^d_\Delta$. Similarly, $[z,v]_{K_\mathbb{C}}$ denotes a point in L_{comp} represented by $(z,v) \in \mathbb{C}^d_\Delta \times \mathbb{C}$.

Next, we define a meromorphic section s^0_{comp} of L_{comp} on M_{comp} by

$$s^0_{\mathrm{comp}}([z]_{K_\mathbb{C}}) = \left[z, \prod_{j=1}^d z_j^{\lambda_j}\right]_{K_\mathbb{C}} \in L_{\mathrm{comp}} \text{ for } z \in \mathbb{C}^d_\Delta.$$

The section s^0_{comp} is holomorphic and non-zero on $M^0_{\mathrm{comp}} = (\mathbb{C}^\times)^d/K_\mathbb{C}$, where $(\mathbb{C}^\times)^d = \{z \in \mathbb{C}^d | z_i \neq 0 \text{ for } i = 1,\ldots,d\} \subset \mathbb{C}^d_\Delta$. So, it induces a holomorphic trivialization of L_{comp} on M^0_{comp}.

For $m \in \Delta \cap (\mathfrak{t}^n)^*_\mathbb{Z}$, we define a holomorphic section σ^m of L_{comp} by

$$(5.5) \qquad \sigma^m([z]_{K_\mathbb{C}}) = \left[z, \prod_{j=1}^d z_j^{l_j(m)}\right]_{K_\mathbb{C}} \in L_{\mathrm{comp}} \text{ for } z \in \mathbb{C}^d_\Delta.$$

It is well known that $\{\sigma^m | m \in \Delta \cap (\mathfrak{t}^n)^*_{\mathbb{Z}}\}$ is a basis of the space of holomorphic sections $H^0(L_{\mathrm{comp}}, \bar{\partial})$.

Next, we introduce a complex coordinate on M^0_{comp}. Fix a $\mathbb{Z}$-basis $p_1, \ldots, p_n \in (\mathfrak{t}^n)^*_{\mathbb{Z}}$ and its dual basis $q_1, \ldots, q_n \in \mathfrak{t}^n_{\mathbb{Z}}$ as in Section 5.1. Then, we define a complex coordinate $\varphi^0_{\mathrm{comp}} \colon M^0_{\mathrm{comp}} \to (\mathbb{C}^\times)^n$ by

$$(5.6) \qquad \varphi^0_{\mathrm{comp}}([z]_{K_{\mathbb{C}}}) = \left(\prod_{j=1}^{d} z_j^{\langle p_1, r_j \rangle}, \ldots, \prod_{j=1}^{d} z_j^{\langle p_n, r_j \rangle} \right),$$

where $r_j \in \mathfrak{t}^n_{\mathbb{Z}}$ is the vector in (5.1) for $j = 1, \ldots, d$. Since $\prod_{j=1}^{d} z_j^{\langle p_i, r_j \rangle}$ is a $K_{\mathbb{C}}$-invariant meromorphic function on $\mathbb{C}^d$, it descends to a meromorphic function on M_{comp}. If we set $(w_1, \ldots, w_n) = \varphi^0_{\mathrm{comp}}([z]_{K_{\mathbb{C}}})$, then we have

$$(5.7) \qquad \sigma^m([z]_{K_{\mathbb{C}}}) = \left(\prod_{i=1}^{n} w_i^{\langle m, q_i \rangle} \right) s^0_{\mathrm{comp}}([z]_{K_{\mathbb{C}}}) \text{ on } M_{\mathrm{comp}}.$$

5.3. Symplectic potentials. In Sections 5.1 and 5.2, starting from a Delzant polytope Δ defined in (5.1), we constructed a symplectic and complex toric manifold, respectively. In this section, we identify them, using symplectic potentials due to [**Gu1, Gu2, Ab1, Ab2**]. We also recall a certain deformation of toric Kähler structures due to [**BFMN**].

The inclusion $\mu_K^{-1}(\iota^* \lambda_\Delta) \subset \mathbb{C}^d_\Delta$ induces a map $\chi_{\mathrm{can}} \colon M_{\mathrm{symp}} \to M_{\mathrm{comp}}$. It is well known that this map is a diffeomorphism. In [**Gu1, Gu2**], Guillemin showed that this map is described by a single function g_{can} as follows.

Fix a $\mathbb{Z}$-basis $p_1, \ldots, p_n \in (\mathfrak{t}^n)^*_{\mathbb{Z}}$ and its dual basis $q_1, \ldots, q_n \in \mathfrak{t}^n_{\mathbb{Z}}$ as in Sections 5.1 and 5.2. Fix $\tilde{q}_i \in \mathfrak{t}^d_{\mathbb{Z}}$ so that $\pi(\tilde{q}_i) = q_i$ for $i = 1, \ldots, n$. Let $(x, [\theta])$ be the symplectic coordinate on M^0_{symp} and $(w_1, \ldots, w_n)$ the complex coordinate on M^0_{comp} induced by $p_1, \ldots, p_n \in (\mathfrak{t}^n)^*_{\mathbb{Z}}$, respectively. If we write $p = \sum_{i=1}^{n} x_i p_i$, then, by (5.3) and (5.6) we have

$$w_i(\chi_{\mathrm{can}}(x, [\theta])) = \prod_{j=1}^{d} \left(\sqrt{\frac{l_j(p)}{\pi}} e^{2\pi\sqrt{-1} \sum_{l=1}^{n} \langle u_j, \tilde{q}_l \rangle \theta_l} \right)^{\langle p_i, r_j \rangle} = e^{2\pi(\frac{\partial g_{\mathrm{can}}}{\partial x_i} + \sqrt{-1}\theta_i)},$$

where $g_{\mathrm{can}} \colon \mathrm{Int}\Delta \to \mathbb{R}$ is a function defined by

$$g_{\mathrm{can}}(p) = \frac{1}{4\pi} \sum_{j=1}^{d} l_j(p) \log l_j(p) + (\text{a linear function on } (\mathfrak{t}^n)^*) \text{ for } p \in \mathrm{Int}\Delta.$$

Note that g_{can} extends continuously to $g_{\mathrm{can}} \colon \Delta \to \mathbb{R}$.

Definition 5.2. A function $g \in C^0(\Delta)$ is a *symplectic potential* if and only if the following holds:

(1) $g - g_{\text{can}} \in C^\infty(\Delta)$,
(2) the Hessian $\text{Hess}_p g$ of g at p is positive definite for any $p \in \text{Int}\Delta$,
(3) there exists a strictly positive function $\alpha \in C^\infty(\Delta)$ such that

$$\det(\text{Hess}_p g) = \left[\alpha(p) \prod_{j=1}^{d} l_j(p) \right]^{-1} \quad \text{for any } p \in \text{Int}\Delta.$$

The set of symplectic potentials is denoted by $SP(\Delta)$.

The following results are due to [**Gu1, Gu2, Ab1, Ab2**], supplemented by Baier *et al.* [**BFMN**].

Theorem 5.3. *Let $\Delta \subset (\mathfrak{t}^n)^*$ be a Delzant polytope. Let $(M_{\text{symp}}, \omega)$ be a symplectic toric manifold and (M_{comp}, J) a complex toric manifold constructed from Δ. Let $(L_{\text{symp}}, h, \nabla)$ be a prequantum line bundle on M_{symp} and $(L_{\text{comp}}, \bar\partial)$ a holomorphic line bundle on M_{comp} constructed from Δ. Fix a $\mathbb{Z}$-basis $p_1, \ldots, p_n \in (\mathfrak{t}^n)^*_{\mathbb{Z}}$. Let $(x, [\theta])$ be the symplectic coordinate on M^0_{symp} and $w = (w_1, \ldots, w_n)$ the complex coordinate on M^0_{comp} induced by $p_1, \ldots, p_n \in (\mathfrak{t}^n)^*_{\mathbb{Z}}$, respectively.*
(A) Each $g \in SP(\Delta)$ defines a T^n-equivariant diffeomorphism $\chi_g \colon M_{\text{symp}} \to M_{\text{comp}}$ and a T^n-equivariant bundle isomorphism $\tilde\chi_g \colon L_{\text{symp}} \to L_{\text{comp}}$ such that the following holds:

(a1) *The following diagram commutes:*

$$
\begin{array}{ccc}
(L_{\text{symp}}, h, \nabla) & \xrightarrow{\tilde\chi_g} & (L_{\text{comp}}, \bar\partial) \\
\downarrow & & \downarrow \\
(M_{\text{symp}}, \omega) & \xrightarrow{\chi_g} & (M_{\text{comp}}, J)
\end{array}
$$

(a2) *$(M_{\text{symp}}, \omega, \chi_g^* J)$ is a Kähler manifold.*
(a3) *∇ is the Chern connection of the Hermitian holomorphic line bundle $(L_{\text{symp}}, h, \tilde\chi_g^* \bar\partial)$.*
(a4) *$\chi_g|_{M^0_{\text{symp}}} \colon M^0_{\text{symp}} \to M^0_{\text{comp}}$ is a diffeomorphism given by*

$$(5.8) \qquad w_i(\chi_g(x, [\theta])) = e^{2\pi\left(\frac{\partial g}{\partial x_i} + \sqrt{-1}\theta_i\right)} \text{ for } i = 1, \ldots, n.$$

The map χ_g is independent of the choice of the basis $p_1, \ldots, p_n \in (\mathfrak{t}^n)^*_{\mathbb{Z}}$. Moreover, if we write $w_i = e^{2\pi(y_i + \sqrt{-1}\theta_i)}$ for $i = 1, \ldots, n$, then the inverse mapping $(\chi_g|_{M^0_{\text{symp}}})^{-1} \colon M^0_{\text{comp}} \to M^0_{\text{symp}}$ is given by

$$(5.9) \qquad x_i(\chi_g^{-1}(w)) = \frac{\partial f}{\partial y_i}, \quad \theta_i((\chi_g^{-1})(w)) = \theta_i \text{ for } i = 1, \ldots, n,$$

where $f(y) = -g(x(y)) + \sum_{i=1}^{n} x_i(y)y_i$.

(a5) $\tilde{\chi}_g^* s_{\text{comp}}^0 = e^{2\pi\left(g - \sum_{i=1}^n x_i \frac{\partial g}{\partial x_i}\right)} s_{\text{symp}}^0$ *on* M_{symp}^0.

(B) *On the other hand, if* $\chi\colon M_{\text{symp}} \to M_{\text{comp}}$ *is a* T^n-*equivariant diffeomorphism such that* $(M_{\text{symp}}, \omega, \chi^* J)$ *is a Kähler manifold and that* χ *is homotopic to* χ_{can}, *then there exists* $g \in SP(\Delta)$ *such that* $\chi = \chi_g$.

In [**BFMN**], the authors considered a certain one-parameter family of symplectic potentials, which provides a one-parameter family of identifications of a symplectic toric manifold with a complex toric manifold. In other words, it provides a deformation of toric Kähler structures. The authors proved the following remarkable property of the deformation.

Proposition 5.4. *Let* $\chi_s\colon M_{\text{symp}} \to M_{\text{comp}}$ *and* $\tilde{\chi}_s\colon L_{\text{symp}} \to L_{\text{comp}}$ *be the diffeomorphism and the bundle isomorphism defined by* $g_s = g_0 + s\nu \in SP(\Delta)$ *for* $s \geq 0$, *respectively, where* $\nu\colon \Delta \to \mathbb{R}$ *is a smooth strictly convex function. Then, for each* $m \in \Delta \cap (\mathfrak{t}^n)_{\mathbb{Z}}^*$, *the section* $\frac{\tilde{\chi}_s^* \sigma^m}{\|\tilde{\chi}_s^* \sigma^m\|_{L^1(M_{\text{symp}})}}$ *converges to a delta-function section supported on the fiber* $\mu_{T^n}^{-1}(m)$ *in the following sense: there exists a covariantly constant section* δ_m *of* $(L_{\text{symp}}, h, \nabla)|_{\mu_{T^n}^{-1}(m)}$ *and a measure* $d\theta_m$ *on* $\mu_{T^n}^{-1}(m)$ *such that, for any smooth section* ϕ *of the dual line bundle* L_{symp}^*, *the following holds:*

$$\lim_{s\to\infty} \int_{M_{\text{symp}}} \left\langle \phi, \frac{\tilde{\chi}_s^* \sigma^m}{\|\tilde{\chi}_s^* \sigma^m\|_{L^1(M_{\text{symp}})}} \right\rangle \frac{\omega^n}{n!} = \int_{\mu_{T^n}^{-1}(m)} \langle \phi, \delta_m \rangle d\theta_m.$$

Note that the authors proved the above results not only for $m \in \text{Int}\Delta \cap (\mathfrak{t}^n)_{\mathbb{Z}}^*$ but also for all $m \in \Delta \cap (\mathfrak{t}^n)_{\mathbb{Z}}^*$. In Proposition 6.6 below, we slightly generalize this proposition.

6. Submanifolds under the deformation due to [BFMN]

In the last section, starting from a Delzant polytope Δ defined by (5.1), we constructed a symplectic toric manifold $(M_{\text{symp}}, \omega)$ and a complex toric manifold (M_{comp}, J). In this section, we study the change of the identification $\chi_s\colon (M_{\text{symp}}, \omega) \to (M_{\text{comp}}, J)$ and its lift $\tilde{\chi}_s\colon (L_{\text{symp}}, h, \nabla) \to (L_{\text{comp}}, \bar{\partial})$ induced by a family of symplectic potentials $g_s = g_0 + s\nu \in SP(\Delta)$ for $s \geq 0$. In Proposition 5.4, $\nu \in C^\infty(\Delta)$ is assumed to be a strictly convex function. Here, we assume that $\nu \in C^\infty(\Delta)$ is a weakly convex function, which will be important in Propositions 6.5 and 6.6. In particular, we study the behavior of submanifolds and of the prequantum line bundles on them under the change of identification of the ambient toric manifolds.

6.1. Identification of submanifolds. Given a complex submanifold V_{comp} of (M_{comp}, J), we consider the change of the identification $\chi_s\colon (M_{\text{symp}}, \omega) \to (M_{\text{comp}}, J)$. This implies that the complex structure of the complex

submanifold remains the same, but the symplectic structure $(\chi_s^{-1})^*\omega$ on it changes. In this subsection, we develop a method to identify $(V_{\text{comp}}, (\chi_s^{-1})^*\omega)$, for different values of s, as symplectic manifolds. We also lift the identification to the prequantum line bundle.

Proposition 6.1. *Let* (V_{comp}, J^V) *be a compact complex submanifold of* (M_{comp}, J) *and* $\rho_{\text{comp}}\colon V_{\text{comp}} \to M_{\text{comp}}$ *the inclusion. Set* $V_{\text{symp}} = \chi_0^{-1}(V_{\text{comp}})$, *and denote the inclusion by* $\rho_0\colon V_{\text{symp}} \to M_{\text{symp}}$.

Then, there exists an inclusion $\rho_s\colon V_{\text{symp}} \to M_{\text{symp}}$ *and a diffeomorphism* $\underline{\chi}_s\colon V_{\text{symp}} \to V_{\text{comp}}$, *for each* $s \geq 0$, *such the following holds:*

(a) $\rho_s^*\omega = \rho_0^*\omega$.
(b) *The following diagram commutes:*

$$
\begin{array}{ccc}
(M_{\text{symp}}, \omega) & \xrightarrow{\chi_s} & (M_{\text{comp}}, J) \\
\uparrow \rho_s & & \uparrow \rho_{\text{comp}} \\
(V_{\text{symp}}, \rho_0^*\omega) & \xrightarrow{\underline{\chi}_s} & (V_{\text{comp}}, J^V).
\end{array}
$$

(c) $(V_{\text{symp}}, \rho_0^*\omega, \underline{\chi}_s^*J^V)$ *is a Kähler manifold.*
(d) *The maps* ρ_s *and* $\underline{\chi}_s$ *are canonically defined and depend smoothly on* $s \geq 0$.

Proof. If we set $\psi_s = \chi_0 \circ (\chi_s)^{-1}\colon M_{\text{comp}} \to M_{\text{comp}}$ and $\omega_s = ((\chi_s)^{-1})^*\omega \in \Omega^2(M_{\text{comp}})$ for each $s \geq 0$, then we have

$$\psi_s^*\omega_0 = \omega_s. \tag{6.1}$$

We show the following.

Claim 6.2. *There exists a diffeomorphism* $\phi_s\colon V_{\text{comp}} \to V_{\text{comp}}$ *for each* $s \geq 0$, *such that* $\phi_0 = \text{id}_{V_{\text{comp}}}$ *and* $\phi_s^*(\rho_{\text{comp}}^*\omega_s) = \rho_{\text{comp}}^*\omega_0$.

Proof of Claim 6.2. Since $\rho_{\text{comp}}^*\omega_s$ and $\rho_{\text{comp}}^*\omega_0$ are cohomologous, Claim 6.2 is a direct consequence of the theorem of Moser. However, we give a proof, because we will use the notation here in the proof of Proposition 6.3.

Define a vector field $X_s \in \mathcal{X}(M_{\text{comp}})$ by

$$(X_s)_{\psi_s(p)} = \frac{d}{dt}\Big|_{t=0}\psi_{s+t}(p) \in T_{\psi_s(p)}M_{\text{comp}} \text{ for } p \in M_{\text{comp}}. \tag{6.2}$$

By (6.1) we have

$$\frac{d\omega_s}{ds} = \psi_s^*(L_{X_s}\omega_0) = d\eta_s, \text{ where } \eta_s = \psi_s^*\{i(X_s)\omega_0\} \in \Omega^1(M_{\text{comp}}). \tag{6.3}$$

Suppose that there exists a diffeomorphism $\phi_s\colon V_{\text{comp}} \to V_{\text{comp}}$ such that $\phi_s^*(\rho_{\text{comp}}^*\omega_s) = \rho_{\text{comp}}^*\omega_0$ for each $s \geq 0$. If we define a vector field

$Y_s \in \mathcal{X}(V_{\text{comp}})$ by $(Y_s)_{\phi_s(p)} = \frac{d}{dt}\big|_{t=0}\phi_{s+t}(p) \in T_{\phi_s(p)}V_{\text{comp}}$ for $p \in V_{\text{comp}}$, then we have

$$0 = \frac{d}{ds}\{\phi_s^*(\rho_{\text{comp}}^*\omega_s)\} = \phi_s^*\left\{L_{Y_s}(\rho_{\text{comp}}^*\omega_s) + \frac{d\rho_{\text{comp}}^*\omega_s}{ds}\right\}$$
$$= \phi_s^*d\left\{i(Y_s)(\rho_{\text{comp}}^*\omega_s) + \rho_{\text{comp}}^*\eta_s\right\}.$$

Therefore, if we define $Y_s \in \mathcal{X}(V_{\text{comp}})$ conversely by

$$(6.4) \qquad\qquad i(Y_s)(\rho_{\text{comp}}^*\omega_s) + \rho_{\text{comp}}^*\eta_s = 0,$$

then we have a desired diffeomorphism $\phi_s \colon V_{\text{comp}} \to V_{\text{comp}}$ by integrating $Y_s \in \mathcal{X}(V_{\text{comp}})$. Moreover, we have $\phi_0 = id_{V_{\text{comp}}}$ from this construction. $\qquad\square$

Since $\rho_{\text{comp}}\circ\chi_0|_{V_{\text{symp}}} = \chi_0\circ\rho_0$, we have $(\chi_0|_{V_{\text{symp}}})^*(\rho_{\text{comp}}^*\omega_0) = \rho_0^*\omega$. Define a smooth map $\rho_s \colon V_{\text{symp}} \to M_{\text{symp}}$ by $\rho_s = (\chi_s)^{-1} \circ \rho_{\text{comp}} \circ \phi_s \circ \chi_0|_{V_{\text{symp}}}$. By Claim 6.2, we have

$$\rho_s^*\omega = (\chi_0|_{V_{\text{symp}}})^*\phi_s^*(\rho_{\text{comp}}^*\omega_s) = (\chi_0|_{V_{\text{symp}}})^*(\rho_{\text{comp}}^*\omega_0) = \rho_0^*\omega.$$

Thus, we proved (a).

Define $\underline{\chi}_s \colon V_{\text{symp}} \to V_{\text{comp}}$ by $\underline{\chi}_s = \phi_s \circ \chi_0|_{V_{\text{symp}}}$. Then, we have $\chi_s \circ \rho_s = \rho_{\text{comp}} \circ \underline{\chi}_s$, which implies (b).

Since $\underline{\chi}_s^*(\rho_{\text{comp}}^*\omega_s) = \rho_s^*(\chi_s^*\omega_s) = \rho_s^*\omega = \rho_0^*\omega$, we see that $(V_{\text{symp}}, \rho_0^*\omega, \underline{\chi}_s^*J^V)$ is isomorphic to $(V_{\text{comp}}, \rho_{\text{comp}}^*\omega_s, J^V)$, which a Kähler manifold. Therefore, (c) follows.

Finally, we prove (d). In the above construction of ϕ_s, there is no ambiguous choice. So, ϕ_s is canonically defined and depends smoothly on $s \geq 0$. Therefore, the maps ρ_s and $\underline{\chi}_s$ are canonically defined and depend smoothly on $s \geq 0$. Thus, we finish the proof of Proposition 6.1. $\qquad\square$

Next, we construct a lift of the maps $\rho_s \colon V_{\text{symp}} \to M_{\text{symp}}$ and $\underline{\chi}_s \colon V_{\text{symp}} \to V_{\text{comp}}$ to the prequantum line bundle.

Proposition 6.3. *In addition to the assumption of Proposition* 6.1, *let* $(L_{\text{symp}}^V, h^V, \nabla^V) = \rho_0^*(L_{\text{symp}}, h, \nabla)$ *and* $(L_{\text{comp}}^V, \overline{\partial}^V) = \rho_{\text{comp}}^*(L_{\text{comp}}, \overline{\partial})$ *be a prequantum line bundle on* $(V_{\text{symp}}, \rho_0^*\omega)$ *and a holomorphic line bundle on* (V_{comp}, J^V), *respectively. Let* $\tilde{\rho}_{\text{comp}} \colon L_{\text{comp}}^V \to L_{\text{comp}}$ *be the canonical lift of the inclusion* $\rho_{\text{comp}} \colon V_{\text{comp}} \to M_{\text{comp}}$.

Then, there exists a lift $\tilde{\rho}_s \colon L_{\text{symp}}^V \to L_{\text{symp}}$ *of* $\rho_s \colon V_{\text{symp}} \to M_{\text{symp}}$ *and a lift* $\tilde{\underline{\chi}}_s \colon L_{\text{symp}}^V \to L_{\text{comp}}^V$ *of* $\underline{\chi}_s \colon V_{\text{symp}} \to V_{\text{comp}}$, *for each* $s \geq 0$, *such that the following holds:*

(a) $\tilde{\rho}_s^*(L_{\text{symp}}, h, \nabla) = (L_{\text{symp}}^V, h^V, \nabla^V)$.

(b) *The following diagram commutes:*

$$\begin{array}{ccc} (L_{\text{symp}}, h, \nabla) & \xrightarrow{\tilde{\chi}_s} & (L_{\text{comp}}, \bar{\partial}) \\ \uparrow \tilde{\rho}_s & & \uparrow \tilde{\rho}_{\text{comp}} \\ (L_{\text{symp}}^V, h^V, \nabla^V) & \xrightarrow{\tilde{\chi}_s} & (L_{\text{comp}}^V, \bar{\partial}^V). \end{array}$$

(c) ∇^V *is the Chern connection of* $(L_{\text{symp}}^V, h^V, \tilde{\underline{\chi}}_s^* \bar{\partial}^V)$.

(d) *The maps* $\tilde{\rho}_s$ *and* $\tilde{\underline{\chi}}_s$ *are canonically defined and depend smoothly on* $s \geq 0$.

Proof. We use the same notation as in the proof of Proposition 6.1.
First, we show the following.

Claim 6.4. *Let* $R \colon V_{\text{symp}} \times [0, \infty) \to M_{\text{symp}}$ *be the map defined by* $R(p, s) = \rho_s(p)$. *Then, the following holds:*

$$i\left(\frac{\partial}{\partial s}\right)(R^*\omega) = 0 \text{ on } V_{\text{symp}} \times [0, \infty).$$

Proof of Claim 6.4. Define $\theta \colon V_{\text{comp}} \times [0, \infty) \to M_{\text{comp}}$ by $\theta(p, s) = \psi_s \circ \rho_{\text{comp}} \circ \phi_s(p)$. Fix any $(p_0, s_0) \in V_{\text{symp}} \times [0, \infty)$ and $\underline{v} \in T_{p_0} V_{\text{symp}}$. We set $q_0 = \underline{\chi}_0(p_0) \in V_{\text{comp}}$, $\underline{w} = (\underline{\chi}_0)_{*p_0}(\underline{v}) \in T_{q_0} V_{\text{comp}}$, and

$$v = (\underline{v}, 0) \in T_{p_0} V_{\text{symp}} \oplus \mathbb{R} \cong T_{(p_0, s_0)}\{V_{\text{symp}} \times [0, \infty)\},$$
$$w = (\underline{w}, 0) \in T_{q_0} V_{\text{comp}} \oplus \mathbb{R} \cong T_{(q_0, s_0)}\{V_{\text{comp}} \times [0, \infty)\}.$$

Since $R(p, s) = ((\chi_0)^{-1} \circ \theta)(\underline{\chi}_0(p), s)$, we have

$$\left\{i\left(\frac{\partial}{\partial s}\right)(R^*\omega)\right\}_{(p_0, s_0)}(v) = \omega_0\left(\theta_{*(q_0, s_0)}\left(\frac{\partial}{\partial s}\right), \theta_{*(q_0, s_0)}(w)\right).$$

By (6.2), we have

$$\theta_{*(q_0, s_0)}\left(\frac{\partial}{\partial s}\right) = \frac{\partial}{\partial s}\bigg|_{s=s_0} (\psi_s \circ \rho_{\text{comp}} \circ \phi_s)(q_0)$$

$$= (X_{s_0})_{\psi_{s_0} \circ \rho_{\text{comp}} \circ \phi_{s_0}(q_0)} + (\psi_{s_0})_*(\rho_{\text{comp}})_*\left(\frac{\partial}{\partial s}\bigg|_{s=s_0} \phi_s(q_0)\right),$$

$$\theta_{*(q_0, s_0)}(w) = (\psi_{s_0})_*(\rho_{\text{comp}})_*(\phi_{s_0})_{*q_0}(\underline{w}).$$

Thus, we have

$$\left\{ i\left(\frac{\partial}{\partial s}\right)(R^*\omega)\right\}_{(p_0,s_0)}(v)$$

$$= \{\psi_{s_0}^*(i(X_{s_0})\omega_0)\}_{\rho_{\mathrm{comp}}\circ\phi_{s_0}(q_0)}((\rho_{\mathrm{comp}})_*(\phi_{s_0})_{*q_0}(\underline{w}))$$

$$+ \{(\rho_{\mathrm{comp}})^*(\psi_{s_0})^*\omega_0\}((Y_{s_0})_{\phi_{s_0}(q_0)},(\phi_{s_0})_{*q_0}(\underline{w}))$$

$$= (\eta_{s_0})_{\rho_{\mathrm{comp}}\circ\phi_{s_0}(q_0)}((\rho_{\mathrm{comp}})_*(\phi_{s_0})_{*q_0}(\underline{w}))$$

$$+ (-\rho_{\mathrm{comp}}^*\eta_{s_0})_{\phi_{s_0}(q_0)}((\phi_{s_0})_{*q_0}(\underline{w}))$$

$$= 0,$$

where we used (6.3) and (6.4) in the second equality. This implies Claim 6.4. □

Consider the line bundle $(L'_{\mathrm{symp}},h',\nabla') = R^*(L_{\mathrm{symp}},h,\nabla)$ on $V_{\mathrm{symp}} \times [0,\infty)$. Let $S' \subset L'_{\mathrm{symp}}$ be the unit sphere bundle and $p\colon S' \to V_{\mathrm{symp}} \times [0,\infty)$ the projection. If we denote the connection form of ∇' by $\alpha \in \Omega^1(S')$, then we have $d\alpha = p^*R^*\omega$. If we denote the horizontal lift of $\frac{\partial}{\partial s} \in \mathcal{X}(V_{\mathrm{symp}} \times [0,\infty))$ by $\xi \in \mathcal{X}(S')$, we have $i(\xi)\alpha = 0$ and $p_*\xi = \frac{\partial}{\partial s}$. So, we have

$$L_\xi\alpha = i(\xi)d\alpha = i(\xi)(p^*R^*\omega) = p^*\{i(p_*\xi)(R^*\omega)\} = p^*\left\{i\left(\frac{\partial}{\partial s}\right)(R^*\omega)\right\} = 0.$$

Thus, the flow defined by the vector field $\xi \in \mathcal{X}(S')$ preserves the connection ∇'. So, it induces a lift $\tilde{\rho}_s\colon L_{\mathrm{symp}}^V \to L_{\mathrm{symp}}$ of the map $\rho_s\colon V_{\mathrm{symp}} \to M_{\mathrm{symp}}$ such that $\tilde{\rho}_s^*(h,\nabla) = (h^V,\nabla^V)$ for $s \geq 0$. This implies (a).

Since $\chi_s \circ \rho_s = \rho_{\mathrm{comp}} \circ \underline{\chi}_s$ holds, $\underline{\tilde{\chi}}_s = \tilde{\rho}_{\mathrm{comp}}^{-1} \circ \tilde{\chi}_s \circ \tilde{\rho}_s\colon L_{\mathrm{symp}}^V \to L_{\mathrm{comp}}^V$ is well defined. Thus, (b) holds from the definition of $\underline{\tilde{\chi}}_s\colon L_{\mathrm{symp}}^V \to L_{\mathrm{comp}}^V$.

Since $\underline{\tilde{\chi}}_s^*\overline{\partial}^V = \underline{\tilde{\chi}}_s^*(\tilde{\rho}_{\mathrm{comp}}^*\overline{\partial}) = \tilde{\rho}_s^*(\tilde{\chi}_s^*\overline{\partial})$, we see that $(L_{\mathrm{symp}}^V,h^V,\underline{\tilde{\chi}}_s^*\overline{\partial}^V)$ is isomorphic to $\tilde{\rho}_s^*(L_{\mathrm{symp}},h,\tilde{\chi}_s^*\overline{\partial})$. Since ∇ is the Chern connection of $(L_{\mathrm{symp}},h,\tilde{\chi}_s^*\overline{\partial})$, $\nabla^V = \tilde{\rho}_s^*\nabla$ is the Chern connection of $(L_{\mathrm{symp}}^V,h^V,\underline{\tilde{\chi}}_s^*\overline{\partial}^V)$. This implies (c).

Finally, (d) holds obviously from the definition of the maps $\tilde{\rho}_s$ and $\underline{\tilde{\chi}}_s$. □

6.2. Toric subvarieties. Let $(M_{\mathrm{symp}},\omega)$ and (M_{comp},J) be a symplectic and complex toric manifold, respectively, constructed from a Delzant polytope Δ defined by (5.1). In this subsection, we study a (possibly singular) toric subvariety V_{comp} of (M_{comp},J) under the deformation of toric Kähler structures of the ambient toric manifold.

Fix a $\mathbb{Z}$-basis $p_1,\ldots,p_n \in (\mathfrak{t}^n)_{\mathbb{Z}}^*$ and its dual basis $q_1,\ldots,q_n \in \mathfrak{t}_{\mathbb{Z}}^n$. This induces symplectic coordinate $(x,[\theta])$ on M_{symp}^0 as in Section 5.1 and complex coordinate $w = (w_1,\ldots,w_n)$ on M_{comp}^0 as in Section 5.2. Note that M_{comp}^0 is the $T_{\mathbb{C}}^n$-orbit through $e = (1,\ldots,1) \in M_{\mathrm{comp}}^0$.

Proposition 6.5. *Let $T_{\mathbb{C}}^l$ be an l-dimensional subtorus of $T_{\mathbb{C}}^n$. Let $\iota^*\colon$ $(\mathfrak{t}^n)^* \to (\mathfrak{t}^l)^*$ be the dual map of the inclusion of the Lie algebra $\iota\colon \mathfrak{t}^l \to \mathfrak{t}^n$. Let $V_{\mathrm{comp}} \subset M_{\mathrm{comp}}$ be a closed l-dimensional (possibly singular) toric subvariety containing $e = (1, \ldots, 1)$. The torus action on V_{comp} is the restriction of the $T_{\mathbb{C}}^l$-action on M_{comp} and its orbit through e is open dense in V_{comp}.*

(1) *Let $\chi_s\colon M_{\mathrm{symp}} \to M_{\mathrm{comp}}$ be the diffeomorphism defined by*

$$g_s = g_0 + s(\underline{\nu} \circ \iota^*) \in SP(\Delta) \quad \text{for } s \geq 0,$$

where $\underline{\nu}\colon \iota^*(\Delta) \to \mathbb{R}$ is a smooth strictly convex function. Set $V_{\mathrm{symp}} = (\chi_0)^{-1}(V_{\mathrm{comp}})$. Then, $\chi_0 \circ \chi_s^{-1}|_{V_{\mathrm{comp}}}\colon V_{\mathrm{comp}} \to V_{\mathrm{comp}}$ is a homeomorphism for each $s \geq 0$.

(2) *Let $\rho_s\colon V_{\mathrm{symp}} \to M_{\mathrm{symp}}$ and $\underline{\chi}_s\colon V_{\mathrm{symp}} \to V_{\mathrm{comp}}$ be the maps constructed in Proposition 6.1. Then, $\rho_s = \rho_0$ and $\underline{\chi}_s = \chi_s|_{V_{\mathrm{symp}}}$ hold for $s \geq 0$. Moreover, their lifts constructed in Proposition 6.3 are given by $\tilde{\rho}_s = \tilde{\rho}_0\colon L_{\mathrm{symp}}^V \to L_{\mathrm{symp}}$ and $\tilde{\underline{\chi}}_s = \tilde{\chi}_s|_{L_{\mathrm{symp}}^V}\colon L_{\mathrm{symp}}^V \to L_{\mathrm{comp}}^V$.*

Proof. (1) Note that $V_{\mathrm{comp}} \cap M_{\mathrm{comp}}^0$ is a connected component of

$$(6.5) \qquad \left\{ w \in M_{\mathrm{comp}}^0 \mid \prod_{i=1}^{n} w_i^{\langle p, q_i \rangle} = 1 \text{ for all } p \in \ker \iota^* \cap (\mathfrak{t}^n)_{\mathbb{Z}}^* \right\},$$

which contains $e = (1, \ldots, 1)$. By (5.8), we see that $(\chi_s)^{-1}(V_{\mathrm{comp}} \cap M_{\mathrm{comp}}^0)$ is a connected component of

(6.6)

$$\left\{ (x, [\theta]) \in M_{\mathrm{symp}}^0 \mid e^{2\pi \sum_{i=1}^{n} \langle p, q_i \rangle \left(\frac{\partial g_s}{\partial x_i} + \sqrt{-1}\theta_i \right)} = 1 \text{ for all } p \in \ker \iota^* \cap (\mathfrak{t}^n)_{\mathbb{Z}}^* \right\}.$$

On the other hand, we have

$$\sum_{i=1}^{n} \langle p, q_i \rangle \frac{\partial g_s}{\partial x_i} = \sum_{i=1}^{n} \langle p, q_i \rangle \frac{\partial g_0}{\partial x_i} + s \sum_{i=1}^{n} \langle p, q_i \rangle \frac{\partial (\nu \circ \iota^*)}{\partial x_i} = \sum_{i=1}^{n} \langle p, q_i \rangle \frac{\partial g_0}{\partial x_i}$$

because $\sum_{i=1}^{n} \langle p, q_i \rangle \frac{\partial}{\partial x_i}$ is a differential in the direction of $\ker \iota^*$ for all $p \in \ker \iota^* \cap (\mathfrak{t}^n)_{\mathbb{Z}}^*$. Therefore, we have $(\chi_s)^{-1}(V_{\mathrm{comp}}) \subset (\chi_0)^{-1}(V_{\mathrm{comp}}) = V_{\mathrm{symp}}$. So, we have an injective continuous map $\chi_0 \circ \chi_s^{-1}|_{V_{\mathrm{comp}}}\colon V_{\mathrm{comp}} \to V_{\mathrm{comp}}$.

Next, we show that the map $\chi_0 \circ \chi_s^{-1}|_{V_{\mathrm{comp}}}\colon V_{\mathrm{comp}} \to V_{\mathrm{comp}}$ is surjective. Let V_{comp}^0 be the $T_{\mathbb{C}}^l$-orbit through e. Note that $\chi_0 \circ \chi_s^{-1}|_{V_{\mathrm{comp}}}$ is T^l-equivariant and injective. If we consider the isotropy subgroup at each point, we have $\chi_0 \circ \chi_s^{-1}(V_{\mathrm{comp}}^0) \subset V_{\mathrm{comp}}^0$ and $\chi_0 \circ \chi_s^{-1}(V_{\mathrm{comp}} \setminus V_{\mathrm{comp}}^0) \subset V_{\mathrm{comp}} \setminus V_{\mathrm{comp}}^0$. Since $\chi_0 \circ \chi_s^{-1}|_{V_{\mathrm{comp}}^0}$ is a C^∞-map and its differential is an isomorphism at each point, $\chi_0 \circ \chi_s^{-1}(V_{\mathrm{comp}}^0)$ is open in V_{comp}^0. On the

other hand, since V_{comp} is compact, $\chi_0 \circ \chi_s^{-1}(V_{\text{comp}})$ is compact. Therefore, we see that $\chi_0 \circ \chi_s^{-1}|_{V_{\text{comp}}^0} \colon V_{\text{comp}}^0 \to V_{\text{comp}}^0$ is surjective. So, we see that $\chi_0 \circ \chi_s^{-1}|_{V_{\text{comp}}} \colon V_{\text{comp}} \to V_{\text{comp}}$ is surjective.

Since $\chi_0 \circ \chi_s^{-1}|_{V_{\text{comp}}} \colon V_{\text{comp}} \to V_{\text{comp}}$ is a bijective continuous map and V_{comp} is a compact Hausdorff space, it is a homeomorphism.

(2) In the proof of Proposition 6.1, we constructed $\phi_s \colon V_{\text{comp}} \to V_{\text{comp}}$ by integrating the time-dependent vector field $Y_s \in \mathcal{X}(V_{\text{comp}})$ defined by

$$i(Y_s)(\rho_{\text{comp}}^* \omega_s) + \rho_{\text{comp}}^* \eta_s = 0, \ \text{ where } \eta_s = \psi_s^* \{i(X_s)\omega_0\} \in \Omega^1(M_{\text{comp}}).$$

In our situation, the vector field Y_s is defined on V_{comp}^0. Since V_{comp}^0 is non-compact, it is not obvious that Y_s is integrated to define the map $\phi_s|_{V_{\text{comp}}^0} \colon V_{\text{comp}}^0 \to V_{\text{comp}}^0$. However, we show that this holds in our case and that $\phi_s|_{V_{\text{comp}}^0}$ extends to a homeomorphism $\phi_s \colon V_{\text{comp}} \to V_{\text{comp}}$.

In the proof of Proposition 6.5 (1), we showed that $\psi_s|_{V_{\text{comp}}^0} = \chi_0 \circ \chi_s^{-1}|_{V_{\text{comp}}^0} \colon V_{\text{comp}}^0 \to V_{\text{comp}}^0$ is a diffeomorphism. Moreover, by (6.2) the restriction $X_s|_{V_{\text{comp}}^0}$ takes its values in the tangent bundle of V_{comp}^0. If we note (6.1), we have

$$\begin{aligned}
\rho_{\text{comp}}^* \eta_s &= \rho_{\text{comp}}^* \psi_s^* \{i(X_s)\omega_0\} \\
&= \rho_{\text{comp}}^* \{i((\psi_s^{-1})_* X_s)\psi_s^* \omega_0\} = i((\psi_s^{-1})_*(X_s|_{V_{\text{comp}}^0}))\rho_{\text{comp}}^* \omega_s.
\end{aligned}$$

Thus, we have

$$0 = i(Y_s)(\rho_{\text{comp}}^* \omega_s) + \rho_{\text{comp}}^* \eta_s = i(Y_s + (\psi_s^{-1})_*(X_s|_{V_{\text{comp}}^0}))\rho_{\text{comp}}^* \omega_s.$$

So, we have

$$Y_s + (\psi_s^{-1})_*(X_s|_{V_{\text{comp}}^0}) = 0 \in \mathcal{X}(V_{\text{comp}}^0).$$

For any $p \in V_{\text{comp}}^0$, we have

$$(Y_s)_p = -\{(\psi_s^{-1})_*(X_s|_{V_{\text{comp}}^0})\}_p = -\frac{d}{dt}\Big|_{t=0} \psi_s^{-1} \circ \psi_{s+t}(p) = \frac{d}{dt}\Big|_{t=0} \psi_{s+t}^{-1} \circ \psi_s(p).$$

Namely, we have $(Y_s)_{\psi_s^{-1}(p)} = \frac{d}{dt}\big|_{t=0} \psi_{s+t}^{-1}(p)$. Thus, the vector field Y_s on V_{comp}^0 is integrated to define $\phi_s|_{V_{\text{comp}}^0} = \psi_s^{-1}|_{V_{\text{comp}}^0} = \chi_s \circ \chi_0^{-1}|_{V_{\text{comp}}^0}$. So, $\phi_s|_{V_{\text{comp}}^0}$ is extended to a homeomorphism $\phi_s = \chi_s \circ \chi_0^{-1}|_{V_{\text{comp}}} \colon V_{\text{comp}} \to V_{\text{comp}}$. So, we have $\underline{\chi}_s = \phi_s \circ \chi_0|_{V_{\text{symp}}} = \chi_s|_{V_{\text{symp}}}$. Then, the rest of the statement is obvious. $\qquad\square$

Recall that we defined a holomorphic section σ^m of L_{comp} for $m \in \Delta \cap (\mathfrak{t}^n)_{\mathbb{Z}}^*$ by (5.5). Then, $\tilde{\chi}_s^* \sigma^m$ is a section of L_{symp}. By Proposition 6.5 (2), the section $\tilde{\rho}_s^*(\tilde{\chi}_s^* \sigma^m) = \tilde{\rho}_0^*(\tilde{\chi}_s^* \sigma^m)$ of L_{symp}^V can be written as $\tilde{\chi}_s^* \sigma^m|_{V_{\text{symp}}}$.

Proposition 6.6. *In addition to the assumptions in Proposition* 6.5, *suppose that* $\iota^*((\mathfrak{t}^n)_{\mathbb{Z}}^*) = (\mathfrak{t}^l)_{\mathbb{Z}}^*$. *Set* $\mu_{T^l} = \iota^* \circ \mu_{T^n} \colon M_{\text{symp}} \to (\mathfrak{t}^l)^*$ *and* $\Delta_V = \mu_{T^l}(V_{\text{symp}})$.

(1) *For* $m, m' \in \Delta \cap (\mathfrak{t}^n)_{\mathbb{Z}}^*$, $\sigma^m|_{V_{\text{comp}}} = \sigma^{m'}|_{V_{\text{comp}}}$ *if* $\iota^*m = \iota^*m'$.

(2) *For* $p \in \text{Int}\Delta_V$, $\mu_{T^l}^{-1}(p) \cap V_{\text{symp}}$ *is a Bohr–Sommerfeld fiber for the pre-quantum line bundle* $(L_{\text{symp}}, h, \nabla)|_{V_{\text{symp}}}$ *if and only if* $p \in \text{Int}\Delta_V \cap (\mathfrak{t}^l)_{\mathbb{Z}}^*$.

3) *Fix any* $m \in \Delta \cap (\mathfrak{t}^n)_{\mathbb{Z}}^*$ *with* $\iota^*m \in \text{Int}\Delta_V \cap (\mathfrak{t}^l)_{\mathbb{Z}}^*$. *Let* B_{ι^*m} *be an open neighborhood of* ι^*m *in* $(\mathfrak{t}^l)^*$. *Then, there exists* $C_0(s) > 0$, *depending continuously on* $s \geq 0$, *such that* $\lim_{s\to\infty} C_0(s) = 0$ *and, for arbitrary* $s \geq 0$

$$\|\tau_s^m\|_{C^0(M_{\text{symp}} \backslash \mu_{T^l}^{-1}(B_{\iota^*m}))} \leq C_0(s),$$

where $\tau_s^m = \dfrac{\tilde{\chi}_s^* \sigma^m}{\|\tilde{\chi}_s^* \sigma^m|_{V_{\text{symp}}}\|_{L^1(V_{\text{symp}})}}$.

(4) *Fix* $m \in \Delta \cap (\mathfrak{t}^n)_{\mathbb{Z}}^*$ *with* $\iota^*m \in \text{Int}\Delta_V \cap (\mathfrak{t}^l)_{\mathbb{Z}}^*$. *The section* $\tau_s^m|_{V_{\text{symp}}}$ *converges to a delta-function section supported on the Bohr–Sommerfeld fiber* $\mu_{T^l}^{-1}(\iota^*m) \cap V_{\text{symp}}$ *in the following sense: there exists a covariantly constant section* δ_{ι^*m} *of* $(L_{\text{symp}}, h, \nabla)|_{\mu_{T^l}^{-1}(\iota^*m)}$ *and a measure* $d\theta_{\iota^*m}$ *on* $\mu_{T^l}^{-1}(\iota^*m) \cap V_{\text{symp}}$ *such that, for any smooth section* ϕ *of the dual line bundle* $(L_{\text{symp}}^V)^*$, *the following holds:*

$$\lim_{s\to\infty} \int_{V_{\text{symp}}} \langle \phi, \tau_s^m|_{V_{\text{symp}}} \rangle \frac{(\rho_0^*\omega)^l}{l!} = \int_{\mu_{T^l}^{-1}(\iota^*m) \cap V_{\text{symp}}} \langle \phi, \delta_{\iota^*m} \rangle d\theta_{\iota^*m}.$$

Proof. (1) By (5.7), we have $\sigma^{m'}/\sigma^m = \prod_{i=1}^n w_i^{\langle m'-m, q_i \rangle}$ on M_{comp}^0. Since $m' - m \in \ker \iota^*$, due to (6.5), we have $\sigma^{m'}/\sigma^m = 1$ on V_{comp}.

(2) Since $\iota^*((\mathfrak{t}^n)_{\mathbb{Z}}^*) = (\mathfrak{t}^l)_{\mathbb{Z}}^*$, we can take $p_1', \dots, p_l' \in (\mathfrak{t}^n)_{\mathbb{Z}}^*$ so that $\iota^*p_1', \dots, \iota^*p_l'$ is a $\mathbb{Z}$-basis of $(\mathfrak{t}^l)_{\mathbb{Z}}^*$. In addition, if we fix a $\mathbb{Z}$-basis $p_{l+1}', \dots, p_n'$ of $(\ker \iota^*) \cap (\mathfrak{t}^n)_{\mathbb{Z}}^*$, then $p_1', \dots, p_n'$ is a $\mathbb{Z}$-basis of $(\mathfrak{t}^n)_{\mathbb{Z}}^*$. It induces the complex coordinate $w' = (w_1', \dots, w_n')$ on M_{comp}^0 and the symplectic coordinate $(x', [\theta'])$ on M_{symp}^0 as in the previous subsections.

By (6.5), we have $V_{\text{comp}} \cap M_{\text{comp}}^0 = \{w' \in M_{\text{comp}}^0 | w_{l+1}' = \cdots = w_n' = 1\}$. So, by (6.6), we see that $\theta_{l+1}', \dots, \theta_n'$ are constant on $V_{\text{symp}} \cap M_{\text{symp}}^0$. Moreover, by (5.4), we have $\mu_{T^l}(x', [\theta']) = \sum_{i=1}^l x_i' p_i'$ for $(x', [\theta']) \in M_{\text{symp}}^0$. For each $p = \sum_{i=1}^l x_i' p_i' \in \text{Int}\Delta_V$, since $\mu_{T^l}^{-1}(p) \cap V_{\text{symp}}$ is a single T^l-orbit, $x_1', \dots, x_n'$ are also constant on $\mu_{T^l}^{-1}(p) \cap V_{\text{symp}}$.

On the other hand, due to Lemma 5.1, we see that $\nabla|_{M^0_{\text{symp}}} = d - 2\pi\sqrt{-1}\sum_{i=1}^n x'_i d\theta'_i$ with respect to the trivialization defined by s^0_{symp}. Therefore, for a fixed $p = \sum_{i=1}^l x'_i \iota^* p'_i \in \text{Int}\Delta_V$, the multi-valued section $\delta_p([\theta']) = e^{2\pi\sqrt{-1}\sum_{i=1}^l x'_i \theta'_i} s^0_{\text{symp}}$ of $(L_{\text{symp}}, h, \nabla)|_{\mu_{T^l}^{-1}(p) \cap V_{\text{symp}}}$ is covariantly constant. Since δ_p is single-valued if and only if $p \in \text{Int}\Delta_V \cap (\mathfrak{t}^l)^*_{\mathbb{Z}}$, we finish the proof.

(3) The following proof is a slight modification of the argument in [**BFMN**]. If we write $m = \sum_{i=1}^n m'_i p'_i \in (\mathfrak{t}^n)^*_{\mathbb{Z}}$, due to (5.7) and Theorem 5.3, we have

$$
\begin{aligned}
\tilde{\chi_s}^* \sigma^m &= \tilde{\chi_s}^* \left\{ \left(\prod_{i=1}^n (w'_i)^{m'_i} \right) s^0_{\text{comp}} \right\} \\
&= \left\{ \prod_{i=1}^n e^{2\pi m'_i \left(\frac{\partial g_s}{\partial x'_i} + \sqrt{-1}\theta'_i \right)} \right\} e^{2\pi \left(g_s - \sum_{i=1}^n x'_i \frac{\partial g_s}{\partial x'_i} \right)} s^0_{\text{symp}} \\
&= e^{2\pi \left(g_s - \sum_{i=1}^n (x'_i - m'_i) \frac{\partial g_s}{\partial x'_i} \right)} e^{2\pi\sqrt{-1}\left(\sum_{i=1}^n m'_i \theta'_i \right)} s^0_{\text{symp}} \\
&= e^{-2\pi s \alpha_m(x')} \varsigma^m,
\end{aligned}
$$

where

$$
\varsigma^m(x', [\theta']) = e^{2\pi \left(g_0 - \sum_{i=1}^n (x'_i - m'_i) \frac{\partial g_0}{\partial x'_i} \right)} e^{2\pi\sqrt{-1}\left(\sum_{i=1}^n m'_i \theta'_i \right)} s^0_{\text{symp}}(x', [\theta']),
$$

$$
\alpha_m(x') = \sum_{i=1}^n (x'_i - m'_i) \frac{\partial(\underline{\nu} \circ \iota^*)}{\partial x'_i}(x') - (\underline{\nu} \circ \iota^*)(x').
$$

If we set $\underline{\alpha}_{\iota^* m}(p) = \sum_{i=1}^l (x'_i - m'_i) \frac{\partial \underline{\nu}}{\partial x'_i}(p) - \underline{\nu}(p)$ for $p = \sum_{i=1}^l x'_i \iota^* p'_i \in \iota^*\Delta \subset (\mathfrak{t}^l)^*$, then we have $\alpha_m(x') = \underline{\alpha}_{\iota^* m} \circ \mu_{T^l}(x', [\theta'])$. As in the argument in Section 4 in [**BFMN**], we have

$$
\begin{aligned}
\underline{\alpha}_{\iota^* m}(p) &= \underline{\alpha}_{\iota^* m}(\iota^* m) + \int_0^1 \frac{d}{dt} \underline{\alpha}_{\iota^* m}(\iota^* m + t(p - \iota^* m)) dt \\
&= -\underline{\nu}(\iota^* m) + \int_0^1 {}^t(p - \iota^* m)(\text{Hess}_{\iota^* m + t(p - \iota^* m)} \underline{\nu})(p - \iota^* m) dt.
\end{aligned}
$$

Since $\underline{\nu} : \iota^*(\Delta) \to \mathbb{R}$ is strictly convex and $\iota^*\Delta$ is compact, if we put $\|p\|^2 = \sum_{i=1}^l (x'_i)^2$ for $p = \sum_{i=1}^l x'_i p'_i = \iota^* x' \in (\mathfrak{t}^l)^*$, there exists $C_1, C_2 > 0$ such that

$$
-\underline{\nu}(\iota^* m) + C_1 \|p - \iota^* m\|^2 \leq \underline{\alpha}_{\iota^* m}(p) \leq -\underline{\nu}(\iota^* m) + C_2 \|p - \iota^* m\|^2 \text{ for } p \in \iota^*\Delta.
$$

So, we have

$$
(6.7) \qquad e^{-s\underline{\alpha}_{\iota^* m}(p)} \leq e^{s\underline{\nu}(\iota^* m) - sC_1 \|p - \iota^* m\|^2} \text{ for } p \in \iota^*\Delta.
$$

On the other hand, there exists $C_3 > 0$, for sufficiently small $r > 0$

$$\int_{\Delta_V} e^{-s\underline{\alpha}_{\iota^*m}(p)} dp$$

$$\geq \int_{B_r(\iota^*m) \cap \Delta_V} e^{s\underline{\nu}(\iota^*m) - sC_2\|p-\iota^*m\|^2} dp \geq C_3 r^l e^{s\underline{\nu}(\iota^*m) - sC_2 r^2}.$$

Since $\alpha_m = \underline{\alpha}_{\iota^*m} \circ \mu_{T^l}$ is a smooth function on M_{symp}, ς^m is a smooth section of L_{symp}. Since ς^m is non-zero on $\mu_{T^l}^{-1}(\iota^*m)$ and independent of $s \geq 0$, there exists $C_4 > 0$ such that

(6.8) $$\|\tilde{\chi}_s^* \sigma^m|_{V_{\mathrm{symp}}}\|_{L^1(V_{\mathrm{symp}})} \geq C_4 r^l e^{s\underline{\nu}(\iota^*m) - sC_2 r^2}.$$

By (6.7) and (6.8), there exists $C_5 > 0$ such that

$$\left|\tau_s^m(x', [\theta'])\right| = \left|\frac{\tilde{\chi}_s^* \sigma^m(x', [\theta'])}{\|\tilde{\chi}_s^* \sigma^m|_{V_{\mathrm{symp}}}\|_{L^1(V_{\mathrm{symp}})}}\right|$$

$$\leq C_5 \frac{e^{s\underline{\nu}(\iota^*m) - sC_1\|\iota^*x' - \iota^*m\|^2}}{r^l e^{s\underline{\nu}(\iota^*m) - sC_2 r^2}} = C_5 r^{-l} e^{-s(C_1\|\iota^*x' - \iota^*m\|^2 - C_2 r^2)}.$$

Since we can take small $r > 0$ so that $C_1\|p - \iota^*m\|^2 - C_2 r^2 > 0$ for any $p \in \iota^*\Delta \setminus B_{\iota^*m}$, we finish the proof.

(4) By the above argument, we also have

$$\lim_{s \to \infty} \frac{e^{-2\pi s\underline{\alpha}_{\iota^*m}(p)}}{\|e^{-2\pi s\underline{\alpha}_{\iota^*m}}\|_{L^1(\Delta_V)}} = \delta(p - \iota^*m)$$

for $\iota^*m \in \mathrm{Int}\Delta_V \cap (\mathfrak{t}^l)_{\mathbb{Z}}^*$, where $\delta(x)$ is the Dirac delta function on $(\mathfrak{t}^l)^*$ supported at the origin. Moreover, the restriction $\varsigma^m|_{\mu_{T^l}^{-1}(\iota^*m) \cap V_{\mathrm{symp}}} = c(e^{2\pi\sqrt{-1}(\sum_{i=1}^l m_i' \theta_i')} s_{\mathrm{symp}}^0)|_{\mu_{T^l}^{-1}(\iota^*m) \cap V_{\mathrm{symp}}}$, where c is a constant, is a covariantly constant section on $\mu_{T^l}^{-1}(\iota^*m) \cap V_{\mathrm{symp}}$, which we denote by $\delta_{\iota^*m}(\theta')$. So, the assertion is proved easily. The details are the same as in [**BFMN**]. $\square$

7. Proof of main result

In this section, we prove Theorem 2.1 by applying the method developed in Section 6. In Section 7.1, we explain how the setting of Theorem 2.1 fits into the framework of Section 6. In Section 7.2, we construct a family of complex structures on the flag manifold, from which (1)–(3) of Theorem 2.1 turn out to be obvious. Finally, we prove Theorem 2.1 (4) in Sections 7.3 and 7.4.

7.1. Set up. In Section 2, we fixed a symplectic structure $\omega_{\mathbb{P}}$ on $\mathbb{P} = \prod_{l=1}^{n-1} \mathbb{P}(\bigwedge^l \mathbb{C}^n)$. We denote the complex structure on $\mathbb{P}$ by $J_{\mathbb{P}}$. Note that

$(\mathbb{P}, \omega_\mathbb{P}, J_\mathbb{P})$ is a toric Kähler manifold, constructed from a Delzant polytope $\Delta_\mathbb{P}$. Moreover, the toric Kähler manifold $(\mathbb{P}, \omega_\mathbb{P}, J_\mathbb{P})$ can be viewed as the identification of a symplectic toric manifold $(\mathbb{P}_\mathrm{symp}, \omega_\mathbb{P})$ with a complex toric manifold $(\mathbb{P}_\mathrm{comp}, J_\mathbb{P})$ by the diffeomorphism $\chi_0 \colon (\mathbb{P}_\mathrm{symp}, \omega_\mathbb{P}) \to (\mathbb{P}_\mathrm{comp}, J_\mathbb{P})$ defined by a symplectic potential $g_0 \in SP(\Delta_\mathbb{P})$, as in Section 5.3. Similarly, the Hermitian line bundle $(L^\mathbb{P}, h^\mathbb{P}, \nabla^\mathbb{P})$ can also be viewed as the identification of the prequantum line bundle $(L^\mathbb{P}_\mathrm{symp}, h^\mathbb{P}, \nabla^\mathbb{P})$ on $(\mathbb{P}_\mathrm{symp}, \omega_\mathbb{P})$ with the holomorphic line bundle $(L^\mathbb{P}_\mathrm{comp}, \overline{\partial}^\mathbb{P})$ on $(\mathbb{P}_\mathrm{comp}, J_\mathbb{P})$ via the bundle isomorphism $\tilde{\chi}_0$, which is a lift of the map $\chi_0 \colon \mathbb{P}_\mathrm{symp} \to \mathbb{P}_\mathrm{comp}$.

The flag manifold $(\mathbb{F}, \omega_\mathbb{F}, J_\mathbb{F})$ in Theorem 2.1 can also be viewed as the identification of $(\mathbb{F}_\mathrm{symp}, \omega_\mathbb{F})$ with $(\mathbb{F}_\mathrm{comp}, J_\mathbb{F})$ as follows. Let us denote the Plücker embedding by $\rho_\mathrm{comp} \colon (\mathbb{F}_\mathrm{comp}, J_\mathbb{F}) \to (\mathbb{P}_\mathrm{comp}, J_\mathbb{P})$. We set $\mathbb{F}_\mathrm{symp} = \chi_0^{-1}(\mathbb{F}_\mathrm{comp})$ and let $\rho_\mathrm{symp} \colon \mathbb{F}_\mathrm{symp} \to \mathbb{P}_\mathrm{symp}$ be the embedding. Note that $\rho_\mathrm{symp}^* \omega_\mathbb{P} = \omega_\mathbb{F}$. We also set $(L^\mathbb{F}_\mathrm{symp}, h^\mathbb{F}, \nabla^\mathbb{F}) = \rho_\mathrm{symp}^*(L^\mathbb{P}_\mathrm{symp}, h^\mathbb{P}, \nabla^\mathbb{P})$ and $(L^\mathbb{F}_\mathrm{comp}, \overline{\partial}^\mathbb{F}) = \rho_\mathrm{comp}^*(L^\mathbb{P}_\mathrm{comp}, \overline{\partial}^\mathbb{P})$. Then, we have the following commutative diagrams:

(7.1)

$$
\begin{array}{ccccccc}
(\mathbb{P}_\mathrm{symp}, \omega_\mathbb{P}) & \xrightarrow{\chi_0} & (\mathbb{P}_\mathrm{comp}, J_\mathbb{P}) & \quad & (L^\mathbb{P}_\mathrm{symp}, h^\mathbb{P}, \nabla^\mathbb{P}) & \xrightarrow{\tilde{\chi}_0} & (L^\mathbb{P}_\mathrm{comp}, \overline{\partial}^\mathbb{P}) \\
\uparrow \rho_\mathrm{symp} & & \uparrow \rho_\mathrm{comp} & & \uparrow \tilde{\rho}_\mathrm{symp} & & \uparrow \tilde{\rho}_\mathrm{comp} \\
(\mathbb{F}_\mathrm{symp}, \omega_\mathbb{F}) & \xrightarrow{\chi_0|_{\mathbb{F}_\mathrm{symp}}} & (\mathbb{F}_\mathrm{comp}, J_\mathbb{F}) & & (L^\mathbb{F}_\mathrm{symp}, h^\mathbb{F}, \nabla^\mathbb{F}) & \xrightarrow{\tilde{\chi}_0|_{L^\mathbb{F}_\mathrm{symp}}} & (L^\mathbb{F}_\mathrm{comp}, \overline{\partial}^\mathbb{F}),
\end{array}
$$

where $\tilde{\rho}_\mathrm{symp}$ and $\tilde{\rho}_\mathrm{comp}$ are the natural embeddings.

In Section 3.1, we constructed a family of varieties $\{Fl_n(t) = M_n(\mathbb{C})//_t B\}_{t \in \mathbb{C}}$. We set $(V_{t,\mathrm{comp}}, J_{V_t}) = Fl_n(t)$ for $t \in [0,1]$ and denote the deformed Plücker embedding by $\rho_{t,\mathrm{comp}} \colon V_{t,\mathrm{comp}} \to \mathbb{P}_\mathrm{comp}$, which is defined in Section 3.1. Let $V_{t,\mathrm{symp}} = \chi_0^{-1}(V_{t,\mathrm{comp}})$ and $\rho_{t,0} \colon V_{t,\mathrm{symp}} \to \mathbb{P}_\mathrm{symp}$ be the embedding. We also set $(L^{V_t}_\mathrm{symp}, h^{V_t}, \nabla^{V_t}) = \rho_{t,0}^*(L^\mathbb{P}_\mathrm{symp}, h^\mathbb{P}, \nabla^\mathbb{P})$ and $(L^{V_t}_\mathrm{comp}, \overline{\partial}^{V_t}) = \rho_{t,\mathrm{comp}}^*(L^\mathbb{P}_\mathrm{comp}, \overline{\partial}^\mathbb{P})$. Then, we have a commutative diagram, which is the case $s = 0$ in the diagram (7.4) below. Note that $V_{1,\mathrm{comp}} = \mathbb{F}_\mathrm{comp}$ and $V_{1,\mathrm{symp}} = \mathbb{F}_\mathrm{symp}$ and that $V_{0,\mathrm{comp}}$ is the Gelfand–Cetlin toric variety $Fl_n(0) \subset \mathbb{P}$. Thus, the above family $\{V_{t,\mathrm{comp}}\}_{t \in [0,1]}$ connects the flag manifold $\mathbb{F}_\mathrm{comp}$ with the Gelfand–Cetlin toric variety $Fl_n(0)$.

For any $t \in [0,1]$, fix a path $\gamma_t \colon [0,1] \to \mathbb{C}^{n-1}$, which is given by straight lines connecting the points

$$\gamma_t(0) = (1, \ldots, 1) \to (1, \ldots, 1, t) \to (1, \ldots, 1, t, t) \to \cdots \to (t, \ldots, t) = \gamma_t(1).$$

Recall that we constructed a family of varieties $\{Fl_n(\tau) = M_n(\mathbb{C})//_\tau B\}$ for $\tau \in (\mathbb{C}^\times)^{n-1}$ in Section 3.2 and that we also constructed a degeneration in stages by extending the family. The path γ_t is an approximation to the path γ_0 for the degeneration in stages. Note that $Fl_n(\gamma_t(1)) = (V_{t,\mathrm{comp}}, J_{V_t})$ for

$t \in [0,1]$. Due to Propositions 4.2 and 4.3, the gradient-Hamiltonian flow along the path γ_t, for $t \in (0,1]$, gives a symplectic diffeomorphism which together with its lift to the prequantum line bundle gives rise to the following diagram:

(7.2)
$$
\begin{array}{ccc}
(L^{\mathbb{F}}_{\text{symp}}, h^{\mathbb{F}}, \nabla^{\mathbb{F}}) & \xrightarrow{\tilde{\Psi}_t} & (L^{V_t}_{\text{symp}}, h^{V_t}, \nabla^{V_t}) \\
\downarrow & & \downarrow \\
(\mathbb{F}_{\text{symp}}, \rho^*_{\text{symp}}\omega_{\mathbb{P}}) & \xrightarrow{\Psi_t} & (V_{t,\text{symp}}, \rho^*_{t,0}\omega_{\mathbb{P}}).
\end{array}
$$

We can also extend $\Psi_t \colon \mathbb{F}_{\text{symp}} \to V_{t,\text{symp}}$ in (7.2) to the case $t = 0$ if we restrict its domain to an open dense subset $\mathbb{F}^\circ_{\text{symp}}$ of $\mathbb{F}_{\text{symp}}$. It is already given by (3.2). Using the notation in this section, it should be written as $\Psi_0 \colon \mathbb{F}^\circ_{\text{symp}} \to V^\circ_{0,\text{symp}}$. We also have its lift to the prequantum line bundle. Thus, we have the following:

(7.3)
$$
\begin{array}{ccc}
(L^{\mathbb{F}}_{\text{symp}}, h^{\mathbb{F}}, \nabla^{\mathbb{F}})|_{\mathbb{F}^\circ_{\text{symp}}} & \xrightarrow{\tilde{\Psi}_0} & (L^{V_0}_{\text{symp}}, h^{V_0}, \nabla^{V_0})|_{V^\circ_{0,\text{symp}}} \\
\downarrow & & \downarrow \\
(\mathbb{F}^\circ_{\text{symp}}, \rho^*_{\text{symp}}\omega_{\mathbb{P}}) & \xrightarrow{\Psi_0} & (V^\circ_{0,\text{symp}}, \rho^*_{0,0}\omega_{\mathbb{P}}).
\end{array}
$$

7.2. Construction of a family of complex structures. On $(\mathbb{P}, \omega_{\mathbb{P}}, J_{\mathbb{P}})$, a $(\frac{1}{2}\dim_{\mathbb{R}}\mathbb{P})$-dimensional torus $T_{\mathbb{P}}$ acts with an open dense orbit. On the Gelfand–Cetlin toric variety $Fl_n(0) = V_{0,\text{comp}} \subset \mathbb{P}$, a $(\frac{1}{2}\dim_{\mathbb{R}}\mathbb{F})$-dimensional torus T_{GC} acts with an open dense subset, as explained in Section 3.2. There is an injective homomorphism $\tilde{\iota}_{\text{GC}} \colon T_{\text{GC}} \to T_{\mathbb{P}}$ such that the embedding $\rho_{0,\text{comp}} \colon V_{0,\text{comp}} \to \mathbb{P}_{\text{comp}}$ is equivariant. It is described explicitly in Section 6 in [**NNU**]. Let $\iota^*_{\text{GC}} \colon \mathfrak{t}^*_{\mathbb{P}} \to \mathfrak{t}^*_{\text{GC}}$ be the dual map of the inclusion of the Lie algebras $\iota_{\text{GC}} \colon \mathfrak{t}_{\text{GC}} \to \mathfrak{t}_{\mathbb{P}}$. From the description of the map $\tilde{\iota}_{\text{GC}} \colon T_{\text{GC}} \to T_{\mathbb{P}}$ given in [**NNU**], we see that $\iota^*_{\text{GC}}((\mathfrak{t}_{\mathbb{P}})^*_{\mathbb{Z}}) = (\mathfrak{t}_{\text{GC}})^*_{\mathbb{Z}}$.

Fix a strictly convex function $\underline{\nu} \colon \mathfrak{t}^*_{\text{GC}} \to \mathbb{R}$ and set $\nu = \underline{\nu} \circ \iota^*_{\text{GC}} \colon \mathfrak{t}^*_{\mathbb{P}} \to \mathbb{R}$. Let us consider the diffeomorphism $\chi_s \colon (\mathbb{P}_{\text{symp}}, \omega_{\mathbb{P}}) \to (\mathbb{P}_{\text{comp}}, J_{\mathbb{P}})$ defined by $g_s = g_0 + s\nu \in SP(\Delta_{\mathbb{P}})$. Due to Propositions 6.1 and 6.3, we have the following commutative diagrams:

(7.4)
$$
\begin{array}{cccccc}
(\mathbb{P}_{\text{symp}}, \omega_{\mathbb{P}}) & \xrightarrow{\chi_s} & (\mathbb{P}_{\text{comp}}, J_{\mathbb{P}}) & (L^{\mathbb{P}}_{\text{symp}}, h^{\mathbb{P}}, \nabla^{\mathbb{P}}) & \xrightarrow{\tilde{\chi}_s} & (L^{\mathbb{P}}_{\text{comp}}, \overline{\partial}^{\mathbb{P}}) \\
\uparrow \rho_{t,s} & & \uparrow \rho_{t,\text{comp}} & \uparrow \tilde{\rho}_{t,s} & & \uparrow \tilde{\rho}_{t,\text{comp}} \\
(V_{t,\text{symp}}, \rho^*_{t,0}\omega_{\mathbb{P}}) & \xrightarrow{\underline{\chi}_{t,s}} & (V_{t,\text{comp}}, J_{V_t}) & (L^{V_t}_{\text{symp}}, h^{V_t}, \nabla^{V_t}) & \xrightarrow{\tilde{\underline{\chi}}_{t,s}} & (L^{V_t}_{\text{comp}}, \overline{\partial}^{V_t}),
\end{array}
$$

where $\underline{\chi}_{t,0} = \chi_0|_{V_{t,\text{symp}}}$ and $\tilde{\underline{\chi}}_{t,0} = \tilde{\chi}_0|_{L^V_{t,\text{symp}}}$. Note that $\rho^*_{t,s}\omega_{\mathbb{P}} = \rho^*_{t,0}\omega_{\mathbb{P}} \in \Omega^2(V_{t,\text{symp}})$.

In the case $(t, s) = (1, 0)$, the diagrams (7.4) are the same as the diagrams (7.1). In the case $t = 0$, the diagrams (7.4) describe the deformation of toric Kähler structures of the Gelfand–Cetlin toric variety $V_{0,\text{comp}}$. The defining equation of the image of the embedding $\rho_{0,\text{comp}} \colon V_{0,\text{comp}} \to \mathbb{P}_{\text{comp}}$ is given by equations (7) in [**NNU**]. From this description, we see that the image $\rho_{0,\text{comp}}(V_{0,\text{comp}})$ contains the point $(1, \ldots, 1) \in \mathbb{P}$ in the notation in Proposition 6.5. So, Proposition 6.6 can be applied to our case. Therefore, the holomorphic sections on $V_{0,\text{comp}}$ converge to delta-function sections supported on Bohr–Sommerfeld fibers as s goes to infinity. Therefore, the holomorphic sections on $V_{t,\text{comp}}$ are close to delta-function sections when t and s go to zero and infinity, respectively, at the same time. So, we make t a function of s as follows: let $t \colon [0, \infty) \to \mathbb{R}_{>0}$ be a strictly decreasing C^∞-function with $t(0) = 1$ and $\lim_{s \to \infty} t(s) = 0$, where $\mathbb{R}_{>0}$ is the set of positive real numbers. (In fact, $t(s)$ should be required to satisfy additional conditions, which will be discussed in Lemma 7.5 below.)

We define a complex structure J_s on $(\mathbb{F}_{\text{symp}}, \rho_{\text{symp}}^* \omega_{\mathbb{P}})$ as the pull back of $J_{V_{t(s)}}$ by the following composition of diffeomorphisms, which appeared in the diagrams (7.2) and (7.4):

$$(\mathbb{F}_{\text{symp}}, \rho_{\text{symp}}^* \omega_{\mathbb{P}}) \xrightarrow{\Psi_t} (V_{t,\text{symp}}, \rho_{t,0}^* \omega_{\mathbb{P}}) \xrightarrow{\chi_{t,s}} (V_{t,\text{comp}}, J_{V_t}).$$

Namely, a family of complex structures $\{J_s\}_{s \in [0, \infty)}$ on $(\mathbb{F}_{\text{symp}}, \rho_{\text{symp}}^* \omega_{\mathbb{P}})$ is defined by

$$(7.5) \qquad\qquad J_s = (\underline{\chi}_{t(s),s} \circ \Psi_{t(s)})^* J_{V_{t(s)}}.$$

Then, (1) and (2) of Theorem 2.1 follow from the construction of $\{J_s\}_{s \in [0, \infty)}$. By Proposition 6.1 (2), $(V_{t,\text{symp}}, \rho_{t,0}^* \omega_{\mathbb{P}}, \underline{\chi}_{t,s}^* J_{V_t})$ is a Kähler manifold. Moreover, $(V_{t(s),\text{symp}}, \rho_{t(s),0}^* \omega_{\mathbb{P}}, \underline{\chi}_{t(s),s}^* J_{V_{t(s)}})$ is isomorphic to $(\mathbb{F}_{\text{symp}}, \rho_{\text{symp}}^* \omega_{\mathbb{P}}, J_s)$ as a Kähler manifold. So, Theorem 2.1 (3) follows as well. Thus, for any $s \in [0, \infty)$, J_s induces the holomorphic structure $\overline{\partial}^s$ of the Hermitian line bundle $(L_{\text{symp}}^{\mathbb{F}}, h^{\mathbb{F}}, \nabla^{\mathbb{F}})$. Note that the map $\tilde{\underline{\chi}}_{t(s),s} \circ \tilde{\Psi}_{t(s)} \colon (L_{\text{symp}}^{\mathbb{F}}, \overline{\partial}^s) \to (L_{\text{comp}}^{V_{t(s)}}, \overline{\partial}^{V_{t(s)}})$ is an isomorphism of holomorphic line bundles.

To prove Theorem 2.1 (4), we have to construct a basis $\{\sigma_s^m | m \in \Delta_{\text{GC}} \cap (\mathfrak{t}_{\text{GC}})_{\mathbb{Z}}^*\}$ of the space of holomorphic sections $H^0(L_{\text{symp}}^{\mathbb{F}}, \overline{\partial}^s)$.

First, we find a basis of the space of holomorphic sections $H^0(L_{\text{comp}}^{V_0}, \overline{\partial}^{V_0})$ in the following way. Recall that the Gelfand–Cetlin polytope Δ_{GC} is considered as a subset of $\mathfrak{t}_{\text{GC}}^*$ as explained in Section 3.2. Since $\iota_{\text{GC}}^*((\mathfrak{t}_{\mathbb{P}})_{\mathbb{Z}}^*) = (\mathfrak{t}_{\text{GC}})_{\mathbb{Z}}^*$, for each $m \in \Delta_{\text{GC}} \cap (\mathfrak{t}_{\text{GC}})_{\mathbb{Z}}^*$, we can choose $\tilde{m} \in \Delta_{\mathbb{P}} \cap (\mathfrak{t}_{\mathbb{P}})_{\mathbb{Z}}^*$ such that $\iota^*(\tilde{m}) = m$. Let $\sigma^{\tilde{m}}$ be the holomorphic section of $(L_{\text{comp}}^{\mathbb{P}}, \overline{\partial}^{\mathbb{P}})$ defined by (5.5). Due to Proposition 6.6 (1), the restriction $(\tilde{\rho}_{0,\text{comp}})^* \sigma_{\tilde{m}}$ to $V_{0,\text{comp}}$ depends only on $m \in \Delta_{\text{GC}} \cap (\mathfrak{t}_{\text{GC}})_{\mathbb{Z}}^*$, not on $\tilde{m}$. Due to Corollary 3.3,

Δ_{GC} is the moment polytope for the action of T_{GC} on the Gelfand–Cetlin toric variety $(V_{0,\mathrm{symp}}, \rho_{0,0}^* \omega_{\mathbb{P}})$, where $\rho_{0,0}^* \omega_{\mathbb{P}}$ is the first Chern form of $(L_{\mathrm{symp}}^{V_0}, h^{V_0}, \nabla^{V_0})$. Since $L_{\mathrm{symp}}^{V_0}$ is naturally identified with $L_{\mathrm{comp}}^{V_0}$ by the map $\tilde{\underline{\chi}}_{0,0}$, we see that $\{(\tilde{\rho}_{0,\mathrm{comp}})^* \sigma_{\tilde{m}} | m \in \Delta_{\mathrm{GC}} \cap (\mathbf{t}_{\mathrm{GC}})_{\mathbb{Z}}^*\}$ is a basis of the space of holomorphic sections $H^0(L_{\mathrm{comp}}^{V_0}, \overline{\partial}^{V_0})$ by the general fact on toric varieties.

Since linearly independence of the restriction of holomorphic sections from $\mathbb{P}$ is an open condition, there exists $s_0 > 0$ such that, for any $s \geq s_0$, $\{(\tilde{\rho}_{t(s),\mathrm{comp}})^* \sigma^{\tilde{m}} | m \in \Delta_{\mathrm{GC}} \cap (\mathbf{t}_{\mathrm{GC}})_{\mathbb{Z}}^*\}$ are liniarly independent. On the other hand, all $(V_{t(s),\mathrm{comp}}, J_{V_{t(s)}})$ and all $\left(L_{\mathrm{comp}}^{V_{t(s)}}, \overline{\partial}^{V_{t(s)}}\right)$ are isomorphic for $s \geq 0$ as complex manifolds and holomorphic line bundles, respectively. Moreover, due to [**GS**], the dimension of the space of holomorphic sections of $L_{\mathrm{comp}}^{V_1} = L_{\mathrm{comp}}^{\mathbb{F}}$ is equal to the numbers of $\Delta_{\mathrm{GC}} \cap (\mathbf{t}_{\mathrm{GC}})_{\mathbb{Z}}^*$. Therefore, we conclude that $\{(\tilde{\rho}_{t(s),\mathrm{comp}})^* \sigma^{\tilde{m}} | m \in \Delta_{\mathrm{GC}} \cap (\mathbf{t}_{\mathrm{GC}})_{\mathbb{Z}}^*\}$ is a basis of the space of holomorphic sections $H^0\left(L_{\mathrm{comp}}^{V_{t(s)}}, \overline{\partial}^{V_{t(s)}}\right)$ for any $s \geq s_0$.

So we define, for $s \geq s_0$,

$$(7.6) \qquad \sigma_s^m = (\tilde{\underline{\chi}}_{t(s),s} \circ \tilde{\Psi}_{t(s)})^* ((\tilde{\rho}_{t(s),\mathrm{comp}})^* \sigma^{\tilde{m}}) \text{ for } m \in \Delta_{\mathrm{GC}} \cap (\mathbf{t}_{\mathrm{GC}})_{\mathbb{Z}}^*.$$

Since all $(V_{t(s),\mathrm{comp}}, J_{V_{t(s)}})$ and all $\left(L_{\mathrm{comp}}^{V_{t(s)}}, \overline{\partial}^{V_{t(s)}}\right)$ are isomorphic for $s \geq 0$ as complex manifolds and holomorphic line bundles, respectively, we can extend a basis $\{\sigma_s^m | m \in \Delta_{\mathrm{GC}} \cap (\mathbf{t}_{\mathrm{GC}})_{\mathbb{Z}}^*\}$ of the space of holomorphic sections $H^0\left(L_{\mathrm{symp}}^{\mathbb{F}}, \overline{\partial}^s\right)$ for all $s \in [0, s_0]$, which depends continuously on s. Thus, we have defined the basis $\{\sigma_s^m | m \in \Delta_{\mathrm{GC}} \cap (\mathbf{t}_{\mathrm{GC}})_{\mathbb{Z}}^*\}$ of the space of holomorphic sections $H^0\left(L_{\mathrm{symp}}^{\mathbb{F}}, \overline{\partial}^s\right)$ for all $s \geq 0$.

7.3. Another gradient-Hamiltonian flow. To prove that the holomorphic sections defined by (7.6) converge to delta-function sections, we introduce another gradient-Hamiltonian flow.

Let us consider the family of varieties $f \colon (M_n(\mathbb{C}) \times \mathbb{C}) /\!/ B \to \mathbb{C}$ constructed in Section 3.1. Put the standard Kähler metric on $\mathbb{C}$. Consider the map $F \colon (M_n(\mathbb{C}) \times \mathbb{C}) /\!/ B \to \mathbb{P}_{\mathrm{symp}} \times \mathbb{C}$ given by $F(x) = (\rho_{t,0}(x), t)$ if $x \in V_{t,\mathrm{symp}} = f^{-1}(t)$. We put the Kähler metric on the smooth part of $(M_n(\mathbb{C}) \times \mathbb{C}) /\!/ B$ by pulling back the metric on $\mathbb{P}_{\mathrm{symp}} \times \mathbb{C}$ by the map F. Consider the gradient-Hamiltonian flow along the straight-line path from 1 to 0 in $\mathbb{C}$. Since $V_{t,\mathrm{symp}}$ are smooth manifolds for all $t \in (0, 1]$, by Lemma 4.1 the vector field and thus the flow are defined on all of $V_{t,\mathrm{symp}}$ for each $t \in (0, 1]$, and also on $V_{0,\mathrm{symp}}^\circ$, where $V_{0,\mathrm{symp}}^\circ$ is the same as in (7.3). Let $V_{t,\mathrm{symp}}^\circ \subset V_{t,\mathrm{symp}}$ denote the image of $V_{0,\mathrm{symp}}^\circ$ under the reverse flow. Then,

due to Propositions 4.2 and 4.3, we have the following symplectic diffeomorphism and its lift defined by the gradient-Hamiltonian flow for $t \in [0,1]$:

$$
\begin{array}{ccc}
(L_{\mathrm{symp}}^{V_t}, h^{V_t}, \nabla^{V_t})|_{V_{t,\mathrm{symp}}^\circ} & \xrightarrow{\tilde{\Phi}_t} & (L_{\mathrm{symp}}^{V_0}, h^{V_0}, \nabla^{V_0})|_{V_{0,\mathrm{symp}}^\circ} \\
\downarrow & & \downarrow \\
(V_{t,\mathrm{symp}}^\circ, \rho_{t,0}^* \omega_{\mathbb{P}}) & \xrightarrow{\Phi_t} & (V_{0,\mathrm{symp}}^\circ, \rho_{0,0}^* \omega_{\mathbb{P}}).
\end{array}
$$

Let $\mu_{T_{\mathbb{P}}} \colon \mathbb{P}_{\mathrm{symp}} \to \mathfrak{t}_{\mathbb{P}}^*$ be the moment map for the $T_{\mathbb{P}}$-action on $(\mathbb{P}_{\mathrm{symp}}, \omega_{\mathbb{P}})$. Set $\mu_{T_{\mathrm{GC}}} = \iota_{\mathrm{GC}}^* \circ \mu_{T_{\mathbb{P}}} \colon \mathbb{P}_{\mathrm{symp}} \to \mathfrak{t}_{\mathrm{GC}}^*$. Fix an open set $B \subset \mathrm{Int}\Delta_{\mathrm{GC}}$ such that $\mathrm{Int}\Delta_{\mathrm{GC}} \cap (\mathfrak{t}_{\mathrm{GC}})_{\mathbb{Z}}^* \subset B$ and $\overline{B} \subset \mathrm{Int}\Delta_{\mathrm{GC}}$. Set $U_0 = \mu_{T_{\mathrm{GC}}}^{-1}(B) \cap V_{0,\mathrm{symp}} \subset V_{0,\mathrm{symp}}^\circ$. Here, we are considering $V_{0,\mathrm{symp}}$ as a subset of $\mathbb{P}_{\mathrm{symp}}$ by the embedding $\rho_{0,0}$. Moreover, we set $U_t = \Phi_t^{-1}(U_0)$ and $U_t^c = V_{t,\mathrm{symp}} \setminus U_t$. We denote the closure of U_t in $V_{t,\mathrm{symp}}$ by $\overline{U_t}$. Note that $\overline{U_t}$ and U_t^c are compact.

Let $d_{\mathbb{P}}(\ ,\ )$ be the distance on $\mathbb{P}$. Then, we have the following.

Lemma 7.1. *For an arbitrary $\epsilon > 0$, there exists $t_1 > 0$ such that $d_{\mathbb{P}}(\rho_{t,0}(x), \rho_{0,0}(\Phi_t(x))) < \epsilon$ for any $0 \leq t \leq t_1$ and $x \in \overline{U_t} \subset V_{t,\mathrm{symp}}$.*

Proof. Fix an arbitrary small $t_1' > 0$. Then, $\overline{U_t}$ consists of regular points of $f \colon (M_n(\mathbb{C}) \times \mathbb{C}) /\!/ B \to \mathbb{C}$ for any $0 \leq t \leq t_1'$ and $\bigcup_{0 \leq t \leq t_1} \overline{U_t}$ is compact. As noted in Lemma 4.1, $|\mathrm{grad}(\Re f)|$ is non-zero at regular points of f. Therefore, there exists $c > 0$ such that $|\mathrm{grad}(\Re f)| \geq c$ on $\overline{U_t}$, for every $t \in [0, t_1']$. Thus, the gradient-Hamiltonian vector field Z satisfies $|Z| \leq \frac{1}{c}$ on $\overline{U_t}$ for $t \in [0, t_1']$. Since Φ_t is the flow of Z over a "time" t, we finish the proof. $\square$

Similarly, we have the following.

Lemma 7.2. *For an arbitrary $\epsilon > 0$, there exists $t_2 > 0$ such that $d_{\mathbb{P}}(\rho_{1,0}(\Psi_t^{-1}(x)), \rho_{1,0}(\Psi_0^{-1} \circ \Phi_t(x))) < \epsilon$ for all $0 \leq t \leq t_2$ and $x \in \overline{U_t} \subset V_{t,\mathrm{symp}}$, where Ψ_t is the map in (7.2) or (7.3).*

Proof. This follows from "smoothness in initial conditions" results in the theory of differential equations. Because the path γ_t is close to the path γ_0 considered in Section 7.1 for small $t > 0$, the resulting diffeomorphisms Ψ_t and Ψ_0 are very close. Combining with Lemma 7.1, we finish the proof. $\square$

7.4. Convergence to delta-function sections. For an $m \in \mathrm{Int}\Delta_{\mathrm{GC}} \cap (\mathfrak{t}_{\mathrm{GC}})_{\mathbb{Z}}^*$, we have chosen $\tilde{m} \in \Delta_{\mathbb{P}} \cap (\mathfrak{t}_{\mathbb{P}})_{\mathbb{Z}}^*$ such that $\iota^*(\tilde{m}) = m$ and defined the holomorphic section σ_s^m by (7.6). From now on, we prove that, if we choose $t(s)$ appropriately for $s \geq 0$, the section $\frac{\sigma_s^m}{\|\sigma_s^m\|_{L^1(\mathbb{F}_{\mathrm{symp}})}}$ converges to a delta-function section supported on the Bohr–Sommerfeld fiber $\mu_{\mathrm{GC}}^{-1}(m)$ as s goes to infinity. Set, for $0 \leq t \leq 1$, $s >> 0$,

$$
\tau_{t,s}^m = \frac{\underline{\tilde{\chi}}_{t,s}^* (\tilde{\rho}_{t,\mathrm{comp}}^* \sigma^{\tilde{m}})}{\|\underline{\tilde{\chi}}_{t,s}^* (\tilde{\rho}_{t,\mathrm{comp}}^* \sigma^{\tilde{m}})\|_{L^1(V_{t,\mathrm{symp}})}} \in H^0(L_{\mathrm{symp}}^{V_t}, \underline{\tilde{\chi}}_{t,s}^* \overline{\partial}^{V_t}).
$$

Since

$$\tilde{\Psi}_t^* \tau_{t,s}^m = \frac{\tilde{\Psi}_t^* \underline{\tilde{\chi}}_{t,s}^* (\tilde{\rho}_{t,\text{comp}}^* \sigma^{\tilde{m}})}{\|\underline{\tilde{\chi}}_{t,s}^* (\tilde{\rho}_{t,\text{comp}}^* \sigma^{\tilde{m}})\|_{L^1(V_{t,\text{symp}})}} = \frac{\tilde{\Psi}_t^* \underline{\tilde{\chi}}_{t,s}^* (\tilde{\rho}_{t,\text{comp}}^* \sigma^{\tilde{m}})}{\|\tilde{\Psi}_t^* \underline{\tilde{\chi}}_{t,s}^* (\tilde{\rho}_{t,\text{comp}}^* \sigma^{\tilde{m}})\|_{L^1(\mathbb{F}_{\text{symp}})}},$$

we have $\tilde{\Psi}_{t(s)}^* \tau_{t(s),s}^m = \frac{\sigma_s^m}{\|\sigma_s^m\|_{L^1(\mathbb{F}_{\text{symp}})}}$, where $t(s)$ will be defined in Lemma 7.5 below.

For a section $\phi \in \Gamma((L_{\text{symp}}^{\mathbb{F}})^*)$, we denote the push-forward of ϕ with respect to the map $\tilde{\Psi}_t$ by $\tilde{\Psi}_{t*}\phi$, which is a section of the line bundle $(L_{\text{symp}}^{V_t})^*$ for $t > 0$ or a section of $(L_{\text{symp}}^{V_0})^*$ restricted to some open dense subset of $V_{0,\text{symp}}$ for $t = 0$. In what follows, we omit the notation for the volume form when integrating on $\mathbb{F}_{\text{symp}}$ or $V_{t,\text{symp}}$, since it is preserved by the maps Ψ_t and Φ_t. First, we have the following:

Lemma 7.3. (1) *For* $m \in \text{Int}\Delta_{\text{GC}}$, $\mu_{T_{\text{GC}}^{-1}}(m) \cap V_{0,\text{symp}}$ *is a Bohr–Sommerfeld fiber for* $(L_{\text{symp}}^{V_0}, h^{V_0}, \nabla^{V_0})$ *if and only if* $m \in (\mathfrak{t}_{\text{GC}})_{\mathbb{Z}}^*$.

(2) *For* $m \in \text{Int}\Delta_{\text{GC}} \cap (\mathfrak{t}_{\text{GC}})_{\mathbb{Z}}^*$, *there exists a covariantly constant section* δ_m *of* $(L_{\text{symp}}^{V_0}, h^{V_0}, \nabla^{V_0})|_{\mu_{T_{\text{GC}}}^{-1}(m) \cap V_{0,\text{symp}}}$ *and a measure* $d\theta_m$ *on* $\mu_{T_{\text{GC}}}^{-1}(m) \cap V_{0,\text{symp}}$ *which satisfy the following: for any* $\phi \in \Gamma((L_{\text{symp}}^{\mathbb{F}})^*)$, *there exists* $C_1(s, \phi) > 0$ *for* $s \geq 0$, *such that* $\lim_{s \to \infty} C_1(s, \phi) = 0$ *and*

$$(7.7) \quad \left| \int_{V_{0,\text{symp}}} \langle \tilde{\Psi}_{0*}\phi, \tau_{0,s}^m \rangle - \int_{\mu_{T_{\text{GC}}}^{-1}(m) \cap V_{0,\text{symp}}} \langle \tilde{\Psi}_{0*}\phi, \delta_m \rangle \, d\theta_m \right| \leq C_1(s, \phi).$$

Proof. (1) follows from Proposition 6.6 (2).

(2) follows from Proposition 6.6 (4). Since the number of points in $\text{Int}\Delta_{\text{GC}} \cap (\mathfrak{t}_{\text{GC}})_{\mathbb{Z}}^*$ is finite, we can choose $C_1(s, \phi)$ independently of $m \in \text{Int}\Delta_{\text{GC}} \cap (\mathfrak{t}_{\text{GC}})_{\mathbb{Z}}^*$. $\square$

We take $U_t \subset V_{t,\text{symp}}$ as in Section 7.3. Then, we have the following.

Lemma 7.4. *For each section* $\phi \in \Gamma((L_{\text{symp}}^{\mathbb{F}})^*)$, *the following holds:*

$$\left| \int_{\mathbb{F}_{\text{symp}}} \langle \phi, \tilde{\Psi}_t^* \tau_{t,s}^m \rangle - \int_{\mu_{T_{\text{GC}}}^{-1}(m) \cap V_{0,\text{symp}}} \langle \tilde{\Psi}_{0*}\phi, \delta_m \rangle \, d\theta_m \right|$$

$$\leq C_1(s, \phi) + \text{vol}(\mathbb{F}_{\text{symp}}) \|\phi\|_{C^0(\mathbb{F}_{\text{symp}})} (\|\tau_{t,s}^m\|_{C^0(U_t^c)} + \|\tilde{\Phi}_t^* \tau_{0,s}^m\|_{C^0(U_t^c)})$$

$$+ \text{vol}(\mathbb{F}_{\text{symp}}) \|\phi\|_{C^0(\mathbb{F}_{\text{symp}})} \|\tau_{t,s}^m - \tilde{\Phi}_t^* \tau_{0,s}^m\|_{C^0(U_t)}$$

$$+ \|\tilde{\Psi}_{t*}\phi - \tilde{\Phi}_t^* \tilde{\Psi}_{0*}\phi\|_{C^0(U_t)}.$$

Proof. Fix arbitrary $\phi \in \Gamma((L_{\mathrm{symp}}^{\mathbb{F}})^*)$. Then, we have

$$
(7.8) \qquad \left| \int_{\mathbb{F}_{\mathrm{symp}}} \langle \phi, \tilde{\Psi}_t^* \tau_{t,s}^m \rangle - \int_{\mu_{T_{\mathrm{GC}}}^{-1}(m) \cap V_{0,\mathrm{symp}}} \langle \tilde{\Psi}_{0*} \phi, \delta_m \rangle \, d\theta_m \right|
$$

$$
= \left| \int_{V_{t,\mathrm{symp}}} \langle \tilde{\Psi}_{t*} \phi, \tau_{t,s}^m \rangle - \int_{\mu_{T_{\mathrm{GC}}}^{-1}(m) \cap V_{0,\mathrm{symp}}} \langle \tilde{\Psi}_{0*} \phi, \delta_m \rangle \, d\theta_m \right|
$$

$$
\leq \left| \int_{V_{t,\mathrm{symp}}} \langle \tilde{\Psi}_{t*} \phi, \tau_{t,s}^m \rangle - \int_{V_{0,\mathrm{symp}}} \langle \tilde{\Psi}_{0*} \phi, \tau_{0,s}^m \rangle \right|
$$

$$
+ \left| \int_{V_{0,\mathrm{symp}}} \langle \tilde{\Psi}_{0*} \phi, \tau_{0,s}^m \rangle - \int_{\mu_{T_{\mathrm{GC}}}^{-1}(m) \cap V_{0,\mathrm{symp}}} \langle \tilde{\Psi}_{0*} \phi, \delta_m \rangle \, d\theta_m \right|.
$$

The second term on the right-hand side of (7.8) is estimated by (7.7). Next, we estimate the first term on the right-hand side of (7.8)

$$
(7.9) \qquad \left| \int_{V_{t,\mathrm{symp}}} \langle \tilde{\Psi}_{t*} \phi, \tau_{t,s}^m \rangle - \int_{V_{0,\mathrm{symp}}} \langle \tilde{\Psi}_{0*} \phi, \tau_{0,s}^m \rangle \right|
$$

$$
= \left| \int_{V_{t,\mathrm{symp}}} \langle \tilde{\Psi}_{t*} \phi, \tau_{t,s}^m \rangle - \int_{V_{t,\mathrm{symp}}} \langle \tilde{\Phi}_t^* \tilde{\Psi}_{0*} \phi, \tilde{\Phi}_t^* \tau_{0,s}^m \rangle \right|
$$

$$
\leq \left| \int_{U_t} \langle \tilde{\Psi}_{t*} \phi, \tau_{t,s}^m \rangle - \langle \tilde{\Phi}_t^* \tilde{\Psi}_{0*} \phi, \tilde{\Phi}_t^* \tau_{0,s}^m \rangle \right|
$$

$$
+ \left| \int_{U_t^c} \langle \tilde{\Psi}_{t*} \phi, \tau_{t,s}^m \rangle - \langle \tilde{\Phi}_t^* \tilde{\Psi}_{0*} \phi, \tilde{\Phi}_t^* \tau_{0,s}^m \rangle \right|
$$

$$
\leq \left| \int_{U_t} \langle \tilde{\Psi}_{t*} \phi, \tau_{t,s}^m \rangle - \langle \tilde{\Phi}_t^* \tilde{\Psi}_{0*} \phi, \tilde{\Phi}_t^* \tau_{0,s}^m \rangle \right|
$$

$$
+ \mathrm{vol}(\mathbb{F}_{\mathrm{symp}}) \|\phi\|_{C^0(\mathbb{F}_{\mathrm{symp}})} (\|\tau_{t,s}^m\|_{C^0(U_t^c)} + \|\tilde{\Phi}_t^* \tau_{0,s}^m\|_{C^0(U_t^c)}).
$$

Finally, we estimate the first term on the right-hand side of (7.9). If we note that

$$
\int_{U_t} |\tilde{\Psi}_{t*} \phi| \leq \mathrm{vol}(U_t) \|\tilde{\Psi}_{t*} \phi\|_{C^0(U_t)} \leq \mathrm{vol}(\mathbb{F}_{\mathrm{symp}}) \|\phi\|_{C^0(\mathbb{F}_{\mathrm{symp}})},
$$

$$
\int_{U_t} |\tilde{\Phi}_t^* \tau_{0,s}^m| = \int_{U_0} |\tau_{0,s}^m| \leq \int_{U_0} \frac{|\underline{\tilde{\chi}}_{0,s}^*(\tilde{\rho}_{0,\mathrm{comp}}^* \sigma^{\tilde{m}})|}{\|\underline{\tilde{\chi}}_{0,s}^*(\tilde{\rho}_{0,\mathrm{comp}}^* \sigma^{\tilde{m}})\|_{L^1(V_{0,\mathrm{symp}})}} \leq 1,
$$

then we have

(7.10)
$$\left| \int_{U_t} \langle \tilde{\Psi}_{t*}\phi, \tau_{t,s}^m \rangle - \langle \tilde{\Phi}_t^* \tilde{\Psi}_{0*}\phi, \tilde{\Phi}_t^* \tau_{0,s}^m \rangle \right|$$

$$\leq \left| \int_{U_t} \langle \tilde{\Psi}_{t*}\phi, \tau_{t,s}^m - \tilde{\Phi}_t^* \tau_{0,s}^m \rangle \right| + \left| \int_{U_t} \langle \tilde{\Psi}_{t*}\phi - \tilde{\Phi}_t^* \tilde{\Psi}_{0*}\phi, \tilde{\Phi}_t^* \tau_{0,s}^m \rangle \right|$$

$$\leq \|\tau_{t,s}^m - \tilde{\Phi}_t^* \tau_{0,s}^m\|_{C^0(U_t)} \int_{U_t} |\tilde{\Psi}_{t*}\phi| + \|\tilde{\Psi}_{t*}\phi - \tilde{\Phi}_t^* \tilde{\Psi}_{0*}\phi\|_{C^0(U_t)} \int_{U_t} |\tilde{\Phi}_t^* \tau_{0,s}^m|$$

$$\leq \|\tau_{t,s}^m - \tilde{\Phi}_t^* \tau_{0,s}^m\|_{C^0(U_t)} \mathrm{vol}(\mathbb{F}_{\mathrm{symp}}) \|\phi\|_{C^0(\mathbb{F}_{\mathrm{symp}})} + \|\tilde{\Psi}_{t*}\phi - \tilde{\Phi}_t^* \tilde{\Psi}_{0*}\phi\|_{C^0(U_t)}.$$

By (7.7)–(7.10), we finish the proof of Lemma 7.4. $\qquad\square$

Next, we introduce a function $t\colon [0,\infty) \to \mathbb{R}$ so that the holomorphic sections σ_s^m converges to delta-function sections as s goes to infinity.

Lemma 7.5. *There exists a strictly decreasing C^∞-function $t\colon [0,\infty) \to \mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ is the set of positive real numbers, with $t(0) = 1$ and $\lim_{s\to\infty} t(s) = 0$ which satisfies the following: for any $\phi \in \Gamma((L_{\mathrm{symp}}^{\mathbb{F}})^*)$, there exists a constant $C_2(s,\phi) > 0$ with $\lim_{s\to\infty} C_2(s,\phi) = 0$ such that*

$$\left| \int_{\mathbb{F}_{\mathrm{symp}}} \langle \phi, \tilde{\Psi}_{t(s)}^* \tau_{t(s),s}^m \rangle - \int_{\mu_{T_{\mathrm{GC}}}^{-1}(m) \cap V_{0,\mathrm{symp}}} \langle \tilde{\Psi}_{0*}\phi, \delta_m \rangle \, d\theta_m \right| \leq C_2(s,\phi).$$

Proof. First, we estimate the term $\|\tilde{\Phi}_t^* \tau_{0,s}^m\|_{C^0(U_t^c)}$ in Lemma 7.4. Due to Proposition 6.6 (3), there exists $C_3(s) > 0$ such that $\lim_{s\to\infty} C_3(s) = 0$ and, for any $t > 0$,

(7.11) $$\|\tilde{\Phi}_t^* \tau_{0,s}^m\|_{C^0(U_t^c)} = \|\tau_{0,s}^m\|_{C^0(U_0^c)} \leq C_3(s).$$

Next, we estimate other terms in Lemma 7.4. In Section 7.3, we fixed an open set $B \subset \mathrm{Int}\Delta_{\mathrm{GC}}$ such that $\mathrm{Int}\Delta_{\mathrm{GC}} \cap (\mathfrak{t}_{\mathrm{GC}})_{\mathbb{Z}}^* \subset B$ and $\overline{B} \subset \mathrm{Int}\Delta_{\mathrm{GC}}$. We set $U_0 = \mu_{T_{\mathrm{GC}}}^{-1}(B) \cap V_{0,\mathrm{symp}} \subset V_{0,\mathrm{symp}}^\circ$ and $U_t = \Phi_t^{-1}(U_0)$. Now, we also take an open set $B_1 \subset \mathrm{Int}\Delta_{\mathrm{GC}}$ such that $\mathrm{Int}\Delta_{\mathrm{GC}} \cap (\mathfrak{t}_{\mathrm{GC}})_{\mathbb{Z}}^* \subset B_1$ and $\overline{B_1} \subset B$. Then, due to Proposition 6.6 (3), there exists $C_4(s) > 0$ such that $\lim_{s\to\infty} C_4(s) = 0$ and, for any $s \geq 0$ and $m \in \mathrm{Int}\Delta_{\mathrm{GC}} \cap (\mathfrak{t}_{\mathrm{GC}})_{\mathbb{Z}}^*$,

(7.12) $$\left\| \frac{\tilde{\chi}_s^* \sigma^{\tilde{m}}}{\|\tilde{\chi}_s^* \sigma^{\tilde{m}}|_{V_{0,\mathrm{symp}}}\|_{L^1(V_{0,\mathrm{symp}})}} \right\|_{C^0(\mathbb{P}_{\mathrm{symp}} \setminus \mu_{T_{\mathrm{GC}}}^{-1}(B_1))} \leq C_4(s).$$

Since $\rho_{0,s} = \rho_{0,0}\colon V_{0,\mathrm{symp}} \to \mathbb{P}_{\mathrm{symp}}$ for $s \geq 0$ by Proposition 6.5 (2), we have

$$\rho_{0,s}(U_0^c) = \rho_{0,0}(U_0^c) \subset \mathbb{P}_{\mathrm{symp}} \setminus \mu_{T_{\mathrm{GC}}}^{-1}(B_1).$$

Note that

$$\lim_{t\to 0}\|\underline{\tilde{\chi}}^*_{t,s}(\tilde{\rho}^*_{t,\text{comp}}\sigma^{\tilde{m}})\|_{L^1(V_{t,\text{symp}})} = \|\underline{\tilde{\chi}}^*_{0,s}(\tilde{\rho}^*_{0,\text{comp}}\sigma^{\tilde{m}})\|_{L^1(V_{0,\text{symp}})} \neq 0.$$

Since U^c_t is compact, for each $n = 1, 2, \ldots$, there exists $t_n \in (0,1]$ which is independent of ϕ and satisfies the following (7.13) holds for each $s \in [n, n+1]$ and $t \in [0, t_n]$:

$$(7.13)\qquad \rho_{t,s}(U^c_t) \subset \mathbb{P}_{\text{symp}} \setminus \mu^{-1}_{T_{\text{GC}}}(B_1), \frac{\|\underline{\tilde{\chi}}^*_{t,s}(\tilde{\rho}^*_{t,\text{comp}}\sigma^{\tilde{m}})\|_{L^1(V_{t,\text{symp}})}}{\|\underline{\tilde{\chi}}^*_{0,s}(\tilde{\rho}^*_{0,\text{comp}}\sigma^{\tilde{m}})\|_{L^1(V_{0,\text{symp}})}} \geq \frac{1}{2}.$$

By (7.12) and (7.13), we have, for each $s \in [n, n+1]$ and $t \in [0, t_n]$

$$(7.14)\qquad C_4(s) \geq \left\|\frac{\tilde{\rho}^*_{t,s}(\tilde{\chi}^*_s\sigma^{\tilde{m}})}{\|\underline{\tilde{\rho}}^*_{0,s}(\tilde{\chi}^*_s\sigma^{\tilde{m}})\|_{L^1(V_{0,\text{symp}})}}\right\|_{C^0(U^c_t)}$$

$$= \left\|\frac{\underline{\tilde{\chi}}^*_{t,s}(\tilde{\rho}^*_{t,\text{comp}}\sigma^{\tilde{m}})}{\|\underline{\tilde{\chi}}^*_{0,s}(\tilde{\rho}^*_{0,\text{comp}}\sigma^{\tilde{m}})\|_{L^1(V_{0,\text{symp}})}}\right\|_{C^0(U^c_t)}$$

$$= \frac{\|\underline{\tilde{\chi}}^*_{t,s}(\tilde{\rho}^*_{t,\text{comp}}\sigma^{\tilde{m}})\|_{L^1(V_{t,\text{symp}})}}{\|\underline{\tilde{\chi}}^*_{0,s}(\tilde{\rho}^*_{0,\text{comp}}\sigma^{\tilde{m}})\|_{L^1(V_{0,\text{symp}})}}\|\tau^m_{t,s}\|_{C^0(U^c_t)}.$$

By (7.13) and (7.14), we have, for each $s \in [n, n+1]$ and $t \in [0, t_n]$

$$(7.15)\qquad \|\tau^m_{t,s}\|_{C^0(U^c_t)} \leq 2C_4(s).$$

Moreover, due to Lemmas 7.1 and 7.2, taking smaller $t_n > 0$ if necessary, we may also conclude that the following (7.16) and (7.17) hold for each $s \in [n, n+1]$ and $t \in [0, t_n]$:

$$(7.16)\quad \|\tau^m_{t,s} - \tilde{\Phi}^*_t\tau^m_{0,s}\|_{C^0(U_t)} \leq \frac{1}{n+2}\quad \text{for any } m \in \text{Int}\Delta \cap (\mathfrak{t}_{\text{GC}})^*_{\mathbb{Z}},$$

$$(7.17)\quad \|\tilde{\Psi}_{t*}\phi - \tilde{\Phi}^*_t\tilde{\Psi}_{0*}\phi\|_{C^0(U_t)} \leq \frac{\|\phi\|_{C^1(\mathbb{F}_{\text{symp}})}}{n+2}\quad \text{for any } \phi \in \Gamma((L^{\mathbb{F}}_{\text{symp}})^*).$$

By Lemma 7.4 together with (7.11), (7.15), (7.16) and (7.17), we have, for each section $\phi \in \Gamma((L^{\mathbb{F}}_{\text{symp}})^*)$, $n = 1, 2, \ldots$, $s \in [n, n+1]$ and $t \in [0, t_n]$

$$\left|\int_{\mathbb{F}_{\text{symp}}}\langle\phi, \tilde{\Psi}^*_t\tau^m_{t,s}\rangle - \int_{\mu^{-1}_{T_{\text{GC}}}(m)\cap V_{0,\text{symp}}}\langle\tilde{\Psi}_{0*}\phi, \delta_m\rangle\, d\theta_m\right|$$

$$\leq C_1(s, \phi) + \text{vol}(\mathbb{F}_{\text{symp}})\|\phi\|_{C^0(\mathbb{F}_{\text{symp}})}(2C_4(s) + C_3(s))$$

$$+ \text{vol}(\mathbb{F}_{\text{symp}})\|\phi\|_{C^0(\mathbb{F}_{\text{symp}})}\frac{1}{s+1} + \frac{\|\phi\|_{C^1(\mathbb{F}_{\text{symp}})}}{s+1}.$$

We can take a continuous decreasing function $t\colon [0,\infty) \to \mathbb{R}$ with $t(0) = 1$ and $\lim_{s\to\infty} t(s) = 0$ such that $t(n) \le t_n$ for $n >> 0$. Thus, we finish the proof of Lemma 7.5. $\qquad\qquad\square$

We use $t(s)$ in Lemma 7.5 to define the complex structure J_s by (7.5) and the holomorphic section σ_s^m by (7.6). If we recall $\tilde{\Psi}_{t(s)}^* \tau_{t(s),s}^m = \frac{\sigma_s^m}{\|\sigma_s^m\|_{L^1(\mathbb{F}_{\mathrm{symp}})}}$, then we have

$$\lim_{s\to\infty} \int_{\mathbb{F}_{\mathrm{symp}}} \left\langle \phi, \frac{\sigma_s^m}{\|\sigma_s^m\|_{L^1(\mathbb{F}_{\mathrm{symp}})}} \right\rangle = \int_{\mu_{T_{\mathrm{GC}}}^{-1}(m)\cap V_{0,\mathrm{symp}}} \langle \tilde{\Psi}_{0*}\phi, \delta_m \rangle \, d\theta_m.$$

Due to Corollary 3.3, if we define a covariantly constant section $\delta_m^{\mathbb{F}}$ of $(L^{\mathbb{F}}, h^{\mathbb{F}}, \nabla^{\mathbb{F}})|_{\mu_{\mathrm{GC}}^{-1}(m)}$ by pulling δ_m on $\mu_{T_{\mathrm{GC}}}^{-1}(m) \cap V_{0,\mathrm{symp}}$ back by $\tilde{\Psi}_0$, then we have the desired convergence in Theorem 2.1 (4).

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