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SPECTRAL MEASURES ON TORIC VARIETIES AND THE ASYMPTOTIC EXPANSION OF TIAN–YAU–ZELDITCH

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We extend a recent result of Burns, Guillemin and Uribe on the asymptotics of the spectral measure for the reduction metric on a toric variety to any toric metric on a toric variety. We show how this extended result together with the Tian–Yau–Zelditch asymptotic expansion can be used to deduce Abreu's formula for the scalar curvature of a toric metric on a toric variety in terms of polytope data.

1. Introduction

Recently, Burns, Guillemin and Uribe have described a procedure to give the asymptotic expansion of the so-called spectral measure sequence on a toric manifold. One obtains this sequence by choosing an orthonormal basis for the space of holomorphic sections of large tensor powers of a quantizing line bundle for the toric manifold and adding the square of the norms of these elements of the basis. More precisely, suppose we are given a toric symplectic manifold (X, ω). Let $L \to X$ be a line bundle whose first Chern class is $[\omega]$. There is a natural metric on X called the reduction metric which is invariant by the torus action on X (see [**G**]). This metric allows us to define a Hermitian metric on the bundle L. We simply ask that this Hermitian metric h satisfies the following: $\mathrm{i}\partial\bar{\partial}\log h$ is the Kähler metric on X. The space of holomorphic sections of L^N which we denote by $H^0(X, L^N)$ inherits a torus action and splits according to the action's weights. Since we have a Hermitian metric on L, we can choose an orthonormal basis $\{s_m\}$ of $H^0(X, L^N)$ which is compatible with the splitting of $H^0(X, L^N)$. The sequence of spectral measures is

$$\mu_N = \sum_m |s_m|^2 \nu,$$

where ν is the Liouville measure. One of the results in [**BGU**] describes the asymptotic behavior of the sequence μ_N in N.

It is well known (see [**G**]) that toric varieties have many toric metrics (i.e., metrics invariant through the torus action) compatible with a given symplectic structure. Each toric metric gives rise to a different Hermitian metric on the bundle L hence to a different orthonormal basis from which it is possible to construct a spectral measure depending on the initial toric metric. It is then natural to ask if the results in [**BGU**] extend to such metrics. The first purpose of this note is to show that this is true. For example, we prove

Theorem 1.1. *Let X be a toric variety with moment polytope $\triangle$ and consider any toric metric on X. Let $\psi \in \mathcal{C}^\infty(\triangle)$, and μ_N be the sequence of spectral measures on X for the chosen toric metric. Then*

$$\int \psi \mu_N = \sum_{i=0}^{N} P_i(\psi) N^{n-i},$$

where ψ can be seen as a function on X via the moment map and the $P_i(\psi)$'s are integrals of differential operators acting on ψ.

The polynomials P_i depend on the polytope $\triangle$ and on the toric metric on X. The pointwise asymptotic behavior of the sequence of functions

$$\sum |s_m|^2$$

has been extensively studied (see [**C**, **L**, **T**, **Z**]). It is known as the Tian–Yau–Zelditch asymptotic expansion. Now when one applies the measures μ_N to functions with compact support on the polytope, one should be able to recover the pointwise asymptotic behavior of the function $\sum |s_m|^2$ (at least in “most” points). In this spirit, we use Theorem 1.1 in a precise version (or rather the method that is used to prove this theorem and which appears in [**BGU**]) to write down the term in N^{n-1} in the expansion. By comparing this term with the corresponding term of the pointwise Tian–Yau–Zelditch asymptotic expansion (which was obtained by Lu in [**L**]) we recover a result of Abreu's (see [**A1**]) which gives a formula for the scalar curvature of any toric metric on a toric variety in terms of polytope data. That is, we give a different proof of the following theorem of Abreu.

Theorem 1.2. *Let X be a toric variety with moment polytope $\triangle$ and consider any toric metric on X with symplectic potential g. The scalar curvature of the metric is given by*

$$-\frac{1}{2}\frac{\partial^2 G^{ij}}{\partial y_i \partial y_j},$$

where y is the moment map coordinate for g on $\triangle$, G_{ij} is the Hessian of g in the y coordinates and G^{ij} is its inverse.

In the above theorem we are using Einstein's convention: repeated indexes mean a sum. We will use this convention throughout the paper unless stated otherwise.

A brief outline of this paper is the following: In the first section we give a very brief review of some old results on the Kähler structure of toric varieties. The results described in that section first appeared in [**A1**, **G**]. The second section deals with the function φ which encodes the information coming from the norms of a special basis of $H^0(X, L^N)$. In this section, we basically prove that the function φ satisfies properties similar to the ones appearing in [**BGU**] which ensure that the sections s_m peak at some fiber over a point in $\triangle$. In the fourth section, we use these properties to write down the asymptotic behavior of $|s_m|$ and of μ_N. Finally, in the fifth section we compare these results with the well-known Tian–Yau–Zelditch asymptotic expansion to get a formula for the scalar curvature of the toric metric in terms of polytope data only. This formula was first obtained in [**A1**] using a different approach.

2. Background

For the sake of completeness we give some background on Kähler toric varieties (see [**A1**, **A2**, **G**] for more details and proofs). We will denote by T^n the n dimensional real torus i.e.

$$T^n = \mathbb{R}^n/\mathbb{Z}^n.$$

2.1. Reduced Kähler toric metrics.

Definition 2.1. A Kähler toric manifold X^{2n} is a closed connected Kähler manifold (X, ω, J) with an effective T^n Hamiltonian action which is also holomorphic.

Such an action admits a moment map $\phi : X \to \mathbb{R}^n$, where we have identified $\mathbb{R}^n$ with its dual.

Definition 2.2. Let (X, ω, J) be a toric manifold with moment map $\phi : X \to \mathbb{R}^n$. The set $\phi(X)$ is a convex polytope in $\mathbb{R}^n$ which we call the Delzant polytope or moment polytope of X.

One should note here that this image is only defined up to translation to start with because ϕ is only defined up to a constant. In fact, the Delzant polytope of a toric manifold is a convex polytope of a special type. Namely, it is what is called a polytope of Delzant type.

Definition 2.3. A convex polytope $\triangle$ in $\mathbb{R}^n$ is of Delzant type if

(1) there are n edges meeting at each vertex,
(2) it is possible to choose a set of primitive exterior normals to facets in $\mathbb{Z}^n$ and
(3) for any vertex, the set of outward normals corresponding to the facets meeting at that vertex forms an integral basis for $\mathbb{Z}^n$.

Given a convex polytope in $\mathbb{R}^n$ which is of Delzant type, Delzant has given a canonical way to associate to it a Kähler toric manifold which we write (X, ω_0, J_0).

Theorem 2.1 (Delzant). *Given a Delzant polytope $\triangle$ in $\mathbb{R}^n$, there is a Kähler manifold (X, ω_0, J_0) and a T^n action on X which is effective, Hamiltonian and holomorphic such that the image of the moment map of the T^n action with respect to ω_0 is $\triangle$.*

The results in [**BGU**] concern this Kähler manifold. The symplectic form ω_0 and the metric associated with ω_0 and J_0 are called reduced. Delzant's construction can be made explicit and shows that one can see X, the toric manifold associated to a Delzant polytope, as a symplectic quotient of a torus action on $\mathbb{C}^d$. The reduced Kähler metric on X comes from the "standard" metric $\mathbb{C}^d$. For more details on this see [**G**]. Note also that Delzant's construction is described in [**BGU**, Section 2]. In fact, Delzant also proved that the Delzant polytope of a toric manifold determines its symplectic type.

Theorem 2.2 (Delzant). *Any two toric Kähler manifolds with the same Delzant polytope (up to translation) are equivariantly symplectomorphic.*

2.2. General Kähler toric metrics. We are interested in more general metrics on toric manifolds which are invariant under the torus action.

Definition 2.4. A toric metric on the toric symplectic manifold (X, ω_0, J_0) is a metric compatible with the symplectic form ω_0 which is invariant under the torus action.

There are two ways of thinking of toric metrics on X starting with (X, ω_0, J_0).

- One can think of equipping X with a different symplectic form ω which is compatible with J_0 and T^n invariant. The pair (ω, J_0) will give rise to a toric metric via the formula $\omega(\cdot, J_0 \cdot)$.
- Alternatively, one can think of fixing ω_0 and equipping X with a different integrable complex structure J which is T^n invariant and compatible with ω_0. Again the formula $\omega_0(\cdot, J\cdot)$ will give rise to a toric metric.

These two points of view are equivalent when ω and ω_0 are cohomologous and J and J_0 are in the same diffeomorphism class. For more on this duality see [**A2**]. For example, assume one starts with a triple (X, ω, J_0). A well-known trick in symplectic geometry, namely Moser's trick, gives a symplectomorphism $F : (X, \omega_0) \to (X, \omega)$. One can use F to pullback J_0 onto an integrable complex structure J which will then be compatible with ω_0 (and will of course be in the same diffeomorphism type as J_0, since in particular F is a diffeomorphism). In this way we get a triple (X, ω_0, J). In particular, this shows that the image of the moment map for (X, ω) is the same as that of the image of the moment map for (X, ω_0) (up to translation). We will give further details on this ahead. One can reverse the above process starting with (X, ω_0, J) as long as J is in the same diffeomorphism class as J_0.

Remark 2.1. Let (X, ω_0, J_0) and (X, ω, J_0) be two toric Kähler manifolds. If ω_0 and ω are cohomologous then the moment maps for the T^n action with respect to ω_0 and ω have the same image up to translation.

Although all Kähler toric manifolds with a given Delzant polytope are symplectomorphic to (X, ω_0) they are by no means Kähler isomorphic to (X, ω_0, J). That is, the symplectic isomorphism between (X, ω) and (X, ω_0) can, in general, not be taken to be holomorphic. In a dual way, we can also say that all Kähler toric manifolds with a given Delzant polytope are biholomorphic to (X, J) but the biholomorphism cannot be taken to be symplectic.

2.3. Complex and symplectic coordinates. Consider a Kähler toric manifold (X, ω, J). One can use J to complexify the torus action so that X also admits a holomorphic $T^n_{\mathbb{C}}$ action. We have

$$T^n = \mathbb{R}^n/\mathbb{Z}^n \subset T^n_{\mathbb{C}}.$$

It is an important feature of toric manifolds that this complexified $T^n_{\mathbb{C}}$ action admits an open orbit which is dense in X. Although we will not take that approach here, toric manifolds are sometimes defined as suitable compactifications of complex tori. Let us call the open dense $T^n_{\mathbb{C}}$ orbit X_0. On X_0 there are two natural sets of coordinates (see [**A2**] for more details):

(1) Complex coordinates coming from the fact that the $T^n_{\mathbb{C}}$ on X_0 is free so that $X_0 \simeq T^n_{\mathbb{C}}$. Let us call these coordinates $z = u + \mathrm{i}v$. These are simply coordinates on $T^n_{\mathbb{C}}$. In these coordinates, the real torus acts by translation in the v variables so that v takes values in $\mathbb{R}^n/\mathbb{Z}^n$
(2) There are also symplectic coordinates on X_0 that make the symplectic form standard. Consider the moment map on X from the torus action. Its image lies in the Delzant polytope of X in $\mathbb{R}^n$ which we call $\triangle$. Let y be the moment map coordinates on this polytope, then, letting v be

as before the real torus coordinate, (y, v) are symplectic coordinates on X_0, that is

$$\omega = \sum dy_i \wedge dv_i.$$

In these coordinates, the complex structure takes a non-standard form

$$\begin{pmatrix} 0 & \operatorname{hess} g \\ -\operatorname{hess} g & 0 \end{pmatrix}.$$

for some function g which is called the symplectic potential. For a proof of this fact see [**A2**]. Even though g is really determining a complex structure we will often refer to g as the symplectic potential for the toric metric associated with ω and J. Note that, since the complex structure is torus invariant, g is really only a function of the y coordinates.

As before, for the first set of coordinates one wants to think of J as being standard, whereas in the second one, one wants to think of ω as being standard. There is a relation between complex and symplectic coordinates (see [**A2**, **G**]), namely the Legendre transform

$$u = g_y. \tag{2.1}$$

Also on the open dense subset X_0, the Kähler form admits a Kähler potential. That is, there exists f, a function of z, such that the Kähler form is $-2\mathrm{i}\partial\bar{\partial}f$. This follows from the $\partial\bar{\partial}$ lemma. The invariance of the Kähler metric with respect to the torus action implies that f is actually only a function of u. The functions f and g are related to each other by the following relations (see [**G**]):

$$f(u) + g(y) = y \cdot u, \qquad y = f_u, \qquad u = g_y. \tag{2.2}$$

2.4. General symplectic potentials. We are going to assume that our polytope is integral, namely, we are going to assume that the polytope $\triangle$ can be described by a set of inequalities

$$\triangle = \{y \in \mathbb{R}^n : y \cdot u_i - c_i \leq 0, \ i = 1, \dots, d\},$$

where c_i is an integer and $u_i \in \mathbb{Z}^n$ is a primitive outward normal to the ith facet of $\triangle$. The integrality condition on the polytope ensures that the cohomology class of $\omega/2\pi$ is in $H^2(X, \mathbb{Z})$. In fact, it is known (see [**G**]) that the cohomology class of ω is given by

$$\frac{[\omega]}{2\pi} = -\sum c_i A_i,$$

where A_i is the cohomology class in $H^2(X,\mathbb{Z})$ dual to the submanifold of X obtained as the pre-image of the ith facet via the moment map. This fact is going to be of importance to us because if $[\omega]/2\pi$ is integral then there is a line bundle $L \to X$ whose first Chern class is $[\omega]$.

Set $l_i(y) = c_i - y \cdot u_i$. In [**G**], Guillemin proves the following.

Theorem 2.3 (Guillemin). *The reduction metric corresponding to ω_0 has symplectic potential*

$$g_0 = \frac{1}{2}\sum_{i=1}^{d}(l_i \log l_i - l_i).$$

This formula has played an important role in studying Kähler toric manifolds.

Abreu showed that the symplectic potential of a general toric metric is given by $g = g_0 + g_r$, where g_r is smooth in a neighborhood of $\triangle$ (see [**A2**]). In particular, the singular behavior of the potential on the faces of the polytope is that of g_0. Not all functions of the form $g = g_0 + g_r$, where g_r is smooth, are symplectic potentials though (see Theorem 2.8 in [**A2**]). In particular, g needs to be such that $\mathrm{hess}(g)$ is positive definite on the interior of $\triangle$. This Hessian will have a certain behavior on the faces. For example, as one reaches a point in the interior of an $(n-1)$-dimensional facet, the inverse matrix of the Hessian converges but acquires a kernel which is generated by the u_i corresponding to that facet. As one approaches an $(n-2)$-dimensional face, the inverse of the Hessian still converges but this time acquires a two-dimensional kernel etc.

2.5. Holomorphic sections on toric manifolds. We will assume from now on that $\triangle$ is an integral polytope and that $[\omega]/2\pi$ is in $H^2(X,\mathbb{Z})$. There is a line bundle $L \to X$ whose first Chern class is $[\omega]$. In the toric case, by using Delzant's explicit construction of X as a symplectic quotient of $\mathbb{C}^d$, L can be obtained in an explicit way as well from the trivial bundle over $\mathbb{C}^d$. The T^n action lifts to L but not in a canonical way. In fact the following lemma holds.

Lemma 2.1. *For each lift of the T^n action to L there is a holomorphic line bundle structure on L and a T^n invariant section $\mathbf{1}$ such that $H^0(X,L)$ is spanned by*

$$\mathrm{e}^{m\cdot z}\mathbf{1},$$

where m is in $\mathbb{Z}^n \cap \triangle$.

For a proof of this lemma see [**BGU**, Section 3]. Even though our form ω may not be ω_0, the construction is exactly the same as that of [**BGU**]. In fact, L and its holomorphic structure only depend on the cohomology class of ω. What will depend on the form ω is the Hermitian structure on L.

Remark 2.2. The above-mentioned T^n invariant section $\mathbf{1}$ is not necessarily holomorphic. If we assume that $0 \in \triangle$ then $\mathbf{1}$ it is indeed holomorphic (we are allowed to choose $m = 0$ then).

The T^n action on X also induces an action on $H^0(X, L^N)$, the space of holomorphic sections on L^N. This vector space must split according to the weights of the action. It is not hard to see that any holomorphic section of L^N can be written as a linear combination of the sections $\mathrm{e}^{m\cdot z}\mathbf{1}^N$ i.e.,

$$H^0(X, L^N) = \mathrm{span}\{\mathrm{e}^{m\cdot z}\mathbf{1}^k, \quad m \in \mathbb{Z}^n \cap N\triangle\}.$$

We set $\mathbb{Z}^n \cap N\triangle = [N\triangle]$. This basis decomposes $H^0(X, L^N)$ into one-dimensional weight spaces for the torus action. Namely,

$$\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{m\cdot z} = \mathrm{e}^{\mathrm{i}m\cdot\theta}\mathrm{e}^{m\cdot z},$$

where θ is in $\mathbb{R}^n$ and we write $\mathrm{e}^{m\cdot z}$ for $\mathrm{e}^{m\cdot z}\mathbf{1}^N$. The set $\{\mathrm{e}^{m\cdot z}, \quad m \in \mathbb{Z}^n \cap N\triangle\}$ forms an orthogonal basis of $H^0(X, L^N)$. This is simply because

$$\int_{T^n} \mathrm{e}^{im\cdot v} dv = 0,$$

unless $m = 0$.

In what follows, we will mostly assume that our toric metrics come from a toric Kähler manifold of the form (X, ω, J_0) where J_0 is the reduced complex structure from Delzant's construction, ω is compatible with J_0 and in the same cohomology class as ω_0, the reduced symplectic form. But in fact we will often use the "symplectic potential" of ω to describe our Kähler toric manifold (X, ω, J_0). This is really an abuse of notation as "symplectic potential" really means the symplectic potential of the complex structure J one gets by applying Moser's trick to (X, ω, J_0). Since the two view points, that of fixing the symplectic structure and varying the complex structure and that of fixing the complex structure and varying the symplectic structure, are equivalent (in the same cohomology class for the symplectic forms and the same diffeomorphism class for complex structures) this does not cause any problems. Also, the reader may think that we are simply varying the symplectic form (and letting the complex structure fixed) but we parametrize such variations in a slightly "exotic" way — namely via symplectic potentials.

3. The function φ

Our setting is almost the same as the setting in [**BGU**]. Let X be a Kähler toric manifold of complex dimension n such that the symplectic form on X, ω has integral cohomology class. For example, one can assume that $\triangle$ is integral and that ω is cohomologous to ω_0 and compatible with J_0.

Some of our results hold in greater generality but we will in general require this assumption. There is a line bundle $L \to X$ whose first Chern class is $[\omega]$. What is more, there is a connection on L whose curvature is ω. Consider the Hermitian metric on L, h associated to that connection (it is unique up to a multiplicative constant). In fact, the Kähler metric on X is given by $\mathrm{i}\partial\bar{\partial}\log h$. Pick an orthonormal basis for $H^0(X, L^N)$ which is an eigenbasis for the torus action, say $\{s_m\}$. As in [**BGU**], we are interested in the asymptotic behavior of the spectral measure

$$\mu_N = \sum_m |s_m|^2 \nu,$$

where ν is the Liouville measure. In [**BGU**], Burns, Guillemin and Uribe consider the case where the symplectic form on X is the so-called reduced symplectic form. Here we are concerned with the general case. To this end, we look at the norms of the sections $\mathrm{e}^{m\cdot z}$ with respect to the Hermitian metric associated to ω and thus define the function φ which encodes the information from all of these norms.

3.1. Definition. Suppose we have a fixed Kähler toric metric on X. This metric allows us to define a Hermitian metric on the bundle L. Simply set

$$\omega = \mathrm{i}\partial\bar{\partial}\log h,$$

where h is the norm of the torus invariant section we have called $\mathbf{1}$. The function h is torus invariant. We note here that the norm of any of the $\mathrm{e}^{m\cdot z}$ is also torus invariant. This is because

$$|\mathrm{e}^{m\cdot z}|_h^2 = \mathrm{e}^{2m\cdot u} h,$$

which does not depend on the v coordinate. We define

$$\varphi\left(\frac{m}{N}, y\right) = \frac{1}{2N}\log|\mathrm{e}^{m\cdot z}|_h^2 \circ \phi^{-1}(y),$$

where $m \in \mathbb{Z}^n \cap N\triangle$ and ϕ is the moment map for the torus action with respect to ω. Even though ϕ^{-1} is not a well-defined function, as a function of y, φ is well defined at least on the interior of $\triangle$ where we know that $\mathrm{e}^{m\cdot z}$ is non-zero. To be more precise, suppose that two points in X have the same image via the moment map in the polytope. Then, they are in the same torus orbit and therefore, since $|\mathrm{e}^{m\cdot z}|_h^2$ is torus invariant, the above quantity is well defined. We will see later that in fact if m/N is in a face of $\triangle$, φ can be extended to the interior of that face. In [**BGU**], Burns, Guillemin and Uribe consider the case where the symplectic form is the reduction symplectic form. Then we have

$$\varphi_0\left(\frac{m}{N}, y\right) = \frac{1}{2N}\log|\mathrm{e}^{m\cdot z}|_{h_0}^2 \circ \phi_0^{-1}(y),$$

where h_0 is the Hermitian metric corresponding to the reduction symplectic form and ϕ_0 is the moment map of the torus action associated with the reduction symplectic form. We will see later that, as φ_0, φ also extends as a function of the first variable to $\mathbb{R}^n$.

3.2. φ and φ_0. We can write down a relation between the functions φ and φ_0. For that we need to consider the map

$$\alpha(y) = \phi_0 \circ \phi^{-1}(y).$$

This is well defined because if two points have the same image via ϕ then they lie in the same torus orbit above that image and therefore they have the same image via ϕ_0. Note that in the interior of $\triangle$ there an explicit expression for α, namely

$$\alpha(y) = f_{0u} \circ f_u^{-1}.$$

Another way to think of α is the following. From Moser's trick, assuming that ω is compatible with J_0, we know that there is an equivariant symplectomorphism $F : (X, \omega_0) \to (X, \omega)$. Let ξ be in T^n and let X_ξ be the vector field induced by the action of ξ on X. Then

$$X_\xi \lrcorner \omega_0 = d\langle \phi_0, \xi \rangle$$

and

$$X_\xi \lrcorner \omega = X_\xi \lrcorner (F^{-1})^* \omega_0 = d\langle \phi, \xi \rangle,$$

which shows that $\phi_0 = \phi \circ F$ (up to translation). So that $\alpha = \phi_0 \circ F \circ \phi_0^{-1}$. For later use we prove the following simple lemma

Lemma 3.1. *The function α sends each codimension r face in $\triangle$ onto itself.*

Proof. The proof is just a generalization of the fact that α takes $\triangle$ onto itself. The point is that each codimension r face is the moment polytope of a toric submanifold of our toric manifold. The image of a toric manifold via its moment map does not depend on the symplectic form but only on its cohomology class up to translation. The restrictions of ω and ω_0 to a toric submanifold are cohomologous so we can assume that the two moment maps have the same image (by normalizing the moment maps "in the same way") and α preserves any given face. □

The forms ω_0 and ω, are cohomologous therefore there is a globally defined function on X, say ρ, such that

$$\omega = \omega_0 + 2\partial\bar{\partial}\rho.$$

Seen as a function of z in the open dense orbit, ρ only depends on u because it must be invariant by the torus action.

Lemma 3.2. *The functions φ and φ_0 are related via*

$$\varphi(x, y) = \varphi_0(x, \alpha(y)) + \rho(\alpha(y)).$$

Proof. This is straightforward. We must have

$$h = \mathrm{e}^{2\rho} h_0,$$

up to a constant, hence

$$|\mathrm{e}^{m\cdot z}|_h^2 = |\mathrm{e}^{m\cdot z}|_{h_0}^2 \mathrm{e}^{2N\rho}$$

and

$$\varphi(x,y) = \frac{1}{2N} \log |\mathrm{e}^{m\cdot z}|_h^2 \circ \phi_0^{-1}(\phi_0 \circ \phi^{-1}(y)),$$

where $x = m/N$. That is

$$\varphi(x,y) = \frac{1}{2N} \log \left(|\mathrm{e}^{m\cdot z}|_{h_0}^2 \mathrm{e}^{2N\rho} \right) \circ \phi_0^{-1}(\alpha(y)),$$

and the result follows. ☐

3.3. φ and g. Let g denote, as before, the symplectic potential for (X, ω, J) where (X, ω, J) is a Kähler toric manifold (i.e., g determines the complex structure J in symplectic coordinates for ω). It is possible to write down φ in terms of g alone.

Lemma 3.3. *Let g be the symplectic potential for (X, ω, J), we have*

$$\varphi(x,y) = g(y) + (x-y)\cdot g_y(y).$$

Proof. We start by determining an expression for the Hermitian metric h. Since both $-2f$ and $\log h$ are potentials for the Kähler metric on X we must have

$$h = \mathrm{e}^{-2f}.$$

Therefore

$$|\mathrm{e}^{m\cdot z}|_h^2 = \mathrm{e}^{2m\cdot u} \mathrm{e}^{-2Nf}.$$

Replacing f by the expression given in Equation (2.2) and using Equation (2.1) we have

$$|\mathrm{e}^{m\cdot z}|_h^2(y) = \mathrm{e}^{2N(g(y) + (\frac{m}{N} - y)\cdot g_y(y))},$$

where $y = \phi(z)$. We have

$$\varphi\left(\frac{m}{N}, y\right) = \frac{1}{2N} \log |\mathrm{e}^{m\cdot z}|_h^2 \circ \phi^{-1}(y)$$

and the result follows. ☐

Let us check that this fits in well with the expression in [**BGU**] for φ_0. Since

$$g_0 = \sum_{i=1}^{d} l_i \log l_i - l_i,$$

we have

$$g_{0y} = -\sum_{i=1}^{d} u_i \log l_i,$$

and therefore

$$g_0 + (x-y)\cdot g_{0y} = \sum l_i(x)\log l_i(y) - l_i(y),$$

because $x - y \cdot u_i = l_i(x) - l_i(y)$. The above expression coincides with the expression appearing in [**BGU**] for φ_0.

Later we are going to be interested in an orthonormal basis for $H^0(X, L^N)$. This is simply the set $\{s_m\}$, where

$$s_m = \frac{\mathrm{e}^{m\cdot z}}{||\mathrm{e}^{m\cdot z}||},$$

where $||.||$ refers to the L^2 norm and we can write

$$|s_m|_h^2(y) = \frac{\mathrm{e}^{2N\varphi(\frac{m}{N},y)}}{\int_\triangle \mathrm{e}^{2N\varphi(\frac{m}{N},y)}dy}, \quad m \in [N\triangle].$$

3.4. Two lemmas on $\boldsymbol{\varphi}$. The point here is that independently of g, the function φ satisfies two lemmas which appear in [**BGU**] for the case $g = g_0$.

Lemma 3.4. *Let x be a point in the interior of $\triangle$ and g be a symplectic potential on $\triangle$. Then, the function φ regarded as a function of $y = f_u(u)$ has a unique critical point at $x = y$ and this unique critical point is the unique global maximum of the function φ on $\triangle$.*

Proof. We first note that as y tends to $\partial\triangle$, $\alpha(y)$ also tends to $\partial\triangle$. Consider the formula

$$\varphi(x,y) = \varphi_0(x,\alpha(y)) + \rho(\alpha(y)).$$

We know that $\varphi_0(x,y)$ tends to $-\infty$ as y tends to $\partial\triangle$ because

$$\varphi_0(x,y) = \sum l_i(x)\log l_i(y) - l_i(y).$$

As for ρ, since it is a globally defined function on X it must have a finite limit as y tends to $\partial\triangle$. We conclude that $\varphi(x,y)$ tends to $-\infty$ as well on $\partial\triangle$. On the other hand, it is bounded from above on $\triangle$ since the l_i and ρ are. Therefore, it has a maximum on the interior of $\triangle$. This maximum is a critical point of φ. Using

$$\varphi(x,y) = g(y) + (x-y)\cdot g_y(y),$$

we see that

$$\frac{\partial\varphi}{\partial y} = \mathrm{hess}(g)(x-y). \tag{3.3}$$

Now from the properties of g mentioned in Section 2.4 we know that hess(g) is positive definite on the interior of $\triangle$ and the result follows. $\square$

A similar result can be proved when x is in the boundary of $\triangle$ namely

Lemma 3.5. *Let x be a point in the interior of a face F of $\triangle$ and g be a symplectic potential on $\triangle$. Then, the restriction to F of the function φ regarded as a function of y has a unique critical point at $x = y$ and this unique critical point is the unique global maximum of the restriction to F of the function φ on $\triangle$. Moreover, the derivatives of φ in the directions normal to F are not zero at this maximum.*

Proof. Let $I = \{i \in \{1, \dots, d\} : x \notin l_i^{-1}(0)\}$, where d is the total number of facets in $\triangle$. That is, I is the set of indexes of the facets to which x does not belong. We start by showing that $\varphi(x, \cdot)$ actually extends to F. This follows from the formula

$$\varphi(x,y) = \sum_{i \in I} l_i(x) \log l_i(\alpha(y)) - \sum_{i=1}^{d} l_i(\alpha(y)) + \rho(\alpha(y))$$

which in turn is a consequence of the expression

$$\varphi_0(x,y) = \sum_{i \in I} l_i(x) \log l_i(y) - \sum_{i=1}^{d} l_i(y),$$

from [**BGU**]. Again as y tends to ∂F, so does $\alpha(y)$ and it follows from the expression above that $\varphi_0(x, \cdot)$ tends to $-\infty$. Therefore $\varphi(x, \cdot)$ is $-\infty$ on the boundary of F. It is also bounded from above on this facet so there must be a maximum on the interior of F and this maximum must be a critical point of the restriction of φ to F as a function of y. Consider the expression

$$x - y = (\text{hess}_y(g))^{-1} \frac{\partial \varphi}{\partial y}, \tag{3.4}$$

which holds true on the interior of $\triangle$. We know from the properties of g described in Section 2.4 and discussed in [**A2**] that $(\text{hess}_y(g))^{-1}$ extends to F with a kernel generated by $\{u_i, i \in I^c\}$. As for $\frac{\partial \varphi}{\partial y}$ it is given by

$$\frac{\partial \varphi}{\partial y}(y) = D\alpha(y) \left(\frac{\partial \varphi_0}{\partial y}(x, \alpha(y)) + \frac{\partial \rho}{\partial y}(\alpha(y)) \right)$$

and

$$\frac{\partial \varphi_0}{\partial y} = -\sum_{i \in I} \frac{l_i(x)}{l_i(y)} u_i + \sum_{i=1}^{d} u_i,$$

which clearly extends to the interior of F. Hence we conclude that $\frac{\partial \varphi}{\partial y}$ itself extends to the interior of F. So Equation (3.4) holds even for y in F. Suppose that the point $y \in F$ is critical for the restriction of φ to F. This means that

$$\frac{\partial \varphi}{\partial y} \in T^{\perp} F = \text{span}\{u_i, i \in I^c\},$$

hence

$$(\text{hess}_y(g))^{-1}\frac{\partial\varphi}{\partial y}=0$$

which implies that $x=y$. Next, we would like to see that $\frac{\partial\varphi}{\partial y}$ cannot be zero at $x=y$. Define

$$g_{0I^c}=\sum_{i\in I^c} l_i\log l_i-l_i.$$

We have $g=g_{0I^c}\circ\alpha+(g_0-g_{0I^c})\circ\alpha+g_r$. The functions g_r and $(g_0-g_{0I^c})\circ\alpha$ extend smoothly to the interior of F so we must check that for some sequence of y's tending to x

$$\text{hess}_y(g_{0I^c}\circ\alpha)(x-y)$$

tends to something which is not zero. We have

$$\text{hess}(g_{0I^C}\circ\alpha)(y)=D\alpha^t(y)\text{hess}(g_{0I^C})(\alpha(y))D\alpha(y)+R(y), \tag{3.5}$$

where the i,j entry of the matrix R is given by

$$R_{ij}(y)=\frac{\partial g}{\partial y_a}(\alpha(y))\frac{\partial\alpha_a}{\partial y_i\partial y_j}(y).$$

We will deal with each of the terms in the sum above separately.

- We start with the second term in the sum (3.5). Now since

$$\frac{\partial g}{\partial y_a}\circ\alpha=\sum_r u_r^a\log(l_r\circ\alpha),$$

we have

$$R_{ij}=\sum_r u_r^a\log(l_r\circ\alpha)\frac{\partial\alpha_a}{\partial y_i\partial y_j}.$$

Also, because $\alpha(x)$ is in F

$$l_r(\alpha(y))=l_r(\alpha(y))-l_r(\alpha(x))=u_r\cdot(\alpha(y)-\alpha(x))$$

so that

$$\log(l_r(\alpha(y)))(x_i-y_i)=\log(u_r\cdot(\alpha(y)-\alpha(x)))(x_i-y_i).$$

The expression for $R(x-y)$ then becomes

$$R_{ij}(x_i-y_i)=\sum_r u_r^a\frac{\partial\alpha_a}{\partial y_i\partial y_j}\log(u_r\cdot(\alpha(y)-\alpha(x)))(x_i-y_i).$$

There is constant A such that for any index i in $\{1,\dots,d\}$

$$|x_i-y_i|\le A|\alpha(x)-\alpha(y)|.$$

Now for the right choice of y (for example as long as $\alpha(y)-\alpha(x)$ remains in a cone with axis u_r)

$$|\alpha(x)-\alpha(y)| \leq B|u_r\cdot(\alpha(x)-\alpha(y))|,$$

for some constant B, so that

$$|R_{ij}(x_i-y_i)| \leq -C\sum_r \log(u_r\cdot(\alpha(y)-\alpha(x)))|u_r\cdot(\alpha(y)-\alpha(x))|,$$

for a constant C. So the second term in the sum (3.5) tends to zero at least for some choice of sequence of y's tending to x.

- We now analyze the first term in the sum (3.5). We have

$$\text{hess}(g_{0I^C}) = \sum_{i\in I^c}\frac{u_i u_i^t}{l_i(y)},$$

hence

$$\text{hess}(g_{0I^C})(\alpha(y))D\alpha(y)(x-y) = \sum_{i\in I^c}\frac{u_i\cdot(D\alpha(x)-D\alpha(y))}{l_i(\alpha(y))}u_i.$$

Now we use the fact that $\alpha(x)-\alpha(y)=D\alpha(x)-D\alpha(y)+O(|x-y|^2)$ so the expression above becomes

$$\sum_{i\in I^c}\frac{u_i\cdot(\alpha(x)-\alpha(y))}{l_i(\alpha(y))}u_i + \text{smaller order terms}.$$

Also

$$u_i\cdot(\alpha(x)-\alpha(y)) = l_i(\alpha(x))-l_i(\alpha(y)) = -l_i(\alpha(y)).$$

Then we have

$$\text{hess}(g_{0I^C}\circ\alpha)(y)D\alpha(y)(x-y) = -\sum_{i\in I^c}u_i + \text{smaller order terms},$$

which is non-zero because of the Delzant condition. Therefore the limit of

$$D\alpha^t(y)\text{hess}(g_{0I^C})(\alpha(y))D\alpha(y)(x-y)$$

is also non-zero as y tends to x. Note that this is true for any sequence of y's tending to x.

We thus conclude that the sum (3.5) tends to a non-zero vector for at least some choices of sequences y tending to x and therefore the derivative cannot be zero. ∎

4. The results

As in [**BGU**] we are interested in the spectral measures of the manifold X. These are defined by

$$\mu_N = \sum |s_m|_h^2 \nu,$$

where ν is the Liouville measure. Let ψ be a smooth function on $\triangle$. Then the goal is to write an asymptotic formula for

$$\int_\triangle \psi \mu_N,$$

in N, as N tends to ∞. We have

$$\int_\triangle \psi \mu_N = \sum_{m \in [N\triangle]} \psi^\sharp \left(\frac{m}{N} \right),$$

where

$$\psi^\sharp(x) = \int_\triangle \frac{\psi(y) \mathrm{e}^{2N\varphi(x,y)}}{\int_\triangle \mathrm{e}^{2N\varphi(x,y)}} dy.$$

The results in [**BGU**] use two main ingredients:

- The first in an asymptotic formula for sums of the form

  $$\sum_{m \in [N\triangle]} \psi \left(\frac{m}{N} \right)$$

  for any continuous function ψ on $\triangle$ (see [**L2**]). This argument clearly does not depend on the symplectic potential g.
- The second ingredient is an application of the Euler–McLaurin formula to the function ψ as it appears in the integral

  $$\int_\triangle \frac{\psi(y) \mathrm{e}^{2N\varphi(x,y)}}{\int_\triangle \mathrm{e}^{2N\varphi(x,y)} dy} dy$$

  around the point x which is the point where φ attains its maximum. Again this argument works for general g since Lemmas 4.1 and 4.2 of [**BGU**] carry over to this case. Their generalizations to this setting are Lemmas 3.4 and 3.5 from the previous section. We will carry out this method explicitly in the next section for the case when ψ has compact support in $\triangle$.

We can summarize the results obtained by applying this method in the following generalization of the main theorem in [**BGU**]:

Theorem 4.1. *Let X be a toric variety with moment polytope $\triangle$ endowed with a toric metric. Let $\psi \in \mathcal{C}^\infty(\triangle)$, and μ_N be the sequence of spectral measures on X for the chosen toric metric. Then*

$$\int \psi \mu_N = \sum_{i=0}^{N} P_i(\psi) N^{n-i},$$

where ψ can be seen as a function on X via the moment map and the $P_i(\psi)$'s are integrals of differential operators acting on ψ.

The other asymptotic results appearing in [**BGU**] hold true in this new setting provided one is careful to note that the coordinates y are now given by the new moment map ϕ corresponding to ω. We have for example:

Theorem 4.2. *Let x be in $\triangle$ and suppose $m = Nx \in \mathbb{Z}^n \cap N\triangle$. The sequence of sections s_m converges to a delta function on the fiber $\phi^{-1}(x)$.*

Proof. This is exactly as in [**BGU**]. We note that

$$\int_\triangle \mathrm{e}^{N\varphi(x,y)} dy \simeq \left(\frac{2\pi}{N}\right)^{n/2} h(x)^{-1/2} \mathrm{e}^{N\varphi(x,x)},$$

where $h(x)$ is the determinant of $\mathrm{hess}(g)$. Therefore,

$$|s_m|^2 \simeq \left(\frac{N}{2\pi}\right)^{n/2} h(x)^{1/2} \mathrm{e}^{N(\varphi(x,y)-\varphi(x,x))}.$$

Since for y not equal to x we have $\varphi(x,y) - \varphi(x,x) < 0$, the above converges to zero except if $x = y$, that is on the x fiber of ϕ. □

5. Abreu's scalar curvature formula

5.1. An explicit calculation using [BGU] approximation method. In the previous section, we roughly described the method first presented in [**BGU**] to obtain the asymptotic behavior of the integral

$$\int_\triangle \psi \mu_N.$$

In the case where ψ has compact support in $\triangle$, the method simplifies considerably. We are going to write down explicitly the first two terms in the expansion and see how, from the second term, we can recover Abreu's formula (see [**A1**]) for the scalar curvature of a toric manifold.

Let ψ be in $\mathcal{C}_0^\infty(\triangle)$. First write

$$\int_\triangle \psi \mu_N = \sum_{m \in [N\triangle]} \psi^\sharp\left(\frac{m}{N}\right).$$

From [**L2**] it is known that

$$\frac{1}{N^n}\sum_{m\in[N\triangle]}\psi^\sharp\left(\frac{m}{N}\right)\sim\tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right)\int_{\triangle_h}\psi^\sharp(h=0), \tag{5.6}$$

where $h\in\mathbb{R}^d$. By $\triangle_h$, we mean the dilated polygon

$$\triangle_h=\{y\in\mathbb{R}^n : y\cdot u_i-c_i\le h_i, i=1,\dots,d\}.$$

The function τ is defined by

$$\tau(s):=\frac{s}{1-e^s}=1+\frac{s}{2}+O(s^2)$$

and

$$\tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right)=\tau\left(\frac{1}{N}\frac{\partial}{\partial h_1}\right)\cdots\tau\left(\frac{1}{N}\frac{\partial}{\partial h_d}\right),$$

so that

$$\tau\left(\frac{1}{N}\frac{\partial}{\partial h}\right)=1+\frac{1}{N}\left(\frac{\partial}{\partial h_1}+\cdots+\frac{\partial}{\partial h_d}\right)+O\left(\frac{1}{N^2}\right).$$

We now move on to write the asymptotics for

$$\psi^\sharp(x)=\int_\triangle\frac{\psi(y)\mathrm{e}^{2N\varphi(x,y)}}{\int_\triangle \mathrm{e}^{2N\varphi(x,y)}dy}dy. \tag{5.7}$$

Proposition 5.1. *The first terms in the asymptotic expansion for $\psi^\sharp$ are given by*

$$\psi^\sharp(x)=\psi(x)+\frac{1}{2N}\left(\frac{1}{2}\frac{\partial^2\psi}{\partial y_a\partial y_b}G^{ab}+\frac{\partial\psi}{\partial y_a}\frac{\partial G^{ab}}{\partial y_b}\right)+O\left(\frac{1}{N^2}\right).$$

Proof. We can write the Taylor expansion for ψ around x

$$\psi(y)=\psi(x)+\frac{\partial\psi}{\partial y_a}(\xi(x,y))(y_a-x_a),$$

where ξ is a smooth function satisfying

$$\xi(x,y)\in\bar{xy},$$
$$\xi(x,x)=x$$

and

$$\xi(x,y)=\xi(y,x).$$

Here, $\bar{xy}$ denotes the set $\{tx+(1-t)y, t\in[0,1]\}$. These properties imply that

$$\frac{\partial\xi_a}{\partial y_b}(x,y)=\frac{\delta_{ab}}{2}.$$

From Equation (3.3) we can write

$$y-x=-(\mathrm{hess}(g))^{-1}\frac{\partial\varphi}{\partial y}$$

or, writing $(\text{hess}(g))^{-1} = (G^{ab})$

$$y_a - x_a = -G^{ab}\frac{\partial \varphi}{\partial y_b},$$

so that in the Taylor expansion for ψ we can write

$$\psi(y) = \psi(x) - \frac{\partial \psi}{\partial y_a}(\xi(x,y))G^{ab}\frac{\partial \varphi}{\partial y_b}.$$

Now

$$\frac{\partial \varphi}{\partial y_b}\mathrm{e}^{2N\varphi(x,y)} = \frac{1}{2N}\frac{\partial \mathrm{e}^{2N\varphi(x,y)}}{\partial y_b}.$$

We can replace for ψ in Equation (5.7) and write

$$\psi^\sharp(x) = \psi(x) - \int_\triangle \frac{1}{\int_\triangle \mathrm{e}^{2N\varphi(x,y)}dy}\frac{\partial \psi}{\partial y_a}(\xi(x,y))G^{ab}\frac{1}{2N}\frac{\partial \mathrm{e}^{2N\varphi(x,y)}}{\partial y_b}dy.$$

Next we integrate by parts. We do not pick any boundary terms. Note that even for y in the boundary of $\triangle$, $\xi(x,y)$ is not necessarily in the boundary so that $\psi(\xi(x,y))$ is not necessarily 0. But the term G^{ab} does vanish on the bth facet of $\triangle$. The easiest way to see this is to choose coordinates so as to standardize the bth facet to have normal $u_b = e_b$. Then the boundary behavior of G^{ab} implies that $G^{ab}u_b = 0$ on the bth facet and therefore $G^{ab} = 0$ on the bth facet. Note also that for fixed b we need only to integrate by parts in the variable b.

Lemma 5.1. *In integrating*

$$\int_\triangle \frac{\partial \psi}{\partial y_a}(\xi(x,y))G^{ab}\frac{\partial e^{2N\varphi(x,y)}}{\partial y_b}dy$$

by parts, we do not pick any boundary terms.

Proof. Fix b. We choose coordinates $y_1, \dots, y_n$ centered at one of the vertices in F_b so that

- In these coordinates, $\triangle$ is a subset of the positive octant of $\mathbb{R}^n$.
- For all $i = 1, \dots, n$, each facet $F_i = l_i^{-1}(0)$ is contained in the set $\{(y_1, \dots, y_n) : y_i = 0\}$.

Let Q denote a rectangle in $\mathbb{R}^n$ of the form $[0, \alpha_1] \times \cdots \times [0, \alpha_n]$ containing $\triangle$. One can find such a Q as long as the α_i's are big enough. Extend ψ by zero to all of Q. This extension is still smooth because the support of ψ is contained in the interior of $\triangle$. We will also need to extend G^{ab} to Q in a smooth way so that it is zero when one of the y_b's is α_b. Now consider

$$\int_{\triangle} \frac{\partial \psi}{\partial y_a}(\xi(x,y)) G^{ab} \frac{\partial \mathrm{e}^{2N\varphi(x,y)}}{\partial y_b} dy. \tag{5.8}$$

Note that in this proof b is fixed, we are not using Einstein's notation and the above does not denote a sum. We can rewrite this integral as

$$\int_{Q} \frac{\partial \psi}{\partial y_a}(\xi(x,y)) G^{ab} \frac{\partial \mathrm{e}^{2N\varphi(x,y)}}{\partial y_b} dy.$$

We will do integration by parts by using a special order in the integration variables namely we are going to integrate with respect to the bth variable first. That is, we choose to write the above integral as

$$\int_{\widehat{Q}_b} \int_0^{\alpha_b} \frac{\partial \psi}{\partial y_a}(\xi(x,y)) G^{ab} \frac{\partial \mathrm{e}^{2N\varphi(x,y)}}{\partial y_b} dy_b d\hat{y},$$

where

$$\widehat{Q}_b = [0,\alpha_1] \times \cdots \times \widehat{[0,\alpha_b]} \times \cdots \times [0,\alpha_n]$$

and

$$d\hat{y} = dy_1 \wedge \cdots \wedge \widehat{dy_b} \wedge \cdots \wedge dy_n.$$

Integrating

$$\int_0^{\alpha_b} \frac{\partial \psi}{\partial y_a}(\xi(x,y)) G^{ab} \frac{\partial \mathrm{e}^{2N\varphi(x,y)}}{\partial y_b} dy_b,$$

by parts we get

$$-\int_0^{\alpha_b} \frac{\partial}{\partial y_b}\left(\frac{\partial \psi}{\partial y_a}(\xi(x,y)) G^{ab}\right) \mathrm{e}^{2N\varphi(x,y)} dy_b$$

and two boundary terms

$$\left[\frac{\partial \psi}{\partial y_a}(\xi(x,y)) G^{ab} \mathrm{e}^{2N\varphi(x,y)}\right]_{y_b=\alpha_b}$$

and

$$-\left[\frac{\partial \psi}{\partial y_a}(\xi(x,y)) G^{ab} \mathrm{e}^{2N\varphi(x,y)}\right]_{y_b=0}.$$

Integral (5.8) must then be equal to

$$-\int_{\triangle} \frac{\partial}{\partial y_b}\left(\frac{\partial \psi}{\partial y_a}(\xi(x,y)) G^{ab}\right) \mathrm{e}^{2N\varphi(x,y)} dy_b,$$

plus two sums of boundary terms

$$\int_{y_b=\alpha_b} \frac{\partial \psi}{\partial y_a}(\xi(x,y))G^{ab}\mathrm{e}^{2N\varphi(x,y)}$$

and

$$-\int_{y_b=0} \frac{\partial \psi}{\partial y_a}(\xi(x,y))G^{ab}\mathrm{e}^{2N\varphi(x,y)}.$$

The first of these boundary terms is zero because $G^{ab}=0$ when $y_b=\alpha_b$. Now consider the second boundary term. The set where $y_b=0$ contains the facet F_b. For this facet, we can take $u_b=e_b$. We know $G^{ab}u_b=0$ on the facet F_b, this means that

$$G^{ab}(y)=0, \quad \forall y \in F_b,$$

and the second boundary term is also zero. Now since for each b we dot pick boundary terms, we do not pick boundary terms when integrating the sum for all b of the above expressions. $\square$

After the integration by parts, Equation (5.7) becomes

$$\psi^\sharp(x)=\psi(x)+\frac{1}{2N}\int_\triangle \frac{1}{\int_\triangle \mathrm{e}^{2N\varphi(x,y)}dy}\frac{\partial}{\partial y_b}\left(\frac{\partial \psi}{\partial y_a}(\xi(x,y))G^{ab}\right)\mathrm{e}^{2N\varphi(x,y)}dy.$$

Now we apply this process again. Set

$$\psi_1(x,y)=\frac{\partial}{\partial y_b}\left(\frac{\partial \psi}{\partial y_a}(\xi(x,y))G^{ab}\right).$$

We write the Taylor expansion for ψ_1 in y around x

$$\psi_1(x,y)=\psi_1(x,x)+\frac{\partial \psi_1}{\partial y_a}(\xi_1(x,y))(y_a-x_a),$$

but we are only interested in the first term of this expansion since the second will bring an $O\left(\frac{1}{N^2}\right)$ term to Equation (5.7). Now

$$\psi_1(x,x)=\frac{1}{2}\frac{\partial^2\psi}{\partial y_a \partial y_b}G^{ab}+\frac{\partial \psi}{\partial y_a}\frac{\partial G^{ab}}{\partial y_b},$$

where we have used the calculation of the derivatives of ξ at points of the form (x,x). Replacing again in Equation (5.7) we find

$$\psi^\sharp(x)=\psi(x)+\frac{1}{2N}\left(\frac{1}{2}\frac{\partial^2\psi}{\partial y_a \partial y_b}G^{ab}+\frac{\partial \psi}{\partial y_a}\frac{\partial G^{ab}}{\partial y_b}\right)+O\left(\frac{1}{N^2}\right). \qquad \square$$

We use the equation in the above lemma to substitute for $\psi^\sharp$ in (5.6). We get the asymptotic behavior we are interested in, up to terms in $O\left(\frac{1}{N^2}\right)$:

$$\left(1+\frac{1}{N}\left(\frac{\partial}{\partial h_1}+\cdots+\frac{\partial}{\partial h_d}\right)\right)\int_{\triangle_h}\left(\psi(x)+\frac{1}{2N}\left(\frac{G^{ab}}{2}\frac{\partial^2\psi}{\partial y_a\partial y_b}+\frac{\partial\psi}{\partial y_a}\frac{\partial G^{ab}}{\partial y_b}\right)\right),$$

evaluated at $h=0$. But since ψ has compact support in $\triangle$ we must have

$$\frac{\partial}{\partial h_i}\int_{\triangle_h}\psi=0$$

because this derivative is calculated as the limit, as h_i tends to zero, of the expression

$$\frac{\int_{\triangle_{h_i}\setminus\triangle}\psi}{h_i}$$

and the numerator is zero since for small enough h_i, ψ is zero on the set $\triangle_{h_i}\setminus\triangle$ (note that we may have to consider the set $\triangle\setminus\triangle_{h_i}$ instead). The term in $\frac{1}{N}$ in the expansion of $\int_\triangle \psi\mu_N$ is therefore

$$\frac{1}{2}\int_\triangle \frac{G^{ab}}{2}\frac{\partial^2\psi}{\partial y_a\partial y_b}+\frac{\partial\psi}{\partial y_a}\frac{\partial G^{ab}}{\partial y_b}dy,$$

and we can integrate by parts. Since ψ and its derivatives have compact support in $\triangle$ we do not pick boundary terms. We get

$$-\frac{1}{4}\int_\triangle \psi\frac{\partial^2 G^{ab}}{\partial y_a\partial y_b}.$$

5.2. The Tian–Yau–Zelditch asymptotic expansion. The pointwise asymptotics of the function $\sum|s_m|^2$, where $\{s_m\}$ is an orthonormal basis for the space $H^0(X,L^N)$ was studied in [**C**, **L**, **T**, **Z**]. The following theorem holds:

Theorem 5.1 (Catlin, Lu, Tian, Zelditch). *Let X be a Kähler manifold whose symplectic form, ω, has integral cohomology class and $L\to X$ a line bundle with a Hermitian metric coming from the metric on X. Consider an orthonormal basis for the space $H^0(X,L^N)$, $\{s_m\}$. Then there is an asymptotic expansion*

$$\sum|s_m|^2\sim A_0(\omega)N^n+A_1(\omega)N^{n-1}+\cdots$$

and

$$A_0(\omega)=1,\qquad A_1(\omega)=\frac{s(\omega)}{2},$$

where s is the scalar curvature. More precisely, there exist constants K_r such that

$$\left\| \sum |s_m|^2 - \sum_{i=0}^{r} A_i(\omega) N^{n-i} \right\|_{\mathcal{C}^0(X)} \leq K_r N^{n-r-1}.$$

Using this result on our toric variety X, since our symplectic coordinates are well defined and smooth on our dense open subset, it is easy to conclude that for a compactly supported ψ

$$\int_\triangle \sum |s_m|^2 \psi dy \sim N^n \int_\triangle \psi + N^{n-1} \int_\triangle \psi \frac{s}{2} + \cdots .$$

We note here that $\sum |s_m|^2$ is actually a function of y only. This is because the torus action on L preserves the Hermitian metric and thus leaves $\sum |s_m|^2$ invariant. Comparing this result with the result obtained in the previous subsection we conclude that

$$\int_\triangle \psi \frac{s}{2} = -\frac{1}{4} \int_\triangle \psi \frac{\partial^2 G^{ab}}{\partial y_a \partial y_b}.$$

Thus, since this holds for all compactly supported ψ, we must have

$$s = -\frac{1}{2} \frac{\partial^2 G^{ab}}{\partial y_a \partial y_b},$$

on X, which is Abreu's formula for scalar curvature from [**A1**].

6. Concluding remark

One could, in principle, use the very explicit method described in the last section to obtain more terms in the asymptotic expansion of the spectral measure. This would allow one to write down formulas for other A_i's appearing in the Catlin, Lu, Tian, Zelditch Theorem and these formulas would be in terms of polytope data only. For example, it is know (see [**L**]) that

$$A_2(\omega) = \frac{1}{3} \triangle s + \frac{1}{24} (|R|^2 - 4|\mathrm{Ric}|^2 + 3s^2),$$

where $\triangle s$ is the Laplacian of the scalar curvature, R is the curvature tensor and Ric is the Ricci curvature. Therefore, the term in N^{n-2} in the measure asymptotics would allow one to write down an expression for the quantity

$$|R|^2 - 4|\mathrm{Ric}|^2$$

in terms of polytope data only.

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