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ON POISSON FUNCTIONS

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In this paper, defining Poisson functions on super manifolds, we show that the graphs of Poisson functions are Dirac structures, and find Poisson functions which include as special cases both quasi-Poisson structures and twisted Poisson structures.

1. Introduction

In this paper, we define Poisson functions on super manifolds as a generalization of Poisson structures on manifolds, and show that quasi-Poisson and twisted Poisson structures are both special cases of Poisson functions on some supermanifolds. Quasi-Poisson structure are introduced by Alekseev, Kosmann-Schwarzbach, and Meinrenken [**AK**, **AKM**]. They are defined by an invariant bivector field π on a manifold M with a group action such that the Schouten bracket $[\pi, \pi]$ equals the trivector field generated by the Cartan 3-tensor Ψ. A twisted Poisson structure is a bivector field π on a manifold M such that the Schouten bracket $[\pi, \pi]$ equals the trivector field associated to a closed 3-form Φ on M [**P**, **KS**, **SW**].

In the work [**SW**], Ševera and Weinstein interpret twisted Poisson structures in terms of Courant algebroid and Dirac structure, and ask whether there is a general notion which incorporates both quasi-Poisson and twisted Poisson structures. In this paper, first, generalizing Theorem 6.1 of Liu–Weinstein–Xu [**LWX**], we show that the graphs of Poisson functions are Dirac structures. Second, we show that the notion of Poisson function includes various notions: Poisson structure, twisted Poisson structure, quasi-Poisson structure, Lie algebra action, Lie bialgebra action, Poisson action, etc. In particular, we find Poisson functions which include as special cases both quasi-Poisson structures and twisted Poisson structures. Moreover, a Lie algebroid structure in Theorem 4.1 of Lu [**L**] associated to a Poisson action of a Poisson Lie group is understood in this more general context.

In the interesting paper [**K**], Kosmann-Schwarzbach, following Roytenberg [**R2**], studies weaker versions of Poisson structures by using Poisson functions as "twistings," and, with many other results, points out a similarity of quasi-Poisson structures and twistings of Lie quasi-bialgebroids. Independently, Bursztyn and Crainic also relate hamiltonian quasi-Poisson structures and twisted Poisson structures in [**BC**], and give a geometric way to construct Lie algebroids associated with quasi-Poisson structures in [**BCS**] with Ševera.

This paper is mainly based on ideas of Vaintrob [**V**] who interprets Lie algebroid structures as homological vector fields on supermanifolds, and Roytenberg [**R1**] who gets Courant algebroids from homological functions on supermanifolds.

2. Poisson functions

For a smooth vector bundle $V \to M$ on a smooth manifold M, we have a supermanifold $T^*\Pi V$ with canonical Poisson bracket $\{\ ,\ \}$. A choice of a local coordinate system (x^i) on M and a local basis (ξ^a) of sections of V^* induces a local coordinate system (x^i, ξ^a) on ΠV and a local coordinate system $(x^i, \xi^a, p_i, \theta_a)$ on $T^*\Pi V$. The ring of functions on the supermanifold $T^*\Pi V$ is equipped with a bidegree which is compatible with the parity, by assigning bidegree $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ to $(x^i, \xi^a, p_i, \theta_a)$, respectively.

Definition 2.1. A *homological function* on a supermanifold with an even Poisson bracket $\{\ ,\ \}$ is an odd function S satisfying $\{S, S\} = 0$.

An impressive result of D. Roytenberg [**R1**] is that for a homological function S of total degree 3 on $T^*\Pi V$ we have a Courant algebroid structure on $V \oplus V^*$ with

- Loday bracket on $\Gamma(V \oplus V^*)$:
$$[a, b]_S := \{\{a, S\}, b\}$$
- anchor map on $\Gamma(V \oplus V^*)$:
$$(\tau(a))(f) := \{\{a, S\}, f\}$$
- inner product on $\Gamma(V \oplus V^*)$:
$$(a, b) := \{a, b\}$$
- map $\epsilon : C^\infty(M) \to \Gamma(V \oplus V^*)$:
$$\epsilon(f) := -\frac{1}{2}\{f, S\},$$

where we identify sections of $V \oplus V^*$ with functions of total degree 1 on $T^*\Pi V$.

For a function σ of degree $(0,2)$, a canonical transformation

$$e^{\sigma}(a) := a + \{a,\sigma\} + \frac{1}{2}\{\{a,\sigma\},\sigma\} + \cdots$$

preserves the total degree and the Poisson bracket $\{\ ,\ \}$:

$$\{e^{\sigma}(a), e^{\sigma}(b)\} = e^{\sigma}\{a,b\}.$$

Therefore, for a homological function S of total degree 3, the function $e^{\sigma}(S)$ is also of total degree 3 and homological.

Definition 2.2 (see [**K**, **P**, **R2**])**.** Let X be a super manifold with an even Poisson bracket and a compatible bidegree, and let S be a homological function on X of total degree 3. A *Poisson function* with respect to S is a function σ of degree $(0,2)$ such that the $(0,3)$-component $(e^{-\sigma}(S))^{0,3}$ of $e^{-\sigma}(S)$ vanishes.

Remark 2.3. This condition is equivalent to the "Maurer–Cartan" equation:

$$S^{0,3} - \{S^{1,2},\sigma\} + \frac{1}{2!}\{\{S^{2,1},\sigma\},\sigma\} - \frac{1}{3!}\{\{\{S^{3,0},\sigma\},\sigma\},\sigma\} = 0.$$

Remark 2.4. As D. Roytenberg [**R2**] observes, this condition gives a quasi-Lie bialgebroid structure on (V, V^*) (see also [**K**, **P**, **HP**]).

Theorem 2.5. *The graph $\Gamma_{\sigma} = \{\alpha + \{\alpha,\sigma\} : \alpha \in \Gamma(V^*)\}$ is an isotropic and integrable subbundle, i.e., a Dirac subbundle, of the Courant algebroid $V \oplus V^*$ if and only if σ is a Poisson function.*

Proof. First, we note that

$$e^{\sigma}(\alpha) = \alpha + \{\alpha,\sigma\}$$

for any $(1,0)$-function α because the bracket $\{\ ,\ \}$ has degree $(-1,-1)$. Then, for any $(1,0)$-functions α,β, we have

$$\begin{aligned}(\alpha + \{\alpha,\sigma\}, \beta + \{\beta,\sigma\}) &= (e^{\sigma}(\alpha), e^{\sigma}(\beta)) \\ &= \{e^{\sigma}(\alpha), e^{\sigma}(\beta)\} \\ &= e^{\sigma}\{\alpha,\beta\} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}[\alpha + \{\alpha,\sigma\}, \beta + \{\beta,\sigma\}]_S &= [e^{\sigma}(\alpha), e^{\sigma}(\beta)]_S \\ &= e^{\sigma}[\alpha,\beta]_{e^{-\sigma}(S)} \\ &= e^{\sigma}[\alpha,\beta]_{(e^{-\sigma}(S))^{0,3}+(e^{-\sigma}(S))^{1,2}},\end{aligned}$$

where we use in the last equation the fact that the bracket $\{\ ,\ \}$ has degree $(-1,-1)$. Therefore, σ is a Poisson function if and only if

$$\begin{aligned}[\alpha+\{\alpha,\sigma\},\beta+\{\beta,\sigma\}]_S &= e^{\sigma}[\alpha,\beta]_{(e^{-\sigma}(S))^{1,2}} \\ &= [\alpha,\beta]_{(e^{-\sigma}(S))^{1,2}}+\{[\alpha,\beta]_{(e^{-\sigma}(S))^{1,2}},\sigma\},\end{aligned}$$

which means that the graph Γ_σ is integrable. This completes the proof of Theorem 2.5. □

When a given homological function S has degree $(1,2)+(2,1)$, this proof gives a proof of Theorem 6.1 in Liu–Weinstein–Xu [**LWX**].

3. Quasi-Poisson and twisted Poisson structures

For a smooth manifold M and a Lie algebra $\mathfrak{g}$ with structure constants f_{ab}^c for a basis (τ_a), we consider the supermanifold $X = T^*(\Pi TM \times \Pi\mathfrak{g}^*)$ with local coordinates (x^i, ξ^i, τ_a) on $\Pi TM \times \Pi\mathfrak{g}^*$ and conjugate local coordinates (p_i, θ_i, η_a). Each 3-form

$$\Phi = \frac{1}{3!}\Phi_{ijk}\xi^i\xi^j\xi^k$$

and each skew-symmetric 3-tensor

$$\Psi = \frac{1}{3!}\Psi^{abc}\tau_a\tau_b\tau_c$$

give a homological function

$$\begin{aligned}S &= S_{\mathfrak{g}} + S_M + \Psi + \Phi \\ &= \frac{1}{2!}f_{ab}^c\eta^a\eta^b\tau_c + \xi^i p_i + \frac{1}{3!}\Psi^{abc}\tau_a\tau_b\tau_c + \frac{1}{3!}\Phi_{ijk}\xi^i\xi^j\xi^k\end{aligned}$$

of degree 3 on X when Φ is closed

$$\{S_M, \Phi\} = 0$$

with respect to S_M and Ψ is closed

$$\{S_{\mathfrak{g}}, \Psi\} = 0$$

with respect to $S_{\mathfrak{g}}$. A $(0,2)$-function

$$\begin{aligned}\sigma &= \pi + \rho \\ &= \frac{1}{2!}\pi^{ij}\theta_i\theta_j + \rho_a^j\eta^a\theta_j\end{aligned}$$

is a Poisson function with respect to S if and only if

- $-\{S_{\mathfrak{g}},\rho\} + \frac{1}{2!}\{\{S_M,\rho\},\rho\} = 0$
- $\{\{S_M,\pi\},\rho\} + \{\{S_M,\rho\},\pi\} = 0$
- $\frac{1}{2!}\{\{S_M,\pi\},\pi\} - \frac{1}{3!}\{\{\{\Psi,\rho\},\rho\},\rho\} - \frac{1}{3!}\{\{\{\Phi,\pi\},\pi\},\pi\} = 0.$

In the special case when $\Phi = 0$, we have

- $-\{S_{\mathfrak{g}},\rho\} + \frac{1}{2!}\{\{S_M,\rho\},\rho\} = 0$

- $\{\{S_M, \pi\}, \rho\} + \{\{S_M, \rho\}, \pi\} = 0$
- $\frac{1}{2!}\{\{S_M, \pi\}, \pi\} - \frac{1}{3!}\{\{\{\Psi, \rho\}, \rho\}, \rho\} = 0.$

These conditions correspond to the following.

- ρ is a representation of: $\rho : \mathfrak{g} \to \Gamma(TM)$
- π is invariant for the action ρ
- π is a *quasi-Poisson structure* with respect to Ψ and ρ,

when there exists an invariant inner product on $\mathfrak{g}$ and Ψ is the associated Cartan 3-tensor. In the special case when $\rho = 0$, we have

$$\frac{1}{2!}\{\{S_M, \pi\}, \pi\} - \frac{1}{3!}\{\{\{\Phi, \pi\}, \pi\}, \pi\} = 0$$

which means that π is a *twisted Poisson structure* with respect to Φ.

Remark 3.1. This interpretation of quasi-Poisson structures gives a clear view to the quasi-Poisson cohomology defined by [**AKM**]. In fact, the differential of the quasi-Poisson cohomology is the restriction to the subspace of G-invariant multivectors $C^\infty(M, \wedge TM)^G$ of the differential

$$\begin{aligned} d &= \{e^{-\sigma}(S))^{1,2}, \cdot\} \\ &= \{-\{S_M, \pi\} - \{S_M, \rho\} + \frac{1}{2}\{\{\Psi, \rho\}, \rho\}, \cdot\} \end{aligned}$$

on the space of $(0,*)$-functions $C^\infty(M, \wedge TM) \otimes \wedge \mathfrak{g}^*$.

4. Lu's Lie algebroid

For a smooth manifold M and a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ with structure constants f_{ab}^c, γ_a^{bc} for a basis $(\tau_a,\, \eta^a)$ we consider the supermanifold $X = T^*(\Pi TM \times \Pi\mathfrak{g}^*)$ with local coordinates (x^i, ξ^i, τ_a) on $\Pi TM \times \Pi\mathfrak{g}^*$ and conjugate local coordinates (p_i, θ_i, η^a). Each 3-form

$$\Phi = \frac{1}{3!}\Phi_{ijk}\xi^i\xi^j\xi^k$$

and each skew-symmetric 3-tensor

$$\Psi = \frac{1}{3!}\Psi^{abc}\tau_a\tau_b\tau_c$$

give a homological function

$$\begin{aligned} S &= S_{\mathfrak{g}} + S_{\mathfrak{g}^*} + S_M + \Psi + \Phi \\ &= \frac{1}{2!}f_{ab}^c\eta^a\eta^b\tau_c + \frac{1}{2!}\gamma_a^{bc}\eta^a\tau_b\tau_c + \xi^i p_i + \frac{1}{3!}\Psi^{abc}\tau_a\tau_b\tau_c + \frac{1}{3!}\Phi_{ijk}\xi^i\xi^j\xi^k \end{aligned}$$

of degree 3 on X when Φ is closed

$$\{S_M, \Phi\} = 0$$

with respect to S_M and Ψ is closed

$$\{S_{\mathfrak{g}} + S_{\mathfrak{g}^*}, \Psi\} = 0$$

with respect to $S_{\mathfrak{g}} + S_{\mathfrak{g}^*}$. A $(0,2)$-function

$$\begin{aligned}\sigma &= \pi + \rho \\ &= \frac{1}{2!}\pi^{ij}\theta_i\theta_j + \rho_a^j\eta^a\theta_j\end{aligned}$$

is a Poisson function with respect to S if and only if

- $-\{S_{\mathfrak{g}}, \rho\} + \frac{1}{2!}\{\{S_M, \rho\}, \rho\} = 0$
- $\{\{S_M, \pi\}, \rho\} + \{\{S_M, \rho\}, \pi\} + \frac{1}{2!}\{\{S_{\mathfrak{g}^*}, \rho\}, \rho\} = 0$
- $\frac{1}{2!}\{\{S_M, \pi\}, \pi\} - \frac{1}{3!}\{\{\{\Psi, \rho\}, \rho\}, \rho\} - \frac{1}{3!}\{\{\{\Phi, \pi\}, \pi\}, \pi\} = 0.$

In the special case when $\Phi = 0$ and $\Psi = 0$, we have

- $-\{S_{\mathfrak{g}}, \rho\} + \frac{1}{2!}\{\{S_M, \rho\}, \rho\} = 0$
- $\{\{S_M, \pi\}, \rho\} + \frac{1}{2!}\{\{S_{\mathfrak{g}^*}, \rho\}, \rho\} = 0$
- $\{\{S_M, \pi\}, \pi\} = 0.$

These conditions correspond to the following.

- ρ is a representation of: $\rho : \mathfrak{g} \to \Gamma(TM)$
- ρ is an infinitesimal Poisson action of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$
- π is a Poisson structure.

For each Poisson function σ, Theorem 2.5 gives a Lie algebroid structure on the graph Γ_σ which is equivalent to the Lie algebroid structure on $T^*M \times \mathfrak{g}$ in Theorem 4.1 of J.-H. Lu [**L**] associated to a Poisson action of a Poisson Lie group.

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