

OZSVÁTH–SZABÓ INVARIANTS AND TIGHT CONTACT 3-MANIFOLDS, III

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We characterize L -spaces which are Seifert fibered over the 2-sphere in terms of taut foliations, transverse foliations and transverse contact structures. We give a sufficient condition for certain contact Seifert fibered 3-manifolds with $e_0 = -1$ to have nonzero contact Ozsváth–Szabó invariants. This yields an algorithm for deciding whether a given small Seifert fibered L -space carries a contact structure with nonvanishing contact Ozsváth–Szabó invariant. As an application, we prove the existence of tight contact structures on some 3-manifolds obtained by integral surgery along a positive torus knot in the 3-sphere. Finally, we prove planarity of every contact structure on small Seifert fibered L -spaces with $e_0 \geq -1$, and we discuss some consequences.

1. Introduction

The Ozsváth–Szabó homology groups of a closed, oriented 3-manifold Y [36, 37] capture important topological information about Y . For example, by [35, Theorem 1.1] the Thurston semi-norm is determined by the evaluation of the first Chern classes of spin^c structures with nontrivial Ozsváth–Szabó homology groups. For rational homology spheres, however, the Thurston norm is trivial, while the Ozsváth–Szabó homology groups are special: the group $\widehat{\text{HF}}(Y, \mathbf{t})$ has odd rank for each spin^c structure $\mathbf{t} \in \text{Spin}^c(Y)$. A rational homology sphere Y shows the simplest possible Heegaard Floer-theoretic behavior if for every spin^c structure $\mathbf{t} \in \text{Spin}^c(Y)$ the group $\widehat{\text{HF}}(Y, \mathbf{t})$ with integral coefficients is isomorphic to \mathbb{Z} , in which case Y is called an L -space. In the present paper, we shall always use $\mathbb{Z}/2\mathbb{Z}$ -coefficients. In this case, if Y is an L -space then $\widehat{\text{HF}}(Y, \mathbf{t}) \cong \mathbb{Z}/2\mathbb{Z}$ for every $\mathbf{t} \in \text{Spin}^c(Y)$. Since $\widehat{\text{HF}}(Y, \mathbf{t}) \cong \widehat{\text{HF}}(-Y, \mathbf{t})$ for each $\mathbf{t} \in \text{Spin}^c(Y)$, the 3-manifold Y is an L -space if and only if $-Y$ is.

If Y is a 3-manifold with elliptic geometry then Y is an L -space [39, Proposition 2.3]. In particular, lens spaces are L -spaces. Seifert fibered L -spaces can be characterized combinatorially using the results of [28, 33].

More generally, by [28] the boundary of a negative definite 4-dimensional plumbing is an L -space if it is the link of a rational surface singularity. In [34, Section 5, Question 11], it was asked whether there is a topological characterization of L -spaces. In the first part of this paper, we study the topological significance of being an L -space for Seifert fibered rational homology 3-spheres.

It is proved in [35, Theorem 1.4] that an L -space admits no taut foliation. Our first result shows that the converse to this statement holds for Seifert fibered rational homology 3-spheres. In addition, we find that it is equivalent to the nonexistence of a transverse foliation (i.e., a foliation transverse to the fibers of the Seifert fibration), as well as to the nonexistence of positive, transverse contact structures for at least one orientation.

Let M be a closed, oriented rational homology 3-sphere which carries a Seifert fibration over S^2 . Then, M is orientation preserving diffeomorphic to the oriented 3-manifold $M(e_0; r_1, r_2, \dots, r_k)$ with surgery presentation given by Figure 1, where $e_0 \in \mathbb{Z}$, $r_i \in (0, 1) \cap \mathbb{Q}$ and $r_1 \geq r_2 \geq \dots \geq r_k$.

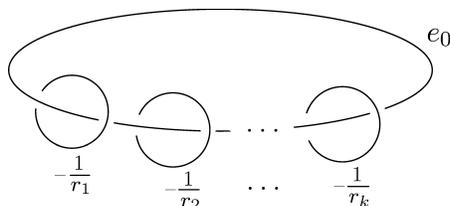


Figure 1. Surgery diagram for $M(e_0; r_1, r_2, \dots, r_k)$.

Moreover, the surgery picture shows that $M(e_0; r_1, \dots, r_k)$ carries a natural Seifert fibration over S^2 , which coincides (up to isotopy) with the image of the fibration on M under the above diffeomorphism. Throughout the paper, we always implicitly refer to this natural fibration on $M(e_0; r_1, \dots, r_k)$.

Define

$$e(M) := e_0(M) + r_1 + r_2 + \dots + r_k.$$

Notice that

$$-M(e_0; r_1, r_2, \dots, r_k) = M(-e_0 - k; 1 - r_1, 1 - r_2, \dots, 1 - r_k),$$

therefore we have $e(-M) = -e(M)$. It is known that M is a rational homology sphere if and only if $e(M) \neq 0$.

Theorem 1.1. *Let us suppose that M is an oriented rational homology 3-sphere which is Seifert fibered over S^2 . Then, the following statements are equivalent:*

- (1) M is an L -space;
- (2) Either M or $-M$ carries no positive, transverse contact structures;
- (3) M carries no transverse foliations;
- (4) M carries no taut foliations.

It was proved in [21] that an oriented Seifert fibered rational homology 3-sphere $M = M(e_0, r_1, r_2, \dots, r_k)$ with $r_1 \geq r_2 \geq \dots \geq r_k$ admits no positive transverse contact structure if and only if

- $e_0(M) \geq 0$, or
- $e_0(M) = -1$ and there are no relatively prime integers $m > a$ such that

$$mr_1 < a < m(1 - r_2) \quad \text{and} \quad mr_i < 1 \quad i = 3, \dots, k.$$

Combined with this result, Theorem 1.1 gives a simple characterization of L -spaces among rational homology 3-spheres of the form $M(e_0; r_1, \dots, r_k)$.

In the second part of the paper, we give a sufficient condition for certain contact Seifert fibered 3-manifolds with $e_0 = -1$ to have nonzero contact Ozsváth–Szabó invariants. Combining this result with Theorem 1.1 and with a result of Paolo Ghiggini (Theorem 1.3 below) yields an algorithm for deciding whether an L -space of the form $M(e_0; r_1, r_2, r_3)$ carries a contact structure with nonvanishing contact Ozsváth–Szabó invariant.

Consider the contact surgery diagram of Figure 2. The diagram should be interpreted as representing all possible contact $(-\frac{1}{r_i})$ -surgeries in case these are not unique (see Section 2 for details). With this convention, Figure 2 represents a family of contact structures on the manifold $M = M(-1; r_1, \dots, r_k)$. Given an oriented contact structure ξ on M , let \mathbf{t}_ξ denote the spin^c structure induced by ξ and $d_3(\xi) \in \mathbb{Q}$ the 3-dimensional invariant determined by the homotopy class of the oriented 2-plane field ξ [17]. Finally, denote by $d(M, \mathbf{t}) \in \mathbb{Q}$ the d -invariant of (M, \mathbf{t}) (cf. [32]).

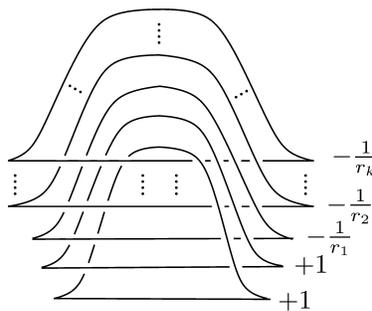


Figure 2. Contact structures on $M(-1; r_1, r_2, \dots, r_k)$.

Theorem 1.2. *Let (M, ξ) be any of the contact, Seifert fibered 3-manifolds given by Figure 2. Suppose that*

$$e(M) = r_1 + r_2 + \cdots + r_k - 1 > 0,$$

and

$$d_3(\xi) = d(M, \mathbf{t}_\xi).$$

Then the contact invariant $c(M, \xi) \in \widehat{\text{HF}}(-M, \mathbf{t}_\xi)$ is nonzero.

Theorem 1.2 applies to some Seifert fibered 3-manifolds which are not L -spaces (such as, e.g., $M(-1; \frac{5}{12}, \frac{1}{3}, \frac{1}{3})$), but it is particularly useful when $k = 3$ and the 3-manifold M is an L -space. To explain why, we need a small digression.

Suppose that (M, ξ) is a contact, Seifert fibered 3-manifold. Then, a Legendrian knot in M smoothly isotopic to a regular fiber admits two framings: one coming from the fibration and another one coming from the contact structure ξ . The difference between the contact framing and the fibration framing is the *twisting number* of the Legendrian curve. We say that ξ has *maximal twisting equal to zero* if there is a Legendrian knot L isotopic to a regular fiber such that L has twisting number zero. According to a result of Ghiggini, if $M = M(e_0; r_1, r_2, r_3)$ admits no transverse contact structure then each tight contact structure on M has maximal twisting equal to zero.

Theorem 1.3 ([13], Corollary 4.6). *$M = M(e_0; r_1, r_2, r_3)$ admits a tight contact structure with negative twisting if and only if M admits a transverse contact structure.*

Thus, combining Theorems 1.1 and 1.3, we see that if $M = M(e_0; r_1, r_2, r_3)$ is an L -space, then each tight contact structure ξ on M has maximal twisting equal to zero. On the other hand, in Section 6 (see Proposition 6.1) we show that every tight contact structure with maximal twisting equal to zero on a 3-manifold of the form $M(e_0; r_1, r_2, r_3)$ is given by one of the diagrams represented by Figure 2. Moreover, the quantities $d_3(\xi)$ and $d(M, \mathbf{t}_\xi)$ can be computed algorithmically from a contact surgery presentation. Therefore Theorem 1.2 yields an algorithm for deciding whether an L -space of the form $M(e_0; r_1, r_2, r_3)$ carries a contact structure with nonvanishing contact Ozsváth–Szabó invariants. (Recall that contact structures with nonvanishing contact Ozsváth–Szabó invariant are tight.) In fact, one can view Theorem 1.2 as a useful tool to attack the classification problem for tight contact structures on small Seifert fibered L -spaces with $e_0 = -1$. (Tight contact structures on small Seifert fibered 3-manifolds with $e_0 \neq -1, -2$ have been classified [14, 44], but the classification for $e_0 = -1, -2$ is expected to be considerably harder, due to the presence of nonfillable structures [11, 22] as well as manifolds with no tight contact structures [10, 25].)

As an application of Theorem 1.2, we prove the existence of tight contact structures on some of the 3-manifolds obtained by integral surgery along a

positive torus knot in S^3 . To put this result in perspective, recall that we showed in [24] that for the positive (p, q) -torus knot $T_{p,q} \subset S^3$ the 3-manifold $S_r^3(T_{p,q})$ obtained by topological r -surgery along $T_{p,q}$ carries tight contact structures for every rational r except possibly when $r = pq - p - q$. In this latter case, the idea used in [24] does not work, since it would require the application of contact 0-surgery, which automatically leads to overtwisted structures. In [25] we analyzed the existence of tight contact structures on the 3-manifold $S_r^3(T_{p,q})$ with r equal to the “critical” surgery coefficient $pq - p - q$. We showed that for $(p, q) = (2, 2n + 1)$ the corresponding manifolds do not admit any tight structures (extending an earlier result of Etnyre and Honda [10], who proved the same result for $(p, q) = (2, 3)$). On the other hand, in the same paper [25] we verified that $S_{p^2n - pn - 1}^3(T_{p, pn + 1})$ does carry tight contact structures for every $n \geq 1$ and $p \geq 3$ odd. The parity assumption on p played a crucial role in the argument, since in this case the candidate contact structure induced a spin structure. This allowed us to use a certain $\mathbb{Z}/2\mathbb{Z}$ symmetry (the so-called J -map) built in Heegaard Floer theory to prove the nonvanishing of the contact invariant. A little computer search using the algorithm outlined above showed that for p even one cannot always find a tight contact structure with nonzero contact Ozsváth–Szabó invariant inducing a spin structure. For example, there is no such structure on $S_{11}^3(T_{4,5})$. This implies that when p is even, a different approach should be used. In Section 5 we use Theorem 1.2 to prove.

Theorem 1.4. *For each $p > 2$, the manifold $S_{p^2 - p - 1}^3(T_{p, p + 1})$ carries tight contact structures.*

Using the fact that every tight contact structure with zero maximal twisting on a 3-manifold of the form $M(e_0; r_1, r_2, r_3)$ is given by a surgery diagram as in Figure 2 (for $k = 3$), we can prove:

Theorem 1.5. *A tight contact structure ξ with zero maximal twisting on a small Seifert fibered 3-manifold $M(e_0; r_1, r_2, r_3)$ is compatible with a planar open book decomposition.*

An immediate corollary of Theorem 1.5 is the following characterization of small Seifert fibered L -spaces:

Corollary 1.6. *A rational homology 3-sphere $M = M(e_0; r_1, r_2, r_3)$ is an L -space if and only if on either M or $-M$ every contact structure is planar.*

Another corollary of Theorem 1.5 is the following.

Corollary 1.7. *If $M = M(e_0; r_1, r_2, r_3)$ with $e_0 \geq -1$ and M is an L -space, then each contact structure on M is planar.*

In view of a recent result of Abbas, Cieliebak and Hofer [1] Corollary 1.7 implies:

Corollary 1.8. *If $M = M(e_0; r_1, r_2, r_3)$ with $e_0 \geq -1$ and M is an L -space, then each contact structure ξ on M satisfies the Weinstein conjecture, that is, every Reeb vector field for (M, ξ) admits a periodic orbit.*

The paper is organized as follows. In Section 2 we collect some background material. In Sections 3, 4 and 5 we prove, respectively, Theorems 1.1, 1.2 and 1.4. Section 6 is devoted to examining compatible open books and establishing Theorem 1.5 and its corollaries.

2. Preliminaries

2.1. Transverse contact structures and transverse foliations. Here we collect some known results on transverse contact structures and foliations on the Seifert fibered 3-manifold

$$M = M(e_0; r_1, r_2, \dots, r_k), \quad r_1 \geq r_2 \geq \dots \geq r_k$$

defined by Figure 1. Consider

$$\Gamma(M) := (r_1, r_2, \dots, r_k) \in (\mathbb{Q} \cap (0, 1))^k.$$

We say that $\Gamma(M)$ is *realizable* if there exist coprime integers $m > a > 0$ such that

$$r_1 < \frac{a}{m}, r_2 < \frac{m-a}{m} \text{ and } r_3, \dots, r_k < \frac{1}{m}.$$

In Section 3 we shall use the following theorem, obtained by combining results of Jankins–Neumann and Naimi, and stated here in our present notation

Theorem 2.1 ([20, 27]). *Let $M = M(e_0; r_1, r_2, \dots, r_k)$ be as above. Then M carries a smooth foliation transverse to the Seifert fibration if and only if one of the following holds:*

- $-k + 2 \leq e_0 \leq -2$;
- $e_0 = -1$ and $\Gamma(M)$ is realizable;
- $e_0 = -k + 1$ and $\Gamma(-M)$ is realizable.

The following theorem is part of a result [21, Theorem 1.3] of the first named author and of Gordana Matić (Ko Honda [19] and Paolo Ghigini [13] also obtained similar results).

Theorem 2.2 ([21]). *Let $M = M(e_0; r_1, r_2, \dots, r_k)$ be as above. Then M carries a positive contact structure transverse to the Seifert fibration if and only if one of the following holds:*

- $k \leq 2$ and $e(M) := e_0 + \sum_i r_i < 0$;
- $e_0 \leq -2$;
- $e_0 = -1$ and $\Gamma(M)$ is realizable.

Recall that the Seifert fibered 3-manifold $M = M(e_0; r_1, r_2, \dots, r_k)$ is defined by the surgery diagram of Figure 1. Applying inverse slam-dunks, the diagram can be turned into a surgery diagram involving only integer coefficients: replace each $(-\frac{1}{r_i})$ -surgery with a sequence of integer surgeries along a chain of unknots, where the integral surgery coefficients are given by the coefficients of the continued fraction expansion

$$-\frac{1}{r_i} = a_0^{(i)} - \frac{1}{a_1^{(i)} - \frac{1}{\dots - \frac{1}{a_k^{(i)}}}}, \quad a_j^{(i)} \leq -2.$$

The integral surgery diagram defines a 4-manifold $W = W(e_0; r_1, \dots, r_k)$. Notice that in the diagram all knots are unknots, they are arranged along a star shaped tree T , and all framings are at most -2 except the central one, which is equal to e_0 . The 4-manifold W is endowed with an S^1 -action obtained by equivariant plumbing according to T , where each vertex corresponds to a disk bundle over a sphere. By [30, Chapter 2] M is isomorphic (as a 3-manifold with S^1 -action) to the boundary of W . Moreover, it follows from the results of [26] that the S^1 -manifold W carries a symplectic form ω such that $\omega \wedge \omega > 0$ and every orbit of the S^1 -action on $\partial W = M$ is tangent to the kernel of $\omega|_{\partial W}$. Therefore, if ξ is a transverse contact structure on M then $\omega|_{\xi} \neq 0$. Recall that a *symplectic filling* of a closed contact 3-manifold (M, ξ) is a symplectic 4-manifold (X, Ω) such that (i) X is oriented by $\Omega \wedge \Omega$, (ii) $\partial X = M$ as oriented manifolds, and (iii) $\Omega|_{\xi} \neq 0$ at every point of M . An immediate corollary of this discussion is the following.

Proposition 2.3. $W = W(e_0; r_1, r_2, \dots, r_k)$ carries a symplectic form ω such that if $M = M(e_0; r_1, r_2, \dots, r_k)$ as above carries a positive transverse contact structure ξ then (W, ω) is a symplectic filling of (M, ξ) . □

It is known [29] that

$$b_2^+(W) = \begin{cases} 1 & \text{if } e(M) > 0, \\ 0 & \text{if } e(M) < 0. \end{cases}$$

The following result is not explicitly stated in [21], but it is implicitly contained in the proof of [21, Theorem 1.3] given in [21, Section 4].

Theorem 2.4 ([21]). *Let $M = M(e_0; r_1, r_2, \dots, r_k)$ be as above. If M carries no positive contact structure transverse to the Seifert fibration then there exists an oriented surface Σ smoothly embedded in W such that $g(\Sigma) > 0$ and*

$$\Sigma \cdot \Sigma > 2g(\Sigma) - 2.$$

2.2. Contact surgery. Suppose that $L \subset (Y, \xi)$ is a Legendrian knot, that is, a knot tangent to the 2-plane field ξ . Oriented normals to ξ along L provide a framing for L , called the *contact framing*. Let $Y_r(L)$ denote the result of r -surgery along L , where the surgery coefficient is measured with respect to the contact framing. Contact structures on $Y_r(L)$ can be defined by taking the restriction $\xi|_{Y-\nu L}$ of ξ to the complement of a standard convex neighborhood of L , and extending it to a contact structure on $Y_r(L)$ which is tight on the glued-up solid torus. By the classification of tight contact structures on $S^1 \times D^2$ [16, 18], such an extension exists if and only if $r \neq 0$ and is unique if and only if $r = \frac{1}{k}$ for some $k \in \mathbb{Z}$. In general, there are many tight structures on $S^1 \times D^2$ satisfying the same boundary condition. As Honda showed, a tight contact $(S^1 \times D^2, \eta)$ can be decomposed as

$$(S^1 \times D^2, \eta) = \bigcup_{i=1}^t (T^2 \times I, \eta_i) \cup (S^1 \times D^2, \eta_0),$$

where $(S^1 \times D^2, \eta_0)$ is the standard neighborhood of a Legendrian knot isotopic to $S^1 \times 0 \subset S^1 \times D^2$ and each $(T^2 \times I, \eta_i)$ is a “continued fraction block.” A continued fraction block $(T^2 \times I, \eta_i)$ admits a further slicing

$$(T^2 \times I, \eta_i) = \bigcup_{j=1}^{s_i} (T^2 \times I, \eta_i(j)),$$

where each $(T^2 \times I, \eta_i(j))$ is a “basic slice” (see [18] for definitions). Basic slices are characterized by a sign $+$ or $-$. The continued fraction block-decomposition of $(S^1 \times D^2, \eta)$ is dictated by the boundary value (the “slope” of $T^2 = \partial(S^1 \times D^2)$), which in the case of surgery is determined by the surgery coefficient r . The signs of the basic slices, however, rely on choices, giving rise to possibly many tight contact structures on $S^1 \times D^2$ with identical boundary condition.

Ding and Geiges [3] showed how to realize each basic slice decomposition of $(S^1 \times D^2, \eta)$ by a contact surgery diagram (cf. also [4]). Let $r = \frac{p}{q}$ and if $r > 0$ choose $k \in \mathbb{N}$ minimal such that $r' = p/(q - kp) < 0$. Consider k Legendrian push-offs L_1, \dots, L_k of L and perform contact $(+1)$ -surgeries along these push-offs. (In this part of the procedure there is no choice.) Next, do contact r' -surgery along L . Suppose that the contact solid torus $(S^1 \times D^2, \eta)$ to be used in the surgery has basic slice decomposition

$$\bigcup_{i=1}^t \bigcup_{j=1}^{s_i} (T^2 \times I, \eta_i(j)) \cup (S^1 \times D^2, \eta_0).$$

After fixing an orientation for L , apply a right stabilizations to L for each positive basic slice in

$$(T^2 \times I, \eta_1) = \bigcup_{j=1}^{s_1} (T^2 \times I, \eta_1(j))$$

and a negative stabilization for each negative basic slice. Denote the result by $L(1)$. Consider a Legendrian push-off $L'(1) = L(2)$ of $L(1)$ and perform contact (-1) -surgery along $L(1)$. Repeat the above procedure for $L(2)$, using the second continued fraction block of the decomposition. After t steps the procedure terminates providing a Legendrian link in (Y, ξ) along which the prescribed contact (± 1) -surgeries give the contact structure on $Y_r(L)$ obtained by applying contact r -surgery with the prescribed extension $(S^1 \times D^2, \eta)$ on the glued-up solid torus.

2.3. Ozsváth–Szabó homologies. In the seminal papers [36, 37] a collection of homology groups — the Ozsváth–Szabó homologies — $\widehat{\text{HF}}(Y, \mathbf{t})$, $\text{HF}^\pm(Y, \mathbf{t})$ and $\text{HF}^\infty(Y, \mathbf{t})$ have been assigned to any closed, oriented spin^c 3-manifold (Y, \mathbf{t}) . For simplicity, *throughout this paper we shall use $\mathbb{Z}/2\mathbb{Z}$ -coefficients*. The groups in question admit a relative $\mathbb{Z}/d(\mathbf{t})\mathbb{Z}$ -grading, where $d(\mathbf{t})$ is the divisibility of the first Chern class $c_1(\mathbf{t})$. When $c_1(\mathbf{t})$ is torsion, then $d(\mathbf{t}) = 0$, and the relative \mathbb{Z} -grading lifts to an absolute \mathbb{Q} -grading. The groups $\text{HF}^\pm(Y, \mathbf{t})$ and $\text{HF}^\infty(Y, \mathbf{t})$ admit $\mathbb{Z}[U]$ -module structures, and multiplication by U decreases grading by 2.

A spin^c cobordism (W, \mathbf{s}) between spin^c 3-manifolds (Y_1, \mathbf{t}_1) and (Y_2, \mathbf{t}_2) induces a $\mathbb{Z}[U]$ -equivariant homomorphism $F_{W, \mathbf{s}}$ between the corresponding groups, and if $\mathbf{t}_1, \mathbf{t}_2$ are both torsion spin^c structures then $F_{W, \mathbf{s}}$ shifts degree by

$$\frac{1}{4}(c_1^2(\mathbf{s}) - 3\sigma(W) - 2\chi(W)).$$

The direct sum of Ozsváth–Szabó homology groups for all spin^c structures is usually denoted by $\widehat{\text{HF}}(Y)$ (respectively, by $\text{HF}^\pm(Y)$, $\text{HF}^\infty(Y)$), while the sum of the maps induced by (W, \mathbf{s}) for all $\mathbf{s} \in \text{Spin}^c(W)$ is denoted by F_W (respectively, F_W^\pm , F_W^∞).

A rational homology sphere Y is called an L -space if $\widehat{\text{HF}}(Y, \mathbf{t})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for all spin^c structure $\mathbf{t} \in \text{Spin}^c(Y)$. Equivalently, the dimension $\dim_{\mathbb{Z}/2\mathbb{Z}} \widehat{\text{HF}}(Y)$ is equal to the order $|H_1(Y; \mathbb{Z})|$. Other equivalent ways to define L -spaces is to require that the U -action on $\text{HF}^+(Y, \mathbf{t})$ is surjective, or that the natural map $\text{HF}^\infty(Y, \mathbf{t}) \rightarrow \text{HF}^+(Y, \mathbf{t})$ is onto. Yet another characterization is that for an L -space Y the group $\widehat{\text{HF}}(Y, \mathbf{t})$ can be identified with the kernel of the U -map

$$U: \text{HF}^+(Y, \mathbf{t}) \rightarrow \text{HF}^+(Y, \mathbf{t}).$$

(The equivalences follow from the long exact sequences connecting the various groups, see [37].) For a rational homology 3-sphere Y and spin^c structure $\mathfrak{t} \in \text{Spin}^c(Y)$ let the d -invariant $d(Y, \mathfrak{t})$ of (Y, \mathfrak{t}) be defined as the absolute degree of the unique nontrivial element $x \in \text{HF}^+(Y, \mathfrak{t})$ which is in the image of the natural map $\text{HF}^\infty(Y, \mathfrak{t}) \rightarrow \text{HF}^+(Y, \mathfrak{t})$ and $Ux = 0$. It is easy to see that for an L -space Y the absolute degree of the generator of $\widehat{\text{HF}}(Y, \mathfrak{t})$ is equal to $d(Y, \mathfrak{t})$. It is known that $d(-Y, \mathfrak{t}) = -d(Y, \mathfrak{t})$.

Suppose that the manifold $Y(K)$ is given as integral surgery along a knot K in Y , while $Y_1(K)$ is defined by an integral surgery along $K \subset Y$ with framing one higher. According to [37, Theorem 9.16] the groups corresponding to these 3-manifolds (together with the maps induced by appropriate cobordisms W_1, W_2, W_3 between the 3-manifolds) fit into the exact triangle:

$$\begin{array}{ccc} \widehat{\text{HF}}(Y) & \xrightarrow{F_{W_1}} & \widehat{\text{HF}}(Y(K)) \\ & \searrow F_{W_3} & \swarrow F_{W_2} \\ & \widehat{\text{HF}}(Y_1(K)) & \end{array}$$

Ozsváth–Szabó homologies are quite hard to compute in general, but for 3-manifolds which can be presented as boundaries of plumbings along negative definite plumbing trees with no “bad” vertices (cf. [33]), such computation has been carried out in [33]. This immediately implies, for example, that a Seifert fibered rational homology 3-sphere with k multiple fibers and $e_0(M) \leq -k$ is an L -space, since it can be presented as the boundary of a plumbing tree without “bad” vertices. Since $e_0(-M) = -k - e_0(M)$ for Seifert fibered spaces, the fact $\dim \widehat{\text{HF}}(Y) = \dim \widehat{\text{HF}}(-Y)$ implies that a Seifert fibered 3-manifold $M = M(e_0; r_1, r_2, \dots, r_k)$ with $e_0(M) \leq -k$ or $e_0(M) \geq 0$ is an L -space.

The following two lemmas can be deduced from the results of [33]. They will be used in Section 3.

Lemma 2.5. *Suppose that W is a negative definite cobordism between the L -space Y_1 and the rational homology 3-sphere Y_2 obtained by attaching 2-handles to Y_1 . Let $\mathfrak{s} \in \text{Spin}^c(W)$ be a spin^c structure such that the degree shift of the maps*

$$F_{W, \mathfrak{s}} : \widehat{\text{HF}}(Y_1, \mathfrak{s}|_{Y_1}) \rightarrow \widehat{\text{HF}}(Y_2, \mathfrak{s}|_{Y_2})$$

and

$$F_{W, \mathfrak{s}}^+ : \text{HF}^+(Y_1, \mathfrak{s}|_{Y_1}) \rightarrow \text{HF}^+(Y_2, \mathfrak{s}|_{Y_2})$$

is

$$d(Y_2, \mathfrak{s}|_{Y_2}) - d(Y_1, \mathfrak{s}|_{Y_1}).$$

Then the maps $F_{W, \mathfrak{s}}$ and $F_{W, \mathfrak{s}}^+$ are injective.

Proof. Since W is negative definite, successive applications of [32, Proposition 9.4] imply that the map

$$F_{W,s}^\infty : \text{HF}^\infty(Y_1, \mathbf{s}|_{Y_1}) \rightarrow \text{HF}^\infty(Y_2, \mathbf{s}|_{Y_2})$$

is an isomorphism for any spin^c structure. This implies that the map $F_{W,s}^+$ is injective provided its degree shift is equal to the difference of the d -invariants. The long exact sequence connecting $\widehat{\text{HF}}(Y, \mathbf{t})$ and $\text{HF}^+(Y, \mathbf{t})$ together with the fact that the natural map

$$\widehat{\text{HF}}(Y_1, \mathbf{s}|_{Y_1}) \rightarrow \text{HF}^+(Y_1, \mathbf{s}|_{Y_1})$$

is injective for the L -space Y_1 imply that if $F_{W,s}^+$ is injective then so is the map $F_{W,s}$ between the $\widehat{\text{HF}}$ -groups, concluding the proof. \square

Lemma 2.6. *Suppose that W is the cobordism from S^3 to an L -space L given by a negative definite plumbing with no “bad” vertex in the sense of [33]. Then for any spin^c structure $\mathbf{t} \in \text{Spin}^c(L)$ there is a spin^c structure $\mathbf{s} \in \text{Spin}^c(W)$ such that the map*

$$F_{W,s} : \widehat{\text{HF}}(S^3) \rightarrow \widehat{\text{HF}}(L, \mathbf{t})$$

is an isomorphism.

Proof. The proof is straightforward using the algorithm of [33] which computes the kernel of the U -map on $\text{HF}^+(L, \mathbf{t})$. Take the negative definite plumbing W for L and choose the characteristic vector $K \in H^2(W; \mathbb{Z})$ with the following three properties:

- K satisfies the starting condition given in the first paragraph of [33, Section 3.1],
- K leads to a final vector with the property given in (16) of the same paragraph, and
- K induces the spin^c structure \mathbf{t} on L .

Since the algorithm computes the Ozsváth–Szabó homology of the boundary L -space L with the given spin^c structure, such a vector clearly exists. By the formula for the grading given in [33], the final vector of this process defines a spin^c structure on the 4-manifold W satisfying the conditions of Lemma 2.5. Since by assumption L is an L -space, this clearly suffices to prove the statement. \square

2.4. Contact Ozsváth–Szabó invariants. A contact structure ξ on Y determines an element $c(Y, \xi) \in \widehat{\text{HF}}(-Y, \mathbf{t}_\xi)$ which has the following crucial properties [38]:

- $c(Y, \xi)$ is an isotopy invariant of the contact 3-manifold (Y, ξ) ;
- $c(Y, \xi) = 0$ if the contact structure ξ is overtwisted;
- $c(Y, \xi) \neq 0$ if (Y, ξ) is Stein fillable;

- if $c_1(\mathbf{t}_\xi)$ is a torsion class and $c(Y, \xi) \neq 0$ then it is of degree $-d_3(\xi)$ in $\widehat{\text{HF}}(-Y, \mathbf{t}_\xi)$, where $d_3(\xi) \in \mathbb{Q}$ is the Hopf invariant of the 2-plane field underlying ξ .

3. The proof of Theorem 1.1

Some of the implications among the equivalent statements of Theorem 1.1 are relatively easy to prove (and have been partly established in the literature), so we will start with those.

(1) \implies (2) Suppose that M carries transverse contact structures with both of its orientations. Since $e(-M) = -e(M)$, we can choose the orientation for which the 4-manifold W given by the integer plumbing representation of M has positive b_2^+ -invariant. According to Proposition 2.3 this manifold carries a symplectic structure which provides a symplectic filling for any transverse contact structure. Since $b_2^+(W) > 0$, by [35, Theorem 1.4] M cannot be an L -space.

(2) \implies (3) Since $M \neq S^1 \times S^2$, according to [6] a transverse foliation can be C^0 -approximated by positive as well as negative transverse contact structures. Therefore, the nonexistence of positive transverse contact structures with one orientation prevents the existence of a transverse foliation on M .

(1) \implies (4) This is part of [35, Theorem 1.4].

(4) \implies (3) Since transverse foliations are taut, this implication is trivial.

(3) \implies (2) Suppose that $M = M(e_0; r_1, r_2, \dots, r_k)$ admits no transverse foliation. Since M is a rational homology sphere, either $e(M) > 0$ or $e(-M) = -e(M) > 0$. Therefore, if $k \leq 2$ the conclusion follows from Theorem 2.2. Now suppose that $k \geq 3$. According to Theorem 2.1, up to changing the orientation of M we may assume that either $e_0(M) \geq 0$ or $e_0(M) = -1$ and $\Gamma(M)$ is not realizable. In both cases, by Theorem 2.2, M carries no positive, transverse contact structures.

In the light of our observations above, to complete the proof of the equivalences we only need to show that (2) implies (1), that is, if the Seifert fibered rational homology 3-sphere M carries no transverse contact structures with one of its orientations then it is an L -space. Observe that if $k \leq 2$ then M is a lens space, hence an L -space, therefore in this case there is nothing to prove. Likewise, if either $e_0(M) \leq -k$ or $e_0(M) \geq 0$, then M is an L -space. Moreover, if $-k + 1 \leq e_0(M) \leq -2$ then by Theorem 2.2 M carries transverse contact structures. Therefore, up to changing the orientation of M we may assume that $e_0(M) = -1$, so to establish Theorem 1.1 it will suffice to prove the following.

Proposition 3.1. *If the rational homology 3-sphere $M = M(-1; r_1, \dots, r_k)$ carries no transverse, positive contact structures, then M is an L -space.*

Before proving Proposition 3.1 we need some preliminaries. Suppose that M does not carry a transverse positive contact structure. By Theorem 2.4, this implies that the associated 4-manifold W contains a surface Σ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma > 2g(\Sigma) - 2$. Let W_0 be the 4-manifold obtained from W by deleting an open 4-ball from its interior. Then, by [24, Proposition 2.1] the map

$$F_{W_0, \mathbf{s}} : \widehat{\text{HF}}(S^3) \longrightarrow \widehat{\text{HF}}(M)$$

vanishes for every spin^c structure $\mathbf{s} \in \text{Spin}^c(W_0)$. Now let us consider the subcobordism W_1 obtained by attaching to S^3 the 2-handles corresponding to the vertices of the plumbing tree with framings at most -2 . This 4-manifold is a cobordism between S^3 and a connected sum of k lens spaces L . By standard properties of lens spaces and connected sums, L is an L -space. Therefore, W satisfies the assumptions of Lemma 2.6. The attachment of the (-1) -framed circle induces a cobordism W_2 from $-L$ to M . Since W_1 is negative definite and $W_0 = W_1 \cup W_2$ is not (because it contains a surface with positive self-intersection), we conclude that $b_2^+(W_2) = 1$.

Lemma 3.2. *For each spin^c structure $\mathbf{s}_2 \in \text{Spin}^c(W_2)$ the map F_{W_2, \mathbf{s}_2} on $\widehat{\text{HF}}(L)$ is zero.*

Proof. Let $x \in \widehat{\text{HF}}(L, \mathbf{t})$ be a given generator and consider $\mathbf{s}_1 \in \text{Spin}^c(W_1)$ provided by Lemma 2.6 with the property that $F_{W_1, \mathbf{s}_1}(g) = x$ for the generator $g \in \widehat{\text{HF}}(S^3)$. By the composition law [40, Theorem 3.4] we have

$$F_{W_2, \mathbf{s}_2}(x) = (F_{W_2, \mathbf{s}_2} \circ F_{W_1, \mathbf{s}_1})(g) = \left(\sum_{\mathbf{t} \in \mathfrak{S}} F_{W_0, \mathbf{t}} \right)(g) = 0,$$

where $\mathfrak{S} = \{\mathbf{t} \in \text{Spin}^c(W_0) \mid \mathbf{t}|_{W_i} = \mathbf{s}_i, i = 1, 2\}$. Since $F_{W_0, \mathbf{t}} = 0$ for all $\mathbf{t} \in \text{Spin}^c(W_0)$, this implies $F_{W_2, \mathbf{s}_2} = 0$. □

Proof of Proposition 3.1. To verify that M is an L -space, consider the surgery exact triangle induced by the cobordism W_2 :

$$\begin{array}{ccc} \widehat{\text{HF}}(-L) & \xrightarrow{F_{W_2}} & \widehat{\text{HF}}(M) \\ & \searrow & \swarrow \\ & \widehat{\text{HF}}(M') & \end{array}$$

Notice that M' is a small Seifert fibered space with $e_0(M') = 0$, and hence M' is an L -space. Since $F_{W_2} = 0$, we have

$$\widehat{\text{HF}}(M') = \widehat{\text{HF}}(L) \oplus \widehat{\text{HF}}(M).$$

According to [29, Theorem 8.1], if $Y = M(e_0; r_1, r_2, \dots, r_k)$ with $r_i = \frac{p_i}{q_i}$, $i = 1, 2, \dots, k$ and $e(Y) \neq 0$, then the order of $H_1(Y; \mathbb{Z})$ is equal to

$$q_1 q_2 \cdots q_k |e(Y)|.$$

Therefore, since $e(M) > 0$,

$$\begin{aligned} |H_1(M; \mathbb{Z})| &= \left(\prod_{i=1}^k q_i \right) \left(\sum_{i=1}^k \frac{p_i}{q_i} - 1 \right) = \sum_{i=1}^k p_i \prod_{j \neq i} q_j - \prod_{i=1}^k q_i \\ &= |H_1(M'; \mathbb{Z})| - |H_1(L; \mathbb{Z})|. \end{aligned}$$

Thus, we obtain

$$\dim \widehat{\text{HF}}(M) = |H_1(M; \mathbb{Z})|,$$

concluding the proof. □

4. The proof of Theorem 1.2

Let $M = M(-1; r_1, r_2, \dots, r_k)$ be a 3-manifold as in the statement of Theorem 1.2, satisfying $e(M) = r_1 + r_2 + \dots + r_k - 1 > 0$. Define the family of contact 3-manifolds $\{(S, \xi_S)\}$ using the contact surgery diagram of Figure 3 with the knot K_0 deleted.

Lemma 4.1. *Each contact 3-manifold (S, ξ_S) is Stein fillable, and each underlying 3-manifold S is an L-space.*

Proof. Let K be a Legendrian unknot with maximal Thurston–Bennequin number in the standard contact 3-sphere. A contact (+1)-surgery along K gives a Stein fillable contact 3-manifold $(S^1 \times S^2, \eta)$. This follows, e.g., from the fact that the resulting contact structure has nonvanishing Ozsváth–Szabó invariant [23, Lemma 2.5] and the classification of tight contact structures on $S^1 \times S^2$. Since (S, ξ_S) is given by Legendrian surgery on $(S^1 \times S^2, \eta)$, ξ_S is Stein fillable. Since $e_0(M) = -1$, a simple calculation shows that

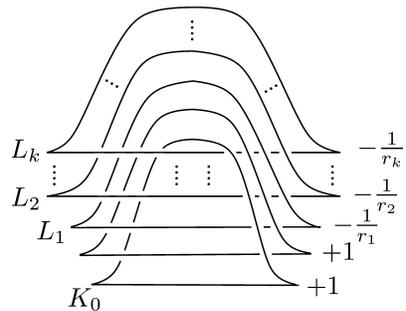


Figure 3. Contact structures on M .

S is a small Seifert fibered 3-manifold with $e_0(S) = 0$, therefore it is an L -space. \square

Let the 4-manifold $-X$ be defined as the cobordism induced by the contact $(+1)$ -surgery along the Legendrian unknot K_0 in Figure 3. Note that $-X$ is a cobordism between S and M . By reversing its orientation, the resulting 4-manifold (a cobordism between $-S$ and $-M$) will be denoted by X .

Lemma 4.2. *The cobordism X is negative definite.*

Proof. Converting the contact framings into smooth framings in Figure 3, reversing orientation, blowing up once and applying three Rolfsen twists one easily checks that the cobordism X is given by Figure 4, where the surgery presentation for $-S$ is drawn using solid curves, and the attaching circle of the 2-handle giving X is drawn using the dashed circle. Using a little bit of Kirby calculus, one can easily see that the cobordism X admits an embedding into the 4-manifold

$$\widehat{W} = W(-k + 1; 1 - r_1, 1 - r_2, \dots, 1 - r_k) \# \overline{\mathbb{C}\mathbb{P}^2}.$$

Since

$$\partial \widehat{W} = M(-k + 1; 1 - r_1, 1 - r_2, \dots, 1 - r_k) = -M$$

and $e(-M) = -e(M) < 0$, we conclude that $W(-k + 1; 1 - r_1, 1 - r_2, \dots, 1 - r_k)$ is negative definite, and hence so is the blow-up \widehat{W} , consequently the cobordism $X \subset \widehat{W}$ is negative definite. \square

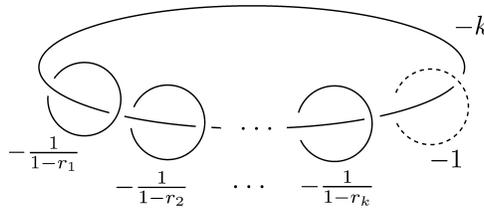


Figure 4. The cobordism X .

Proof of Theorem 1.2. Denote by $c^+(M, \xi)$ the image of $c(M, \xi)$ under the natural homomorphism [36, 37]

$$\widehat{\text{HF}}(-M, \mathbf{t}_\xi) \rightarrow \text{HF}^+(-M, \mathbf{t}_\xi).$$

Clearly, it is enough to show that $c^+(M, \xi) \neq 0$. The contact 3-manifold (M, ξ) is obtained by a contact $(+1)$ -surgery on (S, ξ_S) , for some (S, ξ_S) from the family of Lemma 4.1. Therefore, by [12, Lemma 2.11], there is a spin^c structure \mathbf{s} on X such that $F_{X, \mathbf{s}}^+(c^+(S, \xi_S)) = c^+(M, \xi)$ and

$$-d_3(\xi_S) + \delta(\mathbf{s}) = -d_3(\xi),$$

where

$$\delta(\mathbf{s}) := \frac{1}{4}(c_1^2(\mathbf{s}) - 3\sigma(X) - 2\chi(X)).$$

By assumption, M is a rational homology 3-sphere, and by Lemma 4.1 S is an L -space and (S, ξ_S) is Stein fillable. Therefore $c(S, \xi_S) \neq 0$, and it follows that $d(-S, \mathbf{t}_{\xi_S}) = -d_3(\xi_S)$. By Lemma 4.2, X is negative definite. Moreover, the assumption $d_3(\xi) = d(M, \mathbf{t}_\xi)$ immediately implies that the degree shift of the map $F_{X,\mathbf{s}}^+$ is

$$\delta(\mathbf{s}) = -d_3(\xi) + d_3(\xi_S) = d(-M, \mathbf{t}_\xi) - d(-S, \mathbf{t}_{\xi_S}).$$

Therefore, Lemma 2.5 applies and the map $F_{X,\mathbf{s}}^+$ is injective, so we conclude that $c^+(M, \xi) \neq 0$. □

5. The proof of Theorem 1.4

Let $K \subset S^3$ be a knot, and denote by $S_r^3(K)$ the 3-manifold obtained by r -surgery along K . The proof of Theorem 1.4 will rest on Theorem 1.2, together with Propositions 5.1, 5.2, 5.3 and Lemma 5.4. In the following M_p will denote the 3-manifold $S_{p^2-p-1}^3(T_{p,p+1})$, where $T_{p,p+1}$ is the $(p, p + 1)$ torus knot.

Proposition 5.1. *Let \mathbf{t}_0 denote the spin^c structure induced by the (unique) spin structure on the L -space M_p . Then*

$$d(M_p, \mathbf{t}_0) = \begin{cases} -\frac{1}{4}(3p + 2) & \text{if } p \text{ is even,} \\ -\frac{1}{4}(p + 1) & \text{if } p \text{ is odd.} \end{cases}$$

Proof. If p is even then $-M_p$ is obtained by plumbing according to a negative definite tree with one bad vertex in the sense of [33], and with $3p+2$ vertices having all even weights (see, e.g., [25, Figure 4], where $M_p = E_{p,1}$).

Therefore, the trivial vector $K = 0$ is characteristic and clearly induces \mathbf{t}_0 , so by [33, Corollary 1.5] we have

$$-d(M_p, \mathbf{t}_0) = d(-M_p, \mathbf{t}_0) = \frac{K^2 + 3p + 2}{4} = \frac{3p + 2}{4}.$$

If p is odd, the proof follows easily from [25]. Namely, since $M_p = E_{p,1}$ and $\mathbf{t}_0 = \mathbf{t}_E$ in the notation of [25], by [25, Lemma 6.3, Proposition 6.13] we have

$$d(-M_p, \mathbf{t}_0) = d(-L_p, \mathbf{t}_Z) + \frac{1}{4},$$

where $-L_p$ is the oriented boundary of the smooth 4-manifold Z_p given by attaching 2-handles to B^4 along the framed link of Figure 5, and \mathbf{t}_Z is the spin^c structure induced by the restriction of the unique spin structure on Z_p . A simple computation gives

$$d(-L_p, \mathbf{t}_Z) = \frac{p}{4},$$

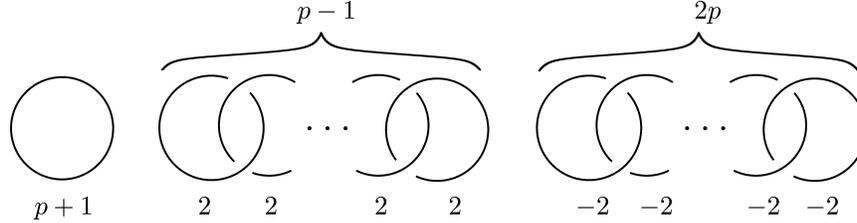


Figure 5. The framed link defining Z_p .

and the conclusion follows. □

Let X_p denote the 4-manifold obtained by attaching a 2-handle to D^4 along $T_{p,p+1} \subset S^3$ with framing $p^2 - p - 1$. In particular, $\partial X_p = M_p$. Choose a generator

$$[\Sigma_p] \in H_2(X_p; \mathbb{Z}) \cong \mathbb{Z}.$$

Since X_p is simply connected, a spin^c structure on X_p is uniquely determined by its first Chern class, so let $\mathbf{s}_k \in \text{Spin}^c(X_p)$ denote the spin^c structure with

$$\langle c_1(\mathbf{s}_k), [\Sigma_p] \rangle = p^2 - p - 1 + 2k.$$

We denote the restriction of \mathbf{s}_k to $\partial X_p = M_p$ by \mathbf{t}_k . Observe that, in accordance with the notation used in Proposition 5.1, \mathbf{t}_0 is the spin^c structure induced by the unique spin structure on M_p . Let

$$\Delta_{T_{p,p+1}}(t) = a_0(p) + \sum_{i=1}^{n_p} a_i(p)(t^i + t^{-i})$$

be the symmetrized Alexander polynomial of the torus knot $T_{p,p+1}$, and define its j th torsion coefficient by

$$r_j(p) = \sum_{i=1}^{n_p} i a_{i+|j|}(p).$$

Proposition 5.2. *For $|k| \leq (p^2 - p - 1)/2$, the invariant $d(M_p, \mathbf{t}_k)$ is equal to*

$$\frac{((p^2 - p - 1) - 2j)^2 - (p^2 - p - 1)}{4(p^2 - p - 1)} - 2r_k(p),$$

where $j \equiv k \pmod{p^2 - p - 1}$ and $0 \leq j < p^2 - p - 1$.

Proof. Recall that $S^3_{pq\pm 1}(T_{p,q})$ are lens spaces. Therefore, combining [32, Theorem 7.2] (together with [32, Remark 7.4]) and [32, Proposition 8.1] (cf. also [41, Theorem 1.2]) we have

$$d(M_p, \mathbf{t}_k) - d(L(p^2 - p - 1, 1), k) = -2r_k(p).$$

On the other hand, by [32, Section 7]

$$d(L(p^2 - p - 1, 1), k) = \frac{((p^2 - p - 1) - 2j)^2 - (p^2 - p - 1)}{4(p^2 - p - 1)},$$

where $j \equiv k \pmod{p^2 - p - 1}$ and $0 \leq j < p^2 - p - 1$. The statement follows immediately. \square

Proposition 5.3. *The coefficients of the Alexander polynomial $\Delta_{T_{p,p+1}}(t)$ satisfy:*

- (i) $a_0(p) = (-1)^{p+1}$,
- (ii) all the $a_i(p)$'s are ± 1 and they alternate in sign, and
- (iii) $a_i(p) = 0$ for $0 < i < \frac{p}{2}$.

Proof. It is known (see, for example, [2, Example 9.15]) that

$$\Delta_{T_{p,p+1}}(t) = \frac{(1 - t^{p(p+1)})(1 - t)}{(1 - t^p)(1 - t^{p+1})} t^{-p(p-1)/2}.$$

Since

$$(1 - t^{p(p+1)}) = (1 - t^p)(1 + t^p + t^{2p} + \dots + t^{p^2}),$$

and

$$\begin{aligned} & (1 + t^p + t^{2p} + \dots + t^{p^2})(1 - t) \\ &= (1 - t^{p+1}) + (t^p - t^{2p+1}) + \dots + (t^{p^2-p} - t^{p^2+1}) + t^{p^2} - t \\ &= (1 - t^{p+1})(1 + t^p + \dots + t^{p^2-p}) + t(t^{p^2-1} - 1) \\ &= (1 - t^{p+1})(1 + t^p + \dots + t^{p^2-p}) \\ &\quad - t(1 - t^{p+1})(1 + t^{p+1} + t^{2(p+1)} + \dots + t^{(p+1)(p-2)}), \end{aligned}$$

we get that the (symmetrized) Alexander polynomial $\Delta_{T_{p,p+1}}$ equals

$$q(t)t^{-p(p-1)/2},$$

where

$$(5.1) \quad q(t) = 1 + t^p + \dots + t^{p^2-p} - t(1 + t^{p+1} + t^{2(p+1)} + \dots + t^{(p+1)(p-2)}).$$

Therefore, $a_0(p)$ is the coefficient of $t^{p(p-1)/2}$ in $q(t)$, which is equal to 1 if $\frac{1}{2}p(p-1)$ is of the form $i \cdot p$ with $1 \leq i \leq p-1$ (this holds if p is odd), and it is equal to -1 if $\frac{1}{2}p(p-1)$ is of the form $1 + k(p+1)$ (which happens with the choice of $k = \frac{1}{2}(p-2)$ for p even). This verifies (i). Part (ii) follows immediately from a close inspection of formula (5.1). To verify (iii), suppose first that p is odd. The first coefficient after $a_0(p)$ which is equal to 1 is clearly $a_p(p)$, while the first -1 is of index $1 + k$ with $k = \frac{1}{2}(p-1)$, showing that $a_{1/2(p-1)+1}(p) = a_{1/2(p+1)}(p)$ is the first nonzero coefficient after $a_0(p)$, verifying (iii) for odd p . For p even, a similar argument shows that the

first coefficient which is equal to -1 is $a_{p+1}(p)$, and the first 1 is $a_{p/2}(p)$, completing the proof. \square

Lemma 5.4. *For $p > 2$ even, we have*

$$d(M_p, \mathbf{t}_{p/2}) = \frac{(p^2 - 2p - 1)^2}{4(p^2 - p - 1)} - \frac{1}{4}(p - 1)^2.$$

Proof. According to Proposition 5.1 we have

$$d(M_p, \mathbf{t}_0) = -\frac{3p + 2}{4},$$

and by Proposition 5.2

$$d(M_p, \mathbf{t}_0) = \frac{p^2 - p - 2}{4} - 2r_0(p).$$

The above identities imply that

$$r_0(p) = \frac{3p + 2}{8} + \frac{p^2 - p - 2}{8} = \frac{p^2 + 2p}{8}.$$

On the other hand, by Proposition 5.2

$$d(M_p, \mathbf{t}_{p/2}) = \frac{(p - (p^2 - p - 1))^2 - (p^2 - p - 1)}{4(p^2 - p - 1)} - 2r_{p/2}(p).$$

To conclude the proof we will express $r_{p/2}(p)$ in terms of $r_0(p)$. It follows from the definition that

$$r_0(p) = r_{p/2}(p) + \frac{p}{2} \sum_{i=0}^{n_p} a_i(p) - \sum_{i=0}^{p/2} \left(\frac{p}{2} - i\right) a_i(p).$$

Since by Proposition 5.3, the nonzero coefficients of $\Delta_{T_{p,p+1}}(t)$ alternate in sign and are all ± 1 , from the assumption that p is even (hence $a_0(p) = -1$) we get

$$\sum_{i=0}^{n_p} a_i(p) = 0.$$

On the other hand, by Proposition 5.3 we know that

$$\sum_{i=0}^{p/2} \left(\frac{p}{2} - i\right) a_i(p) = -\frac{p}{2},$$

hence

$$r_{p/2}(p) = r_0(p) - \frac{p}{2} = \frac{p^2 - 2p}{8}.$$

Substituting this value into the expression for $d(M_p, \mathbf{t}_{p/2})$ given above, the statement follows. \square

Proof of Theorem 1.4. Suppose first that p is even. Define ξ_p by the contact surgery diagram of Figure 6. (In Figures 6 and 7, a coefficient $(+1)$ next to a Legendrian knot K means that contact $(+1)$ -surgery is performed along K , while no coefficient means contact (-1) -surgery). Our plan is to apply Theorem 1.2. Observe that this plan makes sense. In fact, it is easy to check that the proofs of Lemmas 4.1, 4.2 and Theorem 1.2 given in Section 4 apply without modifications to the contact surgery presentation of Figure 6.

Turning the contact framings into smooth framings, a little Kirby calculus (as in [25, Figures 1 and 7]) shows that ξ_p is a contact structure on M_p . Now we want to apply the formula from [4] for the d_3 -invariant of a contact structure defined by a contact (± 1) -surgery diagram. If $c \in H^2(X; \mathbb{Z})$ denotes the 2-cohomology class determined by the rotations numbers (see [4]), $\sigma(X)$ is the signature of X and $b_2(X)$ the second Betti number, a simple computation yields

$$\sigma(X) = 1 - p, \quad b_2(X) = p + 3, \quad c^2 = \frac{(p - (p^2 - p - 1))^2}{p^2 - p - 1} - p(p - 1).$$

Then, [4, Corollary 3.6] (where $b_2(X)$ should be plugged into the formula instead of the Euler characteristic $\chi(X)$ because the 3-dimensional invariant used in Heegaard Floer theory is shifted by $\frac{1}{2}$) gives

$$d_3(\xi_p) = \frac{(p^2 - 2p - 1)^2}{4(p^2 - p - 1)} - \frac{1}{4}(p - 1)^2.$$

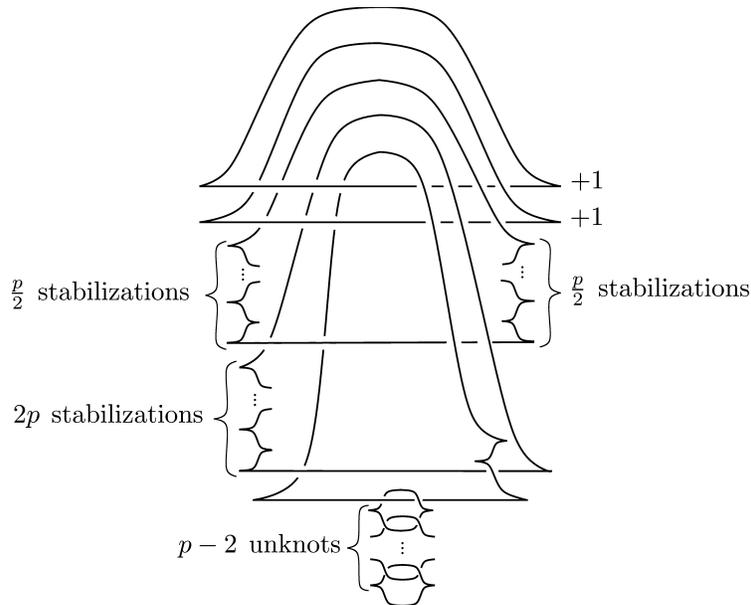


Figure 6. The contact structure ξ_p for p even.

Following the blow-down procedure at the cohomological level, the verification that ξ_p is a contact structure on the given 3-manifold M_p also shows that for $p > 2$

$$\mathbf{t}_{\xi_p} = \mathbf{t}_{p/2}.$$

(For $p = 2$ the inequality $p < p^2 - p - 1$ fails to hold, and we have $\mathbf{t}_{\xi_2} = \mathbf{t}_0$.)

Therefore, by Lemma 5.4 we have

$$d_3(\xi_p) = d(M_p, \mathbf{t}_{\xi_p}).$$

Using Theorem 1.2 we conclude that $c(M_p, \xi_p) \neq 0$, and hence ξ_p is a tight contact structure on M_p .

We now verify the statement for p odd. (This case was already treated in [25, Theorem 1.3].) Let ξ_p denote the contact structure given by the contact surgery diagram of Figure 7. As in the previous case, we can apply Theorem 1.2. A computation as above shows that

$$\sigma(X) = 1 - p, \quad b_2(X) = p + 3, \quad c^2 = -2p$$

and applying [4, Corollary 3.6] we have

$$d_3(\xi_p) = -\frac{1}{4}(p + 1) \quad \text{and} \quad \mathbf{t}_{\xi_p} = \mathbf{t}_0.$$

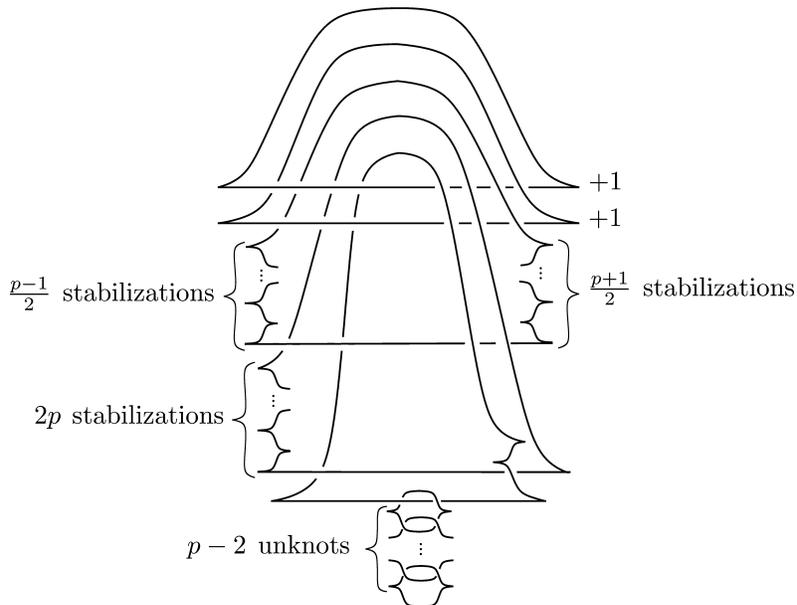


Figure 7. The contact structure ξ_p for p odd.

Using Proposition 5.1 we see that

$$d(M_p, \mathbf{t}_{\xi_p}) = -\frac{1}{4}(p+1) = d_3(\xi_p)$$

which, in view of Theorem 1.2, implies that ξ_p is tight and concludes the proof. \square

6. Planar contact structures

6.1. Surgery diagrams for contact structures with zero twisting. In this section, we show that if ξ is a tight contact structure with maximal twisting equal to zero on the small Seifert fibered 3-manifold $M = M(-1; r_1, r_2, r_3)$, then (M, ξ) is compatible with a planar open book decomposition of M , and we discuss some consequences. We start by establishing the fact that (M, ξ) is given by one of the contact surgeries represented by Figure 3.

Let F_i ($i = 1, 2, 3$) be the three singular fibers of the Seifert fibration on M . We first isotope F_i so that they become Legendrian. Then, since ξ has maximal twisting equal to zero, we can isotope the Seifert fibration further, so that there is a Legendrian regular fiber L with contact framing equal to the framing induced on L by the Seifert fibration.

Let V_i be a standard neighborhood of F_i , $i = 1, 2, 3$. Then $M \setminus (V_1 \cup V_2 \cup V_3)$ can be identified with $\Sigma \times S^1$ where Σ is a pair-of-pants. An identification between $M \setminus (V_1 \cup V_2 \cup V_3)$ and $\Sigma \times S^1$ determines identifications of $-\partial(M \setminus V_i)$ with $\mathbb{R}^2/\mathbb{Z}^2$ so that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the direction of the section $\Sigma \times \{1\}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the direction of the regular fibers. A standard application of convex surface theory (as in, e.g., [15]) shows that the singular fibers F_1, F_2, F_3 admit tubular neighborhoods U_1, U_2, U_3 with minimal convex boundaries such that $-\partial(M \setminus U_i)$ has infinite slope for $i = 1, 2, 3$.

Proposition 6.1. *Let ξ be a tight contact structure with maximal twisting equal to zero on the small Seifert fibered 3-manifold $M = M(-1; r_1, r_2, r_3)$. Then (M, ξ) is given by one of the contact surgeries represented by Figure 3.*

Proof. Let Σ be a pair of pants. Recall that a tight contact structure ξ on $\Sigma \times S^1$ is *appropriate* if there is no contact embedding $(T^2 \times I, \xi_\pi) \hookrightarrow (\Sigma \times S^1, \xi)$, with $T^2 \times \{0\}$ isotopic to a boundary component, where ξ_π is a tight contact structure with convex boundary and twisting π (see [18, § 2.2.1] for the definition of twisting). By the results of [15, Section 2], we are free to change the restriction $\xi|_{M-(U_1 \cup U_2 \cup U_3)}$ without changing the isotopy class of ξ , as long as we replace $\xi|_{M-(U_1 \cup U_2 \cup U_3)}$ by a tight, appropriate contact structure with minimal convex boundaries having the same infinite boundary slopes.

We claim that this contact structure can be chosen to be the contact complement (C, η) of standard neighborhoods of the three Legendrian unknots

L_1, L_2, L_3 in Figure 3. In fact, the contact structure obtained by performing contact (-1) -surgeries along the knots L_1, L_2, L_3 is tight, due to the cancellation of (± 1) -surgeries of opposite signs along Legendrian push-offs [3]. Therefore η is tight, and it is easy to check that η is also appropriate because it extends to a tight contact structure on a closed 3-manifold obtained by filling the neighborhoods of L_1, L_2 and L_3 . It is obvious that the boundary components of (C, η) are minimal and convex. To check that the boundary components have infinite boundary slopes, it is enough to observe that there is a product structure $C \cong \Sigma \times S^1$ such that (i) a fiber F of the projection $C \rightarrow \Sigma$ is Legendrian and has twisting number zero, and (ii) all the L_i 's are Legendrian pushoffs of F . \square

6.2. Open book decompositions. According to a recent result of Giroux [16], isotopy classes of contact structures are in one-to-one correspondence with suitable equivalence classes of open book decompositions of the underlying 3-manifold; cf. also [7]. Recall that an open book decomposition (Y, f) of a closed 3-manifold Y amounts to a link $L \subset Y$ and a fibration $f : Y - L$ such that the closure of every fiber $f^{-1}(t)$ (called a *page*) provides a Seifert surface for L . The open book decomposition is called *planar* if the genus of the page is zero. We will also call *planar* any contact structure compatible with a planar open book decomposition. The following result relates planarity of a contact structure with dynamical properties of its Reeb vector fields.

Theorem 6.2 (Abbas–Cieliebak–Hofer [1]). *If ξ is a contact structure compatible with a planar open book decomposition then it satisfies the Weinstein conjecture, that is, any Reeb vector field of ξ admits a periodic orbit.*

Necessary conditions for a contact structure to be planar were found in [8, 31]. We will prove Theorem 1.5 using Proposition 6.1. Before dwelling into the proof, we describe some consequences of Theorem 1.5, some of which immediately imply Corollaries 1.6, 1.7 and 1.8. First of all, we have the following.

Corollary 6.3. *If $M = M(e_0; r_1, r_2, r_3)$, $e_0 \geq -1$ and M is an L -space, then each contact structure on M is planar and therefore satisfies Weinstein’s conjecture.¹*

Proof. It is known that overtwisted contact structures are planar [8], and hence we can focus on tight structures only. For small Seifert fibered rational homology 3-spheres with $e_0 \geq 0$ it was shown by Wu [44] that each tight contact structure has maximal twisting equal to zero, therefore Theorem 1.5 shows that on those manifolds every contact structure is planar.

¹Added in proof: Cliff Taubes has recently posted a proof of the Weinstein conjecture for every contact 3-manifold [43].

Suppose now that M is an L -space and $e_0(M) = -1$. By Theorem 1.1 the 3-manifold M admits no transverse contact structures, and hence Theorem 1.3 shows that every tight contact structure on M has maximal twisting equal to zero. Theorem 1.5 therefore implies that every tight contact structure on M is planar. \square

Remark 6.4. In [42] Stephan Schönenberger proved that if $e_0 \leq -3$ then every contact structure on $M = M(e_0; r_1, r_2, r_3)$ is planar. Therefore, since in this case M is always an L -space, the statement of Corollary 6.3 holds if $e_0 \leq -3$. On the other hand, the Poincaré sphere admits a Seifert fibration with $e_0 = -2$, is an L -space, but its unique Stein fillable contact structure is not planar because the intersection form of a filling is not diagonalizable [8, 31].

In view of Corollary 6.3, Theorem 1.1 implies the following characterization of L -spaces of the form $M(e_0; r_1, r_2, r_3)$.

Corollary 6.5. *The 3-manifold $M = M(e_0; r_1, r_2, r_3)$ is an L -space if and only if one of the following holds:*

- *each contact structure on M is planar;*
- *each contact structure on $-M$ is planar.*

Proof. If $e_0(M) \geq 0$ then by Corollary 6.3 every contact structure on M is planar. On the other hand, it is known that if $e_0(M) \geq 0$ or $e_0(M) \leq -3$ then M is an L -space. Since $e_0(-M) = -3 - e_0(M)$, this immediately implies the statement if $e_0(M) \geq 0$ or $e_0(M) \leq -3$. Suppose now that M is an L -space and, up to changing its orientation, $e_0(M) = -1$. Then, by Corollary 6.3 every contact structure on M is planar.

To finish the proof, we may assume without loss that $e_0(M) = -1$ and M is not an L -space. According to Theorem 1.1 the manifold M admits a taut foliation, which gives rise to a contact structure ξ having a symplectic semi-filling. By [5, 9] one can use a symplectic cap to construct a symplectic filling of (M, ξ) with $b_2^+ > 0$. Then, according to [8, 31] ξ is not planar. \square

Let $T_{p,np+1}$ denote the positive $(p, np + 1)$ -torus knot, and let

$$M_{p,pn+1} = -S_{p^2n-pn-1}^3(T_{p,pn+1}).$$

Proposition 6.6 below shows that the non-planarity of the Stein fillable contact structure on the Poincaré sphere (cf. Remark 6.4) is a non-isolated phenomenon.

Proposition 6.6. *For $p \geq 2$ and $n \geq 1$ the rational homology 3-sphere $M_{p,np+1}$ is an L -space and it carries non-planar contact structures.*

Proof. Since the slice genus of $T_{p,np+1}$ is $p^2n - pn - 1$, the fact that $M_{p,pn+1}$ is an L -space follows from [24, Proposition 4.1].

Let $W_{p,n}$ denote the canonical plumbing 4-manifold associated to $M_{p,np+1}$ as in Section 2. Since all the weights of the plumbing are at most -2 (and all knots are unknots), $W_{p,n}$ supports Stein structures inducing tight contact structures on $M_{p,n}$. The proof of [8, Theorem 4.1] shows that if any of these structures is planar, then $W_{p,n}$ smoothly embeds in a connected sum of $\overline{\mathbb{C}\mathbb{P}^2}$'s. But the argument given in the proof of [25, Proposition 4.1] shows that the intersection form $Q_{W_{p,n}}$ does not embed into a diagonal lattice. Therefore, none of the contact structures filled by $W_{p,n}$ are planar. \square

Proof of Theorem 1.5. Let ξ be a tight contact structure with twisting number equal to zero on $M(-1; r_1, r_2, r_3)$. By Proposition 6.1 this structure is obtained by performing a (possibly rational) contact surgery along the five-component Legendrian link \mathcal{L} of Figure 2 (for $k = 3$). According to the algorithm outlined in Section 2, ξ is obtained by contact (± 1) -surgery on a Legendrian link $\tilde{\mathcal{L}}$ obtained from \mathcal{L} by successively taking pushoffs and Legendrian stabilizations of (some of) its components. It is well known that performing contact (± 1) -surgery on a Legendrian knot which sits on a page of a compatible open book with contact framing equal to the page framing yields an open book of the same genus compatible with the resulting contact structure. Therefore, it suffices to show that $\tilde{\mathcal{L}}$ sits on a union of pages of a planar open book for S^3 compatible with the standard contact structure. This can be proved by an argument very similar to the one used in [42] to prove that each contact structure on a lens space is planar. From now on, we refer to [7] for standard facts on contact structures and their compatible open books. Start with the open book decomposition of S^3 compatible with the standard contact structure whose page is an annulus and whose binding is a Hopf link. By applying the Legendrian realization principle, a Legendrian unknot with maximal Thurston–Bennequin invariant together with four of its pushoffs can be realized on five distinct pages of this open book, so that contact framings and page framings coincide. Up to positively stabilizing the open book in the sense of Giroux, we can realize in the same way any Legendrian stabilizations of these unknots on distinct pages of a planar open book compatible with the standard contact structure on S^3 . Any number of pushoffs of the stabilized knots can then be realized on distinct pages, any further Legendrian stabilization can be realized on further planar stabilizations of the open book, and so on. This construction clearly establishes what we need. \square

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