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SPIN-QUANTIZATION COMMUTES WITH REDUCTION

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In this paper, we prove that the "quantization commutes with reduction" phenomenon of Guillemin and Sternberg [**12**] applies in the context of the metaplectic correction.

Contents

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1. Introduction

Let K be a compact connected Lie group with Lie algebra $\mathfrak{k}$. An Hamiltonian K-manifold (M,ω,Φ) is Spin-prequantized if M carries an equivariant Spin^c structure P with determinant line bundle being a Kostant–Souriau line bundle over $(M, 2\omega, 2\Phi)$. Let $\mathcal{D}_P$ be the Spin^c Dirac operator attached to P, where M is oriented by its symplectic form. The Spin quantization of (M,ω,Φ) corresponds to the equivariant index of the elliptic operator $\mathcal{D}_P$, and is denoted

$$\mathcal{Q}^K_{\mathrm{spin}}(M) \ \in\ R(K).$$

Let $\widehat{A}(M)(X)$ be the equivariant Â-genus class: it is an equivariant analytic function from a neighborhood of $0 \in \mathfrak{k}$ with value in the algebra of differential forms on M. The Atiyah–Segal–Singer index theorem [**6,** Theorem 8.2] tell us that

$$\mathcal{Q}^K_{\mathrm{spin}}(M)(e^X) := \left(\frac{\mathrm{i}}{2\pi}\right)^{\frac{\dim M}{2}} \int_M \mathrm{e}^{\mathrm{i}(\omega+\langle\Phi,X\rangle)}\widehat{A}(M)(X) \tag{1.1}$$

for $X \in \mathfrak{k}$ small enough. It shows in particular that $\mathcal{Q}^K_{\mathrm{spin}}(M) \in R(K)$ does not depend of the choice of the Spin-prequantum data.

This notion of Spin-quantization is closely related to the notion of *metaplectic correction*. Suppose that (M,ω,Φ) carries a Kostant–Souriau line bundle L_ω, and that the bundle of half-forms $\kappa_J^{1/2}$ associated to an invariant almost complex structure J is well defined. In this case, (M,ω,Φ) is Spin-prequantized by the Spin^c-structure defined by J and twisted by the line bundle $L_\omega \otimes \kappa_J^{1/2}$. The crucial point here is that the corresponding Spin-quantization of (M,ω,Φ) does not depend of the choice of the almost complex structure. Note that the existence of the bundle of half-form $\kappa_J^{1/2}$ is equivalent to the existence of a Spin structure on M [**16**].

The purpose of this paper is to compute geometrically the multiplicities of $\mathcal{Q}^K_{\mathrm{spin}}(M) \ \in\ R(K)$ in a way similar to the famous "quantization commutes with reduction" phenomenon of Guillemin–Sternberg [**12, 14, 18, 19, 21, 24, 27–29**]. This question was first addressed in the work of Cannas–Karshon–Tolman [**9**] and Vergne [**28**] in the case of a circle action. The non-abelian group action case was first studied by Jeffrey-Kirwan [**14**] and by the author [**22**], but both papers made fairly strong assumptions: in [**14**] they suppose that $0 \in \mathfrak{k}^*$ has a big enough neighborhood of regular values of the moment map, and in [**22**] one asks that the infinitesimal stabilizers of the K-action are abelian. In this paper, we obtain a "quantization commutes with reduction" theorem, which holds in the general case. Note that C. Teleman also obtained some results [**26,** Proposition 3.10] in the algebraic setting.

The striking difference with the standard Guillemin–Sternberg phenomenon is the *rho shift* that we explain now. Let T be a maximal torus of K with Lie algebra $\mathfrak{t} \subset \mathfrak{k}$. Let $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ be the closed Weyl chamber. We will look at $\mathfrak{t}^*_+$ as a disjoint union of its open faces, the maximal one being its interior $(\mathfrak{t}^*_+)^{\mathrm{o}}$. Let $\rho \in (\mathfrak{t}^*_+)^{\mathrm{o}}$ be the half sum of the positive roots. At each open face τ of $\mathfrak{t}^*_+$, we associate the term ρ_τ, which is the half sum of the positive roots which are orthogonal to τ. We note that $\rho - \rho_\tau \in \tau$ is equal to the orthogonal projection of ρ on τ.

For any $\xi \in \mathfrak{t}^*_+$ and any face τ containing ξ in its closure, we consider the *shifted symplectic reduction*

$$M^\tau_\xi := \Phi^{-1}(\xi + \rho - \rho_\tau)/K_\tau$$

where K_τ is the common stabilizer of points in τ. Note that $\xi + \rho - \rho_\tau \in \tau$ when $\xi \in \overline{\tau}$.

We are particularly interested to the smallest face σ of the Weyl chamber so that the Kirwan polytope $\Delta(M) := \Phi(M) \cap \mathfrak{t}^*_+$ is contained in the closure of σ. It is not hard to see that the Spin-prequantum data on (M, ω, Φ) descents to the shifted symplectic reduction M^σ_μ when μ is a dominant weight belonging to $\overline{\sigma}$. Then $\mathcal{Q}_{\mathrm{spin}}(M^\sigma_\mu) \in \mathbb{Z}$ is naturally defined when $\mu + \rho - \rho_\sigma$ is a regular value of the moment map. In general, the number $\mathcal{Q}_{\mathrm{spin}}(M^\sigma_\mu)$ is defined by shift-desingularization (see Section 2.4).

By definition $\mathcal{Q}_{\mathrm{spin}}(M^\sigma_\mu)$ vanishes when $\mu + \rho - \rho_\sigma \notin \Delta(M)$, but in fact we can strengthen this vanishing property: $\mathcal{Q}_{\mathrm{spin}}(M^\sigma_\mu) = 0$ if $\mu + \rho - \rho_\sigma$ does not belong to the *relative interior* of the Kirwan polytope $\Delta(M)$.

Recall that the irreducible representations V^K_μ of K are parametrized by their highest weight $\mu \in \widehat{K} \subset \mathfrak{t}^*_+$.

The main result of this paper is the following:

Theorem 1.1. *Let (M, ω, Φ) be a compact Spin-prequantized Hamiltonian K-manifold. Let σ be the smallest face of the Weyl chamber so that $\Delta(M) \subset \overline{\sigma}$. We have*

$$\mathcal{Q}^K_{\mathrm{spin}}(M) = \sum_{\mu \in \widehat{K} \cap \overline{\sigma}} \mathcal{Q}_{\mathrm{spin}}(M^\sigma_\mu)\, V^K_\mu.$$

Let us give some ideas about the proof. The representation V^K_μ is equal to the Spin-quantization of the coadjoint orbit $\mathcal{O}_\mu := K \cdot (\mu + \rho)$. Then the shifting trick tells us that the multiplicity m_μ of V_μ^K in $\mathcal{Q}^K_{\mathrm{spin}}(M)$ is equal to

$$\left[\mathcal{Q}^K_{\mathrm{spin}}(M)\otimes (V_\mu^K)^*\right]^K = \left[\mathcal{Q}^K_{\mathrm{spin}}(M\times\overline{\mathcal{O}}_\mu)\right]^K,$$

where $\overline{\mathcal{O}}_\mu$ is the coadjoint orbit with the opposite symplectic structure. As we did in [**21,22**], we study the expression $\left[\mathcal{Q}^K_{\mathrm{spin}}(M\times\overline{\mathcal{O}}_\mu)\right]^K$ by localizing the Riemann–Roch character on the critical points of the square of the moment map

$$\Phi_\mu : M\times\overline{\mathcal{O}}_\mu \to \mathfrak{k}^*.$$

Here our treatment differs depending on whether the Kirwan polytope $\Delta(M)$ intersects the interior of the Weyl chambers or not (i.e., $\sigma = \mathfrak{t}^*_+$ or not).

When $\sigma = \mathfrak{t}^*_+$, we show that the multiplicity m_μ is calculated using the Riemann–Roch character localized near the zero-level set of the moment map Φ_μ. This case is (more or less) treated in [**22**].

The heart of this paper is when we work out the case $\sigma \neq \mathfrak{t}^*_+$. We have $\Phi_\mu^{-1}(0) = \emptyset$, but we show how to compute m_μ using the Riemann–Roch character localized near

$$K\cdot\left(N^{\rho_\sigma}\cap\Phi_\mu^{-1}(-\rho_\sigma)\right).$$

Here N^{ρ_σ} denotes the submanifold of $N = M\times\mathcal{O}_\mu$ where the infinitesimal action of ρ_σ vanishes.

Notations. *Throughout the paper, K will denote a compact connected Lie group, and $\mathfrak{k}$ its Lie algebra. We let T be a maximal torus in K, and $\mathfrak{t}$ be its Lie algebra. The integral lattice $\wedge\subset\mathfrak{t}$ is defined as the kernel of* $\exp:\mathfrak{t}\to T$, *and the real weight lattice $\wedge^*\subset\mathfrak{t}^*$ is defined by : $\wedge^* := \hom(\wedge, 2\pi\mathbb{Z})$. Every $\mu\in\wedge^*$ defines a one-dimensional T-representation, denoted $\mathbb{C}_\mu$, where $t = \exp(X)$ acts by $t^\mu := \mathrm{e}^{\mathrm{i}\langle\mu,X\rangle}$. We fix a positive Weyl chamber $\mathfrak{t}^*_+\subset\mathfrak{t}^*$. For any dominant weight $\mu\in\widehat{K} := \wedge^*\cap\mathfrak{t}^*_+$, we denote by V_μ^K the irreducible representation with highest weight μ. We denote $R(K)$ the representation ring of K. We denote $R^{-\infty}(K) := \hom_{\mathbb{Z}}(R(K),\mathbb{Z})$ its dual. An element $E\in R^{-\infty}(K)$ can be represented as an infinite sum $E = \sum_{\mu\in\widehat{K}}\mathrm{m}_\mu V_\mu^K$, with $\mathrm{m}_\mu\in\mathbb{Z}$. The multiplicity* m_0 *of the trivial representation is denoted $[E]^K$. If H is a closed subgroup of K, we have the induction map $\mathrm{Ind}_H^K : R^{-\infty}(H)\to R^{-\infty}(K)$ which is the dual of the restriction morphism $R(K)\to R(H)$. We see that $[\mathrm{Ind}_H^K(E)]^K = [E]^H$.*

When K acts on a set X, the stabilizer subgroup of an element $x\in X$ is denoted $K_x := \{k\in K \mid k\cdot x = x\}$. The Lie algebra of K_x is denoted $\mathfrak{k}_x$.

2. Spin-quantization of compact Hamiltonian K-manifolds

Let M be a *compact* Hamiltonian K-manifold with symplectic form ω and moment map $\Phi : M \to \mathfrak{k}^*$ characterized by the relation

$$\iota(X_M)\omega = -d\langle\Phi, X\rangle, \quad X \in \mathfrak{k}, \tag{2.1}$$

where $X_M(m) := \frac{d}{dt}|_{t=0} \mathrm{e}^{-tX} \cdot m$ is the vector field on M generated by $X \in \mathfrak{k}$.

In the Kostant–Souriau framework [**15, 25**], a Hermitian line bundle L_ω with an invariant Hermitian connection ∇ is a prequantum line bundle over (M, ω, Φ) if

$$\mathcal{L}(X) - \nabla_{X_M} = \mathrm{i}\langle\Phi, X\rangle \quad \text{and} \quad \nabla^2 = -\mathrm{i}\omega, \tag{2.2}$$

for every $X \in \mathfrak{k}$. Here $\mathcal{L}(X)$ is the infinitesimal action of $X \in \mathfrak{k}$ on the sections of $L_\omega \to M$. (L_ω, ∇) is also called a Kostant–Souriau line bundle. Remark that conditions (2.2) imply, via the equivariant Bianchi formula, the relation (2.1).

2.1. Spin-quantization: definitions. Let N be a compact even dimensional Riemannian manifold, and let $\mathrm{Cl}(N)$ be its Clifford bundle. A Spin^c structure P on N defines an irreducible Clifford bundle $\mathcal{S}_P \to N$ [**6,** Section 3.3]. If P and P' are two Spin^c structures on N, then we have $\mathcal{S}_{P'} \simeq \mathcal{S}_P \otimes \mathbb{L}_{P,P'}$ where $\mathbb{L}_{P,P'}$ is a line bundle on N defined by the relation

$$\mathbb{L}_{P,P'} := \hom_{\mathrm{Cl}(N)}(\mathcal{S}_P, \mathcal{S}_{P'}). \tag{2.3}$$

If $\mathcal{S} \to N$ is an irreducible Clifford bundle, then its complex dual $\mathcal{S}^* \to N$ is also an irreducible Clifford bundle.

Definition 2.1. The determinant line bundle of a Spin^c structure P on N is the line bundle $\det(P) \to N$ defined by the relation $\det(P) := \hom_{\mathrm{Cl}(N)}(\mathcal{S}_P^*, \mathcal{S}_P)$.

If P and P' are two Spin^c structures on N, we see that

$$\begin{aligned}\det(P') &\simeq \hom_{\mathrm{Cl}(N)}(\mathcal{S}_P^* \otimes \mathbb{L}_{P,P'}^{-1}, \mathcal{S}_P \otimes \mathbb{L}_{P,P'})\\ &\simeq \det(P) \otimes (\mathbb{L}_{P,P'})^2.\end{aligned}$$

On the other hand, we can twist a Spin^c structure P on N by a complex line bundle $\mathbb{L} \to N$: its defines another Spin^c structure $P_{\mathbb{L}}$ such that $\mathcal{S}_{P_{\mathbb{L}}} = \mathcal{S}_P \otimes \mathbb{L}$.

Let us come back to the situation of a K-Hamiltonian manifold (M, ω, Φ). Let J be any invariant almost complex structure on M, not necessarily compatible with the symplectic form ω. Let

$$RR_J^K(M, -)$$

be the corresponding Riemann–Roch character [**21**]. We consider the complex tangent bundle $(\mathrm{T}M, J)$ and its complex dual $\mathrm{T}^*_{\mathbb{C}}M := \hom_{\mathbb{C}}(\mathrm{T}M, \mathbb{C})$. We consider the complex line bundle

$$\kappa_J := \det \mathrm{T}^*_{\mathbb{C}}M.$$

If (M, ω, Φ) is prequantized by L_ω, a standard procedure (called the metaplectic correction in the geometric quantization literature) is to tensor L_ω by the bundle of half-forms $\kappa_J^{1/2}$ [**30**]. We may consider the equivariant index

$$\mathcal{Q}^K_J(M) := \epsilon_J RR^K_J(M, L_\omega \otimes \kappa_J^{1/2}), \tag{2.4}$$

where $\epsilon_J = \pm 1$ is the quotient of the orientations defined by ω and by J. In Proposition 2.3 we check that $\mathcal{Q}^K_J(M)$ has a meaning when the tensor product $\tilde{L} = L_\omega \otimes \kappa_J^{1/2}$ is well defined (even if neither L_ω nor $\kappa_J^{1/2}$ exist).

The almost complex structure J defines a Spin^c structure P_J on M with determinant line bundle $\det P_J = \kappa_J^{-1}$ (see [**16, 22**]). If we twist the Spin^c structure P_J by any complex line bundle $\mathbb{L}$ we get a Spin^c structure $P_{J,\mathbb{L}}$ with determinant line bundle

$$\det(P_{J,\mathbb{L}}) = \kappa_J^{-1} \otimes \mathbb{L}^2.$$

We make the following basic observation.

Proposition 2.2. *Let (M, ω, Φ) be a Hamiltonian K-manifold. The following assertions are equivalent:*

(a) *For any invariant complex structure J there exists a K-equivariant line bundle $\tilde{L}$ such that $\kappa_J^{-1} \otimes \tilde{L}^2$ is a prequantum line bundle over $(M, 2\omega, 2\Phi)$.*
(b) *There exist an invariant complex structure J and a K-equivariant line bundle $\tilde{L}$ such that $\kappa_J^{-1} \otimes \tilde{L}^2$ is a prequantum line bundle over $(M, 2\omega, 2\Phi)$.*
(c) *There exists an equivariant* Spin^c *structure P such that its determinant line bundle* $\det(P)$ *is a prequantum line bundle over $(M, 2\omega, 2\Phi)$.*

When the previous assertions holds, we says that (M, ω, Φ) is Spin-prequantized, either by the Spin^c-structure P, or by the data $(J, \tilde{L})$.

Proposition 2.3. *Let (M, ω, Φ) be a* Spin-*prequantized Hamiltonian K-manifold. The equivariant index $\mathcal{Q}^K_J(M) := \epsilon_J RR^K_J(M, \tilde{L})$ does not depend of the choice of the* Spin-*prequantum data $(J, \tilde{L})$. In fact $\mathcal{Q}^K_J(M)$ coincides with the equivariant index of the* Spin^c *Dirac operator $\mathcal{D}_P$ attached to the* Spin^c-*structure P.*

Definition 2.4. Let (M, ω, Φ) be a Spin-prequantized Hamiltonian K-manifold. The Spin-quantization of (M, ω, Φ) is defined as the equivariant index $\mathcal{Q}^K_J(M)$, and is denoted

$$\mathcal{Q}^K_{\mathrm{spin}}(M) \in R(K).$$

Proof of Propositions 2.2 *and* 2.3. We have obviously $a) \Longrightarrow b)$, and we get $b) \Longrightarrow c)$ by taking the Spin^c structure $P_{J,\tilde{L}}$. Let us prove $c) \Longrightarrow a)$.

Let P be a Spin^c-structure on M such that its determinant line bundle $\det(P)$ is a prequantum line bundle over $(M, 2\omega)$. Let $\mathcal{S}_P$ be the corresponding bundle of spinors. Let P_J and $\mathcal{S}_J$ be respectively the associated Spin^c-structure and the bundle of spinors on M associated to an invariant almost complex structure J on M. Since $\mathcal{S}_P, \mathcal{S}_J$ are irreducible clifford modules, we have

$$\mathcal{S}_P \simeq \mathcal{S}_J \otimes \tilde{L}, \tag{2.5}$$

where $\tilde{L}$ is the line bundle defined by $\tilde{L} := \hom_{\mathrm{Cl}(M)}(\mathcal{S}_J, \mathcal{S}_P)$. From (2.5) we get that the line bundle

$$\begin{aligned}\det(P) &\simeq \det(P_J) \otimes \tilde{L}^2 \\ &\simeq \kappa_J^{-1} \otimes \tilde{L}^2\end{aligned}$$

is a prequantum line bundle over $(M, 2\omega, 2\Phi)$.

Let P be the Spin^c structure attached to a data $(J, \tilde{L})$. The symplectic orientation on M defines a decomposition on the bundle of spinors, $\mathcal{S}_P = \mathcal{S}_P^+ \oplus \mathcal{S}_P^-$, and the corresponding Spin^c Dirac operator $\mathcal{D}_P$ maps $\Gamma(\mathcal{S}_P^+)$ to $\Gamma(\mathcal{S}_P^-)$.

On the other hand, the almost complex structure on M gives the decomposition $\wedge \mathrm{T}^*M \otimes \mathbb{C} = \oplus_{i,j} \wedge^{i,j} \mathrm{T}^*M$ of the bundle of differential form. The corresponding bundle of spinors is $\mathcal{S}_J := \wedge^{0,\bullet} \mathrm{T}^*M$ and the complex orientation induces the splitting $\mathcal{S}_J = \mathcal{S}_J^+ \oplus \mathcal{S}_J^-$ with $\mathcal{S}_J^+ := \wedge^{0,\mathrm{even}} \mathrm{T}^*M$. The Dolbeault Dirac operator $\overline{\partial}_{\tilde{L}} + \overline{\partial}_{\tilde{L}}^*$ maps $\Gamma(\mathcal{S}_J^{\pm} \otimes \tilde{L})$ to $\Gamma(\mathcal{S}_J^{\mp} \otimes \tilde{L})$, and the Riemann–Roch character $RR_J^K(M, \tilde{L})$ is defined as the equivariant index of the elliptic operator

$$\overline{\partial}_{\tilde{L}} + \overline{\partial}_{\tilde{L}}^* : \Gamma(\mathcal{S}_J^+ \otimes \tilde{L}) \longrightarrow \Gamma(\mathcal{S}_J^- \otimes \tilde{L}).$$

If $\epsilon_J = \pm 1$ is the quotient of the orientations defined by ω and by J, one has that

$$\mathcal{S}_P^{\pm} = \mathcal{S}_J^{\pm \epsilon_J} \otimes \tilde{L}.$$

Hence $\mathcal{Q}_J^K(M) = \epsilon_J RR_J^K(M, \tilde{L})$ is defined as the equivariant index of the Dolbeault Dirac operator $\overline{\partial}_{\tilde{L}} + \overline{\partial}_{\tilde{L}}^*$ viewed as an elliptic operator $\mathcal{D}_{\tilde{L}}^+$ from $\Gamma(\mathcal{S}_P^+)$ to $\Gamma(\mathcal{S}_P^-)$.

Finally, we know that $\mathrm{Index}^K(\mathcal{D}_P) = \mathrm{Index}^K(\mathcal{D}_{\tilde{L}}^+)$ since the first-order elliptic operators $\mathcal{D}_P$ and $\mathcal{D}_{\tilde{L}}^+$ have the same principal symbol [**10**]. □

In the remaining part of this paper, we find convenient to work with the following

Definition 2.5. A Hamiltonian K-manifold (M,ω,Φ) is Spin-prequantized by $\tilde{L}$ if there exists an invariant almost complex structure J *compatible* with ω such that $\tilde{L}^2\otimes\kappa_J^{-1}$ is a Kostant–Souriau line bundle over $(M,2\omega,2\Phi)$.

We remark that $\varepsilon_J=1$ when J is *compatible* with ω. Moreover, the Riemann–Roch character $RR_J^K(M,-)$ does not depend [**21**] on the choice of the *compatible* invariant almost complex structure J : we denote it simply by $RR^K(M,-)$.

Finally, when a Hamiltonian manifold (M,ω,Φ) is Spin-prequantized by the line bundle $\tilde{L}$, its Spin-quantization is defined by

$$\mathcal{Q}^K_{\mathrm{spin}}(M):=RR^K(M,\tilde{L}).$$

2.2. Functorial properties. We summarize the functorial properties of $\mathcal{Q}_{\mathrm{spin}}$ in the next

Proposition 2.6.
- *If (M,ω,Φ) is a Spin-prequantized Hamiltonian K-manifold, and H is a closed subgroup of K then the restriction of $\mathcal{Q}^K_{\mathrm{spin}}(M)$ to H is equal to $\mathcal{Q}^H_{\mathrm{spin}}(M)$.*
- *If (M_j,ω_j,Φ_j) are Spin-prequantized Hamiltonian K_j-manifold, for $j=1,2$, then $M_1\times M_2$ is a Spin-prequantized Hamiltonian $K_1\times K_2$-manifold and*

$$\mathcal{Q}^{K_1\times K_2}_{\mathrm{spin}}(M_1\times M_2)=\mathcal{Q}^{K_1}_{\mathrm{spin}}(M_1)\otimes\mathcal{Q}^{K_2}_{\mathrm{spin}}(M_2)$$

  in $R(K_1\times K_2)\simeq R(K_1)\otimes R(K_2)$.
- *If (M,ω_M,Φ_M) and (N,ω_N,Φ_N) are Spin-prequantized Hamiltonian K-manifold, then $M\times N$ is a Spin-prequantized Hamiltonian K-manifold and*

$$\mathcal{Q}^K_{\mathrm{spin}}(M\times N)=\mathcal{Q}^K_{\mathrm{spin}}(M)\cdot\mathcal{Q}^K_{\mathrm{spin}}(N),$$

  where $\cdot$ denotes the product in $R(K)$.
- *A Spin-prequantization on (M,ω,Φ) induces a Spin-prequantization on $\overline{M}:=(M,-\omega,-\Phi)$. The Spin-quantization of $\overline{M}$ corresponds to the dual of the Spin-quantization of M:*

$$\mathcal{Q}^K_{\mathrm{spin}}(\overline{M})=\left[\mathcal{Q}^K_{\mathrm{spin}}(M)\right]^*.$$

Proof. The first three points are direct consequences of the functorial properties of the index map. Let us prove the last point. One see that if $(\tilde{L},J)$ is a Spin-prequantum data for M then $(\tilde{L}^{-1},-J)$ is a Spin-prequantum data for $\overline{M}$. Then we have for $X \in \mathfrak{k}$ small enough

$$\begin{aligned}\mathcal{Q}^K_{\text{spin}}(\overline{M})(\mathrm{e}^X) &= \left(\frac{\mathrm{i}}{2\pi}\right)^{\frac{\dim M}{2}} \int_{\overline{M}} \mathrm{e}^{\mathrm{i}(-\omega-\langle\Phi,X\rangle)}\widehat{A}(M)(X) \\ &= \overline{\left(\frac{\mathrm{i}}{2\pi}\right)^{\frac{\dim M}{2}} \int_{M} \mathrm{e}^{\mathrm{i}(\omega+\langle\Phi,X\rangle)}\widehat{A}(M)(X)} \qquad [1] \\ &= \overline{\mathcal{Q}^K_{\text{spin}}(M)(\mathrm{e}^X)}. \qquad [2]\end{aligned}$$

The relation [1] is due to the fact that the differential form $\widehat{A}(M)(X)$ has real coefficients, and that the quotient of the symplectic orientations on M and $\overline{M}$ is $(-1)^{\frac{\dim M}{2}}$. Since $X \to \mathcal{Q}^K_{\text{spin}}(M)(\mathrm{e}^X)$ are analytic functions, the identity [2] shows that $\mathcal{Q}^K_{\text{spin}}(\overline{M})(k) = \overline{\mathcal{Q}^K_{\text{spin}}(M)(k)}$ for any $k \in K$. In other words the (virtual) representation $\mathcal{Q}^K_{\text{spin}}(\overline{M})$ corresponds to the dual of the (virtual) representation $\mathcal{Q}^K_{\text{spin}}(M)$. □

2.3. Spin-quantization of coadjoint orbits. Let $\mu \in \widehat{K}$ be a dominant weight. Let us denote K_μ its stabilizer subgroup and $\mathfrak{k}_\mu$ its Lie algebra. Let us recall why the Lie algebra morphism $\mathrm{i}\mu : \mathfrak{k}_\mu \to \mathrm{i}\mathbb{R}$ integrates in a character χ_μ of K_μ. The group K_μ, which is connected, decomposes as $K_\mu = [K_\mu, K_\mu]Z_\mu$ where Z_μ is the connected component of the center of K_μ. For the maximal torus T, we have $T = T_\mu Z_\mu$ with $T_\mu = T \cap [K_\mu, K_\mu] = \exp(\mathfrak{t} \cap [\mathfrak{k}_\mu, \mathfrak{k}_\mu])$. We note that $\mathrm{i}\mu : \mathfrak{t} \to \mathrm{i}\mathbb{R}$ integrates in a character χ^T_μ of T which is trivial on T_μ since $\langle \mu, [\mathfrak{k}_\mu, \mathfrak{k}_\mu]\rangle = 0$. Hence, we can define the character χ_μ as being trivial on $[K_\mu, K_\mu]$, and equal to χ^T_μ on Z_μ.

We denote by $\mathbb{C}_\mu$ the one-dimensional representation of K_μ associated to the character χ_μ. Let σ be a face of the Weyl chamber such that $\mu \in \overline{\sigma}$: hence the stabilizer subgroup K_μ contains K_σ. We still denote by $\mathbb{C}_\mu$ the induced one-dimensional representation of the group K_σ.

Let ρ be half the sum of the positive roots, and let ρ_σ be half the sum of the positive roots, which are orthogonal to σ. Note that $\rho - \rho_\sigma$ belongs to σ, hence $\mu + \rho - \rho_\sigma$ belongs also to σ for any $\mu \in \overline{\sigma}$. The coadjoint orbit

$$\mathcal{O}^\sigma_\mu := K \cdot (\mu + \rho - \rho_\sigma) \simeq K/K_\sigma$$

is Spin-prequantized by the compatible complex structure and the line bundle $\tilde{L} = K \times_{K_\sigma} \mathbb{C}_\mu$. We have

$$\begin{aligned}\mathcal{Q}^K_{\text{spin}}(\mathcal{O}^\sigma_\mu) &= RR^K(K/K_\sigma, K \times_{K_\sigma} \mathbb{C}_\mu) \\ &= V^K_\mu\end{aligned}$$

thanks to the Borel–Weil theorem. We know also that $\mathcal{Q}^K_{\text{spin}}\left(\overline{\mathcal{O}^\sigma_\mu}\right) = \left(V^K_\mu\right)^*$, where $\overline{\mathcal{O}^\sigma_\mu}$ be the coadjoint orbit $\mathcal{O}^\sigma_\mu$ with the opposite symplectic form (see Proposition 2.6).

We have seen that the same irreducible representations V_μ^K can be realized as the Spin-quantization of the coadjoint orbits $\mathcal{O}_\mu^\sigma$ where σ is a face of the Weyl chamber containing μ in its closure.

2.4. Spin-prequantization commutes with reduction. We consider first the case of a Hamiltonian H-manifold (N,ω,Φ), not necessarily compact, which is Spin-prequantized by $\tilde{L}$. We suppose that 0 is a regular value of Φ. Let $N_0 := \Phi^{-1}(0)/H$ be the orbifold reduced space with its canonical symplectic structure ω_0.

Lemma 2.7. *The orbifold line bundle* $\tilde{\mathcal{L}}_0 := (\tilde{L}|_{\Phi^{-1}(0)})/H$ *Spin-prequantizes* (N_0,ω_0).

Proof. The fiber $\mathcal{Z} = \Phi^{-1}(0)$ is a smooth H-invariant submanifold of N. Let $\pi : \mathcal{Z} \to \mathcal{Z}/H = N_0$ be the projection. Recall that the symplectic structure ω_0 on N_0 is defined by the relation $\pi^*(\omega_0) = \omega|_{\mathcal{Z}}$. Let $L_{2\omega}$ the Kostant–Souriau line bundle on $(N, 2\omega, 2\Phi)$ such that

$$\tilde{L}^2 = L_{2\omega} \otimes \kappa_J. \tag{2.6}$$

Here J is a compatible invariant almost complex structure on N. We have $\mathrm{T}N|_{\mathcal{Z}} = \mathrm{T}\mathcal{Z} \oplus J(\mathfrak{h}_{\mathcal{Z}})$ where $\mathfrak{h}_{\mathcal{Z}} \subset \mathrm{T}\mathcal{Z}$ is the trivial bundle given by the infinitesimal action of H. Since $\mathrm{T}\mathcal{Z} \simeq \pi^*(\mathrm{T}N_0) \oplus \mathfrak{h}_{\mathcal{Z}}$ we get

$$\mathrm{T}N|_{\mathcal{Z}} \simeq \pi^*(\mathrm{T}N_0) \oplus \mathfrak{h}_{\mathcal{Z}} \oplus J(\mathfrak{h}_{\mathcal{Z}}).$$

Hence J induces a compatible almost complex structure J_0 on (N_0,ω_0), such that $(\kappa_J|_{\mathcal{Z}})/H = \kappa_{J_0}$.

The line bundle $L_{2\omega_0} = (L_{2\omega}|_{\mathcal{Z}})/H$ is a prequantum line bundle on (N_0,ω_0). Finally, if we restrict (2.6) to $\mathcal{Z}$, we get

$$\tilde{\mathcal{L}}_0^2 = L_{2\omega_0} \otimes \kappa_{J_0}.$$

after taking the quotient by H. We have proved that $(J_0, \tilde{\mathcal{L}}_0)$ Spin-prequantizes (N_0,ω_0). □

For the rest of this section we consider a compact Hamiltonian K-manifold (M,ω,Φ), that we suppose Spin-prequantized by the line bundle $\tilde{L}$.

Let τ be a face of the Weyl chamber, and let K_τ be the common stabilizer of points in τ. Following Guillemin–Sternberg [**13**], we introduce the following K_τ-invariant open subset of $\mathfrak{k}_\tau^*$:

$$U_\tau = K_\tau \cdot \{\xi \in \mathfrak{t}_+^* | K_\xi \subset K_\tau\} = K_\tau \cdot \bigcup_{\tau \subset \overline{\sigma}} \sigma.$$

By construction, U_τ is a slice for the coadjoint action: this mean that the map $K \times U_\tau, (k,\xi) \mapsto k\cdot\xi$ factors through an inclusion $K \times_{K_\tau} U_\tau \hookrightarrow \mathfrak{k}^*$.

The symplectic cross-section theorem [**13**] asserts that the pre-image $Y_\tau = \Phi^{-1}(U_\tau)$ is a symplectic submanifold : we denote ω_τ the restriction of ω to Y_τ. The action of K_τ on (Y_τ,ω_τ) is Hamiltonian, where the restriction of Φ

to Y_τ is a moment map. Since $\rho - \rho_\tau$ is a K_τ-invariant element, we can use the translated moment map $\Phi_\tau : Y_\tau \to \mathfrak{k}_\tau^*$ defined by

$$\Phi_\tau = \Phi|_{Y_\tau} - (\rho - \rho_\tau).$$

Lemma 2.8. *The symplectic slice* $(Y_\tau, \omega_\tau, \Phi_\tau)$ *is Spin-prequantized by the line bundle* $\tilde{L}_\tau := \tilde{L}|_{Y_\tau}$.

Proof. We consider the open subset $K \times_{K_\tau} Y_\tau$ of M and the projection $\pi : K \times_{K_\tau} Y_\tau \to K/K_\tau$. We can suppose that the Spin-prequantum data, when restricted to $K \times_{K_\tau} Y_\tau$, is given by $(J, \tilde{L})$ where J is a compatible almost complex structure on $K \times_{K_\tau} Y_\tau$ defined as the "sum" of the compatible almost complex structures J_o and J_τ: J_o on K/K_τ and J_τ on Y_τ. Hence on $K \times_{K_\tau} Y_\tau$ we have

$$\kappa_J = K \times_{K_\tau} (\kappa_{J_\tau}) \otimes \pi^{-1}(\kappa_{J_o})$$

with $\kappa_{J_o} = K \times_{K_\tau} \mathbb{C}_{-2(\rho-\rho_\tau)}$. We see then that the restriction of the K-equivariant line bundle κ_J to the symplectic slice Y_τ is equal to the K_τ-equivariant line bundle $\kappa_{J_\tau} \otimes \mathbb{C}_{-2(\rho-\rho_\tau)}$.

When we restrict the identity $L_{2\omega} = \tilde{L}^2 \otimes \kappa_J^{-1}$ to Y_τ we get

$$L_{2\omega}|_{Y_\tau} = (\tilde{L}|_{Y_\tau})^2 \otimes \kappa_{J_\tau}^{-1} \otimes \mathbb{C}_{2(\rho-\rho_\tau)}. \tag{2.7}$$

We consider the following line bundle on Y_τ:

$$L_{2\omega_\tau} := L_{2\omega}|_{Y_\tau} \otimes \mathbb{C}_{-2(\rho-\rho_\tau)}.$$

The relation (2.7) is then $L_{2\omega_\tau} = (\tilde{L}|_{Y_\tau})^2 \otimes \kappa_{J_\tau}^{-1}$. Since $L_{2\omega_\tau}$ is a K_τ-equivariant prequantum bundle over $(Y_\tau, 2\omega_\tau, 2\Phi_\tau)$, we conclude that $(Y_\tau, \omega_\tau, \Phi_\tau)$ is Spin-prequantized by the data $(J_\tau, \tilde{L}|_{Y_\tau})$. □

Let us consider the case where $\tau = \sigma$ is *the smallest face* of the Weyl chamber so that moment polyhedron $\Delta(M) := \Phi(M) \cap \mathfrak{t}_+^*$ is contained in the closure of σ.

Then the symplectic slice Y_σ is equal to $\Phi^{-1}(\sigma)$, and the action of the subgroup $[K_\sigma, K_\sigma]$ is trivial on it [**17**]. Let Z_σ be the identity component of the center of K_σ. The map $\Phi_\sigma : Y_\sigma \to \mathfrak{k}_\sigma^*$ takes values in $\mathfrak{z}_\sigma^* = \mathbb{R}\sigma \subset \mathfrak{t}^*$ and corresponds to the moment map relative to the action of Z_σ on $(Y_\sigma, \omega_\sigma)$. We know after Lemma 2.8 that $(Y_\sigma, \omega_\sigma, \Phi_\sigma)$ is Spin-prequantized by $\tilde{L}_\sigma := \tilde{L}|_{Y_\sigma}$.

For each dominants weights μ which belongs to the closure of σ, we consider the symplectic reduction

$$\begin{aligned} M_\mu^\sigma &= \Phi^{-1}(\mathcal{O}_\mu^\sigma)/K \\ &= \Phi_\sigma^{-1}(\mu)/Z_\sigma. \end{aligned}$$

For the rest of this section we fix a dominant weight $\mu \in \overline{\sigma}$ *such that* $\mu + \rho - \rho_\sigma \in \Delta(M)$, *and we explain how one defines the Spin-quantization of the (possibly singular) reduced spaces* M_μ^σ.

Let $\overrightarrow{\Delta} \subset \mathfrak{z}_\sigma^*$ be the rational vector subspace generated by $\{a - b \ |a, b \in \Delta(M)\}$. Let $\mathfrak{z}_\sigma^\Delta \subset \mathfrak{z}_\sigma$ be the subspace orthogonal to $\overrightarrow{\Delta}$, and let $Z_\sigma^\Delta \subset Z_\sigma$ be the corresponding subtorus.

Lemma 2.9. *The torus Z_σ^Δ acts trivially on Y_σ and on the line bundle $\tilde{L}_\sigma \otimes \mathbb{C}_{-\mu}$.*

Proof. By definition of $\mathfrak{z}_\sigma^\Delta$, $0 = d\langle \Phi_\sigma, X\rangle = -\iota(X_{Y_\sigma})\omega_\sigma$ on Y_σ for any $X \in \mathfrak{z}_\sigma^\Delta$. Hence the torus Z_σ^Δ acts trivially on Y_σ. Let $L_{2\omega_\sigma}$ be the Kostant–Souriau line bundle over $(Y_\sigma, 2\omega_\sigma, 2\Phi_\sigma)$ so that $\tilde{L}_\sigma^2 = L_{2\omega_\sigma} \otimes \kappa_{J_\sigma}$ (see Lemma 2.8). We have on the section of $L_{2\omega_\sigma}$ the following equality of linear operators:

$$\mathcal{L}(X) - \nabla_{X_M} = \mathrm{i}\langle 2\Phi_\sigma, X\rangle, \quad \forall X \in \mathfrak{z}_\sigma.$$

If one takes $X \in \mathfrak{z}_\sigma^\Delta$, the function $y \in Y_\sigma \mapsto \langle \Phi_\sigma(y), X\rangle$ is constant equal to $\langle \mu, X\rangle$. Finally

$$\mathcal{L}(X) - 2\mathrm{i}\langle \mu, X\rangle = 0, \quad \forall X \in \mathfrak{z}_\sigma^\Delta$$

as an operator on the section of $L_{2\omega_\sigma}$. In other words, the torus Z_σ^Δ acts trivially on $L_{2\omega_\sigma} \otimes \mathbb{C}_{-2\mu} = (\tilde{L}_\sigma \otimes \mathbb{C}_{-\mu})^2 \otimes \kappa_{J_\sigma}^{-1}$. Since Z_σ^Δ acts trivially on κ_{J_σ}, we conclude finally that Z_σ^Δ acts trivially on the line bundle $\tilde{L}_\sigma \otimes \mathbb{C}_{-\mu}$. □

Let $Z'_\sigma \subset Z_\sigma^\Delta$ be another subtorus such that $Z_\sigma = Z_\sigma^\Delta \times Z'_\sigma$: the dual of its Lie algebra $\mathfrak{z}'_\sigma$ is identified with $\overrightarrow{\Delta} \subset \mathfrak{z}_\sigma^*$. We look now at $(Y_\sigma, \omega_\sigma)$ as a Hamiltonian Z'_σ-manifold with moment map

$$\Phi'_\sigma := \Phi_\sigma - \mu = \Phi|_{Y_\sigma} - (\mu + \rho - \rho_\sigma).$$

The Z'_σ-equivariant line bundle $\tilde{L}'_\sigma := \tilde{L}_\sigma \otimes \mathbb{C}_{-\mu}$ Spin-prequantizes the Hamiltonian Z'_σ-manifold $(Y_\sigma, \omega_\sigma, \Phi'_\sigma)$.

If $0 \in \overrightarrow{\Delta}$ is a regular value of Φ'_σ, we know after Lemma 2.7 that the orbifold reduced space $(M_\mu^\sigma, \omega_\mu^\sigma)$ is Spin-prequantized by the line bundle

$$\tilde{\mathcal{L}}_\mu^\sigma := \left(\tilde{L}|_{\Phi_\sigma^{-1}(\mu)} \otimes \mathbb{C}_{-\mu}\right)/Z'_\sigma,$$

and its Spin-quantization $\mathcal{Q}_{\mathrm{spin}}(M_\mu^\sigma)$ is defined like in Definition 2.4. In the general case where $0 \in \overrightarrow{\Delta}$ is not necessarily a regular value of Φ'_σ we proceed by shift desingularization. For $\varepsilon \in \overrightarrow{\Delta}$ small enough and generic we consider the orbifold reduced space

$$M_{\mu+\varepsilon}^\sigma := (\Phi'_\sigma)^{-1}(\varepsilon)/Z'_\sigma = \Phi_\sigma^{-1}(\mu + \varepsilon)/Z'_\sigma$$

and its orbifold line bundle

$$\tilde{\mathcal{L}}_{\mu+\varepsilon}^\sigma := \left(\tilde{L}|_{\Phi_\sigma^{-1}(\mu+\varepsilon)} \otimes \mathbb{C}_{-\mu}\right)/Z'_\sigma.$$

The following crucial fact is proved in Section 3.4.

Theorem 2.10. *The Riemann–Roch number* $RR(M^\sigma_{\mu+\varepsilon}, \tilde{\mathcal{L}}^\sigma_{\mu+\varepsilon}) \in \mathbb{Z}$ *does not depend of the choice of a generic and small enough* $\varepsilon \in \overrightarrow{\Delta}$.

Thanks to the last Theorem we can define the quantization $\mathcal{Q}_{\rm spin}(M^\sigma_\mu) \in \mathbb{Z}$ of the (possibly singular) reduced space M^σ_μ for $\mu \in \widehat{K} \cap \overline{\sigma}$.

Definition 2.11. Let $\mu \in \widehat{K} \cap \overline{\sigma}$.

- If $\mu + \rho - \rho_\sigma \in \Delta(M)$, the integer $\mathcal{Q}_{\rm spin}(M^\sigma_\mu) \in \mathbb{Z}$ is defined as the Riemann–Roch character $RR(M^\sigma_{\mu+\varepsilon}, \tilde{\mathcal{L}}^\sigma_{\mu+\varepsilon})$ for $\varepsilon \in \overrightarrow{\Delta}$ generic and small enough.
- If $\mu + \rho - \rho_\sigma \notin \Delta(M)$, we set $\mathcal{Q}_{\rm spin}(M^\sigma_\mu) = 0$

Remark 2.12. If $\mu + \rho - \rho_\sigma$ does not belongs to the relative interior of $\Delta(M)$, we can choose ε so that $\mu + \rho - \rho_\sigma + \varepsilon \notin \Delta(M)$. Then the reduced space $M^\sigma_{\mu+\varepsilon}$ is empty and the corresponding Riemann–Roch character $RR(M^\sigma_{\mu+\varepsilon}, \tilde{\mathcal{L}}^\sigma_{\mu+\varepsilon})$ vanishes. Hence $\mathcal{Q}_{\rm spin}(M^\sigma_\mu) = 0$.

3. Spin-quantization commutes with reduction

Let (M, ω, Φ) be a compact Hamiltonian K-manifold which is Spin prequantized. We are looking to a geometric interpretation of the multiplicity, denoted m_μ, of the representation V^K_μ into $\mathcal{Q}^K_{\rm spin}(M)$.

The main result of this paper is the following.

Theorem 3.1. *Let* σ *be the smallest face of the Weyl chamber so that* $\Phi(M) \cap \mathfrak{t}^*_+ \subset \overline{\sigma}$. *For* $\mu \in \widehat{K}$, *we have*

$$\mathrm{m}_\mu = \begin{cases} 0 & \text{if} \quad \mu \notin \overline{\sigma}; \\ \mathcal{Q}_{\rm spin}(M^\sigma_\mu) & \text{if} \quad \mu \in \overline{\sigma}. \end{cases}$$

In this section, we introduce the main tools needed for the proof of Theorem 3.1.

In Section 3.1, we recall the notion of *transversally elliptic symbols*.

In Section 3.2, we recall the Witten's way of localization the Riemann–Roch character [**21**]. We recall in Proposition 3.8, the criterium observed in [**22**] for the vanishing of the invariant part of the localized Riemann–Roch character.

In Section 3.3, we recall an induction formula proved in [**21, 22**] for the localized Riemann–Roch character.

In Section 3.4, we prove Theorem 3.1 when K is a torus[1]. We give by the same way a proof of Theorem 2.10 which is essential to the definition of the Spin-quantization of the (possibly singular) reduced spaces M^σ_μ.

[1]This situation was already handled in [**22**].

3.1. Elliptic and transversally elliptic symbols. Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah–Singer in [**1**]. For an axiomatic treatment of the index morphism see Berline–Vergne [**7, 8**] and Paradan–Vergne [**23**]. For a short introduction see [**21**].

Let $\mathcal{X}$ be a *compact* K-manifold. Let $p : \mathrm{T}\mathcal{X} \to \mathcal{X}$ be the projection, and let $(-,-)_{\mathcal{X}}$ be a K-invariant Riemannian metric. If E^0, E^1 are K-equivariant complex vector bundles over $\mathcal{X}$, a K-equivariant morphism $h \in \Gamma(\mathrm{T}\mathcal{X}, \hom(p^*E^0, p^*E^1))$ is called a *symbol* on $\mathcal{X}$. The subset of all $(x,v) \in \mathrm{T}\mathcal{X}$ where[2] $h(x,v) : E^0_x \to E^1_x$ is not invertible is called the *characteristic set* of h, and is denoted by $\mathrm{Char}(h)$.

In the following, the "product" of a symbol h by a complex vector bundle $F \to \mathcal{X}$, is the symbol

$$h \otimes F$$

defined by $h \otimes F(x,v) = h(x,v) \otimes \mathrm{Id}_{F_x}$ from $E^0_x \otimes F_x$ to $E^1_x \otimes F_x$. Note that $\mathrm{Char}(h \otimes F) = \mathrm{Char}(h)$.

Let $\mathrm{T}_K\mathcal{X}$ be the following subset of $\mathrm{T}\mathcal{X}$:

$$\mathrm{T}_K\mathcal{X} = \{(x,v) \in \mathrm{T}\mathcal{X},\ (v, X_{\mathcal{X}}(x))_{\mathcal{X}} = 0, \quad \text{for all } X \in \mathfrak{k}\}.$$

A symbol h is *elliptic* if h is invertible outside a compact subset of $\mathrm{T}\mathcal{X}$ (i.e., $\mathrm{Char}(h)$ is compact), and is *K-transversally elliptic* if the restriction of h to $\mathrm{T}_K\mathcal{X}$ is invertible outside a compact subset of $\mathrm{T}_K\mathcal{X}$ (i.e., $\mathrm{Char}(h) \cap \mathrm{T}_K\mathcal{X}$ is compact). An elliptic symbol h defines an element in the equivariant $\mathbf{K}$-theory of $\mathrm{T}\mathcal{X}$ with compact support, which is denoted by $\mathbf{K}_K(\mathrm{T}\mathcal{X})$, and the index of h is a virtual finite-dimensional representation of K, that we denote $\mathrm{Index}^K_{\mathcal{X}}(h) \in R(K)$ [**2–5**].

Let

$$R^{-\infty}_{\mathrm{tc}}(K) \subset R^{-\infty}(K)$$

be the $R(K)$-submodule formed by all the infinite sum $\sum_{\mu \in \widehat{K}} m_\mu V^K_\mu$ where the map $\mu \in \widehat{K} \mapsto m_\mu \in \mathbb{Z}$ has at most a *polynomial* growth. The $R(K)$-module $R^{-\infty}_{\mathrm{tc}}(K)$ is the Grothendieck group associated to the *trace class* virtual K-representations: we can associate to any $V \in R^{-\infty}_{\mathrm{tc}}(K)$, its trace $k \to \mathrm{Tr}(k, V)$ which is a generalized function on K invariant by conjugation. Then the trace defines a morphism of $R(K)$-module

$$R^{-\infty}_{\mathrm{tc}}(K) \hookrightarrow \mathcal{C}^{-\infty}(K)^{\mathrm{Ad}} \tag{3.1}$$

where $\mathcal{C}^{-\infty}(K)^{\mathrm{Ad}}$ is the space of generalized function on K, which are invariant by conjugation.

A *K-transversally elliptic* symbol h defines an element of $\mathbf{K}_K(\mathrm{T}_K\mathcal{X})$, and the index of h is defined as a trace class virtual representation of K, that we still denote $\mathrm{Index}^K_{\mathcal{X}}(h) \in R^{-\infty}_{\mathrm{tc}}(K)$.

[2] The map $h(x,v)$ will be also denote $h|_x(v)$

Remark that any elliptic symbol of $\mathrm{T}\mathcal{X}$ is K-transversally elliptic, hence we have a restriction map $\mathbf{K}_K(\mathrm{T}\mathcal{X}) \to \mathbf{K}_K(\mathrm{T}_K\mathcal{X})$, and a commutative diagram

$$\begin{array}{ccc} \mathbf{K}_K(\mathrm{T}\mathcal{X}) & \longrightarrow & \mathbf{K}_K(\mathrm{T}_K\mathcal{X}) \\ \downarrow{\scriptstyle \mathrm{Index}^K_{\mathcal{X}}} & & \downarrow{\scriptstyle \mathrm{Index}^K_{\mathcal{X}}} \\ R(K) & \longrightarrow & R^{-\infty}_{\mathrm{tc}}(K)\ . \end{array} \tag{3.2}$$

Using the *excision property*, one can easily show that the index map $\mathrm{Index}^K_{\mathcal{U}} : \mathbf{K}_K(\mathrm{T}_K\mathcal{U}) \to R^{-\infty}_{\mathrm{tc}}(K)$ is still defined when $\mathcal{U}$ is a K-invariant relatively compact open subset of a K-manifold (see [**21,** Section 3.1]).

Suppose that M is a K-manifold equipped with an invariant almost complex structure J. Let us recall the definition of the Riemann–Roch character $RR^K_J(M,-)$.

The complex vector bundle $(\mathrm{T}^*M)^{0,1}$ is K-equivariantly identified with the tangent bundle $\mathrm{T}M$ equipped with the complex structure J. We work with the Hermitian structure on $(\mathrm{T}M, J)$ defined by : $(v,w) := \Omega(v, Jw) - \mathrm{i}\Omega(v,w)$ for $v, w \in \mathrm{T}M$. The symbol

$$\mathrm{Thom}(M,J) \in \Gamma\left(\mathrm{T}M, \hom(p^*(\wedge^{\mathrm{even}}_{\mathbb{C}}\mathrm{T}M),\, p^*(\wedge^{\mathrm{odd}}_{\mathbb{C}}\mathrm{T}M))\right)$$

at $(m,v) \in \mathrm{T}M$ is equal to the Clifford map

$$\mathbf{c}_m(v)\ :\ \wedge^{\mathrm{even}}_{\mathbb{C}}\mathrm{T}_mM \longrightarrow \wedge^{\mathrm{odd}}_{\mathbb{C}}\mathrm{T}_mM, \tag{3.3}$$

where $\mathbf{c}_m(v).w = v \wedge w - \iota(v)w$ for $w \in \wedge^{\bullet}_{\mathbb{C}}\mathrm{T}_mM$. Here $\iota(v) : \wedge^{\bullet}_{\mathbb{C}}\mathrm{T}_mM \to \wedge^{\bullet-1}_{\mathbb{C}}\mathrm{T}_mM$ denotes the contraction map. Since $\mathbf{c}_m(v)^2 = -\|v\|^2\mathrm{Id}$, the map $\mathbf{c}_m(v)$ is invertible for all $v \neq 0$. Hence the characteristic set of $\mathrm{Thom}(M,J)$ corresponds to the 0-section of $\mathrm{T}M$.

Let E be a K-equivariant complex vector bundle over M. It is a classical fact that the principal symbol of the Dolbeault–Dirac operator $\sqrt{2}(\overline{\partial}_E + \overline{\partial}^*_E)$ is equal to the following elliptic symbol[3]

$$\mathbf{c}_E := \mathrm{Thom}(M,J) \otimes E,$$

see [**6,** Prop. 3.67]. Since M is compact, the symbol $\mathbf{c}_E$ is elliptic and then defines an element of the equivariant $\mathbf{K}$-group of $\mathrm{T}M$.

Definition 3.2. The Riemann–Roch character $RR^K_J(M,E) \in R(K)$ is defined equivalently

- as the topological index of $\mathbf{c}_E \in \mathbf{K}_K(\mathrm{T}M)$, or
- as the analytical index of the Dolbeault–Dirac operator $\sqrt{2}(\overline{\partial}_E + \overline{\partial}^*_E)$.

[3]Here we use an identification $\mathrm{T}^*M \simeq \mathrm{T}M$ given by an invariant Riemannian metric.

3.2. Localization of the Riemann–Roch character. Let (M,ω,Φ) a compact Hamiltonian K-manifold Spin-prequantized by $(\tilde{L},J)$ where J is a *compatible* almost complex structure on M. The Riemann–Roch character attached to J is just denoted $RR^K(M,-)$.

By definition the Spin-quantization of (M,ω,Φ) is

$$\mathcal{Q}^K_{\mathrm{spin}}(M) := RR^K(M,\tilde{L}) \;\in\; R(K).$$

We recall the Witten's deformation of the Riemann–Roch character [**21, 22**]. We use in all this paper an isomorphism $\mathfrak{k}^* \simeq \mathfrak{k}$ defined by a K-invariant scalar product on $\mathfrak{k}^*$. In order to simplify the notation, we use the same symbol for $\xi \in \mathfrak{k}^*$ and its corresponding element in $\mathfrak{k}$.

The moment map Φ is seen as en equivariant map from M to $\mathfrak{k}$. We define the *Kirwan vector field* on M:

$$\kappa_m = \left(\Phi(m)\right)_M(m), \quad m \in M. \tag{3.4}$$

Definition 3.3. The symbol $\mathbf{c}_{\tilde{L}} = \mathrm{Thom}(M,J)\otimes\tilde{L}$ pushed by the vector field κ is the symbol $\mathbf{c}^{\kappa}_{\tilde{L}}$ defined by the relation

$$\mathbf{c}^{\kappa}_{\tilde{L}}|_m(v) = \mathrm{Thom}(M,J)\otimes\tilde{L}|_m(v-\kappa_m)$$

for any $(m,v)\in \mathrm{T}M$.

Note that $\mathbf{c}^{\kappa}_{\tilde{L}}|_m(v)$ is invertible except if $v=\kappa_m$. If furthermore v belongs to the subset $\mathrm{T}_K M$ of tangent vectors orthogonal to the K-orbits, then $v=0$ and $\kappa_m=0$. Indeed κ_m is tangent to $K\cdot m$ while v is orthogonal. So we note that $(m,v)\in \mathrm{Char}(\mathbf{c}^{\kappa}_{\tilde{L}})\cap \mathrm{T}_K M$ if and only if $v=0$ and $\kappa_m=0$.

Since κ is the Hamiltonian vector field of the function $\frac{-1}{2}\|\Phi\|^2$, the set of zeros of κ coincides with the set $\mathrm{Cr}(\|\Phi\|^2)$ of critical points of $\|\Phi\|^2$. Finally we have

$$\begin{aligned}\mathrm{Char}(\mathbf{c}^{\kappa}_{\tilde{L}})\cap \mathrm{T}_K M &\simeq \mathrm{Cr}(\|\Phi\|^2)\\ &= \bigcup_{\beta\in\mathcal{B}} \underbrace{K\cdot\Big(M^\beta\cap\Phi^{-1}(\beta)\Big)}_{C_\beta},\end{aligned}$$

where $\mathcal{B}$ is the subset of the Weyl chamber defined by the relation $\beta\in\mathcal{B}\Longleftrightarrow M^\beta\cap\Phi^{-1}(\beta)\neq\emptyset$. Recall the well-known fact.

Lemma 3.4. *The set $\mathcal{B}$ is finite.*

Proof. Let $\Phi_T : M\to\mathfrak{t}^*$ be the Hamiltonian action of a maximal torus of K. We have similarly $\mathrm{Cr}(\|\Phi_T\|^2)=\cup_{\beta\in\mathcal{B}_T}M^\beta\cap\Phi_T^{-1}(\beta)$, with $\mathcal{B}_T\subset\mathfrak{t}^*$. We see that $\mathcal{B}\subset\mathcal{B}_T$, hence it is sufficient to prove that $\mathcal{B}_T$ is finite. Let us consider the collection $\mathcal{B}'$ of polytopes of $\mathfrak{t}^*$ which arise as the image by Φ_T of a connected component of the fixed point set M^H, where H is a subtorus of T. It is easy to see that $\mathcal{B}'$ is finite, and we checks in [**20,** Section 6.1] that any $\beta \in \mathcal{B}_T$ is equal to the orthogonal projection of $0 \in \mathfrak{t}^*$ to the affine space generated by some $\Delta \in \mathcal{B}'$. Hence $\sharp\mathcal{B}_T \leq \sharp\mathcal{B}'$ is finite. □

We are interested to the restriction $\mathbf{c}^{\kappa}_{\tilde{L}}|_U$ of the elliptic symbol on an invariant open subset $U \subset M$. Note that the set $\mathrm{Char}(\mathbf{c}^{\kappa}_{\tilde{L}}|_U) \cap \mathrm{T}_K U \simeq \mathrm{Cr}(\|\Phi\|^2) \cap U$ is compact when

$$\partial\mathcal{U} \cap \mathrm{Cr}(\|\Phi\|^2) = \emptyset. \tag{3.5}$$

When (3.5) holds we denote

$$\mathcal{Q}^K_\Phi(U) := \mathrm{Index}^K_U(\mathbf{c}^{\kappa}_{\tilde{L}}|_U) \quad \in \quad R^{-\infty}_{\mathrm{tc}}(K) \tag{3.6}$$

the equivariant index of the transversally elliptic symbol $\mathbf{c}^{\kappa}_{\tilde{L}}|_U$.

For any $\beta \in \mathcal{B}$, we consider a relatively compact open invariant neighborhood U_β of C_β such that $\mathrm{Cr}(\|\Phi\|^2) \cap \overline{U_\beta} = C_\beta$.

Definition 3.5. We denote

$$\mathcal{Q}^K_\beta(M) \in R^{-\infty}_{\mathrm{tc}}(K)$$

the index of the transversally elliptic symbol $\mathbf{c}^{\kappa}_{\tilde{L}}|_{U_\beta}$.

Everything can be defined if we replace the line bundle $\tilde{L}$ by any equivariant complex vector bundle E. We can consider the pushed symbol $\mathbf{c}^{\kappa}_E$, and the localized Riemann–Roch characters

$$RR^K_\Phi(U, E) := \mathrm{Index}^K_U(\mathbf{c}^{\kappa}_E|_U) \quad \text{and} \quad RR^K_\beta(M, E) := \mathrm{Index}^K_{U_\beta}(\mathbf{c}^{\kappa}_E|_{U_\beta}).$$

A direct application of the excision property [**21,** Section 4] gives that

$$\mathcal{Q}^K_{\mathrm{spin}}(M) = \sum_{\beta \in \mathcal{B}} \mathcal{Q}^K_\beta(M). \tag{3.7}$$

If we work with $RR^K_\Phi(U, E)$, we have

$$RR^K_\Phi(U, E) = \sum_{\beta \in \mathcal{B} \cap \Phi(U)} RR^K_\beta(U, E). \tag{3.8}$$

The decomposition (3.7) and (3.8) will be used in the next chapters when one want to compute the multiplicity, denoted $[\mathcal{Q}^K_{\mathrm{spin}}(M)]^K$, of the trivial representation in $\mathcal{Q}^K_{\mathrm{spin}}(M)$. We have

$$\left[\mathcal{Q}^K_{\mathrm{spin}}(M)\right]^K = \sum_{\beta \in \mathcal{B}} \left[\mathcal{Q}^K_\beta(M)\right]^K.$$

and we finish this section by recalling a *criterium* under which one has $[\mathcal{Q}^K_\beta(M)]^K = 0$.

Let β be a non-zero element in $\mathfrak{k}$: let $\mathbb{T}_\beta \subset K$ be the torus generated by β. For $m \in M^\beta$, let $\alpha^m_1, \ldots, \alpha^m_p$ be the real infinitesimal weights for the action of $\mathbb{T}_\beta$ on the fibers of $\mathrm{T}_m M$ (we equip the fibers of $\mathrm{T}_m M/\mathrm{T}_m M^\beta$ with a $\mathbb{T}_\beta$-invariant complex structure).

Definition 3.6. Let us denote by $\mathbf{Tr}_\beta|\mathrm{T}_m M|$ the following positive number

$$\mathbf{Tr}_\beta|\mathrm{T}_m M| := \sum_{i=1}^{l} |\langle \alpha_i^m, \beta\rangle|.$$

Note that $m \in M^\beta \mapsto \mathbf{Tr}_\beta|\mathrm{T}_m M|$ is constant along a connected component of M^β. We see also that the expression $\mathbf{Tr}_\beta|E|$ is well defined for any H-equivariant real vector bundle $E \to P$, when $\beta \in \mathfrak{h}$ acts trivially on P.

Example 3.7. The map $\beta \in \mathfrak{k} \mapsto \mathbf{Tr}_\beta|\mathfrak{k}|$ is invariant under the adjoint action. When β belongs to the Weyl chamber, one has $\mathbf{Tr}_\beta|\mathfrak{k}| = 2(\rho, \beta)$. Note that $\mathbf{Tr}_\beta|\mathfrak{k}| \leq 2\|\rho\|\,\|\beta\|$ for any $\beta \in \mathfrak{k}$.

We have proved in [**22,** Proposition 3.11] the following useful criterium.

Proposition 3.8. *Let $\beta \neq 0$ in $\mathcal{B}$. The multiplicity of the trivial representation in $\mathcal{Q}^K_\beta(M)$ is equal to zero if*

$$\| \beta \|^2 + \tfrac{1}{2}\,\boldsymbol{Tr}_\beta|\mathrm{T}_m M| > \boldsymbol{Tr}_\beta|\mathfrak{k}|, \qquad \forall\ m \in M^\beta \cap \Phi^{-1}(\beta). \tag{3.9}$$

Remark 3.9. Note that condition (3.9) is equivalent to

$$\| \Phi(m) \|^2 + \tfrac{1}{2}\mathbf{Tr}_{\Phi(m)}|\mathrm{T}_m M| > \mathbf{Tr}_{\Phi(m)}|\mathfrak{k}|, \qquad \forall\ m \in C_\beta. \tag{3.10}$$

If the critical set C_β decomposes in a finite disjoint union of closed K-invariant subset $C_\beta = \cup_j C^j_\beta$, we consider invariant open neighborhood U^j of C^j_β such that $\overline{U^j_\beta} \cap \mathrm{Cr}(\|\Phi\|^2) = C^j_\beta$, and we define

$$\mathcal{Q}^K_{C^j_\beta}(M) := \mathrm{Index}^K_{U^j_\beta}(\mathbf{c}^\kappa|_{U^j_\beta}) \quad \in \quad R^{-\infty}_{\mathrm{tc}}(K).$$

Then the generalized character $\mathcal{Q}^K_\beta(M)$ is equal to the sum $\sum_j \mathcal{Q}^K_{C^j_\beta}(M)$ and Proposition 3.8 tells us that $[\mathcal{Q}^K_{C^j_\beta}(M)]^K = 0$ if (3.10) holds on C^j_β.

3.3. Induction formulas. Let H be a compact connected Lie group. Let $H \cdot a$ be a coadjoint orbit. Let (N, ω_N, Φ_N) be an Hamiltonian H-manifold which is not assumed to be compact. But we assume that Φ_N is *proper* near $H \cdot a$: the pullback $\Phi_N^{-1}(\mathcal{C})$ is compact if $\mathcal{C} \subset \mathfrak{h}^*$ is a small-enough compact invariant neighborhood of $H \cdot a$.

Let H_a be the stabilizer of $a \in \mathfrak{h}^*$, and let Y_a be a symplectic slice near $H \cdot a$: Y_a is a H_a-invariant symplectic manifold of N such that $\Phi_N(Y_a) \subset \mathfrak{h}^*_a$ and such that $H \times_{H_a} Y_a$ is diffeomorphic to an invariant open neighborhood of $\Phi_N^{-1}(H \cdot a)$. We will work with the following moment map on Y_a:

$$\Phi_{Y_a} = \Phi_N|_{Y_a} - a.$$

Let $N \times \overline{H \cdot a}$ be the Hamiltonian H-manifold, with moment map $\Phi(n, \xi) = \Phi_N(n) - \xi$. Let

$$RR^H_0(N \times \overline{H \cdot a},\ -\ )$$

be the Riemann–Roch character localized near the compact subset $\Phi^{-1}(0) \subset N \times \overline{H \cdot a}$. Let

$$RR_0^{H_a}(Y_a,\ -\ )$$

be the Riemann–Roch character localized near the compact subset $\Phi_{Y_a}^{-1}(0) = \Phi_N^{-1}(a) \subset Y_a$.

Let $\mathrm{Ind}_{H_a}^{H} : R^{-\infty}(H_a) \to R^{-\infty}(H)$ be the induction map. If E and F are respectively H-equivariant complex vector bundles on N and $H \cdot a$, we denote $E \boxtimes F$ their product. We have proved in [**21,** Proposition 7.10] (see also [**22,** Proposition 4.13]) the following induction formula.

Proposition 3.10. *For any equivariant complex vector bundles $E \to N$ and $F \to H \cdot a$, we have*

$$RR_0^{H}(N \times \overline{H \cdot a}, E \boxtimes F) = \mathrm{Ind}_{H_a}^{H} \left[RR_0^{H_a}(Y_a, E|_{Y_a} \otimes F|_{\{a\}}) \right].$$

The last Proposition gives in particular that

$$\left[RR_0^{H}(N \times \overline{H \cdot a}, E \boxtimes F) \right]^{H} = \left[RR_0^{H_a}(Y_a, E|_{Y_a} \otimes F|_{\{a\}}) \right]^{H_a}. \tag{3.11}$$

3.4. The torus case. Let T be a compact torus, and let (M, ω, Φ) be a compact Hamiltonian T-manifold which is Spin-prequantized by the data $(J, \tilde{L})$. We suppose that J is compatible with ω. The irreducible representation of T is parametrized by the lattice $\widehat{T} \subset \mathfrak{t}^*$: at each $\mu \in \widehat{T}$ we associate the one-dimensional representation $\mathbb{C}_\mu$.

We write $\mathcal{Q}_{\mathrm{spin}}^{T}(M) = \sum_{\mu \in \widehat{T}} \mathrm{m}_\mu \mathbb{C}_\mu$, and one wants to show that the multiplicity m_μ is equal to the Spin-quantization of the (possibly singular) reduced space $M_\mu := \Phi^{-1}(\mu)/T$.

We fix once for all $\mu \in \widehat{T}$. And we apply the Witten deformation procedure to the Hamiltonian T-manifold $(M, \omega, \Phi - \mu)$ which is Spin-prequantized by $(J, \tilde{L} \otimes \mathbb{C}_{-\mu})$. We have

$$\mathrm{m}_\mu = \sum_{\beta \in \mathcal{B}^\mu} \left[RR_\beta^{T}(M, \tilde{L} \otimes \mathbb{C}_{-\mu}) \right]^{\mathrm{T}},$$

where $\mathcal{B}^\mu$ parametrizes the critical points of $\|\Phi - \mu\|^2$. Here the criterion (3.9) holds for any non-zero β since the Lie algebra $\mathfrak{t}$ is abelian. We have then

$$\mathrm{m}_\mu = \left[RR_0^{T}(M, \tilde{L} \otimes \mathbb{C}_{-\mu}) \right]^{\mathrm{T}}.$$

In particular, $\mathrm{m}_\mu = 0$ if $\mu \notin \Phi(M)$. When $\mu \in \Phi(M)$, we consider a small neighborhood U of $\Phi^{-1}(\mu) \subset M$ so that $\overline{U} \cap \mathrm{Cr}(\|\Phi - \mu\|^2) = \Phi^{-1}(\mu)$. We know then that

$$\mathrm{m}_\mu = \left[RR_{\Phi - \mu}^{T}(U, \tilde{L}|_U \otimes \mathbb{C}_{-\mu}) \right]^{\mathrm{T}}. \tag{3.12}$$

3.4.1. First case: μ is a regular value of Φ. We consider the orbifold reduced space $M_\mu = \Phi^{-1}(\mu)/T$ which is equipped with a canonical symplectic form ω_μ. Let $RR(M_\mu, -)$ be the Riemann–Roch character attached to a compatible almost complex structure. We prove in [**21,** Section 7.1] that for any complex vector bundle $E \to U$

$$\left[RR^T_{\Phi-\mu}(U,E)\right]^T = RR(M_\mu, \mathcal{E}) \tag{3.13}$$

where $\mathcal{E} = E|_{\Phi^{-1}(\mu)}/T$ is the induced orbifold bundle on M_μ. If we take $E = \tilde{L}|_U \otimes \mathbb{C}_{-\mu}$ on sees (thanks to Lemma 2.7) that

$$\tilde{\mathcal{L}}_\mu = (\tilde{L}|_{\Phi^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/T$$

is an orbifold line bundle which Spin-prequantizes (M_μ, ω_μ), and (3.13) gives together with (3.12) that

$$\mathrm{m}_\mu = RR(M_\mu, \tilde{\mathcal{L}}_\mu) = \mathcal{Q}_{\mathrm{spin}}(M_\mu).$$

3.4.2. Second case: μ is a not (necessarily) a regular value of Φ. Let $\overrightarrow{\Delta} \subset \mathfrak{t}^*$ be the rational vector subspace generated by $\{a-b\ |a,b \in \Phi(M)\}$. We work here with a weight $\mu \in \Phi(M)$ so that the polytope $\Phi(M)$ lives in the affine subspace $\mu + \overrightarrow{\Delta}$. Let $\mathfrak{t}_\Delta \subset \mathfrak{t}$ be the subspace orthogonal to Δ, and let $T_\Delta \subset T$ be the corresponding subtorus.

Lemma 3.11. *The group T_Δ acts trivially on M and on the line bundle $\tilde{L} \otimes \mathbb{C}_{-\mu}$.*

Proof. See the proof of Lemma 2.9. □

Let $T' \subset T$ be another subtorus such that $T = T_\Delta \times T'$: the dual of its Lie algebra $\mathfrak{t}'$ is identified with $\overrightarrow{\Delta} \subset \mathfrak{t}^*$. We look now at (M,ω) as a Hamiltonian T'-manifold with moment map

$$\Phi' := \Phi - \mu : M \longrightarrow \overrightarrow{\Delta} = (\mathfrak{t}')^*.$$

The T'-equivariant line bundle $\tilde{L}' := \tilde{L} \otimes \mathbb{C}_{-\mu}$ Spin-prequantizes the Hamiltonian T'-manifold (M,ω,Φ'). Let U be a small neighborhood of $\Phi'^{-1}(0)$ in M. The generalized character $RR^T_{\Phi-\mu}(U, \tilde{L}|_U \otimes \mathbb{C}_{-\mu})$ belongs to $R^{-\infty}(T')$ and corresponds to the localized Riemann–Roch character

$$RR^{T'}_{\Phi'}(U, \tilde{L}'|_U).$$

We deform the moment map Φ' in $\Phi' - \varepsilon$ where ε is a small element in $\overrightarrow{\Delta}$. We have proved in [**22,** Proposition 4.14] the following

Lemma 3.12.
- *If ε is small enough, the critical set of $\|\Phi' - \varepsilon\|^2$ does not intersect ∂U, so that the localized Riemann–Roch character $RR^{T'}_{\Phi'-\varepsilon}(U, -)$ is well defined.*
- *We have $RR^{T'}_{\Phi'}(U, \tilde{L}'|_U) = RR^{T'}_{\Phi'-\varepsilon}(U, \tilde{L}'|_U)$ if ε is small enough.*

Now we are left to the computation of $\mathrm{m}_\mu = \left[RR^{T'}_{\Phi'-\varepsilon}(U, \tilde{L}'|_U)\right]^{T'}$ when $\varepsilon \in \overrightarrow{\Delta}$ is small enough. We start with the decomposition

$$RR^{T'}_{\Phi'-\varepsilon}(U, \tilde{L}'|_U) = \sum_{\beta\in\mathcal{B}_\varepsilon} RR^{T'}_{\Phi'-\varepsilon,\beta}(U, \tilde{L}'|_U)$$

where $RR^{T'}_{\Phi'-\varepsilon,\beta}(U, -)$ denotes the Riemann–Roch character localized near the compact subset $U^\beta \cap (\Phi')^{-1}(\beta+\varepsilon)$. We have proved in [**22,** Lemma 4.16] the following

Lemma 3.13. *If ε is small enough we have $\left[RR^{T'}_{\Phi'-\varepsilon,\beta}(U, \tilde{L}'|_U)\right]^{T'} = 0$ when $\beta \neq 0$.*

At $\varepsilon \in \overrightarrow{\Delta}$ small enough and generic we associate the orbifold $M_{\mu+\varepsilon} = \Phi^{-1}(\mu+\varepsilon)/T'$, which is equipped with the orbifold line bundle

$$\tilde{\mathcal{L}}_{\mu+\varepsilon} = \left(\tilde{L}|_{\Phi^{-1}(\mu+\varepsilon)} \otimes \mathbb{C}_{-\mu}\right)/T'.$$

Let $RR(M_{\mu+\varepsilon}, -)$ be the Riemann–Roch map associated to a compatible almost complex structure. If we use (3.13) together with the Lemmas 3.12 and 3.13 we get:

Theorem 3.14. *The multiplicity m_μ is equal to the Riemann–Roch number $RR(M_{\mu+\varepsilon}, \tilde{\mathcal{L}}_{\mu+\varepsilon}) \in \mathbb{Z}$ where $\varepsilon \in \overrightarrow{\Delta}$ is small and generic.*

We prove here that the quantity $RR(M_{\mu+\varepsilon}, \tilde{\mathcal{L}}_{\mu+\varepsilon})$ does not depend of the choice of ε small and generic: it is the definition of the Spin quantization, denoted $\mathcal{Q}_{\mathrm{spin}}(M_\mu)$, of the (possibly singular) reduced space M_μ.

3.4.3. Proof of Theorem 2.10. The same kind of proof work for Theorem 2.10. We consider an invariant relatively compact neighborhood $U_{\sigma,\mu}$ of $\Phi_\sigma^{-1}(\mu) = \Phi^{-1}(\mu+\rho-\rho_\sigma)$ in the slice Y_σ so that $\mathrm{Cr}(\|\Phi_\sigma - \mu\|^2) \cap \overline{U_{\sigma,\mu}} = \Phi_\sigma^{-1}(\mu)$. Thanks to Lemmas 3.12 and 3.13, we know that the Riemann–Roch character

$$RR^{Z'_\sigma}_{\Phi_\sigma-\mu-\varepsilon}(U_{\sigma,\mu}, \tilde{L}') \in R^{-\infty}(Z'_\sigma)$$

are well defined for $\varepsilon \in \overrightarrow{\Delta}$ small enough, and they do not depend of the choice of ε. If $\varepsilon_1, \varepsilon_2 \in \overrightarrow{\Delta}$ are small enough regular values of $\Phi_\sigma - \mu$ we get thanks to (3.13) that

$$\begin{aligned} RR(M^\sigma_{\mu+\varepsilon_1}, \tilde{\mathcal{L}}^\sigma_{\mu+\varepsilon_1}) &= \left[RR^{Z'_\sigma}_{\Phi_\sigma-\mu-\varepsilon_1}(U_{\sigma,\mu}, \tilde{L}')\right]^{Z'_\sigma} \\ &= \left[RR^{Z'_\sigma}_{\Phi_\sigma-\mu-\varepsilon_2}(U_{\sigma,\mu}, \tilde{L}')\right]^{Z'_\sigma} \\ &= RR(M^\sigma_{\mu+\varepsilon_2}, \tilde{\mathcal{L}}^\sigma_{\mu+\varepsilon_2}). \end{aligned}$$

4. Proof of Theorem 3.1

Let (M, ω, Φ) be a compact Hamiltonian K-manifold which is Spin prequantized. Let σ be the smallest face of the Weyl chamber so that $\Phi(M)\cap \mathfrak{t}^*_+ \subset \overline{\sigma}$. Let μ be a dominant weight, and let m_μ be the multiplicity of V_μ^K in $\mathcal{Q}^K_{\mathrm{spin}}(M)$.

Let $\mathcal{O}_\mu$ be the coadjoint orbit $K\cdot(\mu+\rho)$. Since the dual representation $(V_\mu^K)^*$ can be realized[4] as $\mathcal{Q}^K_{\mathrm{spin}}(\overline{\mathcal{O}_\mu})$, we know by the shifting trick that

$$\mathrm{m}_\mu = \left[\mathcal{Q}^K_{\mathrm{spin}}(M\times\overline{\mathcal{O}_\mu})\right]^K.$$

Now we work with the Hamiltonian K-manifold $N = M\times\overline{\mathcal{O}_\mu}$ with moment map $\Phi_N(m,\xi) = \Phi(m)-\xi$. The Witten's deformation on N gives $\mathcal{Q}^K_{\mathrm{spin}}(M\times\overline{\mathcal{O}_\mu}) = \sum_{\beta\in\mathcal{B}^\mu}\mathcal{Q}^K_\beta(M\times\overline{\mathcal{O}_\mu})$ where $\mathcal{B}^\mu$ is a finite set parametrizing $\mathrm{Cr}(\|\Phi_N\|^2)$, and $\mathcal{Q}^K_\beta(M\times\overline{\mathcal{O}_\mu})$ is an index of a transversally elliptic operator localized near $C_\beta = K(N^\beta\cap\Phi_N^{-1}(\beta))$.

We have then

$$\mathrm{m}_\mu = \sum_{\beta\in\mathcal{B}^\mu}\left[\mathcal{Q}^K_\beta(M\times\overline{\mathcal{O}_\mu})\right]^K. \tag{4.1}$$

We remark that 0 does not appears in $\mathcal{B}^\mu$ when $\sigma\neq\mathfrak{t}^*_+$, since $\mu+\rho\notin\Phi(M)$.

The main point of this section is the following:

Proposition 4.1.
- *If $\mu\notin\overline{\sigma}$, the identity (*3.10*) holds on C_β for any $\beta\in\mathcal{B}^\mu$. Hence $\mathrm{m}_\mu = 0$.*
- *If $\mu\in\overline{\sigma}$, the identity (*3.10*) holds on C_β for any $\beta\neq-\rho_\sigma$. Then*

$$\mathrm{m}_\mu = \left[\mathcal{Q}^K_{-\rho_\sigma}(M\times\overline{\mathcal{O}_\mu})\right]^K.$$

When $\sigma = \mathfrak{t}^*_+$, we have $\rho_\sigma = 0$ and Proposition 4.1 tell us that the multiplicity m_μ id equal to $\left[\mathcal{Q}^K_0(M\times\overline{\mathcal{O}_\mu})\right]^K$ for any $\mu\in\widehat{K}$. In particular, $\mathrm{m}_\mu = 0$ if $\mu+\rho\notin\Phi(M)$.

When $\sigma\neq\mathfrak{t}^*_+$ and $\mu\in\overline{\sigma}$, we precise Proposition 4.1 as follow. The generalized character $\mathcal{Q}^K_{-\rho_\sigma}(M\times\overline{\mathcal{O}_\mu})$ is defined as the index of a transversally elliptic symbol living in a neighborhood of

$$C_{-\rho_\sigma} = K\left(N^{\rho_\sigma}\cap\Phi_N^{-1}(-\rho_\sigma)\right).$$

Let K_{ρ_σ} be the stabilizer subgroup of ρ_σ. Let $W(K_{\rho_\sigma})\subset W$ be the Weyl subgroup of K_{ρ_σ}. A direct computation gives that

$$C_{-\rho_\sigma} = \bigcup_{\bar{w}\in W(K_{\rho_\sigma})\backslash W} C_{-\rho_\sigma,\bar{w}}$$

[4] $\overline{\mathcal{O}_\mu}$ is the coadjoint orbit $\mathcal{O}_\mu$ with the opposite symplectic structure.

with $C_{-\rho_\sigma,\bar{w}} = K\left(M^{\rho_\sigma} \cap \Phi^{-1}(w(\mu+\rho)-\rho_\sigma) \times \{w(\mu+\rho)\}\right)$. We are particularly interested in the component[5]

$$C_{-\rho_\sigma,\bar{e}} := K\left(M^{\rho_\sigma} \cap \Phi^{-1}(\mu+\rho-\rho_\sigma) \times \{\mu+\rho\}\right).$$

Let us denote $C_{-\rho_\sigma,\mathrm{out}}$ the union of the $C_{-\rho_\sigma,\bar{w}}$ for $\bar{w} \neq \bar{e}$. We have a decomposition

$$C_{-\rho_\sigma} = C_{-\rho_\sigma,\bar{e}} \cup C_{-\rho_\sigma,\mathrm{out}} \tag{4.2}$$

into closed invariant disjoint subsets. Then the generalized character $\mathcal{Q}^K_{-\rho_\sigma}(M \times \overline{\mathcal{O}_\mu})$ is equal to the sum

$$\mathcal{Q}^K_{-\rho_\sigma,\bar{e}}(M \times \overline{\mathcal{O}_\mu}) + \mathcal{Q}^K_{-\rho_\sigma,\mathrm{out}}(M \times \overline{\mathcal{O}_\mu})$$

where both terms correspond to the specialization of the transversally elliptic symbol to the neighborhood of each part of the decomposition (4.2).

Proposition 4.2. *Suppose that $\sigma \neq \mathfrak{t}^*_+$ and that $\mu \in \overline{\sigma}$. The identity (3.10) holds on the subset $C_{-\rho_\sigma,out}$, and then*

$$\mathrm{m}_\mu = \left[\mathcal{Q}^K_{-\rho_\sigma,\bar{e}}(M \times \overline{\mathcal{O}_\mu})\right]^K.$$

Note that $C_{-\rho_\sigma,\bar{e}} = \emptyset$ if $\mu+\rho-\rho_\sigma \notin \Phi(M)$. At this stage we know then that $\mathrm{m}_\mu = 0$ if $\mu+\rho-\rho_\sigma$ does not belongs to the image of the moment map.

4.1. Proofs of Propositions 4.1 and 4.2. Let $N = M \times \overline{\mathcal{O}_\mu}$ and let $\|\Phi_N\|^2 : N \to \mathbb{R}$ be the square of the moment map. Recall that we denote by σ the smallest face of the Weyl chamber so that $\Phi(M) \cap \mathfrak{t}^*_+ \subset \overline{\sigma}$.

We want to prove that for any $n = (m,\xi) \in \mathrm{Cr}(\|\Phi_N\|^2)$ the vector $\beta := \Phi(m) - \xi$ satisfies

$$\textbf{(I)} \qquad \|\beta\|^2 + \tfrac{1}{2}\mathbf{Tr}_\beta|\mathrm{T}_n N| \geq \mathbf{Tr}_\beta|\mathfrak{k}|.$$

Afterwards we will discuss the case of equality in **(I)**.

The tangent space $\mathrm{T}_\xi\mathcal{O}_\mu$ is equal to the $\mathfrak{k}_\xi$-module $\mathfrak{k}/\mathfrak{k}_\xi$: then

$$\begin{aligned}\mathbf{Tr}_\beta|\mathrm{T}_\xi\mathcal{O}_\mu| &= \mathbf{Tr}_\beta|\mathfrak{k}| - \mathbf{Tr}_\beta|\mathfrak{k}_\xi| \\ &= \mathbf{Tr}_\beta|\mathfrak{k}|,\end{aligned}$$

since β belongs to the abelian subalgebra $\mathfrak{k}_\xi$. Using that $\mathbf{Tr}_\beta|\mathrm{T}_n N| = \mathbf{Tr}_\beta|\mathrm{T}_m M| + \mathbf{Tr}_\beta|\mathfrak{k}|$, we see that **(I)** is equivalent to

$$\textbf{(II)} \qquad \|\beta\|^2 + \tfrac{1}{2}\mathbf{Tr}_\beta|\mathrm{T}_m M| \geq \tfrac{1}{2}\mathbf{Tr}_\beta|\mathfrak{k}|.$$

The module $\mathfrak{k}/\mathfrak{k}_m$ is naturally a subspace of $\mathrm{T}_m M$. Let E_m be a K_m-equivariant supplement to $\mathfrak{k}/\mathfrak{k}_m$ in $\mathrm{T}_m M$. Using that $\mathbf{Tr}_\beta|\mathrm{T}_m M| = \mathbf{Tr}_\beta|\mathfrak{k}/\mathfrak{k}_m| + \mathbf{Tr}_\beta|E_m|$, we see that **(II)** is equivalent to

$$\textbf{(III)} \qquad \|\beta\|^2 + \tfrac{1}{2}\mathbf{Tr}_\beta|E_m| \geq \tfrac{1}{2}\mathbf{Tr}_\beta|\mathfrak{k}_m|.$$

[5] $\bar{e}$ is the class of the neutral element in $W(K_{\rho_\sigma})\backslash W$.

Since the moment map Φ is equivariant, the Lie algebra stabilizer $\mathfrak{k}_m$ is contained in Lie algebra stabilizer $\mathfrak{k}_{\Phi(m)}$. Finally, we see that **(I)** $\Leftrightarrow$ **(II)** $\Leftrightarrow$ **(III)** are induced by the following inequality

$$\textbf{(IV)} \qquad \| \beta \|^2 \geq \tfrac{1}{2}\mathbf{Tr}_\beta|\mathfrak{k}_{\Phi(m)}|.$$

Lemma 4.3.
- *For any $(m,\xi) \in \mathrm{Cr}(\|\Phi_N\|^2)$ the vector $\beta := \Phi(m) - \xi$ satisfies the inequality **(IV)**.*
- *Let $(m,\xi) \in \mathrm{Cr}(\|\Phi_N\|^2)$ such that $\beta := \Phi(m) - \xi$ satisfies the $\| \beta \|^2 = \frac{1}{2}\boldsymbol{Tr}_\beta|\mathfrak{k}_{\Phi(m)}|$. Then there exists a face τ of σ such that*
  (1) *$\mu \in \overline{\tau}$*
  (2) *(m,ξ) belongs to the K-orbit of $\Phi^{-1}(\mu+\rho-\rho_\tau) \times \{\mu+\rho\} \subset N$.*
  (3) *β belongs to the coadjoint orbit $K\cdot(-\rho_\tau)$.*

Proof. Up to the multiplication of (m,ξ) by an element of K, we can assume that $\beta \in \mathfrak{t}^*$. Up to the multiplication of $n = (m,\xi)$ by an element of the stabilizer subgroup $K_\beta := \{k \in K \,|\, \mathrm{Ad}(k)\beta = \beta\}$ we can assume that $n = (m, w(\mu+\rho))$ with $m \in M^\beta$ and $\Phi(m) = \beta + w(\mu+\rho) \in \mathfrak{t}^*$.

Up to the multiplication of $n = (m, w(\mu+\rho))$ by an element of the Weyl group, we can assume that $\Phi(m)$ belongs to the Weyl chamber: let τ be the face of σ containing $\Phi(m)$ so that $K_{\Phi(m)} = K_\tau$.

So we have to prove that for $\Phi(m) = a \in \tau$ and $w \in W$ the vector $\beta = a - w(\mu+\rho)$ satisfies the relation

$$\| \beta \|^2 \geq \tfrac{1}{2}\mathbf{Tr}_\beta|\mathfrak{k}_\tau|. \tag{4.3}$$

The inequality (4.3) is the consequence of three basic inequalities. The first one is given by the following:

Lemma 4.4. *Let $a \in \mathfrak{t}^*_+$ and b in the interior of $\mathfrak{t}^*_+$. We have*

$$\|a - wb\| \geq \|a - b\| \tag{4.4}$$

for any $w \in W$, and (4.4) is strict unless $w \in W(K_a)$.

Proof. In order to prove (4.4), we consider the function $\xi \in K\cdot b \longmapsto \|\xi - a\|^2 = \|a\|^2 + \|b\|^2 - 2(\xi,a)$. The inclusion $K\cdot b \hookrightarrow \mathfrak{k}^*$ is the moment map relatively to the K-action. The function $\xi \in K\cdot b \mapsto (\xi,a)$, which is the a-th component of the moment map, has a unique *local maximum* on the coadjoint orbit $K\cdot b$ which is reached on an orbit of the stabilizer subgroup K_a (see [**11,** Theorem 5]). Let $\mathfrak{r}$ be a subspace such that $\mathfrak{k} = \mathfrak{k}_a \oplus \mathfrak{r}$. For $X \in \mathfrak{r}$, we compute $(e^X\cdot b, a) = (b,a) + ([X,b],a) + \frac{1}{2}([X,[X,b]],a) + o(\|X\|^2)$. The term $([X,b],a)$ vanishes and $([X,[X,b]],a) = -([X,b],[X,a]) \geq -\mathrm{Cst}\|X\|^2$ since a,b belongs to the Weyl chamber and $\mathfrak{k}_a \cap \mathfrak{r} = \mathfrak{k}_b \cap \mathfrak{r} = \{0\}$. Then we have proved that the local (hence global) minimum of $k\cdot b \mapsto \|a - k\cdot b\|$ is reached on $K_a\cdot b$.

Finally, we have proved that $\|a-wb\| \geq \|a-b\|$ for any $w \in W$, and that the equality $\|a-wb\| = \|a-b\|$ implies that $wb \in K_a \cdot b$, i.e., $w \in W(K_a)$. $\square$

The second inequality is

$$\|\mu+\rho-a\| \geq \frac{(\mu+\rho-a,\rho_\tau)}{\|\rho_\tau\|} \tag{4.5}$$
$$= \frac{1}{\|\rho_\tau\|}\underbrace{(\mu,\rho_\tau)}_{\geq 0} + \frac{1}{\|\rho_\tau\|}\underbrace{(\rho-\rho_\tau-a,\rho_\tau)}_{=0} + \frac{1}{\|\rho_\tau\|}(\rho_\tau,\rho_\tau)$$
$$\geq \|\rho_\tau\|.$$

Note that (4.5) is strict unless $\mu \in \overline{\tau}$ and $\mu+\rho-a = \rho_\tau$. The third inequality is

$$\tfrac{1}{2}\mathbf{Tr}_\beta|\mathfrak{k}_\tau| \leq \|\rho_\tau\|\,\|\beta\|. \tag{4.6}$$

See Example 3.7. If we put (4.4), (4.5) and (4.6) together we have for $\beta = a - w(\mu+\rho)$ the inequalities

$$\|\,\beta\,\|^2 \geq \|\,\beta\,\|\ \|a-(\mu+\rho)\|\ \geq \|\,\beta\,\|\ \|\rho_\tau\|\ \geq\ \tfrac{1}{2}\mathbf{Tr}_\beta|\mathfrak{k}_\tau|,$$

and the equality $\|\,\beta\,\|^2 = \frac{1}{2}\mathbf{Tr}_\beta|\mathfrak{k}_\tau|$ holds if and only if we have the equality in (4.4), (4.5) and (4.6).

However, equalities in (4.4) and (4.5) gives that $w \in W(K_\tau)$, $\mu \in \overline{\tau}$ and $a = \mu+\rho-\rho_\tau \in \tau$. Then $(m, w(\mu+\rho)) = w(m', \mu+\rho)$ with $\Phi(m') = w^{-1}(\mu+\rho-\rho_\tau) = \mu+\rho-\rho_\tau$ and $\beta = \mu+\rho-\rho_\tau - w(\mu+\rho) = -w\rho_\tau$. We have then

$$\tfrac{1}{2}\mathbf{Tr}_\beta|\mathfrak{k}_\tau| = \tfrac{1}{2}\mathbf{Tr}_{\rho_\tau}|\mathfrak{k}_\tau| = \|\rho_\tau\|^2,$$

which is the equality in (4.6). $\square$

Since the strict inequality in **(IV)** implies the strict inequality in **(I)**, Lemma 4.3 tells us that the identity (3.10) holds on C_β for all $\beta \in \mathcal{B}^\mu$ when $\mu \notin \overline{\sigma}$. When $\mu \in \overline{\sigma}$ the identity (3.10) holds

(1) on C_β for the β which are not in $K \cdot (-\rho_\tau)$, where τ is a face of σ such that $\mu \in \overline{\tau}$,
(2) on $C_{-\rho_\sigma,\bar{w}}$ for all the $\bar{w} \neq \bar{e}$.

The proof of Propositions 4.1 and 4.2 is completed by

Lemma 4.5. *Let τ be a face of σ, distinct from σ, such that $\mu \in \overline{\tau}$. Then the identity (*3.10*) holds for C_β for $\beta = -\rho_\tau$.*

Proof. Let $\beta = -\rho_\tau$. The critical set $C_{-\rho_\tau} := K(N^{\rho_\tau} \cap \Phi_N^{-1}(-\rho_\tau))$ admits the decomposition $C_{-\rho_\tau} = \cup_{w \in W(K_\tau)\backslash W} C_{-\rho_\tau,\bar{w}}$ where

$$C_{-\rho_\tau,\bar{w}} = K\left(M^{\rho_\tau} \cap \Phi^{-1}(w(\mu+\rho)-\rho_\tau) \times \{w(\mu+\tau)\}\right).$$

We know then from Lemma 4.3 that the strict inequality in **(IV)** holds on $C_{-\rho_\tau,\bar{w}}$ for $\bar{w} \neq \bar{e}$.

Let us consider now the case where $m \in M^{\rho_\tau} \cap \Phi^{-1}(\mu + \rho - \rho_\tau)$. We know that the equality holds in **(IV)** for $(m, \mu + \rho)$. The equality in **(I)** for $(m, \mu + \rho)$ is then equivalently to

$$\mathbf{Tr}_\beta |E_m| + \mathbf{Tr}_\beta |\mathfrak{k}_\tau / \mathfrak{k}_m| = 0. \tag{4.7}$$

Let us prove that (4.7) can never happen. The image of m by the moment map belongs to τ. Then m belongs to the symplectic slice $Y_\tau \subset M$. A neighborhood m is then $K \times_{K_\tau} Y_\tau$. So the tangent space at m decomposes in two manners

$$\begin{aligned} \mathrm{T}_m M &= \mathfrak{k}/\mathfrak{k}_\tau \oplus \mathrm{T}_m Y_\tau \\ &= \mathfrak{k}/\mathfrak{k}_m \oplus E_m. \end{aligned}$$

If (4.7) holds we see that $\mathbf{Tr}_\beta |\mathrm{T}_m Y_\tau| = \mathbf{Tr}_\beta |E_m| + \mathbf{Tr}_\beta |\mathfrak{k}_\tau/\mathfrak{k}_m| = 0$, which means that $\beta = -\rho_\tau$ acts trivially on the tangent space $\mathrm{T}_m Y_\tau$. Hence, it would implies that ρ_τ acts trivially on the manifold Y_τ. Since $Y_\sigma \subset Y_\tau$, the action of ρ_τ on the principal slice Y_σ is also trivial.

We know that $[\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]$ acts trivially on Y_σ: since $\rho_\sigma \in [\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]$, the infinitesimal action of ρ_σ is trivial on Y_σ. Finally if (4.7) holds, we have that

$$\rho_{\tau/\sigma} := \rho_\tau - \rho_\sigma \in \mathbb{R}\sigma$$

acts trivially on Y_σ. Note that $\rho_{\tau/\sigma}$ is a sum of weights, which are orthogonal to τ.

The moment polytope of M, $\Delta(M)$, which is equal to the closure of $\Phi(Y_\sigma) \subset \sigma$ is a convex polytope. Since the action of $\rho_{\tau/\sigma}$ is trivial on Y_σ we knows that the map $\xi \in \Delta(M) \mapsto (\xi, \rho_{\tau/\sigma})$ is constant.

Finally, we can use the last information in our hands: $\mu + \rho - \rho_\tau = \Phi(m)$ belongs to $\Delta(M)$. Then for $\xi \in \Delta(M)$ we have

$$(\xi, \rho_{\tau/\sigma}) = (\mu + \rho - \rho_\tau, \rho_{\tau/\sigma}) = 0,$$

since $\mu + \rho - \rho_\tau \in \tau$ and $\rho_{\tau/\sigma} \in \tau^\perp$. It is contradictory with the fact that $(\xi, \rho_{\tau/\sigma}) = (\xi, \rho_\tau) > 0$ for any $\xi \in \sigma$.

We have finally proved that when $(m, \xi) \in N^{\rho_\tau} \cap \Phi_N^{-1}(-\rho_\tau)$ the vector $\beta = \Phi(m) - \xi$ satisfies $\| \beta \|^2 + \frac{1}{2}\mathbf{Tr}_\beta |\mathrm{T}_m M| > \mathbf{Tr}_\beta |\mathfrak{k}|$. □

4.2. Computation of the multiplicities when $\boldsymbol{\sigma = \mathfrak{t}^*_+}$. In this section, we suppose that the moment polytope $\Delta(M) = \Phi(M) \cap \mathfrak{t}^*_+$ intersects the interior of the Weyl chamber. Let $\Delta(M)^o \subset (\mathfrak{t}^*_+)^o$ be the relative interior of the moment polytope. We know that $\mathrm{m}_\mu = [\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})]^K$ for any $\mu \in \widehat{K}$. In Definition 2.11, we have defined the number $\mathcal{Q}(M_\mu^{\mathfrak{t}^*_+})$ has follows. If $\mu + \rho \notin \Delta(M)^o$, we set $\mathcal{Q}(M_\mu^{\mathfrak{t}^*_+}) = 0$. If $\mu + \rho \in \Delta(M)^o$, we consider, for ε generic and small enough, the orbifold reduced space $M_{\mu+\varepsilon}^{\mathfrak{t}^*_+} := \Phi^{-1}(\mu + \varepsilon + \rho)/T$

and the orbifold line bundle

$$\tilde{\mathcal{L}}_{\mu+\varepsilon} = \Big(\tilde{L}|_{\Phi^{-1}(\mu+\varepsilon+\rho)} \otimes \mathbb{C}_{-\mu}\Big)/T.$$

The Spin-quantization $\mathcal{Q}_{\rm spin}(M_\mu^{\mathfrak{t}^*_+}) \in \mathbb{Z}$ is defined as the Riemann–Roch number

$$RR(M_{\mu+\varepsilon}^{\mathfrak{t}^*_+}, \tilde{\mathcal{L}}_{\mu+\varepsilon}).$$

The main result of this section is the following

Theorem 4.6. *The number* $\big[\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})\big]^K$ *is equal to* $\mathcal{Q}_{\rm spin}(M_\mu^{\mathfrak{t}^*_+})$.

Proof. When $\mu+\rho \notin \Delta(M)$, we see that $\mathcal{Q}_0^K(M\times\overline{\mathcal{O}_\mu}) = 0$ since the moment map on $M \times \overline{\mathcal{O}_\mu}$ does not goes through $0 \in \mathfrak{k}^*$. We have then $[\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})]^K = \mathcal{Q}_{\rm spin}(M_\mu^{\mathfrak{t}^*_+}) = 0$.

We consider now a dominant weight μ such that $\mu+\rho \in \Delta(M)$. Let $Y = \Phi^{-1}((\mathfrak{t}^*_+)^o)$ be the symplectic slice with its canonical symplectic form ω_Y. The action of T on (Y,ω_Y) is Hamiltonian with moment map $\Phi_Y := \Phi|_Y - \rho$. We know that $\tilde{L}|_Y$ Spin-prequantizes (Y,ω_Y,Φ_Y) (see Lemma 2.8).

We consider the Riemann–Roch character $RR_0^T(Y, \tilde{L}|_Y \otimes \mathbb{C}_{-\mu})$ which is localized near $(\Phi_Y - \mu)^{-1}(0) \subset Y$. Thanks to the induction formula (3.11), we know that

$$\begin{aligned} \mathrm{m}_\mu = \big[\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})\big]^K &= \Big[RR_0^K(M \times \overline{\mathcal{O}_\mu}, \tilde{L} \boxtimes \mathbb{C}_{[-\mu]})\Big]^K \\ &= \Big[RR_0^T(Y, \tilde{L}|_Y \otimes \mathbb{C}_{-\mu})\Big]^{\mathrm{T}} \\ &= \Big[RR^T_{\Phi_Y-\mu}(U, \tilde{L}|_U \otimes \mathbb{C}_{-\mu})\Big]^{\mathrm{T}}, \end{aligned}$$

where U is a small neighborhood of $\Phi_Y^{-1}(\mu)$ in Y.

The computation of the expression $[RR^T_{\Phi_Y-\mu}(U, \tilde{L}|_U \otimes \mathbb{C}_{-\mu})]^{\mathrm{T}}$ is identical to what we have done in Section 3.4. For ε small enough and generic, we get

$$\begin{aligned} [RR^T_{\Phi_Y-\mu}(U, \tilde{L}|_U \otimes \mathbb{C}_{-\mu})]^T &= [RR^T_{\Phi_Y-\mu-\varepsilon}(U, \tilde{L}|_U \otimes \mathbb{C}_{-\mu})]^{\mathrm{T}} \\ &= RR(M_{\mu+\varepsilon}^{\mathfrak{t}^*_+}, \tilde{\mathcal{L}}_{\mu+\varepsilon}) \\ &= \mathcal{Q}_{\rm spin}(M_\mu^{\mathfrak{t}^*_+}). \end{aligned}$$

When $\mu + \rho$ does not belong to the relative interior of $\Delta(M)$, we can choose ε so that $\mu + \rho + \varepsilon \notin \Delta(M)$, and then $RR(M_{\mu+\varepsilon}^{\mathfrak{t}^*_+}, \tilde{\mathcal{L}}_{\mu+\varepsilon}) = \mathcal{Q}_{\rm spin}(M_\mu^{\mathfrak{t}^*_+}) = 0$. □

4.3. Computation of the multiplicities when $\boldsymbol{\sigma \neq \mathfrak{t}^*_+}$. Let $\mu \in \overline{\sigma}$ so that $\mu + \rho - \rho_\sigma \in \sigma$. In the rest of this section the term β is $-\rho_\sigma$.

We have proved in the previous section that $\mathrm{m}_\mu = \left[\mathcal{Q}^K_{\beta,\bar{e}}(M \times \overline{\mathcal{O}_\mu})\right]^K$ where the character $\mathcal{Q}^K_{\beta,\bar{e}}(M \times \overline{\mathcal{O}_\mu})$ corresponds to the Riemann–Roch character $RR^K_{\beta,\bar{e}}(N, \tilde{L}_N)$ localized with the Kirwan vector field near

$$C_{\beta,\bar{e}} = K\left(M^\beta \cap \Phi^{-1}(\mu + \rho - \rho_\sigma) \times \{\mu + \rho\}\right) \subset \mathrm{Cr}(\|\Phi_N\|^2).$$

Look now at N as a K_β-Hamiltonian manifold. Let $\Phi'_N : N \to \mathfrak{k}^*_\beta$ be the corresponding moment map. We are interested in the component[6]

$$\begin{aligned} C'_\beta &:= K_\beta(N^\beta \cap (\Phi'_N)^{-1}(\beta)) \\ &= N^\beta \cap \Phi_N^{-1}(\beta) \\ &= \bigcup_{\bar{w} \in W(K_\beta)\backslash W} K_\beta\left(M^\beta \cap \Phi^{-1}(w(\mu+\rho) - \rho_\sigma) \times \{w(\mu+\rho)\}\right) \end{aligned}$$

of the critical set $\mathrm{Cr}(\|\Phi'_N\|^2)$. Let us consider the Riemann–Roch character

$$RR^{K_\beta}_{\beta,\bar{e}}(N, -)$$

localized with the Kirwan vector field near

$$C'_{\beta,\bar{e}} := K_\beta\left(M^\beta \cap \Phi^{-1}(\mu + \rho - \rho_\sigma) \times \{\mu + \rho\}\right) \subset C'_\beta.$$

We have proved in [**21,** Th. 6.16, Cor. 6.17] that

$$RR^K_{\beta,\bar{e}}(N, \tilde{L}_N) = \mathrm{Ind}^K_{K_\beta}\left(RR^{K_\beta}_{\beta,\bar{e}}(N, \tilde{L}_N) \wedge^\bullet_{\mathbb{C}} (\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}\right), \tag{4.8}$$

where $\mathrm{Ind}^K_{K_\beta} : R^{-\infty}(K_\beta) \to R^{-\infty}(K)$ is the induction map, and $(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}$ is the complexification of the real K_β-module $\mathfrak{k}/\mathfrak{k}_\beta$. It gives that

$$\mathrm{m}_\mu = \left[RR^K_{\beta,\bar{e}}(N, \tilde{L}_N)\right]^K = \left[RR^{K_\beta}_{\beta,\bar{e}}(N, \tilde{L}_N) \wedge^\bullet_{\mathbb{C}} (\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}\right]^{K_\beta}.$$

Let Y_σ be the principal symplectic slice of M. Recall that the subgroup $[K_\sigma, K_\sigma]$ acts trivially on Y_σ and that $\beta = -\rho_\sigma$ belongs to $[\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]$: hence $\Phi^{-1}(\mu + \rho - \rho_\sigma) \subset Y_\sigma \subset M^\beta$ and then

$$C'_{\beta,\bar{e}} = K_\beta\left(\Phi^{-1}(\mu + \rho - \rho_\sigma) \times \{\mu + \rho\}\right).$$

We are looking at a K_β-invariant neighborhood $\mathcal{U}$ of $C'_{\beta,\bar{e}}$ in N^β. We consider the open neighborhood $K \times_{K_\sigma} Y_\sigma$ of $\Phi^{-1}(\mu + \rho - \rho_\sigma)$ in M. Since $K_\beta \cap K_\sigma = T$, one sees that

$$K_\beta \times_T Y_\sigma \subset (K \times_{K_\sigma} Y_\sigma)^\beta$$

[6] In the second equality, we use that β is central in $\mathfrak{k}_\beta$, and that $\Phi'_N = \Phi_N$ on N^β.

is a K_β-invariant neighborhood of Y_σ in M^β. Then we can take

$$\mathcal{U} := (K_\beta \times_T Y_\sigma) \times K_\beta(\mu+\rho) \subset N^\beta.$$

We look at $\mathcal{U}$ as a Hamiltonian K_β-manifold with moment map $\Phi_{\mathcal{U}}([k,y],\xi) = k\Phi(y) - \xi \in \mathfrak{k}_\beta^*$. The set $C'_{\beta,\bar{e}}$ is a connected component of critical points of $\mathrm{Cr}(\|\Phi_{\mathcal{U}}\|^2)$, and we consider the Riemann–Roch character

$$RR_\beta^{K_\beta}(\mathcal{U}, -)$$

localized with the Kirwan vector field near $C'_{\beta,\bar{e}} \subset \mathcal{U}$.

Let $\mathcal{N}$ be the normal bundle of $\mathcal{U}$ in N. We have $\mathcal{N} = \mathcal{N}_1 \boxtimes \mathcal{N}_2$ where $\mathcal{N}_1$ is the normal bundle of $K_\beta \times_T Y_\sigma$ in $K \times_{K_\sigma} Y_\sigma$ and $\mathcal{N}_2$ is the normal bundle of $K_\beta(\mu+\rho)$ in $K(\mu+\rho)$. One computes that $\mathcal{N}_1 = K_\beta \times_T N_1$ where

$$N_1 = \sum_{\substack{\alpha>0 \\ \alpha|_\sigma \neq 0,\, (\alpha,\beta)\neq 0}} \mathfrak{k}(\alpha),$$

and that $\mathcal{N}_2 = K_\beta \times_T N_2$ where

$$N_2 = \sum_{\substack{\alpha<0 \\ (\alpha,\beta)\neq 0}} \mathfrak{k}(\alpha).$$

We decompose $\mathcal{N}$ in the sum of the polarized bundle $\mathcal{N}^{+,\beta}$ and $\mathcal{N}^{-,\beta}$. Similarly let $\mathcal{N}_{\mathbb{C}}$ the complexified bundle, and its polarized β-positive part $\mathcal{N}_{\mathbb{C}}^{+,\beta}$.

Let $S(\mathcal{N}_{\mathbb{C}}^{+,\beta}) = \sum_{k\geq 0} S^k(\mathcal{N}_{\mathbb{C}}^{+,\beta})$ be the symmetric algebra vector bundle associated to $\mathcal{N}_{\mathbb{C}}^{+,\beta}$. Let us compute the rank $n_{+,\beta}$ of the polarized vector bundle vector $\mathcal{N}^{+,\beta}$. We have

$$\begin{aligned} n_{+,\beta} &= \sharp\{\alpha>0 \mid (\alpha,\beta)>0 \text{ and } \alpha|_\sigma \neq 0\} + \sharp\{\alpha<0 \mid (\alpha,\beta)>0\} \\ &= \sharp\{\alpha>0 \mid (\alpha,\beta)>0\} + \sharp\{\alpha<0 \mid (\alpha,\beta)>0\} \qquad (1) \\ &= \frac{1}{2}\dim(K/K_\beta). \end{aligned}$$

In (1) we use that $\alpha|_\sigma = 0$ imposes $(\alpha, \rho-\rho_\sigma) = 0$. Then $(\alpha,\beta) = -(\alpha,\rho) < 0$ for $\alpha > 0$. Let $\det \mathcal{N}^{+,\beta}$ be the determinant line bundle associated to $\mathcal{N}^{+,\beta}$.

Thanks to the results in [**21,** Section 6.3], we know that

$$RR_{\beta,e}^{K_\beta}(N, \tilde{L}_N) = (-1)^{n_{+,\beta}} RR_\beta^{K_\beta}\left(\mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \otimes S(\mathcal{N}_{\mathbb{C}}^{+,\beta})\right). \tag{4.9}$$

Hence we know that $\mathrm{m}_\mu = \left[RR^K_{\beta,\bar{e}}(N,\tilde{L}_N)\right]^K$ is equal to $(-1)^{n_{+,\beta}}$ times

$$\begin{aligned}&\left[RR^{K_\beta}_\beta\left(\mathcal{U},\tilde{L}_N|_{\mathcal{U}}\otimes\det\mathcal{N}^{+,\beta}\otimes S(\mathcal{N}^{+,\beta}_{\mathbb{C}})\right)\wedge^\bullet_{\mathbb{C}}(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}\right]^{K_\beta}\\ &=\sum_{k\geq 0}\left[RR^{K_\beta}_\beta\left(\mathcal{U},\tilde{L}_N|_{\mathcal{U}}\otimes\det\mathcal{N}^{+,\beta}\otimes S^k(\mathcal{N}^{+,\beta}_{\mathbb{C}})\right)\wedge^\bullet_{\mathbb{C}}(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}\right]^{K_\beta}.\end{aligned}\tag{4.10}$$

Let $E\to\mathcal{U}$ be any K_β-equivariant Hermitian vector bundle. Since β acts trivially on $\mathcal{U}$ we can look at the Lie derivative $\mathcal{L}^E(\beta)$ on E. Then $\frac{1}{i}\mathcal{L}^E(\beta)$ defines for each $x\in\mathcal{U}$ a Hermitian endomorphism of E_x. Let us denote introduce Tian–Zhang's positivity condition (see (4.2) in [**27**]): we write

$$\frac{1}{i}\mathcal{L}^E(\beta)>0,$$

when all its eigenvalue on the fibers of E are strictly positive.

We made in [**21,** Lemma 9.4] the crucial observation

Lemma 4.7. *If* $\frac{1}{i}\mathcal{L}^E(\beta)>0$*, then* $\left[RR^{K_\beta}_\beta(\mathcal{U},E)\right]^{K_\beta}=0$.

Let us compute the Lie action $\mathcal{L}(\beta)$ on the fibers of the bundle $\tilde{L}_N|_{\mathcal{U}}\otimes\det\mathcal{N}^{+,\beta}\otimes S^k(\mathcal{N}^{+,\beta}_{\mathbb{C}})$. It is easy to check (see [**22**]) that on $\tilde{L}_N|_{\mathcal{U}}\otimes\det\mathcal{N}^{+,\beta}$ the Lie action $\frac{1}{i}\mathcal{L}(\beta)$ is equal to

$$\|\beta\|^2+\tfrac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}|.$$

Look now at the Lie derivative $\mathcal{L}(\beta)$ on $\wedge^\bullet_{\mathbb{C}}(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}$. As a T-module $\wedge^\bullet_{\mathbb{C}}(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}$ is equal to

$$\begin{aligned}\prod_{(\alpha,\beta)\neq 0}(1-\mathrm{e}^{\mathrm{i}\alpha})&=\prod_{(\alpha,\beta)<0}(1-\mathrm{e}^{\mathrm{i}\alpha})\prod_{(\alpha,\beta)>0}(1-\mathrm{e}^{\mathrm{i}\alpha})\\ &=(-1)^{1/2\dim(K/K_\beta)}\mathrm{e}^{-\mathrm{i}\delta_\beta}\Big(\prod_{(\alpha,\beta)>0}(1-\mathrm{e}^{\mathrm{i}\alpha})\Big)^2\end{aligned}$$

with $\delta_\beta=\sum_{(\alpha,\beta)>0}\alpha$. Note that $\mathrm{e}^{-\mathrm{i}\delta_\beta}$ defines a character of the group K_β that we denote $\mathbb{C}_{-\delta_\beta}$. We have proved then that

$$\wedge^\bullet_{\mathbb{C}}(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}=(-1)^{n_{+,\beta}}\,\mathbb{C}_{-\delta_\beta}\oplus R$$

where the Lie derivative $\frac{1}{i}\mathcal{L}(\beta)$ on $\mathbb{C}_{-\delta_\beta}$ is equal to $-(\delta_\beta,\beta)=-\mathbf{Tr}_\beta|\mathfrak{k}|$ and the Lie derivative $\frac{1}{i}\mathcal{L}(\beta)$ on the $\mathfrak{k}_\beta$-module R is $>-\mathbf{Tr}_\beta|\mathfrak{k}|$.

Since $\|\beta\|^2+\frac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}|=\mathbf{Tr}_\beta|\mathfrak{k}|$, we can conclude that the Lie derivative $\frac{1}{i}\mathcal{L}(\beta)$

(1) is equal to zero on $\tilde{L}_N|_{\mathcal{U}}\otimes\det\mathcal{N}^{+,\beta}\otimes\mathbb{C}_{-\delta_\beta}$,

(2) is >0 on $\tilde{L}_N|_{\mathcal{U}}\otimes\det\mathcal{N}^{+,\beta}\otimes R$,

(3) is >0 on $\tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \otimes S^k(\mathcal{N}^{+,\beta}_{\mathbb{C}}) \otimes \wedge^{\bullet}_{\mathbb{C}}(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}$ for any $k \geq 1$.

With Lemma 4.7, we see that the sum (4.10) restricts to

$$(-1)^{n_{+,\beta}} \left[RR^{K_\beta}_{\beta} \left(\mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \right) \otimes \mathbb{C}_{-\delta_\beta} \right]^{K_\beta}.$$

At this stage we have proved that the multiplicity m_μ is equal to

$$\left[RR^{K_\beta}_{\beta} \left(\mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \right) \otimes \mathbb{C}_{-\delta_\beta} \right]^{K_\beta}. \tag{4.11}$$

On the symplectic slice $(Y_\sigma, \omega_\sigma)$, we have the moment map $\Phi_\sigma - \mu$ relative to the action of Z_σ. The data $(Y_\sigma, \omega_\sigma, \Phi_\sigma - \mu)$ is Spin-prequantized by the line bundle $\tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}$. Let

$$RR^{Z_\sigma}_0(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \in R^{-\infty}(Z_\sigma) \tag{4.12}$$

be the Riemann–Roch character localized near $(\Phi_\sigma - \mu)^{-1}(0) = \Phi^{-1}(\mu + \rho - \rho_\sigma) \subset Y_\sigma$.

We conclude the computation of the multiplicity m_μ with the

Lemma 4.8. *We have*

$$\begin{aligned} \mathrm{m}_\mu &= \left[RR^{K_\beta}_{\beta} \left(\mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \right) \otimes \mathbb{C}_{-\delta_\beta} \right]^{K_\beta} \\ &= \left[RR^{Z_\sigma}_0(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \right]^{Z_\sigma} \qquad (1) \\ &= \mathcal{Q}_{\mathrm{spin}}(M^\sigma_\mu). \qquad (2) \end{aligned}$$

Proof. Let us prove that (1) is a consequence of the induction formula of Proposition 3.10. First, we notice that the data $(Y_\sigma, \omega_\sigma, \Phi_\sigma - \mu, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu})$ is naturally equipped with an action of the maximal torus, but with a trivial action of T/Z_σ. So the generalized character (4.12) coincides with

$$RR^T_0(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \in R^{-\infty}(T).$$

Let us consider the Hamiltonian K_β-manifold $\mathcal{U} := (K_\beta \times_T Y_\sigma) \times \overline{K_\beta(\mu+\rho)}$. Since K_β acts trivially on ρ_σ the map $\xi \mapsto \xi - \rho_\sigma$ realizes a K_β-equivariant symplectomorphic between the coadjoint orbits $\overline{K_\beta(\mu+\rho)}$ and

$$\overline{\mathcal{O}} := \overline{K_\beta(\mu + \rho - \rho_\sigma)}.$$

The manifold $\mathcal{U}$ is then symplectomorphic to $(K_\beta \times_T Y_\sigma) \times \overline{\mathcal{O}}$. Moreover, one sees that the generalized Riemann–Roch character $RR^{K_\beta}_\beta(\mathcal{U}, -)$ coincides with the Riemann–Roch character

$$RR^{K_\beta}_0((K_\beta \times_T Y_\sigma) \times \overline{\mathcal{O}}, -)$$

localized on $C_0 := K_\beta(\Phi^{-1}(\mu + \rho - \rho_\sigma) \times \{\mu + \rho - \rho_\sigma\})$.

Since $K_\beta \cap K_\sigma = T$, the Hamiltonian T-manifold Y_σ corresponds to the symplectic slice of the Hamiltonian K_β-manifold $K_\beta \times_T Y_\sigma$.

The bundle $\det\mathcal{N}^{+,\beta}$ over $(K_\beta \times_T Y_\sigma)\times\overline{\mathcal{O}}$ is equal to the product of $K_\beta\times_T \mathbb{C}_{\delta_1}\to K_\beta\times_T Y_\sigma$ with $K_\beta\times_T\mathbb{C}_{\delta_2}\to\overline{\mathcal{O}}$, where

$$\delta_1=\sum_{\substack{\alpha>0\\(\alpha,\beta)>0}}\alpha \quad\text{and}\quad \delta_2=\sum_{\substack{\alpha<0\\(\alpha,\beta)>0}}\alpha.$$

The line bundle $\tilde{L}_N$ is equal to the product of $\tilde{L}$ with $K\times_T\mathbb{C}_{-\mu}$. Then the restrictions of the lines bundle $\det\mathcal{N}^{+,\beta}$ and $\tilde{L}_N$ to $Y_\sigma\times\{\mu+\rho-\rho_\sigma\}$ are respectively equal to, the trivial line bundle $\mathbb{C}_{\delta_1+\delta_2}=\mathbb{C}_{\delta_\beta}$, and to the line bundle $\tilde{L}|_{Y_\sigma}\otimes\mathbb{C}_{-\mu}$.

Finally, the induction formula of Proposition 3.10 gives that

$$\begin{aligned}RR^{K_\beta}_\beta\left(\mathcal{U},\tilde{L}_N|_{\mathcal{U}}\otimes\det\mathcal{N}^{+,\beta}\right)&=RR^{K_\beta}_0((K_\beta\times_T Y_\sigma)\times\overline{\mathcal{O}},\tilde{L}_N|_{\mathcal{U}}\otimes\det\mathcal{N}^{+,\beta})\\&=\mathrm{Ind}_T^{K_\beta}\left(RR^T_0(Y_\sigma,\tilde{L}|_{Y_\sigma}\otimes\mathbb{C}_{-\mu})\otimes\mathbb{C}_{\delta_\beta}\right)\\&=\mathrm{Ind}_T^{K_\beta}\left(RR^T_0(Y_\sigma,\tilde{L}|_{Y_\sigma}\otimes\mathbb{C}_{-\mu})\right)\otimes\mathbb{C}_{\delta_\beta}.\end{aligned}$$

Hence

$$\begin{aligned}\left[RR^{K_\beta}_\beta\left(\mathcal{U},\tilde{L}_N|_{\mathcal{U}}\otimes\det\mathcal{N}^{+,\beta}\right)\otimes\mathbb{C}_{-\delta_\beta}\right]^{K_\beta}&=\left[RR^T_0(Y_\sigma,\tilde{L}|_{Y_\sigma}\otimes\mathbb{C}_{-\mu})\right]^T\\&=\left[RR^{Z_\sigma}_0(Y_\sigma,\tilde{L}|_{Y_\sigma}\otimes\mathbb{C}_{-\mu})\right]^{Z_\sigma}.\end{aligned}$$

Equality (2), i.e.,

$$\left[RR^{Z_\sigma}_0(Y_\sigma,\tilde{L}|_{Y_\sigma}\otimes\mathbb{C}_{-\mu})\right]^{Z_\sigma}=\mathcal{Q}_{\mathrm{spin}}(M^\sigma_\mu),$$

has been proved in Section 3.4. □

References

[1] M.F. Atiyah, Elliptic operators and compact groups, Springer, 1974. Lecture notes in Mathematics, **401**.

[2] M.F. Atiyah and G.B. Segal, *The index of elliptic operators II*, Ann. Math. **87** (1968), 531–545.

[3] M.F. Atiyah and I.M. Singer, *The index of elliptic operators I*, Ann. Math. **87** (1968), 484–530.

[4] M.F. Atiyah and I.M. Singer, *The index of elliptic operators III*, Ann. Math. **87** (1968), 546–604.

[5] M.F. Atiyah and I.M. Singer, *The index of elliptic operators IV*, Ann. Math. **93** (1971), 139–141.

[6] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren, vol. 298, Springer, Berlin, 1991.

[7] N. Berline and M. Vergne, *The Chern character of a transversally elliptic symbol and the equivariant index*, Invent. Math. **124** (1996), 11–49.

[8] N. Berline and M. Vergne, *L'indice équivariant des opérateurs transversalement elliptiques*, Invent. Math. **124** (1996), 51–101.

[9] A. Cannas da Silva, Y. Karshon and S. Tolman, *Quantization of presymplectic manifolds and circle actions*, Trans. Amer. Math. Soc. **352** (2000), 525–552.

[10] J. J. Duistermaat, *The heat equation and the Lefschetz fixed point formula for the Spin^c-Dirac operator*, Progress in Nonlinear Differential Equation and Their Applications, vol. 18, Birkhauser, Boston, 1996.

[11] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.

[12] V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math., **67** (1982), 515–538.

[13] V. Guillemin and S. Sternberg, *A normal form for the moment map*, in Differential Geometric Methods in Mathematical Physics (S. Sternberg, ed.), Reidel Publishing Company, Dordrecht, 1984.

[14] L. Jeffrey and F. Kirwan, *Localization and quantization conjecture*, Topology **36** (1997), 647–693.

[15] B. Kostant, *Quantization and unitary representations*, in *Modern Analysis and Applications*, Lecture Notes in Math., **170**, Springer-Verlag, 1970, 87–207.

[16] H. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton Math. Series, **38**. Princeton University Press, Princeton, 1989.

[17] E. Lerman, E. Meinrenken, S. Tolman and C. Woodward, *Non-Abelian convexity by symplectic cuts*, Topology, **37**, 1998, 245–259.

[18] E. Meinrenken, *Symplectic surgery and the Spin^c-Dirac operator*, Adv. Math. **134**, (1998), 240–277.

[19] E. Meinrenken and R. Sjamaar, *Singular reduction and quantization*, Topology **38** (1999), 699–762.

[20] P-E. Paradan, *Formules de localisation en cohomologie équivariante*, Composit. Math. **117** (1999), 243–293.

[21] P-E. Paradan, *Localization of the Riemann–Roch character*, J. Funct. Anal. **187** (2001), 442–509.

[22] P-E. Paradan, *Spin^c quantization and the K-multiplicities of the discrete series*, Annal. Sci. N. S. **36** (2003), 805–845.

[23] P-E. Paradan and M. Vergne, *Index of transversally elliptic operators*, Astérique, Soc. Math. Fr. **328** (2009), 297–338.

[24] R. Sjamaar, *Symplectic reduction and Riemann–Roch formulas for multiplicities*, Bull. Amer. Math. Soc. **33** (1996), 327–338.

[25] J.M. Souriau, *Structure des systèmes dynamiques*, Maîtrise de mathématiques, Dunod, 1970.

[26] C. Teleman, *The quantization conjecture revisited*, Annal. Math. **152** (2000), 1–43.

[27] Y. Tian and W. Zhang, *An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg*, Invent. Math. **132** (1998), 229–259.

[28] M. Vergne, *Multiplicity formula for geometric quantization, Part I, Part II, and Part III*, Duke Math. J. **82** (1996), 143–179, 181–194, 637–652.

[29] M. Vergne, *Quantification géométrique et réduction symplectique*, Séminaire Bourbaki **888**, 2001.

[30] N.M.J. Woodhouse, *Geometric quantization*, 2nd edn., Oxford Mathematical Monographs. Clarendon Press, Oxford, 1997.

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