

**A COMPACT SYMPLECTIC FOUR-MANIFOLD ADMITS
 ONLY FINITELY MANY INEQUIVALENT
 TORIC ACTIONS**

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Let (M, ω) be a four-dimensional compact connected symplectic manifold. We prove that (M, ω) admits only finitely many inequivalent Hamiltonian effective 2-torus actions. Consequently, if M is simply connected, the number of conjugacy classes of 2-tori in the symplectomorphism group $\text{Sympl}(M, \omega)$ is finite. Our proof is “soft”. The proof uses the fact that if a symplectic four-manifold admits a toric action, then the restriction of the period map to the set of exceptional homology classes is proper. In an appendix, we present the Gromov–McDuff properness result for a general compact symplectic four-manifold.

Dedicated to Dusa McDuff

1. Introduction

An action of the torus $\mathbb{T}^k = (S^1)^k$ on a symplectic manifold (M, ω) is a homomorphism

$$\rho : \mathbb{T}^k \rightarrow \text{Sympl}(M, \omega)$$

from the torus to the group of symplectomorphisms such that the map $\rho^\dagger : \mathbb{T}^k \times M \rightarrow M$ given by $(a, m) \mapsto \rho(a)(m)$ is smooth. A *sub-torus* of $\text{Sympl}(M, \omega)$ is the image of such a homomorphism. We restrict our attention to actions that are effective (faithful: ρ is one-to-one).

We say that two actions $\rho_1 : \mathbb{T}^k \rightarrow \text{Sympl}(M, \omega)$ and $\rho_2 : \mathbb{T}^k \rightarrow \text{Sympl}(M, \omega)$ are *equivalent* if there exists a symplectomorphism $\psi : M \rightarrow M$ and an automorphism $h : \mathbb{T}^k \rightarrow \mathbb{T}^k$ such that the diagram

$$\begin{array}{ccc} \mathbb{T}^k \times M & \xrightarrow{\rho_1^\dagger} & M \\ (h, \psi) \downarrow & & \psi \downarrow \\ \mathbb{T}^k \times M & \xrightarrow{\rho_2^\dagger} & M \end{array}$$

commutes. Two (effective) torus actions on (M, ω) are equivalent if and only if their images are conjugate sub-tori of $\text{Symp}(M, \omega)$.

A \mathbb{T}^k -action on (M, ω) with generating vector fields ξ_1, \dots, ξ_k is *Hamiltonian* if it has a *moment map*, that is, a map $\Phi: M \rightarrow \mathbb{R}^k$ whose components satisfy Hamilton’s equation $d\Phi_j = -\iota(\xi_j)\omega$. If $H^1(M; \mathbb{R}) = \{0\}$ then every torus action on (M, ω) is Hamiltonian. An (effective) Hamiltonian torus action is *toric* if the dimension of the torus is half the dimension of the manifold; if M is compact and connected, the triple (M, ω, Φ) is called a *symplectic toric manifold*. In this paper we prove the following “finiteness theorem”.

Theorem 1.1. *Let (M, ω) be a four-dimensional compact connected symplectic manifold. Then the set of equivalence classes of toric actions on (M, ω) is finite.*

An important ingredient in the proof of Theorem 1.1 is the following “properness lemma”.

Lemma 1.2. *Let (M, ω) be a symplectic four-manifold with $b_2^+ = 1$. Then the restriction of the period map $E \mapsto \omega(E)$ to the set of classes in $H_2(M; \mathbb{R})$ with self intersection -1 is proper.*

We prove Lemma 1.2 and Theorem 1.1 by “soft” equivariant, algebraic, and combinatorial techniques. Our first proof used “hard” holomorphic techniques, and relied on a “properness lemma”, due to Gromov and McDuff, that applies to general compact symplectic four-manifolds. This earlier approach, although it used stronger machinery than necessary, is still worth presenting, for a number of reasons. First, this approach exhibits the interplay between “soft” and “hard” techniques. (See [19] for another usage of holomorphic techniques in proving results on toric actions.) Second, as far as we know, now these holomorphic techniques *are* necessary for generalizing our “finiteness” results to actions of other groups [33]. Finally, this volume in honour of Dusa McDuff provides an opportunity to present the general Gromov–McDuff properness lemma in a form that should be useful for non-experts. We present this holomorphic approach in an appendix.

Theorem 1.1 has the following corollary.

Corollary 1.3. *Let (M, ω) be a four-dimensional compact connected symplectic manifold with $H^1(M; \mathbb{R}) = \{0\}$. Then the number of conjugacy classes of 2-tori in $\text{Symp}(M, \omega)$ is finite.*

The number of conjugacy classes can be arbitrarily large; see Example 2.6.

If (M, ω) admits a toric action then M is simply connected [15, Corollary 32.2], so $H^1(M; \mathbb{R}) = \{0\}$. We restrict our attention to symplectic manifolds (M, ω) such that M is compact and connected and $H^1(M; \mathbb{R}) = \{0\}$ and, as before, to torus actions that are effective.

Since the orbits of a Hamiltonian torus action are isotropic (see [4, Proposition III.2.12]) and, on an open dense set, the (effective) action is free (see [11, Corollary B.48]), a toric action does not extend to a Hamiltonian action of a larger torus. Because $H^1(M, \mathbb{R}) = 0$, every symplectic torus action is Hamiltonian. Thus, the image of a toric action is a maximal torus in the symplectomorphism group $\text{Symp}(M, \omega)$. If $\dim M = 4$, $H^1(M; \mathbb{R}) = \{0\}$, and $\dim H^2(M; \mathbb{R}) \leq 3$, every Hamiltonian S^1 -action on (M, ω) extends to a toric action [18, Theorem 1], so that in this case every maximal torus in the symplectomorphism group comes from a toric action. However, in general, there exist Hamiltonian S^1 -actions that do not extend to toric actions, so there exist maximal tori that do not come from toric actions. In fact, some symplectic four-manifolds admit S^1 -actions and do not admit any toric actions. For others, the symplectomorphism group contains some maximal tori that are two-dimensional and some that are one-dimensional. See Example 3.5. However, most symplectic four-manifolds do not admit any circle actions (in particular, they do not admit any toric actions); see Example 3.5, and for many more examples see [12].

Maximal tori play a central role in the theory of finite-dimensional Lie groups. In a compact Lie group, every two maximal tori are conjugate. Our work can be viewed as an attempt to find aspects of finite-dimensional Lie theory that carry over to groups of symplectomorphisms. See [6, 21, 34] for related ideas. Homotopy theoretic properties of symplectomorphism groups have been studied in [1, 2, 7, 13, 22, 35]; see the survey [30].

At this point we do not know whether a finiteness result similar to Theorem 1.1 also holds in higher dimensions or for non-Hamiltonian symplectic group actions. The analogous result does *not* hold for contactomorphisms: Eugene Lerman [23] constructed a compact contact three-manifold that admits infinitely many non-conjugate toric actions. “Finiteness” also does not hold for the groups of all diffeomorphisms: for example, $\text{Diff}(S^2 \times S^2)$ contains infinitely many non-conjugate two-dimensional maximal tori; see Example 2.6.

2. Four-dimensional symplectic toric manifolds and corner chopping

The automorphism group of the torus $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$ is the group $\text{GL}(n, \mathbb{Z})$ of $n \times n$ matrices with integer entries and determinant equal to 1 or -1 . If a matrix A belongs to $\text{GL}(n, \mathbb{Z})$ then so do its transpose A^T and its inverse A^{-1} . We also consider the group $\text{AGL}(n, \mathbb{Z})$ of affine transformations of \mathbb{R}^n that have the form $x \mapsto Ax + \alpha$ with $A \in \text{GL}(n, \mathbb{Z})$ and $\alpha \in \mathbb{R}^n$.

A *Delzant polytope* in \mathbb{R}^n is a compact convex polytope Δ such that the edges emanating from each vertex can be generated by a \mathbb{Z} -basis of the lattice \mathbb{Z}^n .

Remark 2.1. Other authors also use the following terms for polytopes with this property: non-singular, primitive, regular, smooth, unimodular and torsion free.

An important model for a toric action is \mathbb{C}^n with the standard symplectic form, the standard \mathbb{T}^n -action given by rotations of the coordinates, and the moment map

$$(2.1) \quad (z_1, \dots, z_n) \mapsto \frac{1}{2}(|z_1|^2, \dots, |z_n|^2).$$

The image of this moment map is the positive orthant,

$$\mathbb{R}_+^n = \{(s_1, \dots, s_n) \mid s_j \geq 0 \text{ for all } j\}.$$

The moment map image of a symplectic toric manifold is a Delzant polytope. A Delzant polytope can be obtained by gluing open subsets of \mathbb{R}_+^n by means of elements of $\text{AGL}(n, \mathbb{Z})$. Similarly, a symplectic toric manifold can be obtained by gluing open \mathbb{T}^n -invariant subsets of \mathbb{C}^n by means of equivariant symplectomorphisms and reparametrizations of \mathbb{T}^n . In fact,

Lemma 2.2. *Let (M, ω) be a symplectic toric manifold with moment map $\Phi: M \rightarrow \text{Lie}(\mathbb{T}^n)^* = \mathbb{R}^n$. Let x be a point in \mathbb{R}^n . Then there exists an open neighborhood of x whose moment map preimage is equivariantly symplectomorphic to an open subset of \mathbb{C}^n with the standard \mathbb{T}^n -action conjugated by some automorphism $\mathbb{T}^n \rightarrow \mathbb{T}^n$.*

Proof. Because the moment map image is closed, the lemma is true if $\Phi^{-1}(x)$ is empty.

Let O be an orbit in $\Phi^{-1}(x)$. By the local normal form for Hamiltonian torus actions, some neighborhood U of O is equivariantly symplectomorphic to an open subset of \mathbb{C}^n . In particular, the level set $\Phi^{-1}(x)$ is a discrete union of orbits. Because the level sets are connected [3, 14], $\Phi^{-1}(x) = O$. Because Φ is proper, the inverse image of a sufficiently small neighborhood of x is contained in U . \square

Delzant [8] classified symplectic toric manifolds up to equivariant symplectomorphism. Applying his theorem to different toric actions on a fixed manifold, we get the following result.

Proposition 2.3. *Let (M, ω) be a $2n$ -dimensional compact connected symplectic manifold.*

- (1) *For any Hamiltonian \mathbb{T}^n -action on (M, ω) , its moment map image is a Delzant polytope in \mathbb{R}^n .*
- (2) *Two Hamiltonian \mathbb{T}^n -actions on (M, ω) are equivalent if and only if their moment map images are $\text{AGL}(n, \mathbb{Z})$ -congruent.*

Proof. By [3, 14], the moment map image is a convex polytope. Lemma 2.2 implies that this polytope is locally $\text{AGL}(n, \mathbb{Z})$ -congruent to \mathbb{R}_+^n ; thus, it is a Delzant polytope.

To prove part (2), let $\rho_1, \rho_2: \mathbb{R}^n \rightarrow \text{Symp}(M, \omega)$ be toric actions with moment maps $\Phi_1, \Phi_2: M \rightarrow \mathbb{R}^n$ and moment map images Δ_1 and Δ_2 .

Suppose that the actions ρ_1 and ρ_2 are equivalent, i.e., there exist $A \in \text{GL}(n, \mathbb{Z})$ and $\psi \in \text{Symp}(M, \omega)$ such that

$$(2.2) \quad \rho_2(A \cdot \lambda) = \psi \circ \rho_1(\lambda) \circ \psi^{-1}$$

for all $\lambda \in \mathbb{T}^n$. A moment map for the \mathbb{T}^n -action $\lambda \mapsto \rho_2(A \cdot \lambda)$ is $A^T \circ \Phi_2$. A moment map for the \mathbb{T}^n -action $\lambda \mapsto \psi \circ \rho_1(\lambda) \circ \psi^{-1}$ is $\Phi_1 \circ \psi^{-1}$. By (2.2), these are moment maps for the same action, so their difference is a constant: for some $\alpha \in \mathbb{R}^n$,

$$\Phi_1 \circ \psi^{-1} = A^T \circ \Phi_2 + \alpha.$$

Comparing the images of these maps, we get $\Delta_1 = A^T \Delta_2 + \alpha$. So Δ_1 and Δ_2 are $\text{AGL}(n, \mathbb{Z})$ -congruent.

Suppose that Δ_1 and Δ_2 are $\text{AGL}(n, \mathbb{Z})$ -congruent. Let $A \in \text{GL}(n, \mathbb{Z})$ and $\alpha \in \mathbb{R}^n$ be such that

$$(2.3) \quad \Delta_2 = A\Delta_1 + \alpha.$$

The map $A \circ \Phi_1 + \alpha$ is a moment map for the action $\rho_1 \circ A^T$. Its image, $A\Delta_1 + \alpha$, coincides with the image Δ_2 of the moment map for the action ρ_2 . By Delzant's theorem [8], toric actions with the same moment map image are equivariantly symplectomorphic. Applying this to the actions $\rho_1 \circ A^T$ and ρ_2 , we find that there exists a symplectomorphism $\psi: M \rightarrow M$ such that $\psi \circ \rho_1(A^T \cdot \lambda) = \rho_2(\lambda) \circ \psi$ for all $\lambda \in \mathbb{T}^n$. That is, the actions ρ_1 and ρ_2 are equivalent. \square

2.4. The *rational length* of an interval d of rational slope in \mathbb{R}^n is the unique number $\ell = \text{length}(d)$ such that the interval is $\text{AGL}(n, \mathbb{Z})$ -congruent to an interval of length ℓ on a coordinate axis. In what follows, intervals are always measured by rational length.

Example 2.5. Figure 1 shows examples of Delzant polygons with three and four edges. On the left there is a *Delzant triangle*,

$$\Delta_\lambda = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq \lambda\}.$$

This is the moment map image of the standard toric action on $\mathbb{C}\mathbb{P}^2$, with the Fubini-Study symplectic form normalized so that the symplectic area of $\mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^2$ is $2\pi\lambda$. The rational lengths of all its edges is λ .

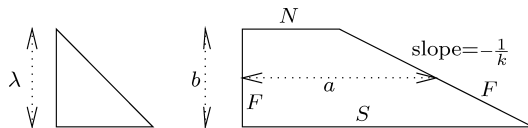


Figure 1. A Delzant triangle, Δ_λ , and a Hirzebruch trapezoid, $H_{a,b,k}$.

On the right there is a *Hirzebruch trapezoid*,

$$H_{a,b,k} = \{(x_1, x_2) \mid -\frac{b}{2} \leq x_2 \leq \frac{b}{2}, 0 \leq x_1 \leq a - kx_2\},$$

where b is the height of the trapezoid, a is its average width, and k is a non-negative integer such that the right edge has slope $-1/k$ or is vertical if $k = 0$. We assume that $a \geq b$ and that $a - k\frac{b}{2} > 0$. This trapezoid is a moment map image of a standard toric action on a Hirzebruch surface. The rational lengths of its left and right edges are b ; the rational lengths of its top and bottom edges are $a \pm kb/2$.

Example 2.6. Consider the manifold $S^2 \times S^2$ with the product symplectic form $\omega_{a,b} = a\tau \oplus b\tau$, where τ is the rotation invariant area form on S^2 with total area 2π . For any toric action on $(S^2 \times S^2, \omega_{a,b})$, its moment map image is $\text{AGL}(2, \mathbb{Z})$ -congruent to the Hirzebruch trapezoid $H_{a,b,k}$ for some non-negative even integer k . See [18]. Given a and b , there exist only finitely many such k 's for which $a - k\frac{b}{2} > 0$. Hence, up to conjugation, there are only finitely many 2-tori in the symplectomorphism group of $(S^2 \times S^2, \omega_{a,b})$.

Different non-negative even integers k correspond to 2-tori in the symplectomorphism group $\text{Sympl}(S^2 \times S^2, \omega_{a,b})$ that are not conjugate. The 2-tori corresponding to different integers k are not even conjugate in the group $\text{Diff}(S^2 \times S^2)$ of *diffeomorphisms* of $S^2 \times S^2$: the value of k can be recovered from the fact that the two-dimensional orbit type strata have self intersection 0 , k , and $-k$. These tori are maximal in $\text{Diff}(S^2 \times S^2)$: because an effective compact group action on a connected manifold is also effective on invariant open subsets, and by local linearization, a torus that acts effectively on a manifold with a fixed point is at most of half the dimension of the manifold. Hence, $\text{Diff}(S^2 \times S^2)$ contains *infinitely many* conjugacy classes of maximal tori.

For each Hirzebruch trapezoid $H_{a,b,k}$ with k even there exists a torus action on $(S^2 \times S^2, \omega_{a,b})$ with moment map image $H_{a,b,k}$. See [18].

2.7. Let d , d' , and d'' be three consecutive edges in a Delzant polygon, ordered counterclockwise. Let u , u' , and u'' be their outward normal vectors, normalized so that they are primitive lattice elements. Then each of (u, u') and (u', u'') is an oriented \mathbb{Z} -basis for \mathbb{Z}^2 . It follows that there exists an integer k such that $u + u'' = ku'$. We define the *combinatorial self intersection number* of d' to be $-k$. We also say that the *combinatorial intersection number* of two different edges is 1 if they are adjacent and 0 if not.

Example 2.8. The edges of the polygon in Figure 2 have the following rational lengths and combinatorial self intersections.

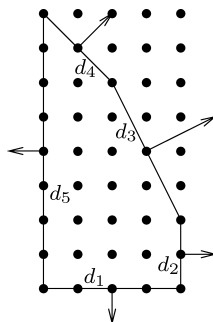


Figure 2. Polygon for Example 2.8.

| | rational length | combinatorial self intersection |
|-------|-----------------|---------------------------------|
| d_1 | 4 | 0 |
| d_2 | 2 | -2 |
| d_3 | 2 | -1 |
| d_4 | 2 | -1 |
| d_5 | 8 | 1 |

2.9. An almost complex structure on a $2n$ -dimensional manifold M is an automorphism of the tangent bundle, $J : TM \rightarrow TM$, such that $J^2 = -\text{id}$. It is *tamed* by a symplectic form ω if $\omega(v, Jv) > 0$ for all $v \neq 0$. The *first Chern class* of the symplectic manifold (M, ω) is defined to be the first Chern class of the complex vector bundle (TM, J) and is denoted $c_1(TM)$. This class is independent of the choice of tamed almost complex structure J [31, § 2.6].

Lemma 2.10. *Let (M, ω) be a compact connected symplectic four-manifold. Let $\Phi : M \rightarrow \mathbb{R}^n$ be the moment map for a toric action, and let Δ be its image.*

- (1) *Let d be an edge of Δ of rational length ℓ . Then its preimage, $\Phi^{-1}(d)$, is a symplectically embedded 2-sphere in M of symplectic area*

$$\int_{\Phi^{-1}(d)} \omega = 2\pi\ell.$$

- (2) *Let d and d' be edges of Δ . Then their combinatorial intersection number (see § 2.7) is equal to the intersection number of their preimages in M .*
- (3) *The preimages of the edges of Δ generate the second homology group of M . The number of vertices of Δ is equal to $\dim H_2(M) + 2$.*

(4) The perimeter of Δ (see § 2.4) is

$$\text{perimeter}(\Delta) = \frac{1}{2\pi} \int_M \omega \wedge c_1(TM).$$

(5) The area of Δ is

$$\frac{1}{(2\pi)^2} \int_M \frac{1}{2!} \omega \wedge \omega.$$

Proof. By Lemma 2.2, the preimage of a sufficiently small subinterval c of the edge d is a disk or an annulus with area $2\pi \text{length}(c)$. Part (1) follows.

When $d \neq d'$, part (2) follows from Lemma 2.2. When $d = d'$, let d_- and d_+ be the edges that are adjacent to d . Let u_- , u , and u_+ be the primitive outward normal vectors to the edges d_- , d , and d_+ . Consider the sub-circle $i: S^1 \rightarrow \mathbb{T}^2$ determined by $u_- \in \text{Lie}(\mathbb{T}^2)$. Then $i(S^1)$ acts on the sphere $\Phi^{-1}(d)$ by rotations with speed 1, it rotates the normal to the sphere at the north pole with speed k , where $-k$ is the combinatorial self intersection of d , and it fixes the normal to the sphere at the south pole. By examining the transition function for a trivializations of the normal bundle over complements of the north and south poles, we see that the Euler number of this normal bundle is $-k$.

By, e.g., Lemma 2.2, a generic component of the moment map is a Morse function with even indices whose critical points are the preimages of the vertices of the polytope. For a critical point of index 2, the descending gradient manifold with respect to an invariant metric is the preimage of an edge of Δ minus its bottom vertex. Part (3) follows from this using Morse theory.

Let C_1, \dots, C_N be the preimages in M of the edges of Δ . Then the homology class $\sum_{i=1}^N [C_i] \in H_2(M; \mathbb{R})$ is the Poincaré dual to the first Chern class $c_1(TM)$. This follows from the fact that the C_i 's generate the second homology group and since

$$\begin{aligned} [C_i] \cdot \sum_{j=1}^N [C_j] &= [C_i] \cdot [C_i] + 2 \quad \text{by part (2)} \\ &= c_1(TM)([C_i]) \quad \text{since } C_i \text{ is an embedded 2-sphere in} \\ &\quad \text{a four-manifold.} \end{aligned}$$

This and part (1) imply part (4).

Part (5) follows from Lemma 2.2. It is a special case of the Duistermaat–Heckman theorem. \square

2.11. Let Δ be a Delzant polytope in \mathbb{R}^n , let v be a vertex of Δ , and let $\delta > 0$ be smaller than the rational lengths of the edges emanating from v . The edges of Δ emanating from v have the form $\{v + s\alpha_j \mid 0 \leq s \leq \ell_j\}$ where the vectors $\alpha_1, \dots, \alpha_n$ generate the lattice \mathbb{Z}^n and $\delta < \ell_j$ for all j . The

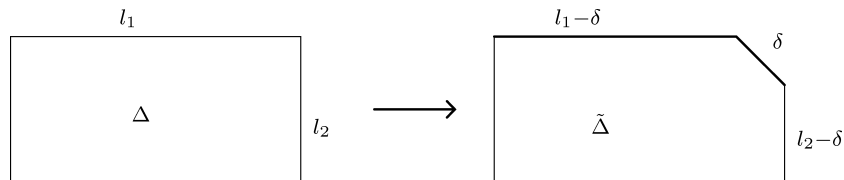


Figure 3. Corner chopping.

corner chopping of size δ of Δ at v is the polytope $\tilde{\Delta}$ obtained from Δ by intersecting with the half-space

$$\{v + s_1\alpha_1 + \cdots + s_n\alpha_n \mid s_1 + \cdots + s_n \geq \delta\}.$$

See, e.g., the chopping of the top right corner in Figure 3. The resulting polytope $\tilde{\Delta}$ is again a Delzant polytope. The corner chopping operation commutes with $\text{AGL}(n, \mathbb{Z})$ -congruence: if $\tilde{\Delta}$ is obtained from Δ by a corner chopping of size $\delta > 0$ at a vertex $v \in \Delta$ then, for any $g \in \text{AGL}(n, \mathbb{Z})$, the polytope $g(\tilde{\Delta})$ is obtained from the polytope $g(\Delta)$ by a corner chopping of size δ at the vertex $g(v)$.

Remark 2.12. Let a torus T act on a $2n$ -dimensional symplectic manifold (M, ω) . By the equivariant Darboux-Weinstein theorem [38], every T -fixed point has a neighborhood U which is equivariantly symplectomorphic to a neighbourhood of the origin in \mathbb{C}^n with T acting through a subgroup of $(S^1)^n$. The symplectic blow-up of size $r^2/2$ is obtained by removing the open ball $B^{2n}(r)$ of radius r about the origin and collapsing its boundary along the Hopf fibration $S^{2n-1} \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ (see [31, Section 7.1]). If M is a toric manifold with moment map image Δ , and $\tilde{\Delta}$ can be obtained from Δ by a corner chopping, then $\tilde{\Delta}$ is the moment map image of a toric manifold obtained from M by an equivariant symplectic blow-up. See [19, Section 3].

The following lemma is an easy consequence of the definitions of corner chopping (§ 2.11) and of combinatorial self intersection (§ 2.7).

Lemma 2.13. *Let d and d' be consecutive edges in a Delzant polygon Δ . Let $\tilde{\Delta}$ be obtained from Δ by a corner chopping at the vertex between d and d' . Let e be the resulting new edge. Let \tilde{d} and \tilde{d}' be the edges of $\tilde{\Delta}$ with the same outward normal vectors as those of d and d' . Then*

- (1) *the combinatorial self intersection of e is -1 ,*
- (2) *the combinatorial self intersection of \tilde{d} is one less than that of d ,*
- (3) *the combinatorial self intersection of \tilde{d}' is one less than that of d' ,*
- (4) *for any of the other edges of $\tilde{\Delta}$, its combinatorial self intersection is the same as that of the edge of Δ with the same outward normal vector.*

Remark 2.14. Let (M, ω, Φ) be a toric manifold with moment map image Δ . Lemma 2.13 is the combinatorial counterpart of the following geometric facts. Let \tilde{M} be the toric manifold obtained from M by equivariant symplectic blow-up centered at a point p . The exceptional divisor in \tilde{M} has self intersection -1 ; a 2-sphere through p in M with self intersection ℓ gives rise to a 2-sphere in \tilde{M} with self intersection $\ell - 1$.

In \mathbb{R}^2 , all Delzant polygons can be obtained by a simple recursive recipe.

Lemma 2.15. (1) *Let Δ be a Delzant polygon with three edges. Then there exists a unique $\lambda > 0$ such that Δ is $\text{AGL}(2, \mathbb{Z})$ -congruent to the Delzant triangle Δ_λ . (See Example 2.5.)*

(2) *Let Δ be a Delzant polygon with four or more edges. Let s be the non-negative integer such that the number of edges is $4 + s$. Then there exist positive numbers $a \geq b > 0$, an integer $0 \leq k \leq 2a/b$, and positive numbers $\delta_1, \dots, \delta_s$, such that Δ is $\text{AGL}(2, \mathbb{Z})$ -congruent to a Delzant polygon that is obtained from the Hirzebruch trapezoid $H_{a,b,k}$ (see Example 2.5) by a sequence of corner choppings of sizes $\delta_1, \dots, \delta_s$.*

For the proof, see [10, Section 2.5 and Notes to Chapter 2].

Lemma 2.16. *Let*

$$\Delta_0, \Delta_1, \dots, \Delta_s$$

be a sequence of Delzant polygons such that, for each i , the polygon Δ_i is obtained from the polygon Δ_{i-1} by a corner chopping of size δ_i . Then

- (1) $\text{area}(\Delta_s) = \text{area}(\Delta_0) - \frac{1}{2}\delta_1^2 - \dots - \frac{1}{2}\delta_s^2$.
- (2) $\text{perimeter}(\Delta_s) = \text{perimeter}(\Delta_0) - \delta_1 - \dots - \delta_s$.
- (3) *For each i , $\delta_i < 2^s \text{perimeter}(\Delta_s)$.*
- (4) *The length of each edge of each Δ_i is a linear combination, with non-negative integer coefficients, of the lengths of the edges of Δ_s .*

Proof. Parts (1) and (2) follow immediately from the definition of corner chopping in §2.11.

Because a corner chopping of size δ_i can only be performed if there are two consecutive edges in Δ_{i-1} of lengths greater than δ_i ,

$$\text{perimeter}(\Delta_{i-1}) > 2\delta_i.$$

Because a corner chopping of size δ_i creates a new edge of size δ_i , shortens by δ_i the lengths of two edges, and the other edges do not change,

$$\text{perimeter}(\Delta_i) = \text{perimeter}(\Delta_{i-1}) - \delta_i.$$

From these two equalities it follows that

$$\text{perimeter}(\Delta_{i-1}) < 2 \text{perimeter}(\Delta_i).$$

By induction, we get

$$\text{perimeter}(\Delta_i) < 2^{s-i} \text{perimeter}(\Delta_s).$$

Because Δ_i contains an edge of size δ_i ,

$$\delta_i < \text{perimeter}(\Delta_i).$$

The last two equalities imply part (3).

Let d be an edge of Δ_{i-1} , let \tilde{d} be the edge of Δ_i with the same outward normal vector, and let e be the edge of Δ_i that was created at the i th corner chopping. If d does not touch the vertex of Δ_{i-1} at which we chop, then $\text{length}(d) = \text{length}(\tilde{d})$. If d touches the vertex of Δ_{i-1} at which we chop, then $\text{length}(d) = \text{length}(\tilde{d}) + \text{length}(e)$. Part (4) of the lemma follows by induction. \square

2.17. For any Delzant polygon Δ , consider the free Abelian group generated by its edges:

$$(2.4) \quad \mathbb{Z}[\text{edges of } \Delta].$$

The *combinatorial intersection pairing* on this group is the \mathbb{Z} -valued bilinear pairing whose restriction to the generators is given by their combinatorial intersection number. The *length functional*

$$\mathbb{Z}[\text{edges of } \Delta] \rightarrow \mathbb{R}$$

is the homomorphism that associates to each basis element its rational length.

Lemma 2.18. *If M is a symplectic toric manifold with moment map image Δ , the homomorphism from $\mathbb{Z}[\text{edges of } \Delta]$ to $H_2(M; \mathbb{Z})$ that sends a generator d to the homology class of the two-sphere $\Phi^{-1}(d) \subset M$ is onto, it carries the combinatorial intersection pairing on $\mathbb{Z}[\text{edges of } \Delta]$ to the ordinary intersection pairing in homology, and it carries the length functional to the functional on homology given by pairing with the symplectic form divided by 2π .*

Proof. The lemma follows from parts (1)–(3) of Lemma 2.10. \square

If Δ_{i+1} is obtained from Δ_i by a corner chopping, we consider the injective homomorphism

$$(2.5) \quad \mathbb{Z}[\text{edges of } \Delta_i] \hookrightarrow \mathbb{Z}[\text{edges of } \Delta_{i+1}]$$

whose restriction to the generators is defined in the following way. If d is an edge of Δ_i that does not touch the corner that was chopped, then d is mapped to the edge of Δ_{i+1} with the same outward normal vector. If d is an edge of Δ_i that touches the corner that was chopped, then d is mapped to $\tilde{d} + e$ where e is the new edge of Δ_{i+1} , created in the chopping, and \tilde{d} is the edge of Δ_{i+1} with the same outward normal vector as d .

Lemma 2.19. *The homomorphism (2.5) respects the combinatorial intersection pairing and the length homomorphism. Identifying edges of Δ_i with their image under (2.5), we get*

$$\mathbb{Z}[\text{edges of } \Delta_{i+1}] = \mathbb{Z}[\text{edges of } \Delta_i] \oplus \mathbb{Z}[e].$$

The combinatorial intersection pairing of e with an edge of Δ_i is zero; the combinatorial self intersection of e is -1 .

Proof. The definitions of corner chopping (§2.11) and of rational length (§2.4) imply that the homomorphism (2.5) respects the length homomorphism. The assertions about the combinatorial intersection pairing follow from Lemma 2.13. \square

Remark 2.20. Suppose that Δ_{i+1} is obtained from Δ_i by a corner chopping. Let M_i and M_{i+1} be toric manifolds with moment map images Δ_i and Δ_{i+1} . Then M_{i+1} can be obtained from M_i by equivariant symplectic blow-up. The homomorphism (2.5) induces the natural inclusion map of $H_2(M_i)$ into $H_2(M_{i+1})$.

Corollary 2.21. *Let (M, ω) be a compact connected symplectic four-manifold that admits a toric action. Then $b_2^+(M) = 1$.*

Recall that $b_2^+(M)$ is the largest dimension of a subspace of $H_2(M; \mathbb{R})$ on which the intersection form is positive definite.

Proof. Let Δ be the moment map image for a toric action on (M, ω) . By Part (1) of Proposition 2.3, Δ is a Delzant polytope. By Lemma 2.18, $b_2^+(M)$ is equal to the largest dimension of a subspace of $H_2(M, \mathbb{R})$ on which the combinatorial intersection pairing is positive definite. By Lemma 2.15, Δ is either $\text{AGL}(2, \mathbb{Z})$ -congruent to a Delzant triangle Δ_λ , or it is $\text{AGL}(2, \mathbb{Z})$ -congruent to a Delzant polytope that is obtained from a Hirzebruch trapezoid $H_{a,b,k}$ by a sequence of corner choppings. By Lemma 2.19, b_2^+ does not change under corner chopping. Thus, it is enough to show that $b_2^+ = 1$ for Δ_λ and for $H_{a,b,k}$. This can be checked by direct computation. \square

3. Soft proofs of properness and finiteness

We now state the key lemma used in the proof of our main theorem.

Lemma 3.1. *Let (M, ω) be a compact connected symplectic four-manifold. Then there exists a finite set $\mathcal{D}(M, \omega)$ of positive numbers such that for any polygon Δ , if Δ is the moment map image of a toric action on (M, ω) , and if Δ can be obtained from a Delzant polygon Δ' by a sequence of corner choppings of sizes $\delta_1, \dots, \delta_s$, then $\delta_i \in \mathcal{D}(M, \omega)$ for all i .*

3.2. The proof of Lemma 3.1 will use the following observation. Let Δ be a polygon that satisfies the assumptions of Lemma 3.1. By part (3) of Lemma

2.10, the number of vertices of Δ is $\dim H_2(M; \mathbb{R}) + 2$, so

$$(3.1) \quad s < \dim H_2(M; \mathbb{R}).$$

By part (3) of Lemma 2.16, by part (4) of Lemma 2.10, and by (3.1), δ_i is smaller than

$$(3.2) \quad 2^{\dim H_2(M; \mathbb{R})} \cdot \frac{1}{2\pi} \int_M \omega \wedge c_1(TM)$$

for all i .

Proof of Lemma 3.1. Suppose that (M, ω) admits a toric action; otherwise take $\mathcal{D}(M, \omega)$ to be empty. By Corollary 2.21, $b_2^+(M) = 1$.

Then, by Lemma 1.2, for any $K > 0$, the set $\{E \in H_2(M; \mathbb{R}) \mid E \cdot E = -1 \text{ and } 0 \leq \omega(E) \leq K\}$ is compact. Since $H_2(M; \mathbb{Z})$ is discrete, its intersection with this compact set,

$$\mathcal{D}_K = \{E \in H_2(M; \mathbb{Z}) \mid E \cdot E = -1 \text{ and } 0 \leq \omega(E) \leq K\},$$

is finite.

Let Δ be a polygon that satisfies the assumptions of Lemma 3.1. Let $\Delta' = \Delta_0, \dots, \Delta_s = \Delta$ be a sequence of Delzant polygons such that each Δ_i is obtained from Δ_{i-1} by a corner chopping of size δ_i . Then δ_i is the rational length of the new edge e_i of Δ_i created in the corner chopping. By part (1) of Lemma 2.13, the combinatorial self intersection of e_i is -1 . By §2.17 and Lemma 2.18, the composed homomorphism

$$\mathbb{Z}[\text{edges of } \Delta_i] \rightarrow \mathbb{Z}[\text{edges of } \Delta_s] \rightarrow H_2(M; \mathbb{Z})$$

sends e_i to a class $S \in H_2(M; \mathbb{Z})$ with $\frac{1}{2\pi}\omega(S) = \delta_i$ and $S \cdot S = -1$.

By §3.2,

$$\delta_i < \frac{1}{2\pi}K,$$

where

$$K = 2^{\dim H_2(M; \mathbb{R})} \int_M \omega \wedge c_1(TM).$$

Also, $0 < \delta_i$ for all i . Thus, the conclusion of Lemma 3.1 holds when $\mathcal{D}(M, \omega)$ is taken to be the set $\{\frac{1}{2\pi}\omega(E) \mid E \in \mathcal{D}_K\}$. \square

Remark 3.3. If there exists a positive number h such that $\int_A \omega$ is an integer multiple of h for all $A \in H_2(M; \mathbb{Z})$, there is an even shorter proof of Lemma 3.1: after replacing ω by $\frac{2\pi}{h}\omega$ (which does not affect the symplectomorphism group) we may assume that $\frac{1}{2\pi} \int_A \omega \in \mathbb{Z}$ for all $A \in H_2(M; \mathbb{Z})$. By part (1) of Lemma 2.10, the lengths of all the edges of Δ are integers. By part (4) of Lemma 2.16, this implies that all the δ_i 's are integers. Thus, the conclusion of Lemma 3.1 holds when $\mathcal{D}(M, \omega)$ is taken to be the set of positive integers that are smaller than (3.2).

We now give an elementary algebraic proof of the properness lemma:

Proof of Lemma 1.2. Let (M, ω) be a symplectic 4-manifold with $b_2^+ = 1$. Let $A \in H_2(M; \mathbb{R})$ be the Poincaré dual to $[\omega]$. Notice that $A \cdot A = \omega(A) = \int_M \omega \wedge \omega$ is positive, and that $A \cdot C = \omega(C)$ for all $C \in H_2(M; \mathbb{R})$. Without loss of generality, $A \cdot A = 1$. Complete A to an orthogonal basis $\mathcal{B} = \{A, B_1, \dots, B_k\}$ of $H_2(M; \mathbb{R})$ with the intersection pairing such that $B_j \cdot B_j = -1$ for all j . The period map $E \mapsto \omega(E)$ takes $E = x_0 A + \sum_{j=1}^k x_j B_j$ to the coefficient x_0 . The restriction of this map to the hyperboloid that consists of the classes in $H_2(M; \mathbb{R})$ with self intersection -1 is proper: if $x_0^2 - \sum_{i=1}^k x_i^2 = -1$ and x_0 is bounded, then so are x_1, \dots, x_k . \square

Remark 3.4. For a manifold obtained from $\mathbb{C}\mathbb{P}^2$ by eight or fewer blow-ups, the Poincaré dual C_1 to the first Chern class $c_1(TM)$ is of positive self intersection. This implies that the set of homology classes $E \in H_2(M; \mathbb{R})$ that satisfy $E \cdot E = -1$ and $c_1(TM)(E) = 1$ is compact. (The argument is similar to the proof of Lemma 1.2, with A replaced by C_1 .) Hence the intersection of this set with the discrete $H_2(M; \mathbb{Z})$ is finite. In particular, there exist only finitely many $E \in H_2(M; \mathbb{Z})$ of self intersection -1 that can be represented by a symplectically embedded sphere. See [9] or [25, ch. IV, § 26].

We can now deduce our main theorem.

Proof of Theorem 1.1. Fix a compact connected symplectic four-manifold (M, ω) .

Let Δ be the moment map image for a toric action on (M, ω) .

By part (1) of Proposition 2.3, Δ is a Delzant polygon. By parts (4) and (5) of Lemma 2.10,

$$(3.3) \quad \text{perimeter}(\Delta) = \frac{1}{2\pi} \int_M \omega \wedge c_1(TM)$$

and

$$(3.4) \quad \text{area}(\Delta) = \frac{1}{(2\pi)^2} \int_M \frac{1}{2!} \omega \wedge \omega.$$

First, suppose that $\dim H_2(M; \mathbb{R}) = 1$. By part (3) of Lemma 2.10, Δ has three edges. By part (1) of Lemma 2.15, Δ is $\text{AGL}(2, \mathbb{Z})$ -congruent to a standard Delzant triangle, Δ_λ . So $\text{perimeter}(\Delta) = 3\lambda$ and $\text{area}(\Delta) = \frac{1}{2}\lambda^2$. By this and (3.3) or (3.4), λ is uniquely determined by the symplectic manifold (M, ω) . By part (2) of Proposition 2.3, up to equivalence, there exists only one toric action on (M, ω) .

Now suppose that $\dim H_2(M; \mathbb{R}) \geq 2$. By part (3) of Lemma 2.10, Δ has $(4 + s)$ vertices, where s is the non-negative integer

$$(3.5) \quad s = \dim H_2(M; \mathbb{R}) - 2.$$

By part (2) of Lemma 2.15, after possibly transforming Δ by an element of $\text{AGL}(2, \mathbb{Z})$, there exist positive numbers $a \geq b > 0$, an integer $0 \leq k < 2a/b$, and (if $s > 0$) positive numbers $\delta_1, \dots, \delta_s > 0$ such that Δ can be obtained from the Hirzebruch trapezoid $H_{a,b,k}$ by a sequence of corner choppings of sizes $\delta_1, \dots, \delta_s$. By part (2) of Lemma 2.16,

$$\text{perimeter}(\Delta) = \text{perimeter}(H_{a,b,k}) - \sum \delta_i = 2(a + b) - \sum \delta_i.$$

By part (1) of Lemma 2.16,

$$\text{area}(\Delta) = \text{area}(H_{a,b,k}) - \frac{1}{2} \sum \delta_i^2 = ab - \frac{1}{2} \sum \delta_i^2.$$

By this, (3.3), and (3.4), the sum and product, $a + b$ and ab , and hence also the numbers a and b themselves, are determined by the symplectic manifold (M, ω) and the values of $\delta_1, \dots, \delta_s$. Because, by Lemma 3.1 and by (3.5), the number of possible tuples $(\delta_1, \dots, \delta_s)$ that can arise in this way is finite, there are only finitely many possibilities for the values of a and b . Because k is an integer and it satisfies

$$0 \leq k < \frac{2a}{b},$$

there are only finitely many possible values for k .

We have shown that, after transforming Δ by an element of $\text{AGL}(2, \mathbb{Z})$, there exists a tuple $(a, b, k; \delta_1, \dots, \delta_s)$ such that Δ can be obtained from $H_{a,b,k}$ by a sequence of corner choppings of sizes $\delta_1, \dots, \delta_s$, and that the number of tuples $(a, b, k; \delta_1, \dots, \delta_s)$ that can arise in this way is finite. For any given polygon, the number of possible corner choppings of a given size is at most the number of vertices, and hence it is finite. It follows that, modulo $\text{AGL}(2, \mathbb{Z})$ -congruence, the number of polygons Δ that can arise as moment map images of toric actions on (M, ω) is finite.

By part (2) of Proposition 2.3, the number of toric actions on (M, ω) is finite. \square

As explained in the introduction, Theorem 1.1 implies that for a four-dimensional compact connected symplectic manifold (M, ω) with $H^1(M; \mathbb{R}) = \{0\}$, the group $\text{Symp}(M, \omega)$ can contain only finitely many non-conjugate two-dimensional maximal tori. We conclude this section with examples of symplectomorphism groups in which not all maximal tori are two-dimensional. In the first symplectomorphism group of Example 3.5, one maximal torus is two-dimensional and another maximal torus is one-dimensional. The second symplectomorphism group contains one-dimensional maximal tori and does not contain any two-dimensional tori. The third symplectomorphism group does not contain any tori. The symplectic manifolds in these examples are obtained from $\mathbb{C}\mathbb{P}^2$ by symplectic blow-ups of equal sizes.

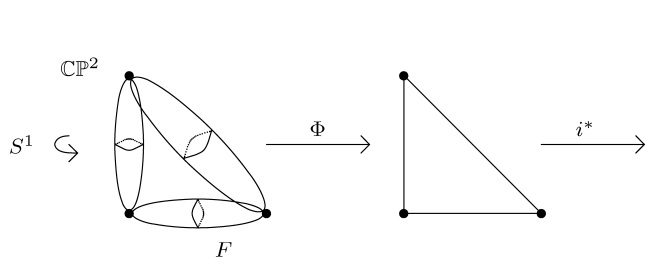


Figure 4. Moment maps for circle and torus actions on $\mathbb{C}\mathbb{P}^2$.

Example 3.5. Take the \mathbb{T}^2 -action on $\mathbb{C}\mathbb{P}^2$ given by $(a, b) \cdot [z_0, z_1, z_2] = [z_0, az_1, bz_2]$. There are exactly three fixed points. If we restrict to the S^1 -action given by $a \cdot [z_0, z_1, z_2] = [z_0, z_1, az_2]$, the S^1 -fixed point set consists of the point $[0, 0, 1]$ and the component $F = \{[z_0, z_1, 0]\} \cong \mathbb{C}\mathbb{P}^1$. The moment map for this S^1 action attains its minimum along F . See Figure ?? (where $\Phi: \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{R}^2$ is the \mathbb{T}^2 moment map, $i: S^1 \rightarrow \mathbb{T}^2$ is the inclusion map, and $i^* \circ \Phi: \mathbb{C}\mathbb{P}^2 \rightarrow \mathbb{R}$ is the S^1 moment map).

Let $T = S^1$ or $T = \mathbb{T}^2$. For any finite set of T -fixed points, we can choose T -equivariant Darboux charts that are centered at these points and are disjoint from each other; if $\epsilon > 0$ is sufficiently small, we can use these Darboux charts to perform T -equivariant symplectic blow-ups of size ϵ centered at the chosen fixed points.

- (1) Choose a sufficiently small positive number ϵ and perform three S^1 -equivariant symplectic blow-ups of size ϵ that are centered at points of F . Each such a blow-up creates an isolated fixed point whose moment map value is equal to that of F plus ϵ . If a Hamiltonian S^1 -action on a symplectic four-manifold has three or more fixed points with the same moment map value, the action does not extend to a Hamiltonian \mathbb{T}^2 -action; see [4] or [17, Proposition 5.21]. Thus, this S^1 -action is a maximal torus. It is a one-dimensional maximal torus in the symplectomorphism group of a symplectic manifold that is obtained from $\mathbb{C}\mathbb{P}^2$ by three symplectic blow-ups of size ϵ .

On the other hand, perform three \mathbb{T}^2 -equivariant symplectic blow-ups of size ϵ , centered at the three \mathbb{T}^2 -fixed points. This yields a two-dimensional maximal torus in the symplectomorphism group of a symplectic manifold that is obtained from $\mathbb{C}\mathbb{P}^2$ by three symplectic blow-ups of size ϵ .

By [29], (also see [5],) any two symplectic manifolds that are obtained from $\mathbb{C}\mathbb{P}^2$ by symplectic blow-ups of the same sizes are symplectomorphic. Therefore, the above constructions give maximal tori in the symplectomorphism group of the same symplectic manifold;

one of these maximal tori is one-dimensional and the other is two-dimensional.

- (2) If ϵ is sufficiently small, we can perform four S^1 -equivariant symplectic blow-ups centered at points of F . Alternatively, we can perform three S^1 -equivariant symplectic blow-ups centered at points of F and one that is centered at the isolated fixed point $[0, 0, 1]$. These two constructions yield two circle actions that are not equivalent (e.g., the lists of moment map values at their fixed points are different). If $1/\epsilon$ is an integer, a manifold that is obtained from $\mathbb{C}\mathbb{P}^2$ by four or more symplectic blow-ups of size ϵ does not admit a toric action; see [19, Theorem 4.1(1)]. Thus, the symplectomorphism group of such a symplectic manifold, when ϵ is sufficiently small and $1/\epsilon$ is an integer, contains at least two non-conjugate one-dimensional maximal tori, and it does not contain any two-dimensional maximal tori.
- (3) If $1/\epsilon$ is an integer and $(k-1)\epsilon \geq 1$, then a symplectic manifold that is obtained from $\mathbb{C}\mathbb{P}^2$ by k symplectic blow-ups of size ϵ does not admit any circle action; see [19, Theorem 4.1(2)]. It is possible to perform four symplectic blow-ups of $\mathbb{C}\mathbb{P}^2$ of size $\epsilon = 1/3$ (see, e.g., [37]). Thus, the resulting symplectic manifold does not admit any circle action.

A. Appendix. J -holomorphic spheres in symplectic four-manifolds

In this appendix, we introduce the non-experts to a beautiful work of McDuff that establishes properness for a general compact symplectic four-manifold.

We then apply this result, together with a lemma on the existence of certain symplectically embedded 2-spheres, to give another proof of Lemma 3.1.

Lemma A.1. *Let (M, ω) be a closed symplectic four-manifold. Let \mathcal{E} denote the set of homology classes $E \in H_2(M; \mathbb{Z})$ that are represented by symplectic exceptional spheres. Then the map $E \mapsto \omega(E)$ from \mathcal{E} to \mathbb{R} is proper.*

When $b_2^+(M) = 1$, Lemma A.1 follows from Lemma 1.2.

A *symplectic exceptional sphere* is a symplectically embedded sphere with self intersection -1 .

Lemma A.1 is proved in [26, Lemma 3.1; 29, § 3] using Gromov’s theory of J -holomorphic curves. We recall relevant parts of this theory.

Let (M, ω) be a compact symplectic manifold. Let $\mathcal{J} = \mathcal{J}(M, \omega)$ be the space of almost complex structures on M that are tamed by ω ; see § 2.9. Given $J \in \mathcal{J}$, a *parametrized J -holomorphic sphere* is a map $u: \mathbb{C}\mathbb{P}^1 \rightarrow M$ such that $du: T\mathbb{C}\mathbb{P}^1 \rightarrow TM$ satisfies the Cauchy–Riemann equation $du \circ i = J \circ du$. Such a u represents a homology class in $H_2(M; \mathbb{Z})$ that we denote $[u]$. The symplectic area $\omega([u]) := \int_{\mathbb{C}\mathbb{P}^1} u^*\omega$ only depends on the homology class $[u] \in H_2(M; \mathbb{Z})$. The pull-back $u^*\omega$ is a positively oriented area form at each point where $du \neq 0$; it follows that the class

$[u]$ is non-zero if and only if u is not constant, if and only if $\omega([u]) > 0$. A J -holomorphic sphere is called *simple* if it cannot be factored through a branched covering of $\mathbb{C}\mathbb{P}^1$. One similarly defines a holomorphic curve in (M, J) whose domain is a Riemann surface other than $\mathbb{C}\mathbb{P}^1$. Gromov, in [13], introduced a notion of “weak convergence” of a sequence of holomorphic curves. This notion is preserved under reparametrization of the curve, and it implies convergence in homology. *Gromov’s compactness theorem* guarantees that, given a converging sequence of almost complex structures, a corresponding sequence of holomorphic curves with bounded symplectic area has a weakly converging subsequence.

The limit under weak convergence might not be a curve; it might be a “cusp curve”, which is a connected union of holomorphic curves. In this paper, we do not need the precise definitions of cusp curve and of weak convergence. Readers who are not experienced with these notions may find this example helpful:

Example A.2. Identify $\mathbb{C}\mathbb{P}^1$ with $\mathbb{C} \sqcup \{\infty\}$. For each $0 < \lambda < \infty$, consider the holomorphic curve in $M = \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 = (\mathbb{C} \sqcup \{\infty\}) \times (\mathbb{C} \sqcup \{\infty\})$ given by $\{(w, z) \mid w = \lambda z\}$. This curve can be parametrized by $z \mapsto (\lambda z, z)$ or, alternatively, by $w \mapsto (w, \lambda^{-1}w)$. As $\lambda \rightarrow 0$, these curves weakly converge to the cusp curve $\{0\} \times \mathbb{C}\mathbb{P}^1 \cup \mathbb{C}\mathbb{P}^1 \times \{\infty\}$.

Lemma A.3. *Let $\{J_n\} \subset \mathcal{J}$ be a sequence of almost complex structures that converges in the C^∞ topology to an almost complex structure $J_\infty \in \mathcal{J}$. For each n , let $f_n: \mathbb{C}\mathbb{P}^1 \rightarrow M$ be a parametrized J_n -holomorphic sphere. Suppose that the set of areas $\omega([f_n])$ is bounded. Then one of the following two possibilities occurs.*

- (1) *There exists a J_∞ -holomorphic sphere $u: \mathbb{C}\mathbb{P}^1 \rightarrow M$ and elements $A_n \in \mathrm{PSL}(2, \mathbb{C})$ such that a subsequence of the $f_n \circ A_n$ ’s converges to u in the C^∞ topology. In particular, there exist infinitely many n ’s for which $[f_n] = [u]$.*
- (2) *There exist two or more non-constant simple J_∞ -holomorphic spheres $u_\ell: \mathbb{C}\mathbb{P}^1 \rightarrow M$ and positive integers m_ℓ , for $\ell = 1, \dots, L$, and infinitely many n ’s for which*

$$[f_n] = \sum_{\ell=1}^L m_\ell [u_\ell] \quad \text{in } H_2(M; \mathbb{Z}).$$

Proof. If there exists a subsequence of the f_n ’s that, after reparametrization, converges in the C^∞ topology to a map $u: \mathbb{C}\mathbb{P}^1 \rightarrow M$, then the map u is J_∞ -holomorphic, and the first of the two possibilities occurs.

Otherwise, by Gromov’s compactness theorem, there exists a subsequence that weakly converges to a cusp curve. See [36, § 5]. This cusp curve has at least two components; its components are non-constant J_∞ -holomorphic

maps $v_\ell: \mathbb{C}\mathbb{P}^1 \rightarrow M$; weak convergence implies convergence in homology, so that $[f_n] = \sum_\ell [v_\ell]$ for all but a finite number of f_n 's in the subsequence.

If v_ℓ is not simple then there exists a simple non-constant J_∞ -holomorphic $u_\ell: \mathbb{C}\mathbb{P}^1 \rightarrow M$ and a positive integer m_ℓ such that $[v_\ell] = m_\ell[u_\ell]$. \square

Corollary A.4. *Let (M, ω) be a closed symplectic manifold. Let K be a positive real number. For each $J \in \mathcal{J}(M, \omega)$, there exist only finitely many homology classes $A \in H_2(M; \mathbb{Z})$ such that $\omega(A) \leq K$ and such that the class A is represented by a J -holomorphic sphere.*

Proof. Otherwise, there exists a sequence of J -holomorphic spheres $f_n: \mathbb{C}\mathbb{P}^1 \rightarrow M$ such that $\omega([f_n]) \leq K$ for each n and such that the classes $[f_n]$ are all different. This contradicts Lemma A.3. \square

Fix a compact symplectic manifold (M, ω) . For any class $A \in H_2(M; \mathbb{Z})$, consider the universal moduli space of simple parametrized holomorphic spheres in the class A ,

$$\mathcal{M}(A, \mathcal{J}) = \{(u, J) \mid J \in \mathcal{J}, u: \mathbb{C}\mathbb{P}^1 \rightarrow M \text{ is simple } J\text{-holomorphic, and } [u] = A\},$$

and the projection map

$$p_A: \mathcal{M}(A, \mathcal{J}) \rightarrow \mathcal{J}.$$

For any positive number K , let

$$\mathcal{N}_K = \{A \in H_2(M; \mathbb{Z}) \mid A \neq 0, c_1(TM)(A) \leq 0, \text{ and } \omega(A) < K\}.$$

The importance of this set lies in the fact that if a homology class E with $\omega(E) \leq K$ and $c_1(TM)(E) \leq 1$ is represented by a J -holomorphic cusp-curve with two or more components then at least one of these components must lie in a homology class in \mathcal{N}_K :

Lemma A.5. *Let $J \in \mathcal{J}$. Let $u_\ell: \mathbb{C}\mathbb{P}^1 \rightarrow M$, $\ell = 1, \dots, L$ be two or more non-constant J -holomorphic spheres, and let m_1, \dots, m_L be positive integers. Consider the homology class $E = \sum m_\ell [u_\ell]$ in $H_2(M; \mathbb{Z})$. Suppose that $c_1(TM)(E) \leq 1$. Let $K = \omega(E)$. Then there exists $1 \leq \ell \leq L$ such that $[u_\ell] \in \mathcal{N}_K$.*

Proof. By assumption,

$$(A.1) \quad \sum_{\ell=1}^L m_\ell \omega([u_\ell]) = K.$$

Because the left-hand side consists of two or more positive summands, each summand is smaller than the sum K :

$$(A.2) \quad \omega([u_\ell]) < K \quad \text{for all } \ell.$$

By assumption,

$$\sum_{\ell=1}^L m_\ell c_1(TM)([u_\ell]) \leq 1.$$

Because this is a positive combination of two or more integers and the sum is no greater than one, the integers cannot all be greater than or equal to one, so there must exist at least one ℓ such that

$$(A.3) \quad c_1(TM)([u_\ell]) \leq 0.$$

By this and (A.2), the class $[u_\ell]$ is in \mathcal{N}_K . \square

The automorphism group $\mathrm{PSL}(2, \mathbb{C})$ of \mathbb{CP}^1 acts on $\mathcal{M}(A, \mathcal{J})$ by reparametrizations. The quotient $\mathcal{M}(A, \mathcal{J})/\mathrm{PSL}(2, \mathbb{C})$ is the space of unparametrized J -holomorphic spheres representing $A \in H_2(M; \mathbb{Z})$. Let

$$U_K = \mathcal{J} \setminus \bigcup_{A \in \mathcal{N}_K} \text{image } p_A.$$

Lemma A.6. *Let $E \in H_2(M; \mathbb{Z})$ be a homology class such that $c_1(TM)(E) \leq 1$. Let $K = \omega(E)$. Then the map*

$$p_E^{-1}(U_K)/\mathrm{PSL}(2, \mathbb{C}) \rightarrow U_K$$

that is induced from the projection map $p_E: \mathcal{M}(E, \mathcal{J}) \rightarrow \mathcal{J}$ is proper.

Proof. The lemma follows from Gromov's compactness in the following way. Let $D \subset U_K$ be a compact subset. We need to show that $p_E^{-1}(D)/\mathrm{PSL}(2, \mathbb{C})$ is compact. Because $\mathcal{M}(E, \mathcal{J})$ is Hausdorff and first countable, it is enough to show that for every sequence $\{(f_n, J_n)\}$ in $p_E^{-1}(D)$ there exists a subsequence that, after reparametrization, has a limit in $p_E^{-1}(D)$ in the C^∞ topology.

Take such a sequence, $\{(f_n, J_n)\}$. Because $J_n \in D$ and D is compact and contained in U_K , after passing to a subsequence we may assume that $\{J_n\}$ converges to $J_\infty \in U_K$.

Suppose that there exists a subsequence that, after reparametrization, converges to some $u: \mathbb{CP}^1 \rightarrow M$ in the C^∞ topology. Because each f_n is a J_n -holomorphic sphere in the class E , the limit u must also be in the class E and it must be J_∞ -holomorphic.

We need to show that u is simple. If not, there is a simple J_∞ -holomorphic curve v and an integer $m \geq 2$ such that $[u] = m[v]$. Because v is simple and J_∞ -holomorphic, the pair (v, J_∞) is in the moduli space $\mathcal{M}(A, \mathcal{J})$ for $A = [v]$, so $J_\infty \in \text{image } p_A$. The inequalities $c_1(TM)([u]) \leq 1$ and $\omega([u]) = K$ imply that $c_1(TM)([v]) \leq 0$ and $\omega([v]) < K$, so that $A = [v]$ is in \mathcal{N}_K . So $J_\infty \in \text{image } p_A$ cannot be in U_K . Therefore, u is simple.

The pair (u, J_∞) is then in the moduli space $\mathcal{M}(E, \mathcal{J})$, and since $J_\infty \in D$, this pair is in $p_E^{-1}(D)$.

Now suppose that there does not exist such a subsequence, and we would like to arrive at a contradiction. By Lemma A.3, there exist two or more non-constant simple J_∞ -holomorphic spheres $u_\ell: \mathbb{C}\mathbb{P}^1 \rightarrow M$ and positive integers m_ℓ such that $\sum m_\ell[u_\ell] = E$. Because $\omega(E) = K$ and $c_1(TM)(E) \leq 1$, by Lemma A.5 at least one of the u_ℓ 's represents a homology class in \mathcal{N}_K . This contradicts the fact that the u_ℓ 's are J_∞ -holomorphic for $J_\infty \in U_K$ and the definition of U_K . \square

Corollary A.7. *Let E be a homology class such that $c_1(TM)(E) = 1$ and $\omega(E) \leq K$. Then image $p_E \cap U_K$ is closed in U_K .*

Lemma A.8. *The subset $U_K \subset \mathcal{J}$ is open.*

Proof. Let $\{J_n\}$ be a sequence of almost complex structures that converges in the C^∞ topology to an almost complex structure J_∞ . We need to show that if each J_n lies in the union

$$\bigcup_{A \in \mathcal{N}_K} \text{image } p_A$$

then so does J_∞ . For each n , let $f_n: \mathbb{C}\mathbb{P}^1 \rightarrow M$ be a J_n -holomorphic sphere such that

$$(A.4) \quad \omega([f_n]) < K \quad \text{and} \quad c_1(TM)([f_n]) \leq 0.$$

By Lemma A.3, there exist one or more non-constant simple J_∞ -holomorphic spheres $u_\ell: \mathbb{C}\mathbb{P}^1 \rightarrow M$ and positive integers m_ℓ such that

$$(A.5) \quad \sum_{\ell=1}^L m_\ell[u_\ell] = [f_n]$$

for infinitely many n 's. By Lemma A.5 there exists at least one u_ℓ whose homology class $[u_\ell]$ is in \mathcal{N}_K . Because u_ℓ is J_∞ -holomorphic and u_ℓ is simple, $J_\infty \in \text{image } p_{[u_\ell]}$. \square

We have the following consequences of the Sard–Smale theorem, the infinite-dimensional inverse mapping theorem, and the ellipticity of the Cauchy–Riemann equations. See the arXiv version of this paper, [20], for a derivation of these consequences, using the results of [32, Chapter 3].

Lemma A.9. (a) *The universal moduli space $\mathcal{M}(A, \mathcal{J})$ and the space \mathcal{J} of compatible almost complex structures are Fréchet manifolds, and the projection map*

$$p_A: \mathcal{M}(A, \mathcal{J}) \rightarrow \mathcal{J}$$

is a Fredholm map of index

$$(A.6) \quad \dim \ker dp_A - \dim \text{coker } dp_A = 2c_1(TM)(A) + 2n,$$

where $2n$ is the dimension of M .

- (b) If (u, J) is a regular value for p_A , then for any neighborhood U of (u, J) in $\mathcal{M}(A, \mathcal{J})$, its image, $p_A(U)$, contains a neighborhood of J in \mathcal{J} .
- (c) The set of J 's that are regular for the map $p_A: \mathcal{M}(A, \mathcal{J}) \rightarrow \mathcal{J}$ for all $A \in H_2(M; \mathbb{Z})$ is dense in \mathcal{J} .
- (d) Let \mathcal{A} be a subset of $H_2(M; \mathbb{Z})$. Let $\{J_t\}_{t \in [0,1]}$ be a C^1 simple path in \mathcal{J} whose endpoints are regular values for p_A for all $A \in \mathcal{A}$. Then there exists a C^1 perturbation $\{\tilde{J}_t\}$ of J_t with the same endpoints which is transversal to p_A for all $A \in \mathcal{A}$.

Lemma A.10. (C_1 lemma). *Let (M, ω) be a compact symplectic four-manifold. The subset $U_K \subset \mathcal{J}$ is dense and path connected.*

Proof. Let $A \in \mathcal{N}_K$. Since $A \neq 0$, the $\mathrm{PSL}(2, \mathbb{C})$ -orbits in $\mathcal{M}(A, J)$ are six-dimensional submanifolds of the level sets $p_A^{-1}(J)$, and so $\dim \ker dp_A|_{(u, J)} \geq 6$ for all $(u, J) \in \mathcal{M}(A, \mathcal{J})$. Since $c_1(TM)(A) \leq 0$ then, by (A.6), $\dim \mathrm{coker} dp_A \geq 2$ for all $(u, J) \in \mathcal{M}(A, \mathcal{J})$.

In particular, dp_A is never onto. So a point in \mathcal{J} is a regular value for p_A if and only if it is not in the image of p_A . The fact that U_K is dense then follows from part (c) of Lemma A.9.

Let J_0 and J_1 be two points in U_K . Let $\{J_t\}_{t \in [0,1]}$ be any simple path in \mathcal{J} connecting J_0 to J_1 . By part (d) of Lemma A.9, there exists a path $\{\tilde{J}_t\}_{t \in [0,1]}$ connecting J_0 to J_1 that is transversal to p_A for all $A \in \mathcal{N}_K$. Because $\dim \mathrm{coker} dp_A \geq 2$ for all $A \in \mathcal{N}_K$, transversality means that the path is disjoint from image p_A . So the path lies in U_K . \square

Lemma A.11. *Let (M, ω) be a compact symplectic four-manifold. Let $A \in H_2(M; \mathbb{Z})$ be a homology class that is represented by an embedded symplectic sphere C . Then for any $J \in \mathcal{J}$ and any simple J -holomorphic $u: \mathbb{C}\mathbb{P}^1 \rightarrow M$ in the class A , the map u is an embedding.*

Proof. Let $u_0: \mathbb{C}\mathbb{P}^1 \rightarrow M$ be a symplectic embedding whose image is C . Construct an almost complex structure $J_0 \in \mathcal{J}$ for which u_0 is a J_0 -holomorphic sphere. (Define $J_0|_{TC}$ such that u_0 is holomorphic; extend it to a compatible fiberwise complex structure on the symplectic vector bundle $TM|_C$; then extend it to a compatible almost complex structure on M . See [31, Section 2.6].)

By the adjunction inequality, for any $(u, J) \in \mathcal{M}(A, \mathcal{J})$,

$$A \cdot A - c_1(TM)(A) + 2 \geq 0,$$

with equality if and only if u is an embedding. See [28, Theorem 2.2.1] or [32, Corollary E.1.7]. Applying this to (u_0, J_0) , we get that the homology class A satisfies $A \cdot A - c_1(TM)(A) + 2 = 0$. Applying the adjunction inequality to any other $(u, J) \in \mathcal{M}(A, \mathcal{J})$, we get that u is an embedding. \square

Lemma A.12. *Let (M, ω) be a compact symplectic four-manifold. Let $E \in H_2(M; \mathbb{Z})$ be a homology class that can be represented by an embedded symplectic sphere and such that $c_1(TM)(E) = 1$. Then*

- (1) *The projection map $p_E: \mathcal{M}(E, \mathcal{J}) \rightarrow \mathcal{J}$ is open.*
- (2) *Let $K \geq \omega(E)$. Then, for any $J \in U_K$, the class E is represented by an embedded J -holomorphic sphere.*

Proof. Let $(u, J) \in \mathcal{M}(E, \mathcal{J})$. By Lemma A.11, the map $u: \mathbb{C}\mathbb{P}^1 \rightarrow M$ is an embedding. By the Hofer–Lizan–Sikorav regularity criterion [16], if u is an immersed J -holomorphic sphere and $c_1([u]) \geq 1$, then (u, J) is a regular point for the map $p_E: \mathcal{M}(E, \mathcal{J}) \rightarrow \mathcal{J}$. By part (b) of Lemma A.9 it follows that the map p_E is open.

Therefore, image p_E is an open subset of \mathcal{J} . Because, by Lemmas A.8 and A.10, U_K is open and dense in \mathcal{J} , the set image $p_E \cap U_K$ is a non-empty open subset of U_K . Because, by Lemma A.7, image $p_E \cap U_K$ is also closed in U_K , and because, by Lemma A.10, the set U_K is connected, we conclude that image $p_E \cap U_K = U_K$. So for every $J \in U_K$ there exists a simple J -holomorphic sphere $u: \mathbb{C}\mathbb{P}^1 \rightarrow M$ in the class E . By Lemma A.11, u is an embedding. □

We can now complete the proof of the general properness lemma.

Proof of Lemma A.1. Let K be a positive real number. Suppose that E_1, E_2, \dots are infinitely many distinct classes in $H_2(M; \mathbb{Z})$ that satisfy the following conditions.

- (1) The class E can be represented by an embedded symplectic sphere in M ;
- (2) The self intersection $E \cdot E$ is -1 ;
- (3) The symplectic area $\omega(E)$ is not larger than K .

For each i , because $E_i \cdot E_i = -1$ and E_i is represented by an embedded symplectic sphere, we have $c_1(TM)(E_i) = 1$. Pick any $J \in U_K$. By part (2) of Lemma A.12, for each i , there exists an embedded J -holomorphic sphere in the class E_i . By Corollary A.4, the E_i 's cannot all be distinct. □

To deduce Lemma 3.1, we will also need the following result, about the existence of certain symplectic exceptional spheres.

Lemma A.13. *Let (M, ω) be a compact symplectic four-manifold. Suppose that (M, ω) is obtained by a sequence of symplectic blow-ups from a symplectic manifold (M_0, ω_0) . Let C_0 be a symplectic exceptional sphere in (M_0, ω_0) . Then (M, ω) contains a symplectic exceptional sphere C such that*

$$\int_C \omega = \int_{C_0} \omega_0.$$

In the proof of Lemma 3.1, Lemma A.13 can be replaced by a “soft” argument; see the arXiv version of this paper, [20, Lemma A.15 and Appendix B].

Sketch of proof of Lemma A.13. Suppose that (M, ω) is obtained from (M_0, ω_0) by a sequence of symplectic blow-ups: $M_0, M_1, \dots, M_s = M$. Let Σ_0 be a symplectic exceptional sphere in (M_0, ω_0) . We claim that there exists a symplectic exceptional sphere in (M, ω) in (the image in $H_2(M)$ of) the class $[\Sigma_0]$.

By induction, it is enough to address the case $s = 1$ of a single blow-up. Let $\iota: B^4(\delta) \hookrightarrow M_0$ be a symplectic embedding defining the blow-up. By a parametrized version of the C_1 lemma, and because exceptional J -holomorphic spheres persist along any deformation of regular pairs $\{(\omega_t, J_t)\}$, see [26, §3], it is enough to show the claim for a deformation equivalent symplectic form on M . Since any two symplectic blow-ups centered at the same fixed point are deformation equivalent, this implies that we can choose the size δ arbitrarily small. If δ is small enough, we can find a Hamiltonian isotopy $\phi_t: M_0 \rightarrow M_0$, $t \in [0, 1]$, such that the image of $\phi_1 \circ \iota$ is disjoint from Σ_0 . Because the embeddings ι and $\phi_1 \circ \iota$ are isotopic, the symplectic blow-up along $\phi_1 \circ \iota$ gives a manifold (M', ω') symplectomorphic to (M, ω) , see [26, §3]. Because the symplectic blow-up operation takes place in an arbitrarily small neighborhood of the embedded closed ball, (M', ω') , and thus (M, ω) , contains an embedded symplectic sphere representing the class $[\Sigma_0]$. \square

Remark A.14. Symplectic exceptional spheres in four-manifolds can be studied using Taubes–Seiberg–Witten theory of Gromov invariants, see [24, 29]. We can apply this theory to reduce the claim in Lemma A.13 on the existence of a symplectic exceptional sphere to a claim on the existence of an embedded smooth sphere in the corresponding homology class, and then deduce the latter claim from the construction of smooth blow-ups.

A.15. We now have holomorphic proof of Lemma 3.1. Let (M, ω) be a compact connected four-manifold. Let Δ be a moment map image for a toric action on (M, ω) , and suppose that Δ can be obtained from a Delzant polygon Δ' by a sequence of corner choppings of sizes $\delta_1, \dots, \delta_s$. By §3.2,

$$\delta_i < \frac{1}{2\pi} K$$

for all i , where

$$K = 2^{\dim H_2(M; \mathbb{R})} \int_M \omega \wedge c_1(TM).$$

By assumption, there exist Delzant polytopes

$$\Delta' = \Delta_0, \Delta_1, \dots, \Delta_s = \Delta$$

such that each Δ_i is obtained from Δ_{i-1} by a corner chopping of size δ_i . By Remark 2.12 and Delzant's theorem (see part (2) of Proposition 2.3), these correspond to a sequence of symplectic toric manifolds such that each manifold in the sequence can be obtained from the previous one by a blow-up of size δ_i , and such that the last manifold in the sequence is (M, ω) . Thus, (M, ω) can be obtained by a sequence of symplectic blow-ups from a symplectic manifold that contains a sphere of size δ_i . By Lemma A.13, (M, ω) contains a symplectic exceptional sphere of size δ_i . Thus, the δ_i 's belong to the set

$$\mathcal{D}(M, \omega) := \left\{ \frac{1}{2\pi} \int_C \omega \mid C \subset M \text{ is a symplectic exceptional sphere} \right. \\ \left. \text{and } \int_C \omega \leq K \right\}.$$

This set is finite, by Lemma A.1.

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