JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 12, Number 1, 1–22, 2014

DEFORMATION QUANTIZATION AND IRRATIONAL NUMBERS

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Diophantine approximation is the problem of approximating a real number by rational numbers. We propose a version of this in which the numerators are approximately related to the denominators by a Laurent polynomial. Our definition is motivated by the problem of constructing strict deformation quantizations of symplectic manifolds. We show that this type of approximation exists for any real number and also investigate what happens if the number is rational or a quadratic irrational.

1. Introduction

Let $\mathcal{M}$ be a manifold with symplectic form $\omega \in \Omega^2(\mathcal{M})$. The starting point of geometric quantization is a complex line bundle $L \to \mathcal{M}$ (with a Hermitian inner product and a compatible connection) whose curvature equals ω.

This can be used to construct a Hilbert space and some correspondence between operators and functions on $\mathcal{M}$. If $\mathcal{M}$ is the phase space of some classical mechanical system, then these are supposed to be the state space of quantum mechanics and a correspondence between quantum and classical observables. However, the rules for how quantum and classical physics should correspond [**3, 7, 11**] are stated in terms of the classical limit in which "Planck's constant" $\hbar$ approaches 0.

Changing Planck's constant is equivalent to rescaling the symplectic form, and this can be achieved by taking tensor powers of the line bundle. The curvature of $L^{\otimes k}$ is $k\omega$.

Using these tensor powers and identifying $\hbar = \frac{1}{k}$, a strict deformation quantization can be constructed. In particular, the commutator of operators corresponds approximately to $i\hbar$ times the Poisson bracket, which is defined by treating ω as a matrix and inverting it.

Unfortunately, this procedure is not always possible. The line bundle L only exists if the symplectic form ω satisfies an integrality condition —

namely, that the integral of ω over any closed surface must be an integral multiple of 2π.

What can we do if ω violates this condition? In particular, what if ω is not even proportional to an integral form? The solution is to take some more general sequence of line bundles, rather than just tensor powers of a fixed line bundle. The sequence of values of $\hbar$ may also be very different.

The point is that the quantum-classical correspondence only refers to the classical limit, so the curvatures of the line bundles only need to *approximate* multiples of ω.

The difficult part of this is topological. The Chern classes of these line bundles are integral cohomology classes, so they lie on a lattice inside $H^2(\mathcal{M}, \mathbb{R})$. The condition on the classical limit means that these lattice points must converge toward a given line in $H^2(\mathcal{M}, \mathbb{R})$, namely, the set of multiples of $[\omega]$.

If $H^2(\mathcal{M}, \mathbb{R}) \cong \mathbb{R}^2$, then this is a matter of approximating a real number by rational numbers — Diophantine approximation. To construct a strict deformation quantization of $\mathcal{M}$, we need a Diophantine approximation to the ratio between the components of $[\omega]$.

This would be enough to satisfy some definitions of strict deformation quantization, but those definitions do not impose very good behavior in the classical limit. In particular, the Jacobi identity for the Poisson bracket is an unnatural and unnecessary condition unless there is some stronger condition on the classical limit. One of us [**7**] has proposed a definition of "order N strict deformation quantization" where $2 \leq N \leq \infty$. This leads to a stronger condition on the sequence of Chern classes and a more restrictive version of Diophantine approximation.

The purpose of this paper is to study this kind of approximation.

The above motivation was based on the standard construction of geometric quantization, but the modified version of geometric quantization in [**6**] only requires a weaker integrality condition: the integral of ω over any $S^2 \subset \mathcal{M}$ should be a multiple of 2π.

On the other hand, it appears that some sort of integrality condition is necessary from first principles, not just for some constrictions. In [**7**], one of us proved this for the symplectic S^2. In [**5**], Fedosov proved an integrality condition for "asymptotic operator representations".

1.1. Outline. In Section 2, Definition 2.1 is our definition for an "order N rational approximation" to a real number $\alpha \in \mathbb{R}$; this is a more restrictive form of Diophantine approximation. After some background on continued fractions, Corollary 2.7 proves that these rational approximations always exist.

In Section 3, we motivate this definition in two ways by considering the strict deformation quantization of a symplectic manifold when the cohomology class of the symplectic form is not a multiple of an integral class. When the cohomology class is integral, then a quantization can be constructed using the tensor powers of a fixed line bundle, but for a more general symplectic form, Theorem 3.4 shows that a quantization can be constructed using a sequence of line bundles; rational approximation arises from the sufficient conditions in this construction. On the other hand, Theorem 3.6 shows that infinite-order rational approximation arises as a necessary condition for the existence of a deformation quantization with some additional conditions.

In Section 4, we consider rational approximations. Proposition 4.1 classifies the rational approximations when α is rational. When α is a quadratic irrational (the solution to a quadratic equation with integer coefficients) Theorem 4.2 classifies some rational approximations and 4.3 shows that they exist. Finally, in Section 5, we discuss unanswered questions.

2. Rational approximation

2.1. Definition. Diophantine approximation is one of the oldest topics in number theory. Given a number $\alpha \in \mathbb{R}$, the problem is to approximate α by rational numbers; that is, we need a set of pairs of integers (r, s), such that

$$\frac{r}{s} \to \alpha \tag{2.1}$$

as s increases. This is a rather weak condition, so one usually considers the stronger condition,

$$r - s\alpha \to 0. \tag{2.2}$$

As we explain in Section 3, the problem of deformation quantization motivates us to define a more restrictive condition:

Definition 2.1. An order $N \in \mathbb{N}$ rational approximation of $\alpha \in \mathbb{R}$ is an infinite subset $\mathcal{R} \subset \mathbb{Z}^2$, such that there exist real numbers $\gamma_1, \ldots, \gamma_N \in \mathbb{R}$ for which

$$\frac{r}{s} = \alpha + \gamma_1 s^{-1} + \gamma_2 s^{-2} + \cdots + \gamma_N s^{-N} + o\left([|r| + |s|]^{-N}\right),$$

as $|r| + |s| \to \infty$, for $(r, s) \in \mathcal{R}$. We will refer to the numbers s as the *denominators*. An *infinite-order rational approximation* of α is a subset $\mathcal{R}$ satisfying this condition for any N.

There is nothing special about the expression $|r|+|s|$ here. It is simply the easiest norm on $\mathbb{R}^2$ to write down. Any other norm would give an equivalent definition.

It is easy to see that the expansion coefficients $\gamma_1, \ldots, \gamma_N$ are uniquely determined by $\mathcal{R}$. In the case of infinite order, this is an asymptotic expansion of r as a function of s, although r need not actually be a function of s.

In terms of this definition, the condition (2.1) is the definition of an order 0 rational approximation, and (2.2) means an order 1 rational approximation with $\gamma_1 = 0$.

2.2. Continued fractions. In our investigation of finite- and infinite-order rational approximations, we will use continued fractions. Every irrational real number α has a *simple continued fraction expansion*

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} =: [a_0; a_1, a_2, a_3, \ldots],$$

where a_0 is an integer and $a_1, a_2, \ldots$ is a sequence of positive integers. The integers $a_0, a_1, \ldots$ are uniquely determined by α and are called the *partial quotients* in this expansion. The rational numbers

$$\frac{p_n}{q_n} := [a_0; a_1, \ldots, a_n], \quad n \geq 0$$

are called the *principal convergents* to α. We will always assume that $q_n > 0$ and $\gcd(p_n, q_n) = 1$ for each n. Finally, for $n \geq 0$ we define the *complete quotients* in the continued fraction expansion of α by

$$\zeta_n := [a_n; a_{n+1}, \ldots],$$

and we also define the quantities

$$\xi_n := \frac{q_{n-1}}{q_n}.$$

The most basic facts about continued fractions are that

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad \text{and} \tag{2.3}$$

$$\frac{1}{2q_nq_{n+1}} \leq \left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{q_nq_{n+1}}. \tag{2.4}$$

In our applications, we will also use the facts that

$$\alpha - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n^2(\zeta_{n+1} + \xi_n)} \quad \text{and} \tag{2.5}$$

$$\xi_n = [0; a_n, a_{n-1}, \ldots, a_1]. \tag{2.6}$$

Proofs of all of these facts can be found in [**12**]. The following proposition gives a representation of natural numbers in terms of denominators of convergents to α. This is known as the *Ostrowski expansion of a natural number* with respect to α.

Proposition 2.1. [**12, Ch. II**] *Suppose $\alpha \in \mathbb{R}$ is irrational. Then for every $s \in \mathbb{N}$ there is a unique integer $M \geq 0$ and a unique sequence $\{c_{n+1}\}_{n=0}^{\infty}$ of integers such that $q_M \leq s < q_{M+1}$ and*

$$s = \sum_{n=0}^{\infty} c_{n+1} q_n, \tag{2.7}$$

with $0 \leq c_1 < a_1$ and $0 \leq c_n \leq a_n$ for all $n \geq 1$,

$$c_{n+1} = a_{n+1} \implies c_n = 0,$$

and

$$c_{n+1} = 0 \quad \text{for all} \quad n > M.$$

We can construct a similar expansion for real numbers. For $n \geq 0$ let

$$D_n := q_n \alpha - p_n. \tag{2.8}$$

By (2.3) these quantities satisfy the identities

$$a_{n+1} D_n = D_{n+1} - D_{n-1} \quad \text{for} \quad n \geq 1, \tag{2.9}$$

and it is also not difficult to show that

$$D_n = (-1)^n \|q_n \alpha\| \quad \text{for} \quad n \geq 1, \tag{2.10}$$

where $\|\cdot\|$ denotes the distance to the nearest integer. The following proposition provides us with a way of expanding real numbers in terms of the quantities D_n. We will call this the *Ostrowski expansion of a real number* with respect to α.

Proposition 2.2. [**12, Ch. II**] *Suppose $\alpha \in [0, 1)$ is an irrational number with continued fraction expansion denoted as above. For any $\gamma \in [-\alpha, 1-\alpha)$ that satisfies*

$$\|s\alpha - \gamma\| > 0 \text{ for all } s \in \mathbb{Z} \tag{2.11}$$

there is a unique sequence $\{b_{n+1}\}_{n=0}^{\infty}$ of integers such that

$$\gamma = \sum_{n=0}^{\infty} b_{n+1} D_n, \tag{2.12}$$

$$\text{with } 0 \leq b_1 < a_1, \quad 0 \leq b_{n+1} \leq a_{n+1} \text{ for } n \geq 1, \qquad \text{and}$$
$$b_n = 0 \quad \text{whenever} \quad b_{n+1} = a_{n+1} \text{ for some } n \geq 1.$$

We point out that (2.3), (2.4) and (2.10) together imply that the series (2.12) is absolutely convergent. The reason for our interest in Ostrowski expansions is that they give us a precise and convenient way of working with the quantities $\|s\alpha - \gamma\|$, $s \geq 1$, as illustrated by the following proposition.

Proposition 2.3 ([2]). *Let $\alpha \in [0,1)$ be irrational and suppose that $\gamma \in [-\alpha, 1-\alpha)$ satisfies* (2.11)*. Choose an integer $s \in \mathbb{N}$ and, referring to the Ostrowski expansions* (2.7) *and* (2.12)*, write $\delta_{n+1} := c_{n+1} - b_{n+1}$ for $n \geq 0$. Let m be the smallest integer for which $\delta_{m+1} \neq 0$. If $m \geq 4$ then*

$$\|s\alpha - \gamma\| = \left|\sum_{n=m}^{\infty} \delta_{n+1} D_n\right| = \operatorname{sgn}(\delta_{m+1} D_m) \cdot \sum_{n=m}^{\infty} \delta_{n+1} D_n. \tag{2.13}$$

Combining this proposition with (2.5) and (2.10) gives us the following corollary.

Corollary 2.4. *With the same notation as in Proposition* 2.3*, if $m \geq 4$ then*

$$\|s\alpha - \gamma\| = (-1)^m \operatorname{sgn}(\delta_{m+1}) \cdot \sum_{n=m}^{\infty} \frac{(-1)^n \delta_{n+1}}{q_n(\zeta_{n+1} + \xi_n)}.$$

The essence of Proposition 2.3 is that when $m \geq 4$ the term $\delta_{m+1} D_m$ dominates the rest of the series in (2.13). This can be exploited to give good estimates for the quantities $\|s\alpha - \gamma\|$. For our purposes, we only need upper bounds, and the following corollary of Proposition 2.3 (proved in [**2**]) will suffice.

Corollary 2.5. *With the same notation as in Proposition* 2.3*, if $m \geq 4$ then*

$$\|s\alpha - \gamma\| \leq (|\delta_{m+1}| + 2)\|q_m \alpha\|.$$

2.3. Existence. Now we return to our problems about finite- and infinite-order rational approximations. First, we show that every irrational number has an infinite-order approximation.

Theorem 2.6. *Let $\Psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function, and suppose that $\alpha \in [0,1)$ is irrational. There exists a real number γ and a strictly increasing sequence $\{s_k\}_{k=1}^{\infty}$, such that*

$$\|s_k \alpha - \gamma\| \leq \Psi(s_k) \quad \text{for} \quad k \geq 1.$$

Proof. First, we construct a sequence $\{n_k\}$ of positive integers by setting $n_1 = 4$ and then, for $k \geq 1$, choosing n_{k+1} to be the smallest integer greater than $n_k + 1$ for which

$$\frac{3}{q_{n_{k+1}}} \leq \Psi(q_{n_k+1}).$$

Let $\gamma \in \mathbb{R}$ be the real number with Ostrowski expansion, in terms of α, given by

$$b_{n_k+1} = 1, \text{ for all } k \in \mathbb{N}, \text{ and}$$
$$b_{n+1} = 0, \text{ for all } n \in \mathbb{N} \smallsetminus \{n_k\},$$

and for each k let s_k be defined by

$$s_k = \sum_{n=0}^{n_k} b_{n+1} q_n = \sum_{m=1}^{k} q_{n_m}.$$

Then by Corollary 2.5 and inequality (2.4) we have that

$$\|s_k \alpha - \gamma\| \leq 3\|q_{n_{k+1}}\alpha\| \leq \frac{3}{q_{n_{k+1}+1}}.$$

Since $s_k \leq q_{n_k+1}$ and Ψ is decreasing, the right-hand side here is less than $\Psi(s_k)$. □

For example, by choosing $\Psi(s) = e^{-s}$ we obtain the following corollary.

Corollary 2.7. *If $\alpha \in \mathbb{R}$ is irrational, then there exists an infinite-order rational approximation $\mathcal{R}$ to α with $\gamma_j = 0$ for all $j \geq 2$. That is, there exists a real number γ_1 such that, for all $N \in \mathbb{N}$,*

$$\frac{r}{s} = \alpha + \gamma_1 s^{-1} + o\left([|r| + |s|]^{-N}\right), \quad \textit{as} \quad |s| \to \infty, \ (r,s) \in \mathcal{R}.$$

Proof. With s_k defined by Theorem 2.6, let r_k be the nearest integer to αs_k. The rational approximation is then

$$\mathcal{R} = \{(r_k, s_k) \mid k \in \mathbb{N}\}.$$

□

The proof of Theorem 2.6 tells us how to construct infinite-order approximations to any irrational number. A more subtle problem is to try to construct infinite-order approximations where the denominators do not grow too quickly. In Section 6, we will demonstrate a construction for quadratic irrationals which produces infinite-order approximations with denominators that grow at most exponentially. By contrast, for the integers s_k constructed in the proof of Theorem 2.6 with $\Psi(s) = e^{-s}$, we have that $s_k \geq e^{e^{\cdot^{\cdot^{\cdot}}}}$ (k times).

3. Quantization

3.1. Definition. The idea of strict deformation quantization was conceived by Rieffel [**11**]. There are several variations on his definition and several ways of describing the structure. We will use a continuous field of C*-algebras and a quantization map.

This is the definition of quantization given in [**7**]:

Definition 3.1. Let $\mathcal{A}_0$ be a Poisson *-algebra of continuous functions on a Poisson manifold $\mathcal{M}$, large enough to separate points. An *order N strict deformation quantization* (I, A, Q) of $\mathcal{M}$ consists of: a locally compact subset

$I \subseteq \mathbb{R}$ with $0 \in I$ an accumulation point, a continuous field of C*-algebras A over I, and a *-linear map $Q : \mathcal{C}_0^\infty(\mathcal{M}) \to \Gamma(I, A)$ such that:

(1) At $0 \in I$, this map is an inclusion $Q_0 : \mathcal{A}_0 \hookrightarrow A_0 \subset \mathcal{C}_b(\mathcal{M})$ of $\mathcal{A}_0$ as a dense *-subalgebra;
(2) for $f, g \in \mathcal{A}_0$, there exist functions $C_1(f,g), \ldots, C_N(f,g) \in \mathcal{A}_0$ such that
$$Q_\hbar(f)Q_\hbar(g) = Q_\hbar(fg + \hbar C_1(f,g) + \cdots + \hbar^N C_N(f,g)) + o(\hbar^N);$$
(3) for $f, g \in \mathcal{A}_0$,
$$C_1(f,g) - C_1(g,f) = i\{f,g\}.$$

An *infinite-order strict deformation quantization* satisfies these conditions for any N.

It is easy to check that $C_j(f,g)$ is uniquely determined by f and g, which justifies this cumbersome notation. This uniqueness implies that C_j is bilinear, but it is not necessarily bidifferential.

In practice, the given structure is usually the collection of algebras and maps $Q_\hbar : \mathcal{A}_0 \to A_\hbar$ for $\hbar \neq 0 \in I$. The continuous field structure is then constructed from this.

3.2. A construction. Let $\mathcal{M}$ be a compact, Kähler manifold with symplectic form $\omega \in \Omega^2(\mathcal{M})$. Suppose that $L \to \mathcal{M}$ is a Hermitian, holomorphic line bundle with curvature, $\operatorname{curv} L = \omega$. The space $L^2_{\text{hol}}(\mathcal{M}, L)$ of holomorphic sections of L is finite-dimensional, so it is automatically a closed subspace of the Hilbert space $L^2(\mathcal{M}, L)$ of square-integrable sections of L (defined using the Hermitian inner product and the Kähler volume form). Let
$$\Pi_L : L^2(\mathcal{M}, L) \to L^2_{\text{hol}}(\mathcal{M}, L)$$
be the orthogonal projection onto this subspace. There is an obvious representation of the algebra of continuous functions $\mathcal{C}(\mathcal{M})$ on $L^2(\mathcal{M}, L)$, defined by pointwise multiplication. In particular, if $f \in \mathcal{C}(\mathcal{M})$ and $\psi \in L^2_{\text{hol}}(\mathcal{M}, L)$, then the product $f\psi \in L^2(\mathcal{M}, L)$ is square-integrable, so we can construct a vector $\Pi_L(f\psi) \in L^2_{\text{hol}}(\mathcal{M}, L)$. This construction defines a map,
$$T_L : \mathcal{C}(\mathcal{M}) \to \mathcal{L}[L^2_{\text{hol}}(\mathcal{M}, L)],$$
$$T_L(f)\psi := \Pi_L(f\psi).$$

The tensor powers $L^{\otimes k}$ are also positive, Hermitian, holomorphic line bundles, but with the curvature rescaled:
$$\operatorname{curv} L^{\otimes k} = k\omega.$$

For any smooth functions, f and g, the product $T_{L^{\otimes k}}(f)T_{L^{\otimes k}}(g)$ can be asymptotically expanded in k. To be precise:

Theorem 3.1 ([13]). *For any $f, g \in \mathcal{C}^\infty(\mathcal{M})$, there exist unique functions $C_j^\omega(f,g) \in \mathcal{C}^\infty(\mathcal{M})$ for $j = 0, 1, 2, \dots$ starting with $C_0^\omega(f,g) = fg$ such that for any $N \in \mathbb{N}$,*

$$T_{L^{\otimes k}}(f) T_{L^{\otimes k}}(g) = \sum_{j=0}^{N} k^{-j}\, T_{L^{\otimes k}}[C_j^\omega(f,g)] + \mathcal{O}(k^{-N-1}) \tag{3.1}$$

for all $k \in \mathbb{N}$. Antisymmetrizing C_1^ω gives,

$$C_1^\omega(f,g) - C_1^\omega(g,f) = i\{f,g\}_\omega,$$

the Poisson bracket determined by ω as a symplectic form. The norm of $T_{L^{\otimes k}}(f)$ converges to,

$$\lim_{k\to\infty} \|T_{L^{\otimes k}}(f)\| = \|f\|. \tag{3.2}$$

Equation (3.1) for $N = 0$ and equation (3.2) imply that there exists a unique minimal continuous field A of C*-algebras over $\{0, \dots, \frac{1}{3}, \frac{1}{2}, 1\} \subset \mathbb{R}$ with $A_0 := \mathcal{C}(\mathcal{M})$ and $A_{1/k} := \mathcal{L}[L^2_{\mathrm{hol}}(\mathcal{M}, L^{\otimes k})]$, such that for any $f \in \mathcal{C}^\infty(\mathcal{M})$,

$$Q_0(f) = f,$$
$$Q_{1/k}(f) = T_{L^{\otimes k}}(f)$$

defines a continuous section $\hbar \mapsto Q_\hbar(f)$. The rest of the theorem shows that this is an infinite-order quantization of $\mathcal{M}$.

Theorem 3.2. *For each j, $C_j^\omega(f,g)$ is a bidifferential operator on f and g, determined by ω. The value of $C_j^\omega(f,g)$ at each point of $\mathcal{M}$ depends continuously upon ω in the $\mathcal{C}^\infty$ Fréchet topology (or in the $\mathcal{C}^m$ topology for some m). As a function of ω, it is homogeneous of degree $-j$.*

Proof. This follows from the results of [**9**], where it is shown that these are the terms of a "star product with separation of variables" (although with the order of multiplication reversed) and that the formal 2-form classifying this product is simply constructed from ω. By the results of [**8**], this implies that C_j^ω can (in principle) be constructed from the complex structure, ω, and finitely many of its derivatives. This implies the stated continuity.

The homogeneity is because the deformation parameter $\hbar = 1/k$ and symplectic form only enter the formal 2-form in the combination $\hbar^{-1}\omega$, and the star product can be constructed from the formal 2-form. □

This homogeneity implies that equation (3.1) can be stated more directly in terms of $\operatorname{curv} L^{\otimes k} = k\omega$ as,

$$T_{L^{\otimes k}}(f) T_{L^{\otimes k}}(g) = \sum_{j=0}^{N} T_{L^{\otimes k}}[C_j^{k\omega}(f,g)] + \mathcal{O}(k^{-N-1}). \tag{3.3}$$

This approach of taking tensor powers of a fixed line bundle gives a quantization of $\mathcal{M}$ with the symplectic form ω, which is by definition a closed, type $(1,1)$ differential form, but it is not arbitrary. Since the cohomology class $[\frac{\omega}{2\pi}] = c_1(L) \in H^{1,1}(\mathcal{M})$ is the first Chern class of L, it must be integral.

Constructing a quantization for a symplectic form ω without this integrality property is more subtle. Instead of taking tensor powers of a fixed line bundle we can take a more general sequence of line bundles, and instead of identifying $\hbar$ with $\frac{1}{k}$, we can take a more general sequence of values.

Definition 3.2. Let $\{L_k\}_{k=1}^\infty$ be some sequence of holomorphic, Hermitian line bundles over M with positive curvatures $\omega_k := \operatorname{curv} L_k$. Let $\{\hbar_k \in \mathbb{R}\}_{k=1}^\infty$ be some non-repeating sequence with $\lim_{k\to\infty} \hbar_k = 0$. Define

$$I := \{0, \hbar_k \mid k \in \mathbb{N}\} \subset \mathbb{R},$$

$A_0 := \mathcal{C}(\mathcal{M})$, $A_{\hbar_k} := \mathcal{L}[L^2_{\text{hol}}(\mathcal{M}, L_k)]$, and for $f \in \mathcal{C}^\infty(\mathcal{M})$,

$$\begin{aligned} Q_0(f) &:= f, \\ Q_{\hbar_k}(f) &:= T_{L_k}(f). \end{aligned} \tag{3.4}$$

Let A be the minimal continuous field of C*-algebras over with fibers $A_\hbar$ such that for any f, $Q(f)$ is a continuous section (if such a continuous field exists).

If the sequence of curvatures is reasonably well behaved, then the leading order approximation to the commutator will be,

$$[T_{L_k}(f), T_{L_k}(g)] \approx iT_{L_k}(\{f, g\}_{\omega_k}).$$

We want this to be (approximately) $i\hbar_k T_{L_k}(\{f, g\}_\omega)$, therefore we need (for any $f, g \in \mathcal{C}^\infty(\mathcal{M})$)

$$\{f, g\}_\omega = \lim_{k\to\infty} \hbar_k^{-1} \{f, g\}_{\omega_k},$$

or equivalently,

$$\omega = \lim_{k\to\infty} \hbar_k \omega_k,$$

where the topology on $\Omega^2(\mathcal{M})$ is the $\mathcal{C}^0$ topology given by a sup-norm defined with an arbitrary metric. This means that the Kähler metrics given by these curvatures must — after rescaling — converge to the Kähler metric given by ω.

Recall that the Riemann and Ricci tensors of a Kähler manifold are determined by the metric and are invariant under rescaling the metric.

Lemma 3.3. *If*

- *in the* $\mathcal{C}^0$ *topology*

$$\omega = \lim_{k\to\infty} \hbar_k \omega_k, \tag{3.5}$$

- *the magnitude of the Riemann tensor of the Kähler structure* $\hbar_k\omega_k$ *is of order* $o(\hbar_k^{-1})$*,*
- *and the magnitude of the derivative of the Ricci tensor is of order* $o(\hbar_k^{-2})$*,*

then for all $f \in \mathcal{C}^\infty(M)$*,*

$$\lim_{k\to\infty} \|T_{L_k}(f)\| = \|f\| \tag{3.6}$$

and Definition 3.2 *defines an order* 1 *strict deformation quantization of* $\mathcal{M}$ *with the symplectic structure* ω*. In particular, if* (3.5) *holds in the* $\mathcal{C}^3$ *topology, then these hypotheses are true and the conclusion holds.*

Proof. The calculations in [**6,** Lemma 4.8] show that for any $f, g \in \mathcal{C}^\infty(\mathcal{M})$,

$$T_{L_k}(f)T_{L_k}(g) = T_{L_k}\left[fg + C_1^{\omega_k}(f,g)\right] + o(\hbar_k), \tag{3.7}$$

where $C_1^{\omega_k}(f,g)$ is the contraction of the holomorphic derivative of f with the antiholomorphic derivative of g using the Kähler metric defined by ω_k. In the notation of [**6**], $s = \hbar_k^{-1}$, the norms are taken using the rescaled Kähler structure $\hbar_k\omega_k \approx \omega$, $\hat{K}$ is constructed from the Ricci tensor, and K_2 is constructed from the Riemann tensor.

(Alternately, we can take $s = 1$. In that case, the norms are taken with respect to the Kähler structure determined by ω_k. This means that the norm of the derivative of f is of order $\mathcal{O}(\hbar_k)$, the norm of the second derivative is of order $\mathcal{O}(\hbar_k^2)$, and the norms of the Riemann tensor and its derivative are rescaled by $\hbar_k$ and $\hbar_k^2$, respectively.)

Since we are assuming (equation (3.5)) that $\hbar_k\omega_k = \omega + o(1)$, the approximation (3.7) is equivalent to

$$T_{L_k}(f)T_{L_k}(g) = T_{L_k}\left[fg + \hbar_k C_1^{\omega}(f,g)\right] + o(\hbar_k). \tag{3.8}$$

In particular,

$$T_{L_k}(f)T_{L_k}(g) = T_{L_k}(fg) + \mathcal{O}(\hbar_k), \tag{3.9}$$

and

$$[T_{L_k}(f), T_{L_k}(g)] = i\hbar_k T_{L_k}(\{f,g\}_\omega) + o(\hbar_k). \tag{3.10}$$

By the reasoning in [**6,** Lemma 7.9], equation (3.10) implies that the normalized trace of $T_{L_k}(f)$ converges to the normalized integral of f. By the reasoning in [**6,** Theorem 7.10], this and equation (3.9) imply that equation (3.4) does define sections of a unique minimal continuous field over I, and that equation (3.6) is true. Finally, equation (3.10) is the statement that this is a quantization for the symplectic structure ω, and equation (3.8) is the statement that this is a first-order quantization.

In particular, if $\hbar_k\omega_k$ converges in the $\mathcal{C}^3$ topology, this implies that the Riemann tensor and its derivative converge (and are bounded) in the $\mathcal{C}^0$ topology, so all three hypotheses are satisfied. □

The question now is how well behaved the sequence $\{\omega_k\}_{k=1}^\infty$ must be to give an order N quantization.

The obvious generalization of equation (3.3) would be

$$T_{L_k}(f)T_{L_k}(g) = \sum_{j=0}^{N} T_{L_k}[C_j^{\omega_k}(f,g)] + o(\hbar_k^N), \tag{3.11}$$

where $C_j^{\omega_k}$ is as defined in Theorem 3.1. Unfortunately, the proof of Theorem 3.1 in [**13**] uses functions on a circle bundle over M. This is a good way of working with all powers of a fixed line bundle, but it does not apply to a more general sequence of line bundles, so it is not clear how to prove equation (3.11).

Theorem 3.4. *If (for some $N \geq 2$) there exists a Laurent polynomial $\rho(\hbar) \in \omega\hbar^{-1} + \Omega^2(M)[\hbar]$ such that*

$$\omega_k = \rho(\hbar_k) + o(\hbar_k^{N-2}) \tag{3.12}$$

in the $\mathcal{C}^\infty$ topology, and equation (3.11) *is true, then Definition* 3.2 *gives an order N strict deformation quantization.*

Likewise, if there is an asymptotic expansion $\omega_k \sim \omega\hbar_k^{-1} + \cdots$ in the $\mathcal{C}^\infty$ topology and equation (3.11) *is true, then Definition* 3.2 *gives an infinite-order strict deformation quantization.*

Proof. This implies in particular that

$$\omega = \lim_{k\to\infty} \hbar_k\omega_k$$

in the $\mathcal{C}^\infty$ topology, so the hypotheses of Lemma 3.3 are satisfied and Definition 3.2 at least gives an order 1 strict deformation quantization.

Next, note that because $\rho(\hbar)$ begins with an $\hbar^{-1}$ term, its reciprocal begins with an $\hbar$ term, and ω_k^{-1} is actually approximated with an error of order $o(\hbar_k^N)$. By Theorem 3.2, for any $f, g \in \mathcal{C}^\infty(M)$ and $j \in \mathbb{N}$,

$$C_j^{\omega_k}(f,g) = C_j^{\rho(\hbar_k)}(f,g) + o(\hbar_k^N).$$

Inserting this into equation (3.11) and using equation (3.6) gives

$$T_{L_k}(f)T_{L_k}(g) = \sum_{j=0}^{N} T_{L_k}[C_j^{\rho(\hbar_k)}(f,g)] + o(\hbar_k^N).$$

Each $C_j^{\rho(\hbar_k)}(f,g)$ can be expanded in powers of $\hbar_k$, and this gives an expansion,

$$T_{L_k}(f)T_{L_k}(g) = \sum_{j=0}^{N} \hbar_k^j T_{L_k}[C_j(f,g)] + o(\hbar_k^N),$$

for some functions $C_j(f,g) \in \mathcal{C}^\infty(M)$. This shows that the quantization is of order N.

Finally, if there is an asymptotic expansion for ω_k, then the hypotheses are satisfied for any N. The quantization constructed by Definition 3.2 is of order N for any N — and so it is an infinite-order strict deformation quantization. □

If the hypotheses of Theorem 3.4 are satisfied, then the sequence of integral Dolbeault cohomology classes

$$c_1(L_k) = [\tfrac{\omega_k}{2\pi}] \in H^{1,1}(\mathcal{M})$$

has the property that it can be approximated to order $o(\hbar_k^{N-2})$ by a Laurent polynomial in $H^{1,1}(\mathcal{M})[\hbar_k^{-1}, \hbar_k]$ with leading term $[\frac{\omega}{2\pi}]\hbar_k^{-1}$.

(Note that the sequence of numbers $\hbar_k$ does not really carry any additional information here. It is the ratios between components of $c_1(L_k)$ that are interesting here.)

The simplest non-trivial case occurs when $\dim H^{1,1}(\mathcal{M}) = 2$, so let us consider that case and identify $H^{1,1}(\mathcal{M}) = \mathbb{R}^2$. The integral part of Dolbeault cohomology is identified with $\mathbb{Z}^2 \subset \mathbb{R}^2$.

Suppose that $[\frac{\omega}{2\pi}] = (\alpha, 1)$ for some real number $\alpha \in \mathbb{R}$. Denote the Chern classes by $c_1(L_k) = (r_k, s_k)$. The condition on s_k is that there exist some expansions

$$s_k = \hbar_k^{-1} + \cdots + o(\hbar_k^{N-2}),$$

and this is easily satisfied by choosing $\hbar_k = s_k^{-1}$. With this choice, the condition on r_k becomes

$$\begin{aligned} r_k &= \alpha\hbar_k^{-1} + \cdots + o(\hbar_k^{N-2}) \\ &= \alpha s_k + \gamma_1 + \gamma_2 s_k^{-1} + \cdots + \gamma_{N-1} s_k^{-N+1} + o(s_k^{-N+2}), \end{aligned}$$

for some real numbers $\gamma_1, \ldots, \gamma_{N-1} \in \mathbb{R}$. Equivalently,

$$\frac{r_k}{s_k} = \alpha + \gamma_1 s_k^{-1} + \cdots + \gamma_{N-1} s_k^{-N+1} + o(s_k^{-N+1}).$$

In other words, the set of pairs (r_k, s_k) must be an order $N-1$ rational approximation to the real number α.

Likewise, for an infinite-order quantization, we need an infinite-order rational approximation.

There is a plausible converse construction. The cohomology of a compact manifold is finite-dimensional, so there exists a finite set of line bundles whose Chern classes generate the integral part of $H^{1,1}(M)$. If we begin with a sequence of integral cohomology classes that can be expanded to order $N-1$, then this generating set of line bundles can be used to construct a corresponding sequence of line bundles. However, it is not clear whether equation (3.11) would be satisfied.

3.3. An obstruction.

Definition 3.3. A *formal deformation quantization* [**1, 14**] of a Poisson manifold $\mathcal{M}$ is an associative $\mathbb{C}[[\hbar]]$-linear product on the space of formal power series $\mathcal{C}^\infty(\mathcal{M})[[\hbar]]$ of the form

$$f * g = fg + \sum_{j=1}^{\infty} \hbar^j C_j(f,g)$$

such that $C_1(f,g) - C_1(g,f) = i\{f,g\}$.

Note that we are not imposing any other conditions, such as requiring C_j to be bidifferential.

Suppose that $\mathcal{A}_0 \subseteq \mathcal{C}_b^\infty(\mathcal{M})$ is a Poisson subalgebra of bounded smooth functions, whose restriction to any compact coordinate patch gives all smooth functions there. Any infinite-order strict deformation quantization (I, A, Q) of $\mathcal{A}_0$ determines a formal deformation quantization: For any $f, g \in \mathcal{A}_0$,

$$Q_\hbar(f)Q_\hbar(g) \sim Q_\hbar(f * g),$$

where $\sim$ means that for any $N \in \mathbb{N}$ if the formal power series on the left is truncated at order $\hbar^N$, then the norm of the difference of the two sides is bounded by a multiple of $\hbar^{N+1}$.

When $\mathcal{M}$ is symplectic, any formal deformation quantization determines [**4,5,10**] a characteristic cohomology class $\theta \in \hbar^{-1}H^2(\mathcal{M})[[\hbar]]$ which is given to leading order by the symplectic form as

$$\theta = \frac{[\omega]}{2\pi\hbar} + \cdots .$$

(This is related to Fedosov's notation by $\theta = -\frac{\Omega}{2\pi\hbar}$.) Two formal deformation quantizations determine the same cohomology class if and only if they are isomorphic by an isomorphism that reduces modulo $\hbar$ to the identity on $\mathcal{C}^\infty(\mathcal{M})$.

Let $n := \frac{1}{2}\dim\mathcal{M}$. Any formal deformation quantization of a symplectic manifold admits a natural $\mathbb{C}[[\hbar]]$-linear trace

$$\operatorname{Tr} : \mathcal{C}_c^\infty(\mathcal{M})[[\hbar]] \to \hbar^{-n}\mathbb{C}[[\hbar]].$$

This is given to leading order by the symplectic volume form,

$$\operatorname{Tr} f = \frac{1}{n!\hbar^n}\int_M f\,\omega^n + \cdots .$$

This trace is the subject of the algebraic index theorem. Let

$$e_0 = e_0^2 \in \operatorname{Mat}_m[\mathcal{C}^\infty(\mathcal{M})]$$

be an idempotent matrix of smooth functions. Under the $*$-product, it is only approximately idempotent (modulo $\hbar$). However, this can be corrected to a $*$-product idempotent $e_\hbar$, such that $e_\hbar \equiv e_0 \mod \hbar$.

Suppose, for simplicity, that $\mathcal{M}$ is compact. The trace of the $*$-product naturally extends to matrices, so $\operatorname{Tr} e_\hbar$ is a meaningful expression, and this is what the algebraic index theorem computes. To state it, we need one more definition: since e_0 is an idempotent matrix of functions, it determines a vector subbundle of $\mathbb{C}^m \times \mathcal{M}$, whose fiber at $x \in \mathcal{M}$ is the image $e_0(x)\mathbb{C}^m$; write $\operatorname{ch} e_0$ for the Chern character of this bundle.

Theorem 3.5 ([5, 10]). *Let $*$ be any formal deformation quantization of a compact symplectic manifold $\mathcal{M}$, with characteristic class θ. Let $e_0 \in \operatorname{Mat}_m[\mathcal{C}^\infty(\mathcal{M})]$ be any idempotent. For any $*$-idempotent, $e_\hbar \equiv e_0 \mod \hbar$, the trace is*

$$\operatorname{Tr} e_\hbar = \int_M \operatorname{ch} e_0 \wedge e^\theta \wedge \hat{A}(TM).$$

Fedosov [**5**] has applied this theorem to find a constraint on "asymptotic operator representations" of formal deformation quantizations when $\theta = [\frac{\omega}{2\pi}]$. His notion of an asymptotic operator representation of a formal deformation quantization is almost equivalent to an infinite-order strict deformation quantization corresponding to the given formal deformation quantization. The following is a simple adaptation of Fedosov's result.

Theorem 3.6. *Let $\mathcal{M}$ be a compact symplectic manifold and (I, A, Q) an infinite-order strict deformation quantization of $\mathcal{M}$; let θ and Tr be the characteristic class and trace of the corresponding formal deformation quantization; let $c_1(\omega)$ be the first Chern class of the holomorphic tangent bundle determined by any almost complex structure compatible with the symplectic form. If, for each $\hbar \neq 0 \in I$, $A_\hbar$ is represented on a finite-dimensional Hilbert space such that the operator trace tr in those representations is related to the formal trace by, for any $f \in \mathcal{C}^\infty(\mathcal{M})$,*

$$\operatorname{tr} Q_\hbar(f) \sim \operatorname{Tr} f, \tag{3.13}$$

then

$$\theta + \tfrac{1}{2} c_1(\omega) \in \hbar^{-1} H^2(\mathcal{M})[[\hbar]]$$

is the asymptotic expansion of a map from $I \setminus \{0\}$ to integral de Rham cohomology.

Proof. If $e_0 \in \operatorname{Mat}_m[\mathcal{C}^\infty(\mathcal{M})]$ is any idempotent, then there exists [**7,** Lemma 5.3] an idempotent section $e = e^2 \in \operatorname{Mat}_m[\Gamma(I, A)]$ such that $e(0) = e_0$ and which has an asymptotic expansion $e_\hbar \in \mathcal{C}^\infty(\mathcal{M})[[\hbar]]$:

$$Q_\hbar[e(\hbar)] \sim Q_\hbar(e_\hbar).$$

The condition (3.13) implies that

$$\operatorname{tr} e(\hbar) \sim \operatorname{Tr} e_\hbar.$$

The matrix $e_\hbar$ is automatically an idempotent with $e_\hbar \equiv e_0 \mod \hbar$, so Theorem 3.5 applies and tells us that

$$\operatorname{rk} e(\hbar) = \operatorname{tr} e(\hbar) \sim \int_M \operatorname{ch} e_0 \wedge e^\theta \wedge \hat{A}(TM).$$

The left side is obviously integer-valued.

Let J be an almost complex structure compatible with ω. Let $T_J\mathcal{M}$ be the corresponding holomorphic tangent bundle, so that $c_1(T_JM) = c_1(\omega)$. The $\hat{A}$ class can be factorized as $\hat{A}(TM) = e^{\frac{1}{2}c_1(\omega)} \wedge \operatorname{td}(T_JM)$, so for any idempotent e_0,

$$\int_M \operatorname{ch} e_0 \wedge e^\theta \wedge \hat{A}(TM) = \int_M \operatorname{ch} e_0 \wedge e^{\theta+\frac{1}{2}c_1(\omega)} \wedge \operatorname{td}(T_JM)$$

is asymptotically integral.

The bundle $\Lambda^* T_J^*\mathcal{M}$ is a spinor bundle and defines a Spin^c-structure on $\mathcal{M}$, which defines an orientation class $\varepsilon \in K_0(\mathcal{M})$. By the Atiyah-Singer index theorem, $\int_M \cdots \wedge \operatorname{td}(\omega)$ is $\operatorname{ch}\varepsilon$, the Chern character of ε.

The Picard group (of complex line bundles) $\operatorname{Pic}(\mathcal{M})$ is a multiplicative subgroup of the ring $K^0(\mathcal{M})$. It can also be identified with $H^2(\mathcal{M};\mathbb{Z})$. Taking the Chern character is equivalent to exponentiating; i.e., there is a commutative diagram:

$$\begin{array}{ccc} H^2(\mathcal{M};\mathbb{Z}) & = & \operatorname{Pic}(\mathcal{M}) \\ \downarrow{\scriptstyle \exp} & & \downarrow \\ H^{\mathrm{ev}}(\mathcal{M}) & \xleftarrow{\operatorname{ch}} & K^0(\mathcal{M}). \end{array}$$

So, for any $\sigma \in H^2(\mathcal{M}) = \operatorname{Pic}(\mathcal{M}) \otimes \mathbb{R}$,

$$\begin{aligned} \int_M \operatorname{ch} e_0 \wedge e^\sigma \wedge \operatorname{td}(T_JM) &= \langle \operatorname{ch} e_0 \wedge e^\sigma, \operatorname{ch}\varepsilon\rangle = \langle \operatorname{ch} e_0 \wedge \operatorname{ch}\sigma, \operatorname{ch}\varepsilon\rangle \\ &= \langle [e_0] \cup \sigma, \varepsilon\rangle = \langle \sigma, [e_0] \cap \varepsilon\rangle. \end{aligned}$$

Since $\varepsilon \in K_0(\mathcal{M})$ is an orientation, by Poincaré duality, any class in $K_0(\mathcal{M})$ is the cap product of ε with a class in $K^0(\mathcal{M})$, and any class in $K^0(\mathcal{M})$ is a formal difference of projections. Furthermore, $\sigma \in H^2(\mathcal{M}) = \operatorname{Pic}(\mathcal{M}) \otimes \mathbb{R}$ is integral if and only if it pairs integrally with any class in $K_0(\mathcal{M})$.

Now, for any $N \in \mathbb{N}$, let σ_N be the partial sum of $\theta + \frac{1}{2}c_1(\omega)$ up to order $\hbar^N$. This shows that $\|\langle \sigma_N, [e_0] \cap \varepsilon\rangle\| = \mathcal{O}(\hbar^{N+1})$, for $\hbar \in I \smallsetminus \{0\}$. (The double bars again denote the distance from the integers.) Since this is true for any e_0, this implies that the distance from $\sigma_N(\hbar)$ to the nearest integral cohomology class is of order $\mathcal{O}(\hbar^{N+1})$.

For any $\hbar \in I \smallsetminus \{0\}$, define $\rho(\hbar)$ to be the nearest integral class to $\sigma_0(\hbar)$. Because the difference $\sigma_N(\hbar) - \sigma_0(\hbar)$ is of order $\mathcal{O}(\hbar)$, for $\hbar$ sufficiently small, $\rho(\hbar)$ is also the nearest integral class to $\sigma_N(\hbar)$. Therefore $\sigma_N(\hbar) - \rho(\hbar) =$

$\mathcal{O}(\hbar^{N+1})$, for any N. This shows that $\theta+\frac{1}{2}c_1(\omega)$ is the asymptotic expansion of ρ. □

This is not a completely general result, because of the assumption that each $A_{\hbar\neq 0}$ is represented on a finite-dimensional Hilbert space. This is not true for the example of the non-commutative torus. However, this does give a non-trivial asymptotic integrality condition on the characteristic class θ; if a formal deformation quantization does not satisfy this condition, then it cannot come from a strict deformation quantization with these properties.

To see how this relates to rational approximations, again consider the simplest case, when $H^2(\mathcal{M};\mathbb{Z})\cong\mathbb{Z}^2$ and suppose that $[\frac{\omega}{2\pi}]=(\alpha,1)$. Theorem 3.6 tells us that

$$\theta+c_1(\omega)=(\alpha,1)\hbar^{-1}+\cdots$$

is the asymptotic expansion of some map $(r,s):I\smallsetminus\{0\}\to\mathbb{Z}^2$. The second component of $\theta+c_1(\omega)$ is a formal Laurent series consisting of $\hbar^{-1}$ and non-negative powers of $\hbar$. This can be functionally inverted and inserted into the first component. That is, r can be written as a formal power series in s. This power series is the asymptotic expansion of r in terms of s, so the range of (r,s) is an infinite-order rational approximation to α.

4. Examples

4.1. The rational case. Suppose that $\alpha=a/b$ where $a,b\in\mathbb{Z}$ and $\gcd(a,b)=1$.

The obvious infinite-order rational approximation to this is $\{(ka,kb)\mid k\in\mathbb{Z}\}$. This can also be modified by adding integer constants. In fact, that is all that we can do.

Proposition 4.1. *Let $\alpha=a/b$ with $a,b\in\mathbb{Z}$. If $\mathcal{R}$ is a first-order rational approximation to α, then there exists $d\in\mathbb{Z}$ such that*

$$br=as+d \tag{4.1}$$

for all but finitely many $(r,s)\in\mathcal{R}$. Moreover, $\mathcal{R}$ is an infinite-order rational approximation.

Proof. Being a first-order rational approximation means that there exists a real number $\gamma_1\in\mathbb{R}$ such that for all $(r,s)\in\mathcal{R}$,

$$\frac{r}{s}=\frac{a}{b}+\frac{\gamma_1}{s}+o(s^{-1}).$$

Multiplying by s and b gives,

$$br=as+\gamma_1 b+o(1). \tag{4.2}$$

Since the first two terms are integers, this means that

$$\|\gamma_1 b\|=o(1),$$

where $\|\cdot\|$ again denotes the distance from $\mathbb{Z}$. However, since the left-hand side is a constant, this shows that $\|\gamma_1 b\| = 0$, that is, $d := \gamma_1 b \in \mathbb{Z}$.

Inserting this back into equation (4.2) gives that $br = as + d + o(1)$, but since the first three terms are integers, the error $o(1)$ must be 0 for $|r| + |s|$ sufficiently large. This gives equation (4.1).

Since

$$\frac{r}{s} = \frac{a}{b} + \frac{d}{bs},$$

this satisfies the definition of an infinite-order rational approximation, with coefficients $\gamma_j = 0$ for $j \geq 2$. □

4.2. Quadratic irrationals. First consider the "golden ratio" $\phi := \frac{1+\sqrt{5}}{2}$. Its continued fraction expansion is simply $\phi = [1; 1, 1, \dots]$. The partial quotients are $a_n = 1$ for all n, so equation (2.3) shows that the principal convergents are given by Fibonacci numbers,

$$p_n = F_{n+2} \quad \text{and} \quad q_n = F_{n+1},$$

which are defined recursively by $F_0 = 0$, $F_1 = 1$, and

$$F_n = F_{n-1} + F_{n-2},$$

or explicitly as

$$F_n = \tfrac{1}{\sqrt{5}}[\phi^n - (-\phi)^{-n}]. \tag{4.3}$$

The golden ratio is a root of the polynomial equation $\phi^2 - \phi - 1 = 0$, so consider the related homogeneous polynomial $r^2 - rs - s^2$. Equation (4.3) shows that consecutive Fibonacci numbers satisfy

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (F_{n+1} - \phi F_n)(F_{n+1} + \phi^{-1}F_n) = (-1)^n.$$

This shows that,

$$\left|\frac{F_{n+1}}{F_n} - \phi\right| \leq \frac{1}{\phi F_n^2},$$

for $n \geq 1$, so the set of principal convergents gives a first order rational approximation. However, it is not a second-order rational approximation, because

$$\frac{F_{n+1}}{F_n} - \phi \approx \frac{(-1)^n}{\sqrt{5}F_n^2}$$

is not a nice function of F_n.

Instead, this alternates between two nice functions of F_n, so let

$$\mathcal{R} := \{(F_{2k+1}, F_{2k}) \mid k \in \mathbb{N}\}.$$

These pairs of numbers are generated by starting from $(2, 1)$ and applying the recursion

$$(r, s) \mapsto (2r + s, r + s). \tag{4.4}$$

These satisfy

$$r^2 - rs - s^2 = 1,$$

so this set is just

$$\mathcal{R} = \{(r,s) \in \mathbb{N}^2 \mid r^2 - rs - s^2 = 1\}.$$

In this case, r is an algebraic function of s,

$$r = \frac{s + \sqrt{5s^2+4}}{2} = \frac{1+\sqrt{5+4s^{-2}}}{2}s.$$

For $s > \frac{\sqrt{5}}{2}$, this is given exactly by a Laurent series,

$$r = \phi s + \frac{\sqrt{5}}{2}\sum_{j=1}^{\infty} \frac{(-\frac{8}{5})^j}{j!(2j-1)!!} s^{1-2j},$$

therefore this $\mathcal{R}$ is an infinite-order rational approximation to ϕ.

The growth of the denominators F_{2k} as $k \to \infty$ is extremely different from the rational case. Instead of growing linearly with k, they grow exponentially: $F_{2k} \approx \frac{1}{\sqrt{5}}\phi^{2k}$.

For any $d \neq 0 \in \mathbb{Z}$, there exist natural numbers $r, s \in \mathbb{N}$ with $r^2 - rs - s^2 = d$. The recursion (4.4) preserves this polynomial, and therefore the set

$$\mathcal{R}_d := \{(r,s) \in \mathbb{N}^2 \mid r^2 - rs - s^2 = d\}$$

is infinite, and for the same reasons, it is an infinite-order rational approximation to ϕ.

In general, the behavior for quadratic irrationals is similar.

Theorem 4.2. *If $\mathcal{R}$ is any second-order rational approximation to a quadratic irrational α with $\gamma_1 = 0$, then $\mathcal{R}$ is actually an infinite-order rational approximation, and there exist $a, b, c, d \in \mathbb{Z}$ such that*

$$ar^2 + brs + cs^2 = d$$

for all but finitely many $(r,s) \in \mathcal{R}$.

Proof. Being a quadratic irrational means that there exist $a, b, c \in \mathbb{Z}$ such that $0 = a\alpha^2 + b\alpha + c$. Inserting $(r,s) \in \mathcal{R}$ into the corresponding homogeneous polynomial gives

$$\begin{aligned} ar^2 + brs + cs^2 &= a(\alpha s + \gamma_2 s^{-1} + o(s^{-1}))^2 + b(\alpha s + \gamma_2 s^{-1} + o(s^{-1}))s + cs^2 \\ &= (2a\alpha + b)\gamma_2 + o(1). \end{aligned}$$

Since the left side is always an integer, this implies that $\|(2a\alpha + b)\gamma_2\| = o(1)$, but since this is a constant, that implies that $d := (2a\alpha + b)\gamma_2 \in \mathbb{Z}$. Now, the integers $ar^2 + brs + cs^2 - d$ converge to 0 as $|r| + |s| \to \infty$, which means that they must almost all equal 0.

This shows in particular that for large enough s

$$r = \frac{-bs \pm \sqrt{(b^2-4ac)s^2+4ad}}{2a},$$

where the sign is chosen such that $\alpha = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Then, we have that

$$\frac{r}{s} = \frac{-b \pm \sqrt{b^2-4ac}\sqrt{1+\frac{4ad}{s}}}{2a},$$

and expanding $\sqrt{1+\frac{4ad}{s}}$ as a power series in $4ad/s$ thus exhibits that $\mathcal{R}$ is an infinite-order rational approximation to α. □

Theorem 4.3. *Every quadratic irrational real number α has an infinite-order approximation with denominators that grow at most exponentially.*

Proof. A quadratic irrational real number has an eventually periodic continued fraction expansion (see [**12,** Theorem III.1.2]). Therefore, we write

$$\alpha = [0; a_1, \ldots, a_K, \overline{a_{K+1}, \ldots, a_{K+L}}],$$

for some integers K and L, and let

$$\gamma_2 := \frac{(-1)^{K+1}}{\zeta_{K+1} + [0; \overline{a_{K+L}, \ldots, a_{K+1}}]}.$$

Now by equations (2.5) and (2.6), for any positive integer k,

(4.5)

$$\begin{aligned}
&\left|\alpha - \frac{p_{K+2kL}}{q_{K+2kL}} + \frac{\gamma_2}{q^2_{K+2kL}}\right| \\
&\quad= \left|\frac{(-1)^{K+2kL}}{q^2_{K+2kL}(\zeta_{K+2kL+1}+\xi_{K+2kL})} + \frac{(-1)^{K+1}}{q^2_{K+2kL}(\zeta_{K+1}+[0;\overline{a_{K+L},\ldots,a_{K+1}}])}\right| \\
&\quad= \frac{1}{q^2_{K+2kL}}\left|\frac{1}{\zeta_{K+1}+[0;a_{K+2kL},\ldots,a_1]} - \frac{1}{\zeta_{K+1}+[0;\overline{a_{K+L},\ldots,a_{K+1}}]}\right|.
\end{aligned}$$

Now since

$$\lim_{k\to\infty}[0; a_{K+2kL}, \ldots, a_1] = [0; \overline{a_{K+L}, \ldots, a_{K+1}}],$$

this proves that the quantity in (4.5) is $o(q^{-2}_{K+2kL})$ as $k \to \infty$. In other words, the set

$$\mathcal{R} = \{(p_{K+2kL}, q_{K+2kL}) \mid k \in \mathbb{N}\}$$

is an order 2 rational approximation to α, with $\gamma_1 = 0$ and γ_2 as above. By Theorem 4.2, it is actually an infinite-order approximation.

Since the continued fraction for α is periodic, there is a constant M such that $a_n \le M$ for all n. By (2.3) we have that

$$q_{K+2kL} = \mathcal{O}(M^{K+2kL}),$$

which verifies that the denominators in $\mathcal{R}$ grow no more than exponentially. □

Note that in the proof of this theorem we used equation (2.5), which corresponded in our situation with taking $\gamma = 0$ in Corollary 2.4. It may be the case that using the full generality of Corollary 2.4 could produce infinite-order approximations to other real numbers that grow more slowly than those constructed in the proof of Theorem 2.6.

5. Further questions

This was only a beginning at investigating this topic. Although we have shown that infinite-order rational approximation always exist, there are other basic questions that remain to be answered.

5.1. Growth. Given a number α, how fast do its rational approximations grow? That is, if an order N rational approximation to α is arranged into a sequence, then how fast must the numbers grow?

We have seen that a rational number has infinite-order approximations that grow linearly and a quadratic irrational has approximations that grow exponentially. Does the existence of a linearly or exponentially growing approximation imply that α is rational or quadratic?

5.2. Uniqueness. To what extent are the expansion coefficients $\gamma_1, \gamma_2, \ldots$ restricted by α?

Corollary 2.7 shows that there always exists at least one approximation with $\gamma_j = 0$ for $j \geq 2$. Proposition 4.1 shows that, for α rational, the expansion must be of this form, and γ_1 is greatly restricted. Theorem 4.2 shows that if α is quadratic and $\gamma_1 = 0$, then the other expansion coefficients are determined by a single integer.

5.3. Generalization. Theorems 3.4 and 3.6 actually motivate a more general definition. Rather than considering only a single real number, we could take a point $\alpha \in \mathbb{R}P^n$ and look for a sequence in $\mathbb{Z}^{n+1}$ that converges modulo $\mathbb{R}^\times$ to α. All of the questions about rational approximations can be asked again in this more general context.

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Received 07/22/2011, accepted 03/19/2013

We would like to thank Eric Leichtnam and Alan Weinstein for their questions and comments. A. H. was supported by EPSRC grant F/027028/1.