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COHOMOLOGICALLY SYMPLECTIC SOLVMANIFOLDS ARE SYMPLECTIC

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We consider aspherical manifolds with torsion-free virtually polycyclic fundamental groups, constructed by Baues. We prove that if those manifolds are cohomologically symplectic then they are symplectic. As a corollary we show that cohomologically symplectic solvmanifolds are symplectic.

1. Introduction

A $2n$-dimensional compact manifold M is called cohomologically symplectic (c-symplectic) if we have $\omega \in H^2(M, \mathbb{R})$ such that $\omega^n \neq 0$. A compact symplectic manifold is c-symplectic but the converse is not true in general. For example $\mathbb{C}P^2 \# \mathbb{C}P^2$ is c-symplectic but not symplectic. But for some class of manifolds these two conditions are equivalent. For examples, nilmanifolds, i.e., compact homogeneous spaces of nilpotent simply connected Lie groups. In [**7**], for a nilpotent simply connected Lie group G with a cocompact discrete subgroup Γ (such subgroup is called a lattice), Nomizu showed that the de Rham cohomology $H^*(G/\Gamma, \mathbb{R})$ of G/Γ is isomorphic to the cohomology $H^*(\mathfrak{g})$ of the Lie algebra of G. By the application of Nomizu's theorem, if G/Γ is c-symplectic then G/Γ is symplectic (see [**3, p. 191**]). Every nilmanifold can be represented by such G/Γ (see [**6**]).

Consider solvmanifolds, i.e., compact homogeneous spaces of solvable simply connected Lie groups. Let G be a solvable simply connected Lie group with a lattice Γ. We assume that for any $g \in G$ the all eigenvalues of the adjoint operator Ad_g are real. With this assumption, in [**5**] Hattori extended Nomizu's theorem. By Hattori's theorem, for such case, without difficulty, we can similarly show that if G/Γ is c-symplectic, then G/Γ is symplectic. But the isomorphism $H^*(G/\Gamma, \mathbb{R}) \cong H^*(\mathfrak{g})$ fails to hold for general solvable Lie groups, and not all solvmanifolds can be represented by G/Γ. Thus it is a considerable problem whether every c-symplectic solvmanifold is symplectic.

Let Γ be a torsion-free virtually polycyclic group. In [**1**] Baues constructed the compact aspherical manifold M_Γ with $\pi_1(M_\Gamma) = \Gamma$. Baues proved that every infra-solvmanifold (see [**1**] for the definition) is diffeomorphic to M_Γ. In particular, the class of such aspherical manifolds contains the class of solvmanifolds. We prove that if M_Γ is c-symplectic then M_Γ is symplectic. In other words, for a torsion-free virtually polycyclic group Γ with $2n = \operatorname{rank}\Gamma$, if there exists $\omega \in H^2(\Gamma, \mathbb{R})$ such that $\omega^n \neq 0$ then we have a symplectic aspherical manifold with the fundamental group Γ.

2. Notation and conventions

A general reference here is [**2**]. Let k be a subfield of $\mathbb{C}$. A group $\mathbf{G}$ is called a k-algebraic group if $\mathbf{G}$ is a Zariski-closed subgroup of $GL_n(\mathbb{C})$ which is defined by polynomials with coefficients in k. Let $\mathbf{G}(k)$ denote the set of k-points of $\mathbf{G}$ and $\mathbf{U}(\mathbf{G})$ the maximal Zariski-closed unipotent normal k-subgroup of $\mathbf{G}$ called the unipotent radical of $\mathbf{G}$. Let $U_n(k)$ denote the $n \times n$ k-valued upper triangular unipotent matrix group.

3. Aspherical manifolds with torsion-free virtually polycyclic fundamental groups

Definition 3.1. *A group Γ is polycyclic if it admits a sequence*

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{e\}$$

of subgroups such that each Γ_i is normal in Γ_{i-1} and Γ_{i-1}/Γ_i is cyclic. We denote $\operatorname{rank}\Gamma = \sum_{i=1}^{i=k} \operatorname{rank}\Gamma_{i-1}/\Gamma_i$.

Proposition 3.1 [8, Proposition 3.10]. *The fundamental group of a solvmanifold is torsion-free polycyclic.*

Let k be a subfield of $\mathbb{C}$. Let Γ be a torsion-free virtually polycyclic group. For a finite index polycyclic subgroup $\Delta \subset \Gamma$, we denote $\operatorname{rank}\Gamma = \operatorname{rank}\Delta$.

Definition 3.2. *We call a k-algebraic group $\boldsymbol{H}_\Gamma$ a k-algebraic hull of Γ if there exists an injective group homomorphism $\psi : \Gamma \to \boldsymbol{H}_\Gamma(k)$ and $\boldsymbol{H}_\Gamma$ satisfies the following conditions:*

(1) *$\psi(\Gamma)$ is Zariski-dense in $\mathbf{H}_{\boldsymbol{\Gamma}}$.*
(2) *$Z_{\boldsymbol{H}_\Gamma}(\boldsymbol{U}(\boldsymbol{H}_\Gamma)) \subset \boldsymbol{U}(\boldsymbol{H}_\Gamma)$ where $Z_{\boldsymbol{H}_\Gamma}(\boldsymbol{U}(\boldsymbol{H}_\Gamma))$ is the centralizer of $\boldsymbol{U}(\boldsymbol{H}_\Gamma)$.*
(3) *$\dim \boldsymbol{U}(\boldsymbol{H}_\Gamma) = \operatorname{rank}\Gamma$.*

Theorem 3.1 [1, Theorem A.1]. *There exists a k-algebraic hull of Γ and a k-algebraic hull of Γ is unique up to k-algebraic group isomorphism.*

Let Γ be a torsion-free virtually polycyclic group and $\mathbf{H}_\Gamma$ the $\mathbb{Q}$-algebraic hull of Γ. Denote $H_\Gamma = \mathbf{H}_\Gamma(\mathbb{R})$. Let U_Γ be the unipotent radical of H_Γ

and T a maximal reductive subgroup. Then H_Γ decomposes as a semi-direct product $H_\Gamma = T \ltimes U_\Gamma$. Let $\mathfrak{u}$ be the Lie algebra of U_Γ. Since the exponential map $\exp : \mathfrak{u} \longrightarrow U_\Gamma$ is a diffeomorphism, U_Γ is diffeomorphic to $\mathbb{R}^n$ such that $n = \operatorname{rank}\Gamma$. For the semi-direct product $H_\Gamma = T \ltimes U_\Gamma$, we denote $\phi : T \to \operatorname{Aut}(U_\Gamma)$ the action of T on U_Γ. Then we have the homomorphism $\alpha : H_\Gamma \longrightarrow \operatorname{Aut}(U_\Gamma) \ltimes U_\Gamma$ such that $\alpha(t, u) = (\phi(t), u)$ for $(t, u) \in T \ltimes U_\Gamma$. By the property (2) in Definition 3.2, ϕ is injective and hence α is injective.

In [**1**] Baues constructed a compact aspherical manifold $M_\Gamma = \alpha(\Gamma)\backslash U_\Gamma$ with $\pi_1(M_\Gamma) = \Gamma$. We call M_Γ a standard Γ-manifold.

Theorem 3.2 [1, Theorem 1.2, 1.4]. *A standard Γ-manifold is unique up to diffeomorphism. A solvmanifold with the fundamental group Γ is diffeomorphic to the standard Γ-manifold M_Γ.*

Let $A^*(M_\Gamma)$ be the de Rham complex of M_Γ. Then $A^*(M_\Gamma)$ is the set of the Γ-invariant differential forms $A^*(U_\Gamma)^\Gamma$ on U_Γ. Let $(\bigwedge \mathfrak{u}^*)^T$ be the left-invariant forms on U_Γ which are fixed by T. Since $\Gamma \subset H_\Gamma = T \ltimes U_\Gamma$, we have the inclusion

$$\left(\bigwedge \mathfrak{u}^*\right)^T = A^*(U_\Gamma)^{H_\Gamma} \subset A^*(U_\Gamma)^\Gamma = A^*(M_\Gamma).$$

Theorem 3.3 [1, Theorem 1.8]. *This inclusion induces an isomorphism on cohomology.*

By the application of the above facts, we prove the main theorem of this paper.

Theorem 3.4. *Suppose M_Γ is c-symplectic. Then M_Γ admits a symplectic structure. In particular, cohomologically symplectic solvmanifolds are symplectic.*

Proof. Since we have the isomorphism $H^*(M_\Gamma, \mathbb{R}) \cong H^*((\bigwedge \mathfrak{u}^*)^T)$, we have $\omega \in (\bigwedge^2 \mathfrak{u}^*)^T$ such that $0 \neq [\omega]^n \in H^{2n}((\bigwedge \mathfrak{u}^*)^T)$. This gives $0 \neq \omega^n \in (\bigwedge \mathfrak{u}^*)^T$ and hence $0 \neq \omega^n \in \bigwedge \mathfrak{u}^*$. Since ω^n is a non-zero invariant $2n$-form on U_Γ, we have $(\omega^n)_p \neq 0$ for any $p \in U_\Gamma$. Hence by the inclusion $(\bigwedge \mathfrak{u}^*)^T \subset A^*(U_\Gamma)^T = A^*(M_\Gamma)$, we have $(\omega^n)_{\Gamma p} \neq 0$ for any $\Gamma p \in \Gamma\backslash U_\Gamma = M_\Gamma$. This implies that ω is a symplectic form on M_Γ. Hence, we have the theorem. $\square$

4. Remarks

Let $G = \mathbb{R} \ltimes_\phi U_3(\mathbb{C})$ such that

$$\phi(t) \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{e}^{\mathrm{i}\pi t} \cdot x & z \\ 0 & 1 & \mathrm{e}^{-\mathrm{i}\pi t} \cdot y \\ 0 & 0 & 1 \end{pmatrix},$$

and $D = \mathbb{Z} \ltimes_{\phi} D'$ with

$$D' = \left\{ \begin{pmatrix} 1 & x_1 + \mathrm{i}x_2 & z_1 + \mathrm{i}z_2 \\ 0 & 1 & y_1 + \mathrm{i}y_2 \\ 0 & 0 & 1 \end{pmatrix} : x_1, y_2, z_2 \in \mathbb{Z},\, x_2, y_1, z_1 \in \mathbb{R} \right\}.$$

Then D is not discrete and G/D is compact. We have $D/D_0 \cong \mathbb{Z} \ltimes_{\varphi} U_3(\mathbb{Z})$ such that

$$\varphi(t) \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (-1)^t x & z \\ 0 & 1 & (-1)^{-t} y \\ 0 & 0 & 1 \end{pmatrix},$$

where D_0 is the identity component of D. Denote $\Gamma = D/D_0$. We have the algebraic hull $H_\Gamma = \{\pm 1\} \ltimes_{\psi} (U_3(\mathbb{R}) \times \mathbb{R})$ such that

$$\psi(-1) \cdot \left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, t \right) = \left(\begin{pmatrix} 1 & -x & z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}, t \right).$$

The dual of the Lie algebra $\mathfrak{u}$ of $U_3(\mathbb{R}) \times \mathbb{R}$ is given by $\mathfrak{u}^* = \langle \alpha, \beta, \gamma, \delta \rangle$ such that the differential is given by

$$d\alpha = d\beta = d\delta = 0,$$
$$d\gamma = -\alpha \wedge \beta,$$

and the action of $\{\pm 1\}$ is given by

$$(-1) \cdot \alpha = -\alpha, (-1) \cdot \beta = -\beta,$$
$$(-1) \cdot \gamma = \gamma,\, (-1) \cdot \delta = \delta.$$

Then we have a diffeomorphism $M_\Gamma \cong G/D$ and an isomorphism $H^*(M_\Gamma, \mathbb{R}) \cong H^*((\bigwedge \mathfrak{u}^*)^{\{\pm 1\}})$. By simple computations, $H^2((\bigwedge \mathfrak{u}^*)^{\{\pm 1\}}) = 0$ and hence the solvmanifold G/D is not symplectic.

Remark 1. The proof of the Theorem 3.4 contains a proof of the following proposition.

Proposition 4.1. *If M_Γ admits a symplectic structure, then U_Γ has an invariant symplectic form.*

Otherwise for the above example, $U_\Gamma = U_3(\mathbb{R}) \times \mathbb{R}$ has an invariant symplectic form but M_Γ is not symplectic. Thus the converse of this proposition is not true. If Γ is nilpotent, then T is trivial and any invariant symplectic form on U_Γ induces the symplectic form on M_Γ. Hence for nilmanifolds, the converse of Proposition 4.1 is true.

Remark 2. Γ is a finite extension of a lattice of $U_\Gamma = U_3(\mathbb{R}) \times \mathbb{R}$. Hence M_Γ is finitely covered by a Kodaira–Thurston manifold (see [**9**], [**3, p. 192**]). M_Γ is an example of a non-symplectic manifold finitely covered by a symplectic manifold.

Let $H = G \times \mathbb{R}$. Then the dual of the Lie algebra $\mathfrak{h}$ of H is given by $\mathfrak{h}^* = \langle \sigma, \tau, \zeta_1, \zeta_2, \eta_1, \eta_2, \theta_1, \theta_2 \rangle$ such that the differential is given by

$$d\sigma = d\tau = 0,$$
$$d\zeta_1 = \tau \wedge \zeta_2,\ d\zeta_2 = -\tau \wedge \zeta_1,$$
$$d\eta_1 = \tau \wedge \eta_2,\ d\eta_2 = -\tau \wedge \eta_1,$$
$$d\theta_1 = -\zeta_1 \wedge \eta_1 + \zeta_2 \wedge \eta_2,\ d\theta_2 = -\zeta_1 \wedge \eta_2 - \zeta_2 \wedge \eta_1.$$

By simple computations, any closed invariant 2-form $\omega \in \bigwedge^2 \mathfrak{h}^*$ satisfies $\omega^4 = 0$. Hence H has no invariant symplectic form. Otherwise we have a lattice $\Delta = 2\mathbb{Z} \ltimes U_3(\mathbb{Z} + i\mathbb{Z}) \times \mathbb{Z}$ which is also a lattice of $\mathbb{R}^2 \times U_3(\mathbb{C})$. Thus H/Δ is diffeomorphic to a direct product of a two-dimensional torus and an Iwasawa manifold (see [**4**]). Since an Iwasawa manifold is symplectic (see [**4**]), H/Δ is also symplectic. By this example we can say:

Remark 3. For a simply connected nilpotent Lie group G with a lattice Γ, if the nilmanifold G/Γ is symplectic then G has an invariant symplectic form. But suppose G is solvable we have an example of a symplectic solvmanifold G/Γ such that G has no invariant symplectic form.

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