DISCRETE HAMILTON–PONTRYAGIN MECHANICS AND GENERATING FUNCTIONS ON LIE GROUPOIDS

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We present a discrete analog of the recently introduced Hamilton–Pontryagin variational principle in Lagrangian mechanics. This unifies two, previously disparate approaches to discrete Lagrangian mechanics: either using the discrete Lagrangian to define a finite version of Hamilton's action principle, or treating it as a symplectic generating function. This is demonstrated for a discrete Lagrangian defined on an arbitrary Lie groupoid; the often encountered special case of the pair groupoid (or Cartesian square) is also given as a worked example.

1. Introduction

In a recent paper, Yoshimura and Marsden [20] introduced the Hamilton–Pontryagin variational principle for Lagrangian mechanics. Given a smooth configuration manifold Q and a Lagrangian $L : TQ \to \mathbb{R}$, this principle defines an action for paths in the so-called Pontryagin bundle $TQ \oplus T^*Q$, whose elements are written (q, v, p). Critical paths for this action, in addition to satisfying the usual Euler–Lagrange equations $\dot{p} = \partial L(q, v)/\partial q$, also satisfy the the Legendre transform $p = \partial L(q, v)/\partial v$ and the second-order curve condition $\dot{q} = v$. This can be seen as a unification of two equivalent, but previously disparate, approaches to Lagrangian mechanics, where the Lagrangian is used either (a) to define Hamilton's action functional, studying its critical paths, or (b) to define the Legendre transform, using it to pull back the canonical symplectic structure from the cotangent bundle to the tangent bundle.

There is a similar dilemma in the approach to discrete Lagrangian mechanics. Given a discrete Lagrangian $L_h : Q \times Q \to \mathbb{R}$ (or more generally, $L_h : G \to \mathbb{R}$, where $G \rightrightarrows Q$ is a Lie groupoid), one can either (a) use L_h to define a discrete version of Hamilton's action principle, or (b) treat L_h as a symplectic generating function. While some progress has been made

towards combining these approaches into a discrete Hamilton–Pontryagin principle for certain important special cases — namely, the pair groupoid $Q \times Q$ when Q is either a vector space [5, 6] or a Lie group [1] — this problem has not yet been resolved for general configuration manifolds Q, nor for the even more general case of a Lie groupoid $G \rightrightarrows Q$.

In this paper, we present a discrete Hamilton–Pontryagin principle, which is shown to unify these two disparate approaches (variational principles vs. generating functions) for discrete Lagrangian mechanics on arbitrary Lie groupoids. We begin, in Section 2, by giving a brief review of Lagrangian mechanics, including the continuous Hamilton–Pontryagin principle, as well as summarizing the existing frameworks for discrete Lagrangian mechanics on $Q \times Q$ and on Lie groupoids $G \rightrightarrows Q$. Next, in Section 3, we introduce the discrete Hamilton–Pontryagin principle, which is defined with respect to paths in the cotangent groupoid $T^*G \rightrightarrows A^*G$ beginning at the zero section; the approach is related to that used by Milinković [11] in studying the Morse homology of generating functions. This variational principle and its solutions, which imply those of the previous approaches to discrete Lagrangian mechanics, are derived first in cotangent bundle coordinates and then given intrinsically. Finally, in Section 4, we work out the special case of the pair groupoid $Q \times Q$, in particular showing how this corresponds to the formulation of Leok and Ohsawa [6] when Q is a vector space.

2. Lagrangian mechanics, continuous and discrete

2.1. Lagrangian and Hamilton–Pontryagin mechanics. Let Q be a smooth configuration manifold, and $L : TQ \to \mathbb{R}$ be a Lagrangian on its tangent bundle. There are two theoretically equivalent, but conceptually distinct, approaches to the Lagrangian mechanics of this system. The first, which we will call the variational approach, is to study critical paths $q : [a, b] \to Q$ for the action functional

$$S(q) = \int_a^b L\left(q(t), \dot{q}(t)\right) dt.$$

Such a path is critical if and only if it satisfies the Euler–Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

The second approach, which we will call the symplectic approach, is to use the Legendre transform $\mathbb{F}L : TQ \to T^*Q$, which takes $(q, v) \mapsto (q, \partial L/\partial v)$, to pull back the canonical symplectic structure from T^*Q to TQ. If ω is the canonical symplectic 2-form on T^*Q, then one can define a 2-form $\omega_L = (\mathbb{F}L)^* \omega$ on TQ. (The form ω_L is symplectic when L is a hyperreg-

ular Lagrangian, and presymplectic more generally.) Next, one defines the energy function $E : TQ \to \mathbb{R}$ given by

$$E(q, v) = \mathbb{F}L(q, v) v - L(q, v).$$

Finally, then, one looks for vector fields $X \in \mathfrak{X}(TQ)$ that satisfy

$$i_X \omega_L = dE,$$

which is essentially the tangent bundle version of Hamilton's equations on T^*Q. (For further background, see [9, Chapter 7].)

Yoshimura and Marsden [20] showed that these two approaches can be unified through an expanded variational principle, which they call the *Hamilton–Pontryagin principle*. Given a path $(q, v, p) : [a, b] \to TQ \oplus T^*Q$, the Hamilton–Pontryagin action is given by

$$\tilde{S}(q, v, p) = \int_a^b \left[L(q(t), v(t)) + p(t)(\dot{q}(t) - v(t)) \right] dt.$$

This is essentially the usual action functional — except, rather than simply prescribing the second-order curve constraint $\dot{q} = v$, one treats q and v as independent variables and then uses p as a Lagrange multiplier to enforce this constraint. Varying over paths with prescribed endpoints, so that $\delta q(a) = 0$ and $\delta q(b) = 0$, the variation of the action is

$$d\tilde{S}(q, v, p)(\delta q, \delta v, \delta p)$$
$$= \int_a^b \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial v} \delta v + p(\delta \dot{q} - \delta v) + \delta p(\dot{q} - v) \right] dt$$
$$= \int_a^b \left[\left(\frac{\partial L}{\partial q} - \dot{p} \right) \delta q + \left(\frac{\partial L}{\partial v} - p \right) \delta v + \delta p(\dot{q} - v) \right] dt.$$

Therefore, (q, v, p) is a critical path if and only if it solves the so-called *implicit Euler–Lagrange equations*

$$\dot{p} = \frac{\partial L}{\partial q}, \quad p = \frac{\partial L}{\partial v}, \quad \dot{q} = v,$$

which combine the Euler–Lagrange equations, the Legendre transform, and the second-order curve condition into a single set of equations. It should be noted that this principle is especially useful for studying constrained and other degenerate systems, and is closely connected with the generalized Legendre transform of Tulczyjew [15, 16], the generalized Hamiltonian dynamics formalism of Skinner and Rusk [13], and the Dirac structures of Courant [3].

2.2. Discrete Lagrangian mechanics. The idea of discrete Lagrangian mechanics was put forward in seminal papers by Suris [14] and Moser and Veselov [12], among others, and a general theory was developed over the subsequent decade. (See [10], for a comprehensive survey.) This work was motivated by the need to develop structure-preserving (e.g., symplectic) numerical integrators for Lagrangian mechanical systems on general configuration manifolds; the methods developed using this discrete Lagrangian framework are called *variational integrators*.

For most of the work in this field, the starting point is to replace the Lagrangian $L : TQ \to \mathbb{R}$ by a *discrete Lagrangian* $L_h : Q \times Q \to \mathbb{R}$, which approximates the contribution to the action integral,

$$L_h(q_0, q_1) \approx \int_{t_0}^{t_1} L(q(t), \dot{q}(t))\, dt,$$

for a time step of size $h = t_1 - t_0$. As with continuous Lagrangian mechanics, there are two typical ways to proceed, following either the variational or the symplectic point of view.

The variational approach to discrete Lagrangian mechanics is as follows. Suppose that we specify a sequence of time steps $a = t_0 < t_1 < \cdots < t_N = b$, with equal step size $h = t_{n+1} - t_n$ for $n = 0, \ldots, N - 1$. We then define a discrete path to be a sequence of configuration points $q_0, \ldots, q_N \in Q$; this can be thought of as approximating a continuous path $q : [a, b] \to Q$, with $q_n \approx q(t_n)$. Given the discrete Lagrangian $L_h : Q \times Q \to \mathbb{R}$, the *discrete action sum* is defined to be

$$S_h(q_0, \ldots, q_N) = \sum_{n=0}^{N-1} L_h(q_n, q_{n+1}) \approx \int_a^b L(q(t), \dot{q}(t))\, dt.$$

Next, taking fixed-endpoint variations of the discrete path, so that $\delta q_0 = 0$ and $\delta q_N = 0$, it follows that

$$dS_h(q_0, \ldots, q_N)(\delta q_0, \ldots, \delta q_N) = \sum_{n=1}^{N-1} [\partial_0 L_h(q_n, q_{n+1}) + \partial_1 L_h(q_{n-1}, q_n)]\delta q_n.$$

Therefore, a discrete path is critical if and only if it satisfies the *discrete Euler–Lagrange equations*

$$\partial_0 L_h(q_n, q_{n+1}) + \partial_1 L_h(q_{n-1}, q_n) = 0, \quad n = 1, \ldots, N - 1.$$

This implicitly defines a two-step numerical integrator on $Q \times Q$, which (given suitable assumptions of nondegeneracy) maps $(q_{n-1}, q_n) \mapsto (q_n, q_{n+1})$.

On the other hand, the symplectic approach to discrete Lagrangian mechanics is to view $L_h : Q \times Q \to \mathbb{R}$ as the generating function for a

symplectic map on T^*Q. To do this, one defines the *discrete Legendre transforms* $\mathbb{F}^{\pm} L_h : Q \times Q \to T^*Q$ by

$$\mathbb{F}^- L_h (q_0, q_1) = -\partial_0 L_h (q_0, q_1), \quad \mathbb{F}^+ L_h (q_0, q_1) = \partial_1 L_h (q_0, q_1).$$

Therefore, we can implicitly define a map on T^*Q by

$$p_0 = \mathbb{F}^- L_h (q_0, q_1), \quad p_1 = \mathbb{F}^+ L_h (q_0, q_1),$$

and if $\mathbb{F}^- L_h$ is invertible, this defines one step of the symplectic integrator

$$\mathbb{F}^+ L_h \circ \left(\mathbb{F}^- L_h\right)^{-1} : T^*Q \to T^*Q, \quad (q_0, p_0) \mapsto (q_1, p_1).$$

More precisely, the discrete Legendre transforms define the Lagrangian submanifold of $(T^*Q, -\omega) \times (T^*Q, \omega)$ generated by L_h. From this perspective, if the invertibility condition holds, then this submanifold is the graph of a symplectic map on T^*Q [see 17, 18].

Note that, if we perform the composition in the opposite order, we get the previously derived two-step method,

$$\left(\mathbb{F}^- L_h\right)^{-1} \circ \mathbb{F}^+ L_h : Q \times Q \to Q \times Q, \quad (q_{n-1}, q_n) \mapsto (q_n, q_{n+1}),$$

where the discrete Euler–Lagrange equations follow automatically from the fact that $\mathbb{F}^- L_h (q_n, q_{n+1}) = \mathbb{F}^+ L_h (q_{n-1}, q_n)$.

2.3. Discrete Lagrangian mechanics and Lie groupoids. Weinstein [19] observed that both approaches in the previous section can be generalized using Lie groupoids. Let $G \rightrightarrows Q$ be a given Lie groupoid, and define a discrete Lagrangian $L_h : G \to \mathbb{R}$. The earlier formulations then coincide with the special case $G = Q \times Q$, which is called the *pair groupoid*. This perspective has continued to bear fruit in recent years, being further developed by Marrero *et al.* [8] and extended to discrete nonholonomic Lagrangian mechanics by Iglesias *et al.* [4].

In the variational approach, one begins by taking a fixed element $g \in G$ and considering the space of *admissible sequences*, which consist of composable elements $g_1, \ldots, g_N \in G$ such that $g_1 \cdots g_N = g$. The discrete action for an admissible sequence is then taken to be

$$S_h (g_1, \ldots, g_N) = \sum_{n=1}^{N} L_h (g_n),$$

and discrete Euler–Lagrange equations are obtained by finding sequences that are critical for this action function. In the case of the pair groupoid $Q \times Q$, fixing an element of the groupoid corresponds simply to fixing the endpoints $g = (q_0, q_N)$, while the set of admissible sequences $(q_0, q_1), \ldots, (q_{N-1}, q_N) \in Q \times Q$ can be identified with the sequence of configuration points $q_0, \ldots, q_N \in Q$.

Marrero *et al.* [8] showed that the discrete Euler–Lagrange equations can be expessed in terms of left- and right-invariant vector fields on G, each of which can be identified with sections of the Lie algebroid $AG \to Q$ associated to G. To describe this, we must first introduce some notation. Let $\alpha, \beta : G \to Q$ denote the source and target maps, $\epsilon : Q \to G$ denote the identity section, and $i : G \to G$ denote the inversion map on G. For any $g \in G$, define the left- and right-translation maps, respectively, by

$$\ell_g : \alpha^{-1}\left(\beta(g)\right) \to \alpha^{-1}\left(\alpha(g)\right) \qquad r_g : \beta^{-1}\left(\alpha(g)\right) \to \beta^{-1}\left(\beta(g)\right)$$
$$g' \mapsto gg' \qquad\qquad\qquad\qquad g' \mapsto g'g.$$

Then, given a section $X \in \Gamma\left(AG\right)$, the left-invariant vector field $\overleftarrow{X} \in \mathfrak{X}(G)$ is defined by $\overleftarrow{X}(g) = T_{\epsilon(\beta(g))}\ell_g\left(X\left(\beta(g)\right)\right)$. Similarly, the right-invariant vector field $\overrightarrow{X} \in \mathfrak{X}(G)$ is given by $\overrightarrow{X}(g) = -\left(T_{\epsilon(\alpha(g))}\left(r_g \circ i\right)\right)\left(X\left(\alpha(g)\right)\right)$. This implies the following relationship between the brackets $[\![\cdot, \cdot]\!]$ on $\Gamma\left(AG\right)$ and $[\cdot, \cdot]$ on $\mathfrak{X}(G)$:

$$\overleftarrow{[\![X, Y]\!]} = \left[\overleftarrow{X}, \overleftarrow{Y}\right], \quad \overrightarrow{[\![X, Y]\!]} = -\left[\overrightarrow{X}, \overrightarrow{Y}\right].$$

With these definitions, the discrete Euler–Lagrange equations are

$$\overleftarrow{X}\left[L_h\right]\left(g_n\right) = \overrightarrow{X}\left[L_h\right]\left(g_{n+1}\right), \quad n = 1, \ldots, N-1$$

for all sections $X \in \Gamma\left(AG\right)$. In the pair groupoid case, one sees that

$$\overleftarrow{X}\left[L_h\right]\left(q_{n-1}, q_n\right) = \partial_1 L_h(q_{n-1}, q_n)X\left(q_n\right),$$
$$\overrightarrow{X}\left[L_h\right]\left(q_n, q_{n+1}\right) = -\partial_0 L_h(q_n, q_{n+1})X\left(q_n\right)$$

for any $X \in \mathfrak{X}(Q)$, so this formulation agrees with the earlier expression of the discrete Euler–Lagrange equations on $Q \times Q$.

For the symplectic approach, we begin by defining the *cotangent groupoid* $T^*G \rightrightarrows A^*G$, where the base $A^*G \to Q$ is the dual of the Lie algebroid $AG \to Q$. This is a *symplectic groupoid*, with a canonical symplectic structure on T^*G and Poisson structure on A^*G; moreover, the source and target maps are anti-Poisson and Poisson, respectively [2, 7]. Explicitly, the source map $\tilde\alpha : T^*G \to A^*G$ and target map $\tilde\beta : T^*G \to A^*G$ may be defined such that, taking $\mu \in T^*_g G$,

$$\tilde\alpha\left(\mu\right)X\left(\alpha(g)\right) = \mu\overrightarrow{X}(g), \quad \tilde\beta\left(\mu\right)X\left(\beta(g)\right) = \mu\overleftarrow{X}(g),$$

for every section $X \in \Gamma\left(AG\right)$. (The multiplicative structure of T^*G will not be necessary here, so we omit discussion of it.)

Now, the discrete Lagrangian $L_h : G \to \mathbb{R}$ generates a Lagrangian submanifold of the cotangent groupoid, $dL_h(G) \subset T^*G$. If L_h is suitably nondegenerate, it follows that this submanifold determines a Poisson automorphism $A^*G \to A^*G$ [19, 8]. Specifically, define the discrete Legendre transforms $\mathbb{F}^{\pm}L_h : G \to A^*G$ by

$$\mathbb{F}^- L_h = \tilde{\alpha} \circ dL_h, \quad \mathbb{F}^+ L_h = \tilde{\beta} \circ dL_h.$$

Therefore, $\{(\mathbb{F}^- L_h(g), \mathbb{F}^+ L_h(g)) \mid g \in G\}$ is a coisotropic relation on A^*G, and if $\mathbb{F}^- L_h$ is a diffeomorphism, then this relation is the graph of the discrete flow map

$$\mathbb{F}^+ L_h \circ \left(\mathbb{F}^- L_h\right)^{-1} : A^*G \to A^*G.$$

To see that this is compatible with the discrete Euler–Lagrange equations, observe that for any $X \in \Gamma(AG)$,

$$(\tilde{\alpha} \circ dL_h)(g)X(\alpha(g)) = dL_h(g)\overrightarrow{X}(g) = \overrightarrow{X}[L_h](g)$$

$$\left(\tilde{\beta} \circ dL_h\right)(g)X(\beta(g)) = dL_h(g)\overleftarrow{X}(g) = \overleftarrow{X}[L_h](g).$$

Thus, the discrete Euler–Lagrange equations correspond to the condition $\mathbb{F}^+ L_h(g_n) = \mathbb{F}^- L_h(g_{n+1})$. Equivalently, this means that $g_1, \ldots, g_N$ is a solution of the discrete Euler–Lagrange equations when $dL_h(g_1), \ldots, dL_h(g_N)$ is a composable sequence in T^*G.

3. The discrete Hamilton–Pontryagin principle

For the continuous Hamilton–Pontryagin principle, the key idea was to relax the condition for curves in TQ to be second-order, but to enforce this condition weakly using Lagrange multipliers in T^*Q. Analogously, we will develop a discrete Hamilton–Pontryagin principle by relaxing the requirement that the sequence $g_1, \ldots, g_N \in G$ is admissible — requiring only that it be *admissible up to homotopy* — and will weakly enforce the condition that this homotopy is, in fact, constant. To allow for the "Lagrange multipliers," these homotopies will be given by paths in T^*G rather than in G.

To simplify the exposition, we will first sketch this formulation, in Section 3.1, using cotangent bundle coordinates $(g, \mu) \in T^*G$. The fully intrinsic treatment on T^*G will be given subsequently, in Section 3.2, along with definitions and the proof of the main theorem.

3.1. Formulation in cotangent bundle coordinates. Suppose that $(g_n, \mu_n) : [0, 1] \to T^*G$, for $n = 1, \ldots, N$, is a sequence of paths in the cotangent groupoid. Furthermore, it is required that the initial point $(g_n(0), \mu_n(0))$ of each path lie in the zero section, so that $\mu_n(0) = 0_{g_n(0)}$, and that $g_1(0), \ldots, g_N(0)$ be an admissible sequence for a fixed element $g \in G$.

Given such a sequence of paths, define the discrete Hamilton–Pontryagin action to be

$$\tilde{S}_h\left(g_1, \mu_1, \ldots, g_N, \mu_N\right) = \sum_{n=1}^{N} \int_0^1 \left[L_h\left(g_n(s)\right) + \mu_n(s)g_n'(s)\right]ds.$$

Variations of this action are

$$d\tilde{S}_h\left(g_1, \mu_1, \ldots, g_N, \mu_N\right)\left(\delta g_1, \delta\mu_1, \ldots, \delta g_N, \delta\mu_N\right)$$

$$= \sum_{n=1}^{N} \int_0^1 \left[dL_h(g_n(s))\delta g_n(s) + \mu_n(s)\delta g_n'(s) + \delta\mu_n(s)g_n'(s)\right]ds$$

$$= \sum_{n=1}^{N} \left(\int_0^1 \left[(dL_h\left(g_n(s)\right) - \mu_n'(s))\delta g_n(s) + \delta\mu_n(s)g_n'(s)\right]ds\right.$$

$$\left. + \mu_n(1)\delta g_n(1)\right).$$

Now, the integral terms vanish when (g_n, μ_n) is a solution of

$$\mu_n'(s) = dL_h\left(g_n(s)\right), \quad g_n'(s) = 0.$$

The second equation states that $g_n(s) = g_n$ is constant, so $\mu_n(s)$ moves vertically along the fiber $T_{g_n}^* G$, beginning at $\mu_n(0) = 0_{g_n}$. We use this fact to solve the remaining equation,

$$\mu_n(s) = \mu_n(0) + s\,dL_h\left(g_n\right) = s\,dL_h\left(g_n\right),$$

so in particular, $\mu_n(1) = dL_h\left(g_n\right)$. Additionally, $g_1, \ldots, g_N$ is an admissible sequence, since $g_n = g_n(0)$, which was assumed to be admissible. Finally, restricting to (g_n, μ_n) where these equations are satisfied, the variation of the action is given by the remaining boundary terms

$$\sum_{n=1}^{N} \mu_n(1)\delta g_n(1) = \sum_{n=1}^{N} dL_h\left(g_n\right)\delta g_n.$$

However, the restricted variations δg_n are no longer completely arbitrary, since they must be tangent to the space of admissible sequences $g_1, \ldots, g_N$. Therefore, we are back in the case considered by Marrero *et al.* [8], so these terms vanish precisely when g_n is a solution of the discrete Euler–Lagrange equations — or equivalently, when $\mu_1(1), \ldots, \mu_N(1) \in T^*G$ is a composable sequence.

In summary, $d\tilde{S}_h = 0$ implies that $g_1(1), \ldots, g_N(1)$ is an admissible sequence in G, that $\mu_n(1) = dL_h\left(g_n(1)\right)$, and that $\mu_1(1), \ldots, \mu_N(1)$ is a composable sequence in T^*G.

3.2. Intrinsic formulation. We will now make precise the approach sketched out in the previous section, doing this intrinsically on T^*G rather than in cotangent bundle coordinates. Let $\pi_G : T^*G \to G$ be the cotangent bundle projection, $\tilde{\theta}$ be the canonical 1-form on T^*G, and $\tilde{\omega} = -d\tilde{\theta}$ be the canonical symplectic 2-form.

Given a Lie groupoid $G \rightrightarrows Q$, define the fibration $E \to G$ consisting of paths in T^*G beginning at the zero section,

$$E = \left\{ \gamma : [0,1] \to T^*G \mid \gamma(0) \in 0_G \right\},$$

where the projection is given by $\gamma \mapsto (\pi_G \circ \gamma)(1)$.

Definition 3.1. Given a fixed $g \in G$, a sequence of paths $\gamma_1, \ldots, \gamma_N \in E$ is said to be *admissible* if the initial elements $(\pi_G \circ \gamma_1)(0), \ldots, (\pi_G \circ \gamma_N)(0)$ form an admissible sequence in G.

This implies that the projected sequence $(\pi_G \circ \gamma_1)(1), \ldots, (\pi_G \circ \gamma_N)(1) \in G$ is only admissible up to homotopy, which is weaker than the usual assumption that $g_1, \ldots, g_N$ must actually be an admissible sequence.

Definition 3.2. Let $L_h : G \to \mathbb{R}$ be a discrete Lagrangian, and $\gamma_1, \ldots, \gamma_N \in E$ be an admissible sequence of paths. Then the *discrete Hamilton–Pontryagin action* of this sequence is

$$\tilde{S}_h (\gamma_1, \ldots, \gamma_N) = \sum_{n=1}^{N} \left[\int_0^1 (L_h \circ \pi_G)(\gamma_n(s))\, ds + \int_{\gamma_n} \tilde{\theta} \right],$$

The sequence is said to satisfy the *discrete Hamilton–Pontragin principle* if

$$d\tilde{S}_h (\gamma_1, \ldots, \gamma_N) = 0.$$

Theorem 3.1. *Let $\gamma_1, \ldots, \gamma_N \in E$ be an admissible sequence of paths, and denote $\mu_n = \gamma_n(1) \in T^*G$ and $g_n = \pi_G(\mu_n) \in G$ for $n = 1, \ldots, N$. If the sequence satisfies the Hamilton–Pontryagin principle, then*

(i) $g_1, \ldots, g_N$ is an admissible sequence in G,
(ii) $\mu_n = dL_h(g_n)$ for $n = 1, \ldots, N$,
(iii) $\mu_1, \ldots, \mu_N$ is a composable sequence in T^*G.

These properties may be thought of as, respectively, (i) the discrete second-order curve condition; (ii) the discrete Legendre transform; and (iii) the discrete Euler–Lagrange equations.

Proof. Take a variation of the Hamilton–Pontryagin action to obtain

$$d\tilde{S}_h (\gamma_1, \ldots, \gamma_N)(\delta\gamma_1, \ldots, \delta\gamma_N) = \sum_{n=1}^{N} \left[\int_0^1 (\pi_G^* dL_h)\delta\gamma_n(s)\, ds + \int_{\gamma_n} \mathfrak{L}_{\delta\gamma_n} \tilde{\theta} \right].$$

Applying Cartan's "magic formula," the Lie derivative terms become

$$\int_{\gamma_n} \mathfrak{L}_{\delta\gamma_n} \tilde{\theta} = \int_{\gamma_n} \left(i_{\delta\gamma_n} d\tilde{\theta} + d i_{\delta\gamma_n} \tilde{\theta} \right)$$

$$= -\int_{\gamma_n} i_{\delta\gamma_n} \tilde{\omega} + \int_{\partial\gamma_n} \tilde{\theta}\delta\gamma_n$$

$$= \int_0^1 i_{\gamma_n'(s)} \tilde{\omega}\delta\gamma_n(s)\, ds + \tilde{\theta}\delta\gamma_n(1).$$

Therefore, the variation of the action is

$$d\tilde{S}_h\, (\gamma_1, \ldots, \gamma_N)(\delta\gamma_1, \ldots, \delta\gamma_N)$$

$$= \sum_{n=1}^{N} \left[\int_0^1 \left(\pi_G^* dL_h + i_{\gamma_n'(s)} \tilde{\omega} \right) \delta\gamma_n(s)\, ds + \tilde{\theta}\delta\gamma_n(1) \right].$$

For variations taken along the fibers of E, this is stationary when

$$i_{\gamma_n'(s)} \tilde{\omega} = -\pi_G^* dL_h, \quad n = 1, \ldots, N,$$

i.e., the paths γ_n are solutions to Hamilton's equations for the singular Hamiltonian $\tilde{H} = -L_h \circ \pi_G : T^*G \to \mathbb{R}$. However, since $\tilde{H}$ is constant on fibers of T^*G (which form a Lagrangian foliation), the Hamiltonian flow is fiber-preserving, and hence the projection $(\pi_G \circ \gamma_n)(s) = g_n$ is constant. This proves (i), since $g_n = (\pi_G \circ \gamma_n)(0)$, which was assumed to be an admissible sequence in G. Furthermore, the Hamiltonian vector field is constant on fibers of T^*G, and it follows that $\gamma_n(s) = s\, dL_h(g_n)$. Therefore, $\mu_n = \gamma_n(1) = dL_h(g_n)$, which proves (ii).

Finally, restricting the action to these solutions, the integral terms vanish, and the remaining boundary terms are

$$\sum_{n=1}^{N} \tilde{\theta}\delta\gamma_n(1) = \sum_{n=1}^{N} \mu_n \delta g_n = \sum_{n=1}^{N} dL_h(g_n)\delta g_n.$$

This is precisely the variation of the usual discrete action, so these terms vanish when the discrete Euler–Lagrange equations are satisfied. Therefore, $\tilde{\beta}(\mu_n) = \tilde{\alpha}(\mu_{n+1})$ for $n = 1, \ldots, N-1$, so $\mu_1, \ldots, \mu_N$ is a composable sequence in T^*G, which completes the proof of (iii). $\qquad\square$

Remark 3.1. For a single time step $N = 1$, the action

$$\tilde{S}_h : E \to \mathbb{R}, \qquad \gamma \mapsto \int_0^1 (L_h \circ \pi_G)(\gamma(s))ds + \int_\gamma \tilde{\theta}$$

can be understood as a *Morse family*, which generates the usual Lagrangian submanifold $dL_h(G) \subset T^*G$. This is closely related to the work of Milinković [11], who studied a similar action principle on T^*Q, in connection with the Morse homology of generating functions for Lagrangian submanifolds. In

Milinković's formulation, the Lagrangian submanifold is determined by the time-1 Hamiltonian isotopy of $\tilde{H}$, which, in this case, takes the zero section to $dL_h(G)$.

4. Example: the pair groupoid

Let $G = Q \times Q$ be the pair groupoid, and $L_h : Q \times Q \to \mathbb{R}$ be a discrete Lagrangian. Fixing the endpoints $g = (q_0, q_N)$, suppose that $\gamma_1, \ldots, \gamma_N \in E$ is an admissible sequence of paths, where $\gamma_n(s) = \left(q_n^0(s), p_n^0(s), q_n^1(s), p_n^1(s) \right)$. Since the sequence is admissible, this implies in particular that $q_0 = q_1^0(0)$, $q_n^1(0) = q_{n+1}^0(0)$ for $n = 1, \ldots, N - 1$, and $q_N^1(0) = q_N$. The discrete Hamilton–Pontryagin action of this sequence is then

$$\tilde{S}_h (\gamma_1, \ldots, \gamma_N) = \sum_{n=1}^{N} \int_0^1 \left[L_h \left(q_n^0(s), q_n^1(s) \right) - p_n^0(s) q_n^{0\prime}(s) + p_n^1(s) q_n^{1\prime}(s) \right] ds.$$

Taking variations of this action gives

$$d\tilde{S}_h (\gamma_1, \ldots, \gamma_N)(\delta\gamma_1, \ldots, \delta\gamma_N)$$
$$= \sum_{n=1}^{N} \int_0^1 \left[\partial_0 L_h \left(q_n^0(s), q_n^1(s) \right) \delta q_n^0(s) + \partial_1 L_h \left(q_n^0(s), q_n^1(s) \right) \delta q_n^1(s) \right.$$
$$\left. - p_n^0(s) \delta q_n^{0\prime}(s) + p_n^1(s) \delta q_n^{1\prime}(s) - \delta p_n^0(s) q_n^{0\prime}(s) + \delta p_n^1(s) q_n^{1\prime}(s) \right] ds,$$

and integrating by parts, this simplifies to

$$d\tilde{S}_h (\gamma_1, \ldots, \gamma_N)(\delta\gamma_1, \ldots, \delta\gamma_N)$$
$$= \sum_{n=1}^{N} \left(\int_0^1 \left[\left(\partial_0 L_h \left(q_n^0(s), q_n^1(s) \right) + p_n^{0\prime}(s) \right) \delta q_n^0(s) \right. \right.$$
$$+ \left(\partial_1 L_h \left(q_n^0(s), q_n^1(s) \right) - p_n^{1\prime}(s) \right) \delta q_n^1(s)$$
$$\left. - \delta p_n^0(s) q_n^{0\prime}(s) + \delta p_n^1(s) q_n^{1\prime}(s) \right] ds$$
$$\left. - p_n^0(1) \delta q_n^0(1) + p_n^1(1) \delta q_n^1(1) \right).$$

For variations taken along the fibers of E, this is stationary when the integral terms vanish, so

$$p_n^{0\prime}(s) = -\partial_0 L_h \left(q_n^0(s), q_n^1(s) \right), \quad q_n^{0\prime}(s) = 0,$$
$$p_n^{1\prime}(s) = \partial_1 L_h \left(q_n^0(s), q_n^1(s) \right), \quad q_n^{1\prime}(s) = 0,$$

for $n = 1, \ldots, N$. Therefore, $q_n^0(s) = q_n^0$ and $q_n^1(s) = q_n^1$ are constant, so

$$p_n^0(1) = -\partial_0 L_h \left(q_n^0, q_n^1 \right), \quad p_n^1(1) = \partial_1 L_h \left(q_n^0, q_n^1 \right), \quad n = 1, \ldots, N.$$

Furthermore, the admissibility assumption implies that $q_0 = q_1^0$, $q_n^1 = q_{n+1}^0$ for $n = 1, \ldots, N-1$, and $q_N^1 = q_N$.

Finally, restricting to these solutions, the restricted variations must then satisfy $\delta q_1^0 = 0$, $\delta q_n^1 = \delta q_{n+1}^0 =$ for $n = 1, \ldots, N-1$, and $\delta q_N^1 = 0$. Therefore, the remaining terms of the action are

$$\sum_{n=1}^{N} \left(-p_n^0(1)\delta q_n^0 + p_n^1(1)\delta q_n^1 \right) = \sum_{n=1}^{N-1} \left(-p_{n+1}^0(1) + p_n^1(1) \right)\delta q_n^1,$$

which vanish when $p_n^1(1) = p_{n+1}^0(1)$ for $n = 1, \ldots, N-1$.

In summary, if $\gamma_1, \ldots, \gamma_N$ satisfies the discrete Hamilton–Pontryagin principle on $Q \times Q$, then the following is true:

(i) $q_0 = q_1^0$, $q_n^1 = q_{n+1}^0$ for $n = 1, \ldots, N-1$, and $q_N^1 = q_N$,
(ii) $p_n^0(1) = -\partial_0 L_h \left(q_n^0, q_n^1 \right)$ and $p_n^1(1) = \partial_1 L_h \left(q_n^0, q_n^1 \right)$ for $n = 1, \ldots, N$,
(iii) $p_n^1(1) = p_{n+1}^0(1)$ for $n = 1, \ldots, N-1$.

These are, respectively, the discrete second-order curve condition, the discrete Legendre transform, and the discrete Euler–Lagrange equations for systems on $Q \times Q$.

Remark 4.1. The cotangent paths p_n^0, p_n^1 can be seen as Lagrange multipliers, which serve to enforce the composability conditions $q_n^1 = q_{n+1}^0$, as well as the endpoint constraints $q_1^0 = q_0$ and $q_N^1 = q_N$. Leok and Ohsawa [6] showed that, when Q is a vector space (or given a particular local coordinate neighborhood), this can be done even more directly, by simply subtracting these points rather than taking a path between them. This leads to an alternative choice of the action sum,

$$\sum_{n=1}^{N} L_h \left(q_n^0, q_n^1 \right) + p_0 \left(q_1^0 - q_0 \right) + \sum_{n=1}^{N-1} p_n \left(q_{n+1}^0 - q_n^1 \right) + p_N \left(q_N^1 - q_N \right),$$

whose variations are computed to be

$$\sum_{n=1}^{N} \left[\left(\partial_0 L_h \left(q_n^0, q_n^1 \right) + p_{n-1} \right)\delta q_n^0 + \left(\partial_1 L_h \left(q_n^0, q_n^1 \right) - p_n \right)\delta q_n^1 \right]$$

$$+ \delta p_0 \left(q_1^0 - q_0 \right) + \sum_{n=1}^{N-1} \delta p_n \left(q_{n+1}^0 - q_n^1 \right) + \delta p_N \left(q_N^1 - q_N \right).$$

Therefore, this vanishes when

$$p_{n-1} = -\partial_0 L_h \left(q_n^0, q_n^1 \right), \quad p_n = \partial_1 L_h \left(q_n^0, q_n^1 \right), \quad n = 1, \ldots, N$$

and when

$$q_1^0 = q_0, \quad q_N^1 = q_N, \quad q_n^1 = q_{n+1}^0, \quad n = 1, \ldots, N-1,$$

which are consistent with the results obtained in this section.

Effectively, the Leok–Ohsawa approach can be interpreted as using a smaller fibration $E' = Q \times Q \times T^*Q \times T^*Q$, consisting only of the endpoints of the paths in E, to define a Morse family over $Q \times Q$. However, when Q is not a vector space, there is no intrinsic, global meaning to subtracting two points in Q. This action may still be defined locally, in a neighborhood of the diagonal of $Q \times Q$, given a choice of local coordinates or a retraction. In general, though, one requires the larger space of paths E in order to define the discrete Hamilton–Pontryagin principle globally.

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