

THE SYMPLECTIC GEOMETRY OF THE
GEL'FAND–CETLIN–MOLEV BASIS FOR
REPRESENTATIONS OF $Sp(2n, \mathbb{C})$

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Gel'fand and Cetlin [I. Gel'fand and M. Tsetlin, *Finite-dimensional representations of the group of orthogonal matrices*, Dokl. Akad. Nauk SSSR **17** (1950), 1017–1020; I. Gel'fand and M. Tsetlin, *Finite-dimensional representations of the group of unimodular matrices*. Dokl. Akad. Nauk SSSR **71** (1950), 825–828.] constructed in the 1950s a canonical basis for a finite-dimensional representation $V(\lambda)$ of $U(n, \mathbb{C})$ by successive decompositions of the representation by a chain of subgroups. Guillemin and Sternberg constructed in the 1980s the *Gel'fand–Cetlin integrable system* on the coadjoint orbits of $U(n, \mathbb{C})$, which is the symplectic-geometric version, via geometric quantization, of the Gel'fand–Cetlin construction. (Much the same construction works for representations of $SO(n, \mathbb{R})$.) Molev [A. Molev, *A basis for representations of symplectic Lie algebras*, Comm. Math. Phys. **201**(3) (1999), 591–618.] in 1999 found a Gel'fand–Cetlin-type basis for representations of the symplectic group, using essentially new ideas. An important new role is played by the Yangian $Y(2)$, an infinite-dimensional Hopf algebra, and a subalgebra of $Y(2)$ called the twisted Yangian $Y^-(2)$. In this paper, we use deformation theory to give the analogous symplectic-geometric results for the case of $U(n, \mathbb{H})$, i.e., we construct a completely integrable system on the coadjoint orbits of $U(n, \mathbb{H})$. We call this the *Gel'fand–Cetlin–Molev integrable system*.

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1. Introduction

Symplectic geometry and representation theory can be related by the theory of *geometric quantization*. In this theory, a symplectic manifold M equipped with a Hamiltonian G -action has an associated linear representation V of G , and symplectic reductions by G of M translate to taking G -isotypic components V^λ of V . This correspondence has served as an underlying theme in much work in modern symplectic geometry, and the present paper is no exception.

There are two parallel theories which, as a pair, serve as motivation for the work in this paper. These are the *Gel'fand–Cetlin basis* for representations of $U(n, \mathbb{C})$, and the *Gel'fand–Cetlin integrable system* on coadjoint orbits of $U(n, \mathbb{C})$. The names are no coincidence: via geometric quantization, the Gel'fand–Cetlin system on a coadjoint orbit can be seen to be the symplectic-geometric analog of the Gel'fand–Cetlin basis for an appropriate representation. This parallel works out beautifully in the case of $U(n, \mathbb{C})$ or $SO(n, \mathbb{R})$, but not for other groups. (For simplicity in the discussion below, we refer only to the group $U(n, \mathbb{C})$.)

In both of these parallel theories, the underlying goal is to produce a “large” torus action on the relevant space (in the case of representation theory, a vector space, and in the case of symplectic geometry, a symplectic manifold). On the representation-theoretic side, the torus is “maximal” in the sense that it completely decomposes the representation into 1-dimensional eigenspaces. On the symplectic side, the torus is “maximal” in the well-known sense that the torus is half the dimension of the symplectic manifold, thus making an integrable system.

A finite-dimensional irreducible representation V_λ of $U(n, \mathbb{C})$, when considered as a representation of the subgroup $U(n-1, \mathbb{C})$, decomposes with

multiplicity 1. The representation V_λ can be decomposed successively by a chain of subgroups

$$U(1, \mathbb{C}) \subset U(2, \mathbb{C}) \subset \cdots \subset U(n-1, \mathbb{C}).$$

Since the final subgroup $U(1, \mathbb{C})$ is abelian and because of the multiplicity-free decomposition at each step, one obtains a canonical (up to a choice of this chain of subgroups) basis for any finite-dimensional $U(n, \mathbb{C})$ -representation V_λ . This basis is called the *Gel'fand–Cetlin basis* for V_λ [5]. Its construction is briefly summarized in Section 2.1.

Guillemin and Sternberg [9] showed in the 1980s that a special set of functions on a coadjoint orbit \mathcal{O}_λ of the unitary group $U(n, \mathbb{C})$ give the maximal possible number of Poisson-commuting functions on \mathcal{O}_λ . In other words, they show that the coadjoint orbit \mathcal{O}_λ is a completely integrable system. (The coadjoint orbits are *not* toric varieties; this is because the aforementioned functions are smooth only on an open dense subset of \mathcal{O}_λ .) The coadjoint orbit, equipped with these Poisson-commuting functions, is called the *Gel'fand–Cetlin system* on \mathcal{O}_λ . By geometric quantization, the existence of this maximal set of Poisson-commuting functions is the geometric analog of the multiplicity-free decomposition of V_λ . This integrable system is explained in Section 2.2.

For other groups, finding a Gel'fand–Cetlin basis proves to be much more difficult. In particular, for the case of $U(n, \mathbb{H})$ (the compact form of $Sp(2n, \mathbb{C})$), a difficulty arises in that the finite-dimensional irreducible representations V_λ of $U(n, \mathbb{H})$ decompose *with multiplicity* as representations of $U(n-1, \mathbb{H})$. Similarly, from the symplectic-geometric standpoint, the symplectic reductions of \mathcal{O}_λ by $U(n-1, \mathbb{H})$ are not just points (as they are for the $U(n, \mathbb{C})$ case), but are non-trivial symplectic manifolds.

Nevertheless, Molev [13] found a Gel'fand–Cetlin-type basis for finite-dimensional irreducible representations of $U(n, \mathbb{H})$, which are constructed in the spirit of the original work of Gel'fand–Cetlin. His methods required the use of new tools, including an infinite-dimensional algebra called the Yangian. Molev's theorems are recounted in Section 2.3.

This history is summarized in the following table. The essence of this paper is to answer the following

Question: What is the “Gel'fand–Cetlin-type” integrable system on coadjoint orbits of $U(n, \mathbb{H})$ corresponding (via geometric quantization) to Molev's canonical bases of the representations? In other words, what goes in the bottom right-hand corner of the table below?

	Representation theory	Symplectic geometry
$U(n, \mathbb{C})$	Gel'fand–Cetlin canonical basis for finite-dimensional irreducible representations $V(\lambda)$ [5]	Gel'fand–Cetlin integrable system on coadjoint orbits \mathcal{O}_λ [9]
$U(n, \mathbb{H})$	Gel'fand–Cetlin-type canonical basis for finite-dimensional irreducible representations $V(\lambda)$ [13]	???

It turns out that the answer to this question, which is the analogous theorem to Guillemin and Sternberg's for the case of the group $G = U(n, \mathbb{H})$, is remarkably easy to state. For simplicity, we always assume that \mathcal{O}_λ is a *generic* coadjoint orbit of $U(n, \mathbb{H})$. For convenience, we first state the result in terms of an intermediate geometric object, namely the symplectic reductions of \mathcal{O}_λ by the subgroup $U(n-1, \mathbb{H})$.

Theorem 1.1. *Let $\mathcal{O}_\lambda \cong U(n, \mathbb{H})/T^n$ be a coadjoint orbit of $U(n, \mathbb{H})$. Let Ψ be the n -th component of the T^n moment map on \mathcal{O}_λ . Let $g_{n,m}$, for $1 \leq m \leq n-1$ be defined by*

$$g_{n,m}([A]) = |a_{nm}|^2,$$

where $A = (a_{ij}) \in U(n, \mathbb{H})$ and $a_{ij} \in \mathbb{H}$. Then the functions $\{g_{n,m}\}_{m=1}^{n-1}$ and Ψ descend to a completely integrable system on the reduced space $\mathcal{O}_\lambda //_\mu U(n-1, \mathbb{H})$ for μ a regular value.

Using these functions, plus an inductive construction, one can show that the original coadjoint orbit \mathcal{O}_λ is also an integrable system.

Theorem 1.2. *Let \mathcal{O}_λ be a (generic) coadjoint orbit of $U(n, \mathbb{H})$. Then there exists a maximal set of Poisson-commuting functions on \mathcal{O}_λ , making it an integrable system.*

We call this the Gel'fand–Cetlin–Molev integrable system. The proofs of these main results are in Section 3. Although the formulæ for the $g_{n,m}$ given above are remarkably simple, it is nevertheless instructive to reveal how these formulæ were derived, via the theory of deformation quantization, from Molev's work. In particular, the technical heart of the derivation of the formulæ used in Theorem 1.1 lies in the remarkable fact that certain algebra automorphisms of the Yangian $Y(2n)$ degenerate, in the classical limit, to be *trivial*. This story is presented in Section 4.

Finally, we note that this manuscript is in no way unique in its use of Yangians and related quantum group constructions to shed light on integrable systems; see, for instance, [6, 7] (and references therein), for applications of

Yangians and Gel'fand–Cetlin–type constructions to questions in quantum integrable systems.

2. History

2.1. The Gel'fand–Cetlin basis for $U(n, \mathbb{C})$ -representations. We now briefly recall the construction of the Gel'fand–Cetlin basis [5]. Let $U(n, \mathbb{C})$ be the standard unitary group, acting on \mathbb{C}^n equipped with the standard hermitian form and standard orthonormal basis denoted by $\{e_1, \dots, e_n\}$.

We consider the chain of subgroups

$$U(1, \mathbb{C}) \subset U(2, \mathbb{C}) \subset \dots \subset U(n-1, \mathbb{C}) \subset U(n, \mathbb{C}), \quad (2.1)$$

where $U(k, \mathbb{C})$ is the subgroup of $U(n, \mathbb{C})$ fixing $\{e_{k+1}, \dots, e_n\}$.

Let $V(\lambda)$ be the irreducible representation of $U(n, \mathbb{C})$ of highest weight $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n) \in (\mathfrak{t}^n)_{\mathbb{Z}}^*$. Considered as a $U(n-1, \mathbb{C})$ -representation, $V(\lambda)$ may not be irreducible. Indeed, it will decompose as

$$V(\lambda) \cong \bigoplus_{\mu} V(\lambda)^{\mu} \quad \text{as } U(n-1, \mathbb{C})\text{-representations,}$$

where W^{μ} denotes the μ -isotypic component. We denote by $\mu = (\mu_1, \dots, \mu_{n-1}) \in (\mathfrak{t}^{n-1})_{\mathbb{Z}}^*$ a dominant weight for $U(n-1, \mathbb{C})$.

The μ -isotypic component of $V(\lambda)$ may, a priori, contain many copies of the irreducible representation $V(\mu)$ of $U(n-1, \mathbb{C})$. In other words, we may also write

$$V(\lambda)^{\mu} \cong M_{\lambda}^{\mu} \otimes V(\mu), \quad \text{as } U(n-1, \mathbb{C})\text{-representations,}$$

where $U(n-1, \mathbb{C})$ acts trivially on M_{λ}^{μ} , and the multiplicity space M_{λ}^{μ} is the subspace of high-weight vectors in the μ -isotypic component, i.e.,

$$M_{\lambda}^{\mu} := (V(\lambda)^{\mu})^+.$$

Thus we have

$$V(\lambda) \cong \bigoplus_{\mu} (M_{\lambda}^{\mu} \otimes V(\mu)) \quad \text{as } U(n-1, \mathbb{C})\text{-representations.}$$

The following two facts are crucial. First, it turns out that $\dim(M_{\lambda}^{\mu}) \neq 0$ if and only if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n. \quad (2.2)$$

Second, for μ that do appear in the decomposition, $\dim(M_{\lambda}^{\mu}) = 1$. In other words, the decomposition is multiplicity-free.

Recall that the maximal torus T^{n-1} of $U(n-1, \mathbb{C})$, the subgroup of diagonal matrices in $U(n-1, \mathbb{C})$, naturally acts on $V(\lambda)$ by restriction. We now define a new T^{n-1} -action on $V(\lambda)$, different from, though related to, the above action. Namely, we define the new torus T^{n-1} to act on each μ -isotypic component as scalar matrices, so that each non-zero vector

behaves as a T^{n-1} -weight vector of weight μ . In other words, for $v \in V(\lambda)^\mu$, we define

$$t \cdot v := \mu(t)v.$$

We call this the Gel'fand–Cetlin T^{n-1} -action on $V(\lambda)^\mu$. We now repeat this process, using the subgroup $U(n-2, \mathbb{C})$ of $U(n-1, \mathbb{C})$ in (2.1). The key observation now is that the new T^{n-2} -action, defined in the same fashion, commutes with the T^{n-1} -action defined previously. This is because the T^{n-1} acts by scalar matrices on each component $V(\mu)$. Since $V(\lambda)$ is a sum of the $V(\mu)$, the two tori commute on $V(\lambda)$. Thus we have now a $T^{n-1} \times T^{n-2}$ -action on $V(\lambda)$. By continuing this process for the whole chain of subgroups, we obtain at each step a T^k -action commuting with the previous T^{k+1} . Hence, when we reach the last subgroup $U(1, \mathbb{C})$, we have obtained a $T^{n-1} \times T^{n-2} \times \dots \times T^2 \times T^1 \cong T^{n(n-1)/2}$ -action on $V(\lambda)$. This decomposition is schematically illustrated in Figure 1.

Now consider the decomposition of $V(\lambda)$ into $T^{n(n-1)/2}$ -weight spaces. Since the decomposition is multiplicity-free at each step, and since the last group $U(1, \mathbb{C}) \cong S^1$ is abelian, each $T^{n(n-1)/2}$ -weight space is 1-dimensional. Hence, this torus completely decomposes the representation $V(\lambda)$ into 1-dimensional subspaces, thus providing (up to a choice of scalar in each weight space) a canonical basis for $V(\lambda)$. This basis is called the *Gel'fand–Cetlin basis* for $V(\lambda)$.

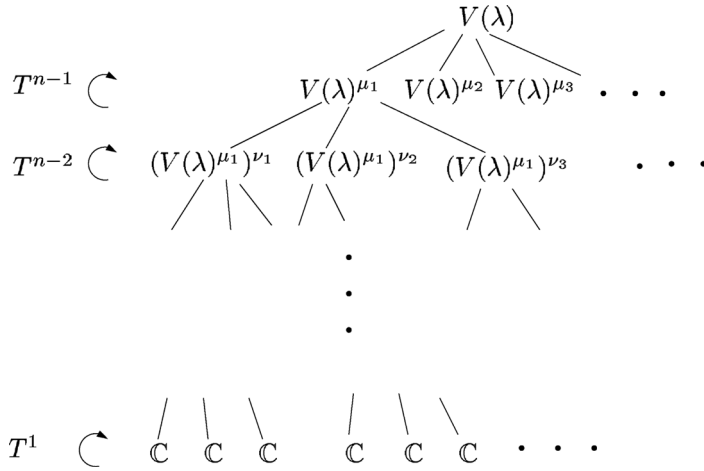


Figure 1. At each step in the successive decomposition, we get a T^{n-k} -action for appropriate k . Since at each step the T^{n-k} act as scalars on each isotypic component, all the T^{n-k} actions commute with each other. Hence we get a $T^{n(n-1)/2}$ -action.

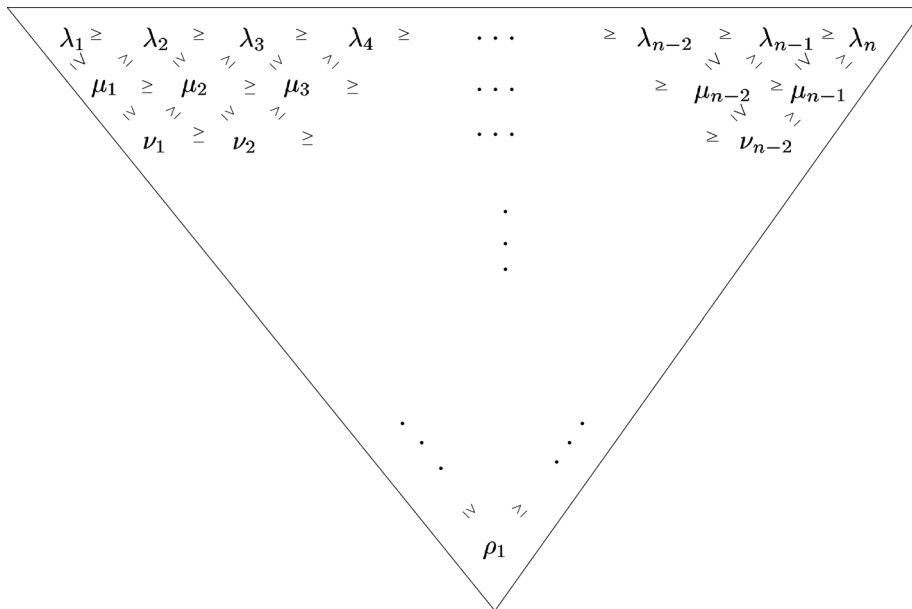


Figure 2. The integer arrays parametrizing the Gel'fand–Cetlin basis for $U(n, \mathbb{C})$ -representations. The top row is fixed, given by the highest weight of the irreducible representation $V(\lambda)$.

It is pleasant to note that this construction also gives a *combinatorial* algorithm for counting the dimension of any finite-dimensional irreducible representation of $V(\lambda)$. As a result of the facts that $\dim(M_\lambda^\mu) \neq 0$ if and only if the inequalities (2.2) are satisfied and that the decomposition is multiplicity-free, the dimension of $V(\lambda)$ is given by the number of integer fillings of the triangle in Figure 2 with specified top row, satisfying the given inequalities.

2.2. The Gel'fand–Cetlin integrable system for $U(n, \mathbb{C})$. We now explain the symplectic-geometric side of the Gel'fand–Cetlin story for $U(n, \mathbb{C})$ [9]. By geometric quantization, the symplectic-geometric object corresponding to an irreducible representation $V(\lambda)$ is the coadjoint orbit \mathcal{O}_λ of $U(n, \mathbb{C})$ through the point $\lambda \in \mathfrak{t}_{\mathbb{Z}}^* \subseteq \mathfrak{t}^* \subseteq \mathfrak{g}^*$ [10]. Here we use the Killing form to identify $\mathfrak{g} \cong \mathfrak{g}^*$ and will think of \mathfrak{t}^* as a subspace of \mathfrak{g}^* . Such a coadjoint orbit \mathcal{O}_λ is a symplectic manifold, generically of (real) dimension $n(n-1)$. Hence, the result analogous to the existence of a Gel'fand–Cetlin basis for $V(\lambda)$ will be the existence of a $\frac{n(n-1)}{2}$ -dimensional torus acting in a Hamiltonian fashion on \mathcal{O}_λ .

We construct this large torus action on \mathcal{O}_λ as follows. As in the representation-theoretic construction above, we will consider the actions of

the subgroups $U(n-k, \mathbb{C})$ in the chain (2.1) and use the tori T^{n-k} in each $U(n-k, \mathbb{C})$. The coadjoint orbit \mathcal{O}_λ is a Hamiltonian $U(n, \mathbb{C})$ -manifold with moment map $\Phi : \mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$ the inclusion. We may restrict to the subgroups $U(n-k, \mathbb{C})$, which also act in a Hamiltonian fashion on \mathcal{O}_λ with moment maps given by composing Φ with the projections

$$\pi_{n-k} : \mathfrak{u}(n, \mathbb{C})^* \rightarrow \mathfrak{u}(n-k, \mathbb{C})^*.$$

Thus, we have a collection of moment maps $\Phi_{n-k} := \pi_{n-k} \circ \Phi$, as shown below:

$$\begin{array}{c} \Phi_{n-2} \\ \curvearrowright \\ \mathcal{O}_\lambda \xrightarrow{\Phi} \mathfrak{u}(n, \mathbb{C})^* \xrightarrow{\Phi_{n-1}} \mathfrak{u}(n-1, \mathbb{C})^* \xrightarrow{\Phi_{n-2}} \mathfrak{u}(n-2, \mathbb{C})^* \longrightarrow \cdots \end{array} \quad (2.3)$$

given by successively projecting onto smaller $\mathfrak{u}(n-k, \mathbb{C})^*$.

The large torus action is obtained by using the *Thimm trick* on each $U(n-k, \mathbb{C})$ moment map. By taking the projections to the positive Weyl chamber for each moment map Φ_k , we get a sequence of maps to smaller and smaller Weyl chambers.

$$\begin{array}{c} \mathcal{O}_\lambda \hookrightarrow \mathfrak{u}(n, \mathbb{C})^* \longrightarrow \mathfrak{u}(n-1, \mathbb{C})^* \longrightarrow \mathfrak{u}(n-2, \mathbb{C})^* \longrightarrow \cdots \\ \downarrow /U(n-1, \mathbb{C}) \qquad \downarrow /U(n-2, \mathbb{C}) \\ (\mathfrak{t}^{n-1})_+^* \qquad (\mathfrak{t}^{n-2})_+^* \qquad \cdots \end{array} \quad (2.4)$$

In linear-algebraic terms, each of these projections is given by diagonalizing a matrix in $\mathfrak{u}(k, \mathbb{C})^* \cong \mathfrak{u}(k, \mathbb{C})$ and reading off the diagonal entries (arranged to be in non-increasing order). The first projection from $\mathfrak{u}(n, \mathbb{C})^* \rightarrow (\mathfrak{t}^n)_+^*$ is omitted since it is trivial when restricted to the fixed coadjoint orbit \mathcal{O}_λ .

By the Thimm trick, these functions to the positive Weyl chambers give rise to a torus action on an open dense set in \mathcal{O}_λ . The action of the Thimm torus on M is, heuristically, given by “moving to the symplectic slice, acting by the (usual) torus, then moving back.” This is analogous to having the tori T^{n-k} act as *scalars* on the whole $V(\mu)$ instead of just on the high-weight vectors. We will call these the *Gel'fand–Cetlin tori* T^{n-k} acting on \mathcal{O}_λ for each k , in analogy with the representation-theoretic situation. These tori also commute with one another since their moment maps are Casimirs (i.e., constant on symplectic leaves). Thus, there is an action of $T^{n-1} \times T^{n-2} \times \cdots \times T^1 \cong T^{n(n-1)/2}$, as advertised, on \mathcal{O}_λ . Since this is the maximal possible dimension of a torus acting Hamiltonianly on \mathcal{O}_λ , this is called the *Gel'fand–Cetlin integrable system*. Note that the T -action on \mathcal{O}_λ coming from the maximal torus $T \subset U(n, \mathbb{C})$ is a subtorus action of the Gel'fand–Cetlin torus action. The analogous statement will *not* be true in the $U(n, \mathbb{H})$ case.

We summarize, in the “Rosetta Stone” below, the correspondences between specific objects arising in the two related constructions. (Here “G–C” stands for “Gel’fand–Cetlin.”)

Symplectic geometry	Representation theory
Coadjoint orbit \mathcal{O}_λ	Irreducible representation $V(\lambda)$
$U(n, \mathbb{C})$ action on \mathcal{O}_λ	$U(n, \mathbb{C})$ action on $V(\lambda)$
$U(n-1, \mathbb{C}) \subset U(n, \mathbb{C})$ action on \mathcal{O}_λ	$U(n-1, \mathbb{C}) \subset U(n, \mathbb{C})$ action on $V(\lambda)$
$\Phi^{-1}(\mathcal{O}_\mu)/\text{Stab}(\mu)$	μ -isotypic component $V(\lambda)^\mu$
Symplectic slice $S = \Phi^{-1}((\mathfrak{t}_+^*)_0)$	High-weight vectors $V(\lambda)^+$
Thimm torus action T^{n-1} on \mathcal{O}_λ	G–C torus action T^{n-1} on $V(\lambda)$
Symplectic reduction $\mathcal{O}_\lambda //_\mu U(n-1, \mathbb{C})$	Multiplicity space $((V(\lambda))^\mu)^+ \cong M_\lambda^\mu$

The last correspondence between the symplectic reduction and the multiplicity space is the content of the “quantization-commutes-with-reduction” theorem in [8]. In particular, in the case of the Gel’fand–Cetlin system for $U(n, \mathbb{C})$, the fact that the multiplicity spaces M_λ^μ are dimension 1 corresponds to the symplectic-geometric fact that the symplectic reductions $\mathcal{O}_\lambda //_\mu U(n-1, \mathbb{H})$ are just *points*.

2.3. The Gel’fand–Cetlin basis for $U(n, \mathbb{H})$ -representations. Molev’s construction in [13] of the analogous Gel’fand–Cetlin basis for finite-dimensional irreducible representations $V(\lambda)$ is phrased in terms of the complex group $Sp(2n, \mathbb{C})$, and in this section we do the same.¹ We will now briefly recount his results, following his convention of using the complex group $Sp(2n, \mathbb{C})$.

As in the construction of the Gel’fand–Cetlin basis for $GL(n, \mathbb{C})$ -representations, we first fix a choice of chain of subgroups

$$Sp(2, \mathbb{C}) \subset Sp(4, \mathbb{C}) \subset \cdots \subset Sp(2(n-1), \mathbb{C}) \subset Sp(2n, \mathbb{C}). \tag{2.5}$$

Let $V(\lambda)$ be a finite-dimensional irreducible representation of $Sp(2n, \mathbb{C})$, of highest weight $\lambda \in \mathfrak{t}_{\mathbb{Z}}^*$. Since the Weyl group of $Sp(2n, \mathbb{C})$ is the group of *signed* permutations, we follow Molev’s conventions in [13] and choose the positive Weyl chamber so that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$, and

$$0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n. \tag{2.6}$$

¹We will abuse notation throughout and use the same notation for objects associated to $U(n, \mathbb{C})$ and the corresponding objects associated to $U(n, \mathbb{H})$. We hope the context will make clear the group under discussion.

We now restrict to the action of the subgroup $Sp(2(n-1), \mathbb{C})$ on $V(\lambda)$. As in the $GL(n, \mathbb{C})$ case, we obtain a decomposition

$$V(\lambda) \cong \bigoplus_{\mu} (M_{\lambda}^{\mu} \otimes V(\mu)) \quad \text{as } Sp(2(n-1), \mathbb{C})\text{-representations,} \quad (2.7)$$

where $V(\mu)$ is a $Sp(2(n-1), \mathbb{C})$ -irreducible representation of highest weight μ .

The main difficulty in the $Sp(2n, \mathbb{C})$ case is that the decomposition in (2.7) is *not* multiplicity-free, i.e., $\dim(M_{\lambda}^{\mu})$ is not necessarily equal to 1. Thus, the Thimm torus $T^{n(n-1)/2}$, constructed exactly as in the case of $U(n, \mathbb{C})$, acting on the decomposition by the chain (2.5) will decompose $V(\lambda)$ into smaller subspaces, but *not* into 1-dimensional pieces. Hence, in order to obtain a complete decomposition, we must find an additional action on the multiplicity spaces M_{λ}^{μ} .

We need not look very far to find an algebra acting on M_{λ}^{μ} . Since the multiplicity spaces M_{λ}^{μ} are the subspaces of high-weight vectors $(V(\lambda)^{\mu})^{+}$, the centralizer $U(\mathfrak{sp}(2n, \mathbb{C}))^{\mathfrak{sp}(2(n-1), \mathbb{C})}$ of $\mathfrak{sp}(2(n-1), \mathbb{C})$ in $U(\mathfrak{sp}(2n, \mathbb{C}))$, acts on each of the highest weight spaces $V(\lambda)_{\mu}^{+}$. In fact, it is known that it acts *irreducibly* [3, Section 9.1]. However, it is difficult to find explicitly the weights of vectors in M_{λ}^{μ} for an appropriate commuting subalgebra of $U(\mathfrak{sp}(2n, \mathbb{C}))^{\mathfrak{sp}(2(n-1), \mathbb{C})}$, thus extending the Thimm torus action.

Molev finds another approach in [13]. There is an algebra map

$$\Psi : Y^{-}(2) \rightarrow U(\mathfrak{sp}(2n, \mathbb{C}))^{\mathfrak{sp}(2(n-1), \mathbb{C})}, \quad (2.8)$$

where $Y^{-}(2)$ is an infinite-dimensional algebra called the *twisted Yangian*. Molev then shows that the induced action of $Y^{-}(2)$ on $(V(\lambda)^{\mu})^{+}$ is still irreducible. This map Ψ , originally used by Ol'shanskii in [16] and simplified by Molev and Ol'shanskii in [15], is the key new ingredient to Molev's construction of a Gel'fand–Cetlin basis for $Sp(2n, \mathbb{C})$. This is because the representations of Yangians and twisted Yangians are well-understood, and a Gel'fand–Cetlin-type basis for representations of Yangians is constructed in [12]. Molev explicitly identifies $(V(\lambda)^{\mu})^{+}$, and therefore M_{λ}^{μ} , with known representations of the Yangian. He then combines the Thimm action on $V(\lambda)$ with the Yangian action on the multiplicity spaces to construct a Gel'fand–Cetlin basis for representations of $Sp(2n, \mathbb{C})$. He finds that, as in the $U(n, \mathbb{C})$ case, the basis vectors are parametrized by patterns of integer arrays as in Figure 3. The fact that the decompositions into irreducible $V(\mu)$ under the action of $Sp(2(n-1), \mathbb{C})$ are *not* necessarily multiplicity-free is reflected by the presence of the additional integer parameters $\lambda' = (\lambda'_1, \dots, \lambda'_n)$ “in between” the $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_{n-1})$, etc.

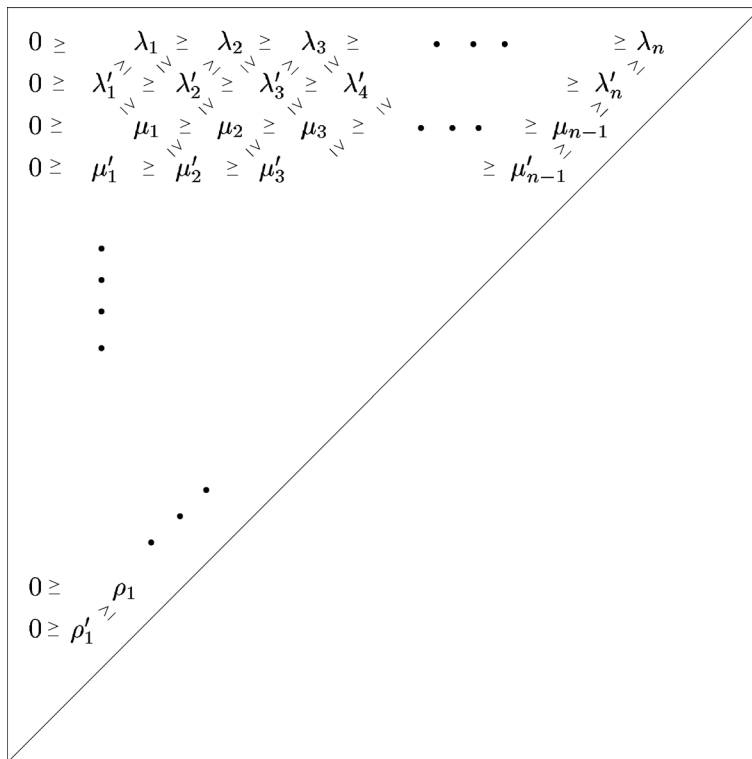


Figure 3. The integer arrays parametrizing the Gel'fand–Cetlin basis for representations of $U(n, \mathbb{H})$ (or $Sp(2n, \mathbb{C})$). The top row $\lambda = (\lambda_1, \dots, \lambda_n)$ is fixed and is given by the highest weight of the irreducible $V(\lambda)$.

3. The Gel'fand–Cetlin–Molev integrable system

In this section, we construct the Gel'fand–Cetlin–Molev integrable system on coadjoint orbits \mathcal{O}_λ of $U(n, \mathbb{H})$, which will be the $U(n, \mathbb{H})$ -analog of the Gel'fand–Cetlin system described in Section 2.2 and is the answer to the Question posed in the Introduction. In the Guillemin–Sternberg construction of the Gel'fand–Cetlin system on $U(n, \mathbb{C})$ coadjoint orbits, the Thimm functions, obtained by projections to smaller and smaller Weyl chambers, provide enough functionally independent Poisson-commuting functions to produce a half-dimensional torus action on the coadjoint orbits. A simple dimension count reveals that, in the case of $U(n, \mathbb{H})$ -coadjoint orbits, this is simply not possible: the rank of the maximal torus in $U(n, \mathbb{H})$ is too small in comparison to the dimension of the group. This problem is the symplectic-geometric manifestation of the fact that the decomposition of

$V(\lambda)$ by $Sp(2(n-1), \mathbb{C})$ is not multiplicity-free. See the bottom line in the “Rosetta Stone” in Section 2.2.

We construct the Gel’fand–Cetlin–Molev system on \mathcal{O}_λ in Section 3.1. We will explain the interpretations in terms of the non-trivial reduced spaces $\mathcal{O}_\lambda //_\mu U(n-1, \mathbb{H})$ in Section 3.2.

3.1. The construction. We refer the reader to Appendix A for reminders on linear algebra over \mathbb{H} and Lie-group-theoretic facts about $U(n, \mathbb{H})$.

The first part of the construction of an integrable system on a coadjoint orbit \mathcal{O}_λ of $U(n, \mathbb{H})$ follows exactly the procedure used to construct the Gel’fand–Cetlin system on orbits of $U(n, \mathbb{C})$. Again, we choose a chain of subgroups

$$U(1, \mathbb{H}) \subset U(2, \mathbb{H}) \subset \cdots \subset U(n-1, \mathbb{H}) \subset U(n, \mathbb{H}), \quad (3.1)$$

where $U(k, \mathbb{H})$ is the subgroup of upper left $k \times k$ matrices in $U(n, \mathbb{H})$. Recall that the Thimm functions for the Gel’fand–Cetlin system are obtained by taking projections at each step to the positive Weyl chamber. The same method works for the $U(n, \mathbb{H})$ case to produce $n(n-1)/2$ Poisson-commuting, independent functions on \mathcal{O}_λ . We have the diagram

$$\begin{array}{ccccccc} \mathcal{O}_\lambda \hookrightarrow & \mathfrak{u}(n, \mathbb{H})^* & \longrightarrow & \mathfrak{u}(n-1, \mathbb{H})^* & \longrightarrow & \mathfrak{u}(n-2, \mathbb{H})^* & \longrightarrow \cdots \\ & & & \downarrow /U(n-1, \mathbb{H}) & & \downarrow /U(n-2, \mathbb{H}) & \\ & & & (\mathfrak{t}^{n-1})_+^* & & (\mathfrak{t}^{n-2})_+^* & \cdots \end{array} \quad (3.2)$$

analogous to (2.4). In linear-algebraic terms, these Thimm functions are obtained by diagonalizing a matrix in $\mathfrak{u}(k, \mathbb{H})^* \cong \mathfrak{u}(k, \mathbb{H})$ by an element of $U(n, \mathbb{H})$ to a diagonal matrix of the form (A.8) and reading off the diagonal entries (ignoring factors of i). Again, the first projection from $\mathfrak{u}(n, \mathbb{H})^* \rightarrow (\mathfrak{t}^n)_+^*$ is omitted since it is trivial when restricted to \mathcal{O}_λ .

The main obstacle in the $U(n, \mathbb{H})$ case is that these $n(n-1)/2$ functions do not suffice to completely integrate a generic coadjoint orbit of $U(n, \mathbb{H})$, since for such an \mathcal{O}_λ we have

$$\dim(\mathcal{O}_\lambda) = \dim(U(n, \mathbb{H})) - \dim(T) = 2n^2.$$

Thus, to completely integrate a generic \mathcal{O}_λ , it is necessary to find an *additional* $n^2 - n(n-1)/2 = n(n+1)/2$ independent, Poisson-commuting functions on \mathcal{O}_λ . This will be our main task in this section.

It will turn out that these new functions are obtained by augmenting the diagram (3.2) with new functions G_{n-k} as follows.

$$\begin{array}{ccccccc}
 & \mathbb{R}^n & & \mathbb{R}^{n-1} & & \mathbb{R}^{n-2} & \cdots & (3.3) \\
 & \uparrow G_n & & \uparrow G_{n-1} & & \uparrow G_{n-2} & & \\
 \mathcal{O}_\lambda \hookrightarrow & \mathfrak{u}(n, \mathbb{H})^* & \longrightarrow & \mathfrak{u}(n-1, \mathbb{H})^* & \longrightarrow & \mathfrak{u}(n-2, \mathbb{H})^* & \longrightarrow & \cdots \\
 & & & \downarrow /U(n-1, \mathbb{H}) & & \downarrow /U(n-2, \mathbb{H}) & & \\
 & & & (\mathfrak{t}^{n-1})_+^* & & (\mathfrak{t}^{n-2})_+^* & & \cdots
 \end{array}$$

Since each G_{n-k} has $n-k$ components, this is exactly the number that we need to completely integrate \mathcal{O}_λ . The motivation behind the definitions of these functions G_{n-k} is the subject of Section 4. In this section, we will simply take the G_{n-k} as defined and concentrate on showing that they (along with the Thimm functions) integrate \mathcal{O}_λ .

We will now define these new functions G_{n-k} . Since they are all defined analogously, for concreteness we define G_n . We denote the components of G_n by $G_n = (g_{n,1}, g_{n,2}, \dots, g_{n,n})$. We begin with the first $n-1$ components $(g_{n,1}, \dots, g_{n,n-1})$. Let $X \in \mathfrak{u}(n, \mathbb{H})^* \cong \mathfrak{u}(n, \mathbb{H})$ be an element in the $U(n, \mathbb{H})$ -orbit of $(\mathfrak{t}_+^*)_0$. Then, there exists a unique diagonal matrix $D_\lambda := (i\lambda_1, \dots, i\lambda_n)$, where

$$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_n,$$

such that X is conjugate by $U(n, \mathbb{H})$ to D_λ . Let

$$X = AD_\lambda A^*$$

for an element $A \in U(n, \mathbb{H})$. Note this equation defines A up to right multiplication by the maximal torus T^n of $U(n, \mathbb{H})$. We take

$$g_{n,m}(X) := |a_{n,m}|^2, \quad 1 \leq m \leq n-1. \quad (3.4)$$

So the $g_{n,m}$ just takes a norm-square of an entry in the bottom row of the matrix A . Since the norm-squares are T^n -invariant, the $g_{n,m}$ are well defined.

Remark 3.1. *From this description, it is clear that we cannot define the n -th component $g_{n,n}$ of G_n in the same way as the first $n-1$ components, because (since A is unitary) the matrix entries always satisfy*

$$\sum_{m=1}^n |a_{n,m}|^2 = 1.$$

Thus the components in G_n would not be functionally independent if $g_{n,n}(X)$ were also defined to be $|a_{n,n}|^2$.

We now define the n -th component of G_n , which has a qualitatively different description. Recall that the maximal torus T^n also acts on the coadjoint orbit \mathcal{O}_λ , with moment map induced by the inclusion $\iota : \mathfrak{t}^n \hookrightarrow \mathfrak{u}(n, \mathbb{H})$:

$$\mathcal{O}_\lambda \hookrightarrow \mathfrak{u}(n, \mathbb{H})^* \xrightarrow{\iota^*} (\mathfrak{t}^n)^* \cong \mathbb{R}^n. \quad (3.5)$$

Here we take the standard identification of \mathfrak{t}^n with its dual to identify $(\mathfrak{t}^n)^*$ with \mathbb{R}^n . We define the n -th component $g_{n,n}$ of G_n to be the n -th component of the moment map (3.5), i.e., for ϵ_n the standard n -th basis vector in $\mathbb{R}^n \cong \mathfrak{t}^n$, we have

$$g_{n,n}(X) := \langle \iota^*(X), \epsilon_n \rangle. \quad (3.6)$$

Note that in the $U(n, \mathbb{H})$ case, the components of the moment map for the action of the maximal torus are functionally independent of the Thimm trick functions, in contrast to the $U(n, \mathbb{C})$ case. Hence, it makes sense to use them as components of the G_{n-k} .

The functions G_{n-k} , as mentioned above, are defined analogously. Before stating the main results, we make a remark on notation. From the sequence of subgroups (3.1), we get a sequence of moment maps

$$\begin{array}{c} \Phi_{n-2} \\ \curvearrowright \\ \mathcal{O}_\lambda \xrightarrow{\Phi} \mathfrak{u}(n, \mathbb{H})^* \xrightarrow{\Phi_{n-1}} \mathfrak{u}(n-1, \mathbb{H})^* \xrightarrow{\Phi_{n-2}} \mathfrak{u}(n-2, \mathbb{H})^* \longrightarrow \dots \end{array} \quad (3.7)$$

By abuse of notation, we will sometimes denote the pullbacks $\Phi_{n-k}^* G_{n-k}$ by G_{n-k} . Similarly, we will sometimes refer to the Thimm functions $\mathfrak{u}(n-k, \mathbb{H})^* \rightarrow (\mathfrak{t}^{n-k})_+^*$ as functions on \mathcal{O}_λ , by pulling back via Φ_{n-k} . With this notation in place, the main theorem of this paper may now be stated.

Theorem 3.2. *Let \mathcal{O}_λ be a generic coadjoint orbit of $U(n, \mathbb{H})$. The functions $\{G_{n-k}\}_{k=0}^{n-1}$ as defined in (3.4) and (3.6), plus the Thimm functions defined in (3.2), pull back via the diagram (3.3) to give a completely integrable system on an open dense set of \mathcal{O}_λ .*

We call this the *Gelfand–Cetlin–Molev system* on the coadjoint orbit \mathcal{O}_λ of $U(n, \mathbb{H})$. To prove Theorem 3.2, there are two things to check: that the functions above Poisson-commute and that they are functionally independent. Thus, Theorem 3.2 follows immediately from the following two propositions.

Proposition 3.3. *Let \mathcal{O}_λ be a generic coadjoint orbit of $U(n, \mathbb{H})$. Let $\{G_{n-k}\}_{k=0}^{n-1}$ be defined as in (3.4) and (3.6) and the Thimm functions defined in (3.2). Then these functions Poisson-commute on an open dense subset of \mathcal{O}_λ .*

Proposition 3.4. *Let \mathcal{O}_λ be a generic coadjoint orbit of $U(n, \mathbb{H})$. Let $\{G_{n-k}\}_{k=0}^{n-1}$ be defined as in (3.4) and (3.6) and the Thimm functions defined as in (3.2). Then these functions are independent on an open dense subset of \mathcal{O}_λ .*

In the course of the proofs of both Propositions, it will turn out to be convenient to replace the functions G_{n-k} with functions $F_{n-k} = (f_{1,n-k}, \dots, f_{n-k,n-k})$ defined as follows. Again, for concreteness, we define F_n , but the others are defined similarly.

$$f_{n,m}(X) := \sum_{\ell=1}^n (-1)^m \lambda_\ell^{2m} |a_{n,\ell}|^2. \quad (3.8)$$

The motivation behind the definitions of F_{n-k} will be explained fully in Section 4. Indeed, it is the F_{n-k} , and not the G_{n-k} , which are obtained directly, in Theorem 4.16, as classical limits of generators of an *abelian* subalgebra in a (non-commutative) algebra $Y^-(2)$.

Going back to (3.8), we first note that the functions $f_{n,m}$ are obtained from the $g_{n,m}$ and from the components of the Thimm function to $(\mathfrak{t}^n)_+^*$. Since the matrix

$$\begin{bmatrix} (-1)\lambda_1^2 & (-1)^2\lambda_1^4 & \cdots & (-1)^n(\lambda_1)^{2n} \\ (-1)\lambda_2^2 & (-1)^2\lambda_2^4 & & (-1)^n(\lambda_2)^{2n} \\ \vdots & & & \vdots \\ (-1)\lambda_n^2 & (-1)^2\lambda_n^4 & \cdots & (-1)^n(\lambda_n)^{2n} \end{bmatrix}.$$

is invertible when the λ_ℓ are distinct, the $g_{n,m}$ may also be obtained from the $f_{n,m}$ plus the Thimm functions. Thus, for the purposes of showing independence and Poisson-commutativity, it is equivalent to use the F_{n-k} .

Remark 3.5. *Remark 3.1 also applies to the F_{n-k} , in that the n functions $\{f_{1,n}, \dots, f_{n,n}\}$ are not independent, but only give $n - 1$ independent functions. In either case, it is necessary to also include the “extra” S^1 moment map as given in equation (3.6).*

It will also be useful to have in hand another description of the functions F_{n-k} . Again, for simplicity, we take the case $k = 0$. Given an element $X = AD_\lambda A^* \in \mathfrak{u}(n, \mathbb{H})^* \cong \mathfrak{u}(n, \mathbb{H})$, it is a straightforward computation to verify that

$$f_{n,m}(X) = \text{rtr}(X^{2m} E_{nn}) = \text{rtr}(AD_\lambda^{2m} A^* E_{nn}) \quad (3.9)$$

is equivalent to the formula given in (3.8). Here, rtr denotes the reduced trace pairing on $\mathfrak{u}(n, \mathbb{H})$ defined in (A.9). We will use this form in the proofs below.

Proof of Proposition 3.3. The symplectic leaves on the dual of a Lie algebra \mathfrak{g}^* are the orbits under the coadjoint action of G , so any G -invariant function

on \mathfrak{g}^* is a Casimir. Since the Thimm function on $\mathfrak{u}(n-k, \mathbb{H})^*$ is by construction $U(n-k, \mathbb{H})$ -invariant, this implies that the components of the Thimm function from $\mathfrak{u}(n-k, \mathbb{H})^*$ to $(\mathfrak{t}^k)_+^*$ Poisson-commutes with any component of G_{n-k} . Similarly, they Poisson-commute with anything “to the right” in the diagram (3.3), i.e., any component of G_{n-p} or of the Thimm functions from $\mathfrak{u}(n-p, \mathbb{H})^*$ for any $k < p \leq n-1$. By a similar argument, since the components of G_{n-k} are $U(n-k-1, \mathbb{H})$ -invariant, they Poisson-commute with any component of the projection to $\mathfrak{u}(n-k-1, \mathbb{H})^*$.

It remains to show that, at each step, the components of G_{n-k} Poisson-commute with each other. For concreteness, we consider the case $k=0$. The other steps may be argued similarly. In fact, as remarked above, it will be more convenient here to use the function F_n as defined in (3.9) rather than the G_{n-k} .

We will first show that $g_{n,n}$, the S^1 moment map, commutes with all $f_{n,m}$, $1 \leq m \leq n$. Note that it suffices to check that they commute on a fixed symplectic leaf \mathcal{O} . For the case $k=0$ which we consider, the only relevant symplectic leaf is the original coadjoint orbit \mathcal{O}_λ . Here and below, we will use for convenience the projection map $\pi : U(n, \mathbb{H}) \rightarrow U(n, \mathbb{H})/T^n \cong \mathcal{O}$ to pull back calculations to $U(n, \mathbb{H})$. Let $\bar{f}_{m,n}$ denote the pullback $\pi^* f_{n,m}$ for $1 \leq m \leq n$. Let Y^\sharp denote the vector field on \mathcal{O} generated by the S^1 -action, and let \bar{Y}^\sharp denote the corresponding vector field on $U(n, \mathbb{H})$ generated by the S^1 (by left multiplication). By construction, $d\pi(\bar{Y}^\sharp) = Y^\sharp$. By definition of the Poisson bracket, it suffices to show that

$$df_{n,m}(Y^\sharp) \equiv \{f_{n,m}, g_{n,n}\} \equiv 0,$$

for any $1 \leq m \leq n$. Lifting to $U(n, \mathbb{H})$, it suffices to show that

$$d\bar{f}_{m,n}(\bar{Y}^\sharp) = 0.$$

Let $A \in U(n, \mathbb{H})$. We trivialize the tangent bundle to $U(n, \mathbb{H})$ by right multiplication. Let W_A^\sharp denote the tangent vector at A corresponding to $W \in \mathfrak{u}(n, \mathbb{H}) \cong T_1(U(n, \mathbb{H}))$. Using the definition of $\bar{f}_{m,n}$, it is straightforward to compute that

$$d\bar{f}_{m,n}(W_A^\sharp) = -\text{rtr}([(AD_\lambda A^*)^{2k}, E_{nn}]W). \quad (3.10)$$

Since the Hamiltonian vector field Y^\sharp associated to the S^1 -action is generated by the element $Y = iE_{nn} \in \mathfrak{u}(n, \mathbb{H})$, we immediately find that

$$d\bar{f}_{m,n}(\bar{Y}^\sharp) = 0.$$

Therefore, by definition, $g_{n,n}$ Poisson-commutes with any $f_{n,m}$, $1 \leq m \leq n$.

Finally, it remains to show that the $f_{n,m}$, for $1 \leq m \leq n$, Poisson-commute among themselves. This fact follows from the construction of the $f_{n,m}$ as classical limits of generators of an abelian subalgebra contained in $Y^-(2)$.

Since the corresponding generators commute in the quantization, the classical limits automatically *Poisson-commute*. \square

Remark 3.6. It is possible to prove directly, using the standard Kostant–Kirillov Poisson structure on $\mathfrak{u}(n, \mathbb{H})^*$, that the $f_{n,m}$ Poisson-commute. However, the calculation is long, and we thus prefer to invoke the deformation theory.

We must now show that the functions in (3.3) are independent. In fact, we will show more: they induce independent functions on the reduced spaces. This interpretation will be further discussed in Section 3.2.

Proof of Proposition 3.4. As in the proof of Proposition 3.3, we will occasionally use the F_{n-k} instead of the G_{n-k} . We first claim that the $\{f_{n,m}\}_{m=1}^{n-1}$ are independent. This is clear, since they are norm-squares of different matrix entries of elements in $U(n, \mathbb{H})$. Similarly, the components $\{f_{n-k,m}\}_{m=1}^{n-k-1}$ are also independent for any $1 \leq k \leq n-1$. Second, we claim the components of G_{n-k} for a fixed k , $1 \leq k \leq n-1$ are also independent of the components of the Thimm function $\mathfrak{u}(n-k, \mathbb{H})^* \rightarrow (\mathfrak{t}^{n-k})_+^*$. This is because on the open dense set $U(n-k, \mathbb{H}) \cdot ((\mathfrak{t}^{n-k})_+^*)_0 \cong U(n-k, \mathbb{H})/T^{n-k} \times ((\mathfrak{t}^{n-k})_+^*)_0 \subset \mathfrak{u}(n-k, \mathbb{H})^*$, the Thimm function simply reads off the second factor, whereas the components of G_{n-k} are functions on the first factor. Third, for a fixed k , $1 \leq k \leq n-1$, the last component $g_{n-k, n-k}$ is also independent of the Thimm functions. This is because $g_{n-k, n-k}$ generates, by definition, a non-trivial action on each generic symplectic leaf in $\mathfrak{u}(n-k, \mathbb{H})^*$, and in particular is non-constant on those leaves, whereas the Thimm functions are constant on leaves. Fourth, the components of the Thimm function $\mathfrak{u}(n-k, \mathbb{H})^* \rightarrow (\mathfrak{t}^{n-k})_+^*$ are independent of any component of the projection $\iota^* : \mathfrak{u}(n-k, \mathbb{H})^* \rightarrow \mathfrak{u}(n-k-1, \mathbb{H})^*$. To see this, note that the restriction of ι^* to a coadjoint orbit \mathcal{O} in $\mathfrak{u}(n-k, \mathbb{H})^*$ gives the moment map $\mu_{\mathcal{O}}$ for the $U(n-k-1, \mathbb{H})$ -action on \mathcal{O} . We assume \mathcal{O} is an orbit through a point ν in the interior of $(\mathfrak{t}^{n-k})_+^*$. For any $X \in \mathfrak{u}(n-k-1, \mathbb{H})$, there exists an element $Z \in \mathfrak{u}(n-k, \mathbb{H})$ such that

$$d\mu_{\mathcal{O}}^X(Z^\sharp) = \omega_\nu(X^\sharp, Z^\sharp) = \langle \nu, [X, Z] \rangle \neq 0.$$

This implies that $\mu_{\mathcal{O}}^X$ is non-constant on \mathcal{O} for all $X \in \mathfrak{u}(n-k, \mathbb{H})$. In particular, any component is functionally independent of the Thimm functions (which are constant on \mathcal{O}).

It remains now to show (and this is the bulk of the proof) two things: that the last component $g_{n-k, n-k}$ of G_{n-k} is independent of the first $n-k-1$ components and that all components of G_{n-k} are independent of any component of the projection $\mathfrak{u}(n-k, \mathbb{H})^* \rightarrow \mathfrak{u}(n-k-1, \mathbb{H})^*$. Without loss of generality, we consider the case $k=0$.

In order to show that the last component $g_{n,n}$ is independent of the first $n-1$ components of G_n , it suffices to show that there exists an element

$Z \in \mathfrak{u}(n, \mathbb{H})^*$ such that on \mathcal{O}_λ we have

$$df_{n,m}(Z^\sharp) \equiv 0$$

for all $1 \leq m \leq n-1$, but

$$dg_{n,n}(Z^\sharp) \neq 0.$$

Consider the subgroup $U(1, \mathbb{H})$ sitting in $U(n, \mathbb{H})$ as the “bottom right” (n, n) -th entry. For any $v \in \text{Im}(\mathbb{H})$, where $a, b, c \in \mathbb{R}$ and $\|v\| = 1$, there is a corresponding S^1 subgroup $\{e^{i\theta} | \theta \in \mathbb{R}\} \subset U(1, \mathbb{H})$. Let Z_v^\sharp denote the vector field generated on \mathcal{O}_λ by this copy of S^1 . Then at a point $\xi \in \mathcal{O}_\lambda$,

$$\begin{aligned} d(g_{n,n})_\xi(Z_v^\sharp) &= \omega_\xi(Y^\sharp, Z_v^\sharp) \\ &= \langle \xi, [Y, Z] \rangle. \end{aligned}$$

Here $Y = iE_{n,n}$ is the element in $\mathfrak{u}(n, \mathbb{H})$ generating the S^1 -action corresponding to $g_{n,n}$, and Y^\sharp is the corresponding vector field on \mathcal{O}_λ . By using the action of $U(n-1, \mathbb{H})$, we may take the point $\xi \in \mathfrak{u}(n, \mathbb{H})^*$ to be of the form

$$\xi = \begin{bmatrix} D_\mu & \star \\ \star & z \end{bmatrix}, \quad (3.11)$$

for $z \in \text{Im}(\mathbb{H})$, and a diagonal matrix D_μ of the form (A.8). Since $\xi \in \mathcal{O}_\lambda$, the (n, n) -th entry z cannot be equal to zero. In particular, there exists $v \in \text{Im}(\mathbb{H})$, $\|v\| = 1$, for which $Z = vE_{n,n} \in \mathfrak{u}(n, \mathbb{H})$ has the property that

$$\begin{aligned} d(g_{n,n})_\xi(Z_v^\sharp) &= \langle \xi, [Y, Z] \rangle \\ &= \text{rtr}(\xi[Y, Z]) \\ &= \text{Re}(z(i \cdot v - v \cdot i)) \in \mathbb{H} \\ &\neq 0. \end{aligned}$$

On the other hand, from equation (3.10), we conclude that for $v \in \text{Im}(\mathbb{H})$, we have that

$$df_{n,m}(Z_v^\sharp) = 0$$

for $1 \leq m \leq n-1$. Thus $g_{n,n}$ is independent from the $\{f_{n,m}\}_{m=1}^{n-1}$. In fact, this argument additionally shows that $g_{n,n}$ is independent of any component of the moment map $\Phi_{n-1} : \mathcal{O}_\lambda \rightarrow \mathfrak{u}(n-1, \mathbb{H})^*$, since by construction, $[W, Z_v] = 0$ for any $W \in \mathfrak{u}(n-1, \mathbb{H})$.

Our last task is to show that the components of F_n are independent from the components of the moment map $\Phi_{n-1} : \mathcal{O}_\lambda \rightarrow \mathfrak{u}(n-1, \mathbb{H})^*$. As in the proof of Proposition 3.3, we will do calculations on $U(n, \mathbb{H})$ instead of $U(n, \mathbb{H})/T$. Define $\bar{\Phi}_{n-1} := \pi^*(\Phi_{n-1})$ and $\bar{F}_n := \pi^*(F_n)$. It will suffice to show that there exists some $A \in U(n, \mathbb{H})$ such that the linear equations defining the kernels of both $d\bar{\Phi}_{n-1}$ and $d\bar{F}_n$ are linearly independent at A .

Let $A \in U(n, \mathbb{H})$. We trivialize $TU(n, \mathbb{H})$ by right translation. Let X_A^\sharp denote the right translate of X to $T_A U(n, \mathbb{H})$. Let j denote the map

$\mathfrak{u}(n, \mathbb{H}) \rightarrow \mathfrak{u}(n-1, \mathbb{H})$ given by taking the upper left $(n-1) \times (n-1)$ submatrix. Then the pullback of the moment map $\bar{\Phi}_{n-1}$ can be expressed as

$$\bar{\Phi}_{n-1} : A \mapsto j(AD_\lambda A^*)$$

and the derivative by

$$(d\bar{\Phi}_{n-1})_A(X_A^\sharp) = j([X, AD_\lambda A^*]). \quad (3.12)$$

Since j is the map which takes the upper left submatrix, it is convenient to write an element $X \in \mathfrak{u}(n, \mathbb{H})$ as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ -X_{12}^* & X_{22} \end{bmatrix}, \quad (3.13)$$

where $X_{11} \in \mathfrak{u}(n-1, \mathbb{H})$, $X_{12} \in \mathbb{H}^{n-1}$, $X_{22} \in \text{Im}(\mathbb{H})$. Moreover, as in (3.11), we may assume that A is such that

$$AD_\lambda A^* = \begin{bmatrix} D_\mu & W \\ -W^* & z \end{bmatrix},$$

for $W \in \mathbb{H}^{n-1}$ and z non-zero. Then from (3.12) we see that the linear equations (over \mathbb{R}) defining $\ker(d\bar{\Phi}_{n-1}) \in T_A(U(n, \mathbb{H})) \cong \mathfrak{u}(n, \mathbb{H})$ are given by the single matrix equation

$$X_{11}D_\mu - X_{12}W^* = D_\mu X_{11} + WX_{21}.$$

In other words, the matrix $X_{11}D_\mu - X_{12}W^*$ must be \mathbb{H} -hermitian.

Now we compute the linear equations for $\ker(\bar{f}_{m,n})$. The pullbacks $\bar{f}_{m,n}$ are given by

$$\bar{f}_{m,n} : A \mapsto -\text{rtr}(AD_\lambda^{2m} A^* E_{nn}). \quad (3.14)$$

Let X be written as in (3.13), and write $X_{12} = ((X_{12})_1, \dots, (X_{12})_{n-1})^t \in \mathbb{H}^{n-1}$, where $(X_{12})_i \in \mathbb{H}$. Then

$$\begin{aligned} d\bar{f}_{m,n}(X_A^\sharp) &= -\text{rtr}([AD_\lambda^{2m} A^*, E_{nn}]X) \\ &= -2 \cdot \text{Re} \left(\sum_{i=1}^{n-1} (-1)^m \left(\sum_{\ell=1}^n \lambda_\ell^{2m} a_{i\ell} \bar{a}_{n\ell} \right) (-\overline{(X_{12})_i}) \right) \\ &\quad + 2 \cdot \text{Re} \left(\sum_{i=1}^{n-1} (-1)^m \left(\sum_{\ell=1}^n \lambda_\ell^{2k} a_{n\ell} \bar{a}_{i\ell} \right) (X_{12})_i \right), \end{aligned}$$

so $\ker d\bar{f}_{m,n}$ is given by the condition that the expression above is 0. Denote the (real) variables in X_{11} by $\{z_a\}$, and the (real) variables in X_{12} by $\{w_b\}$. Note that the linear equations defining the $\ker(\bar{f}_{m,n})$, as seen above, involve only the w_b . The linear equations defining $\ker(d\bar{\Phi}_{n-1})$ involve both z_a and w_b , but they are also linearly independent modulo $\langle w_b \rangle$. This implies that the $d\bar{f}_{m,n}$ are also linearly independent on $\ker(d\bar{\Phi}_{n-1})$. Thus the $\{f_{n,m}\}$ are independent of any component of Φ_{n-1} , as desired. \square

3.2. Interpretation on the reduced spaces. We now take a moment to interpret the results of the previous section in terms of the reduced spaces $\mathcal{O}_\lambda //_\mu U(n-1, \mathbb{H})$ and comment on the differences between the cases of $U(n, \mathbb{C})$ and $U(n, \mathbb{H})$.

We already mentioned in the beginning of Section 3.1 that the essential new problem in the $U(n, \mathbb{H})$ case is that the Thimm functions do not give “enough” functions to completely integrate a generic coadjoint orbit of $U(n, \mathbb{H})$. We now briefly review the Thimm trick construction of torus actions and explicitly see that this deficiency is due to the presence of non-trivial symplectic reductions.

Suppose a compact Lie group G acts on a symplectic manifold M with moment map μ . Let $T \subset G$ be a maximal torus and identify $\mathfrak{g} \cong \mathfrak{g}^*$ so that $\mathfrak{t}^* \subset \mathfrak{g}^*$. We assume $\mu(M) \cap (\mathfrak{t}_+^*)_0 \neq \emptyset$. It is shown in [11] that the preimage $S := \mu^{-1}((\mathfrak{t}_+^*)_0)$ is a symplectic submanifold of M and the restriction of μ to S is a moment map for the T -action on S . This submanifold S is called a *symplectic slice*. Note that for a regular value $\alpha \in (\mathfrak{t}_+^*)_0$, we have $M //_\alpha G = S //_\alpha T$. The Thimm torus is then defined to act on $G \cdot S$ as follows: for any $g \cdot p \in G \cdot S$ and $t \in T$, define

$$t \cdot (g \cdot p) := g \cdot (t \cdot p),$$

where by $t \cdot p$ we mean the *original* T -action on M , restricted to the slice S .

Suppose that the symplectic manifold is a coadjoint orbit of $U(n, \mathbb{C})$ or $U(n, \mathbb{H})$ and G is the first subgroup in the chain (2.1) or (3.1), respectively. Let $p \in S$. (We consider points in the slice without loss of generality; for any other point in $G \cdot S$, we could repeat the argument with a conjugate torus.) The presence of a completely integrable system translates to the presence of a half-dimensional Lagrangian subspace $L_p \subset T_p M$ spanned by the Hamiltonian vector fields of the Poisson-commuting functions. By the above description of the Thimm torus action, we see that the Hamiltonian vector fields arising from the Thimm functions are exactly the X_p^\sharp for $X \in \mathfrak{t} \subset \mathfrak{g}$, so the span is exactly $T_p(T \cdot p) \subset T_p S$, giving an isotropic subspace of $T_p S$. There are two reasons why $T_p(T \cdot p)$ may not be a Lagrangian subspace of $T_p M$. First, perhaps most obviously, $T_p S$ is not all of $T_p M$. There is a complementary symplectic subspace of $T_p S$ in $T_p M$, mapping isomorphically under $d\mu$ to the tangent space to the coadjoint orbit $\mathcal{O}_{\mu(p)}$, which is not accounted for by $T_p(T \cdot p)$. Second, $T_p(T \cdot p)$ may not be Lagrangian even in $T_p S$. There is a symplectic subspace in $T_p S$ mapping isomorphically to the tangent space of the reduced space $T_{[p]}(M//T)$ which is also not accounted for by the $T_p(T \cdot p)$.

The first reason mentioned above, the presence of a subspace isomorphic to $T_{\mu(p)}\mathcal{O}_{\mu(p)}$, is partially resolved by the inductive step. Namely, $\mathcal{O}_{\mu(p)}$ is itself a symplectic manifold with respect to H , where H is now either $U(n-2, \mathbb{C})$ or $U(n-2, \mathbb{H})$. By considering the Thimm functions arising from

the action of H , the subspace for $T\mathcal{O}_{\mu(p)}$ will in turn break up into pieces, part of which will be spanned by the Hamiltonian vector fields arising from the Thimm functions from H . Hence, it is the second reason mentioned above which is the essential obstacle to having the Thimm functions completely integrate the original manifold $M = \mathcal{O}_\lambda$. At each inductive step, if the subspace corresponding to the reduced space is non-trivial and thus has positive-dimension, then it is impossible for the Thimm functions to integrate \mathcal{O}_λ .

We may now compare the cases of $U(n, \mathbb{C})$ and $U(n, \mathbb{H})$ in this light. The Gel'fand–Cetlin construction given in Section 2.2 works exactly because the symplectic reductions $\mathcal{O} // U(n-1, \mathbb{C})$ are *trivial*. The analogous construction for the $U(n, \mathbb{H})$ case does not work, and we need more functions, precisely because the symplectic reductions $\mathcal{O} // U(n-1, \mathbb{H})$ are *not* trivial. Indeed, generically they are dimension $2n$. This is in exact correspondence with the positive-dimensionality of the multiplicity spaces M_λ^μ in Section 2.3. We invite the reader to take another look at the Rosetta Stone in Section 2.1 with these interpretations in mind.

As advertised in the previous section, the functions G_n may be viewed as an integrable system on the reduced spaces. We record the following, which we already stated in the Introduction as Theorem 1.1.

Theorem 3.7. *Let $\mathcal{O}_\lambda \cong U(n, \mathbb{H})/T^n$ be a coadjoint orbit of $U(n, \mathbb{H})$. Let G_n be defined as in equations (3.4) and (3.6). Then the components of G_n descend to functionally independent, Poisson-commuting functions on the reduced space $\mathcal{O}_\lambda //_\mu U(n-1, \mathbb{H})$ for μ a regular value.*

Proof. Since the G_n defined by (3.4) and (3.6) are $U(n-1, \mathbb{H})$ -invariant, and because they are shown in the proof of Proposition 3.4 to be functionally independent of any component of Φ_{n-1} , they automatically induce functionally independent functions on the reduced spaces $\mathcal{O} // U(n-1, \mathbb{H})$. Moreover, by the definition of the symplectic structure on the reduced space, they also automatically Poisson-commute on the reduced space. Thus we have n independent Poisson-commuting functions, and therefore a completely integrable system, on the reduced space. \square

4. The classical limits

In this section, we explain our deformation-theoretic derivation of the formulæ for the functions $f_{n,m}$ used in the construction of the Gel'fand–Cetlin–Molev integrable system. Some comments are in order: first of all, as seen in the previous section, once the formulæ are given, it is possible to prove directly that the functions $f_{n,m}$ give a completely integrable system on $\mathcal{O}_\lambda //_\mu U(n-1, \mathbb{H})$. However, the way in which these formulæ were derived, via the use of the theory of deformation quantization and classical limits, is a beautiful story in its own right. Hence we present it in this section using

this perspective. Since the construction is the same at each step in (3.3), we concentrate here solely on the first step, i.e., the derivation of the $\{f_{n,m}\}_{m=1}^n$.

We now recall briefly the basic general philosophy underlying the computations below. We refer the reader to [1] for details. The central theme is that the classical limit of a non-commutative algebra A_h is a commutative algebra A_0 equipped with a Poisson bracket. The Poisson bracket is the “first-order term” in the parameter h , and the commutative algebra A_0 can then be viewed as a space of functions $\text{Fun}(M)$ on a Poisson space M . Following this general recipe, our task in this section is as follows. We will first determine in Section 4.1 the classical limits of the non-commutative algebras involved in the Molev construction as recounted in Section 2.3. In Section 4.2, we take the classical limit of the algebra map Ψ used in Section 2.3 and get a map between Poisson spaces. The technical heart of the calculation lies in Theorem 4.7, which allows us to obtain in Theorem 4.16 the explicit formulæ for the $f_{n,m}$.

Some basic necessary definitions and constructions regarding Yangians, twisted Yangians, and their deformations are recounted briefly in Appendix B.

4.1. The classical limits of the algebras. In order to describe the classical limit of the Yangian, following the standard methods in deformation quantization, we must first exhibit the Yangian as a deformation. There is a family of topological Hopf algebras $Y_h(2n)$ for $h \in \mathbb{C} \setminus \{0\}$, where $Y_1(2n) = Y(2n)$, and in fact $Y_h(2n) \cong Y(2n)$ for $h \neq 0$. We then set $h = 0$ in the formulæ for the algebra and coalgebra structures for $Y_h(2n)$ to obtain a classical limit.

We define $Y_h(2n)$ as follows. We denote the generators of a fixed $h \in \mathbb{C} \setminus \{0\}$ by $\tilde{t}_{ij}^{(M)}$, for i, j in \mathcal{I} as in (A.2) and $M \geq 0$. We define the algebra structure by

$$[\tilde{t}_{ij}^{(M)}, \tilde{t}_{kl}^{(N)}] = h \cdot \left(\sum_{r=0}^{\min(M,L)-1} \left(\tilde{t}_{kj}^{(r)} \tilde{t}_{il}^{(M+L-1-r)} - \tilde{t}_{kj}^{(M+L-1-r)} \tilde{t}_{il}^{(r)} \right) \right). \quad (4.1)$$

Note that this is the same formula as for $Y(2n)$ except that we multiply by a factor of h . The coproduct, antipode, and coidentity structures are defined by the same formulæ as for $Y(2n)$ with *no* additional factor of h . One can check that these definitions give $Y_h(2n)$ the structure of a Hopf algebra. We have, by definition, $Y_1(2n) = Y(2n)$. In fact, for $h \neq 0$, $Y_h(2n)$ is isomorphic to $Y(2n)$ as a Hopf algebra.

Lemma 4.1. *Let $Y_h(2n)$ be defined as above. Then for $h \neq 0$,*

$$Y_h(2n) \cong Y(2n),$$

as Hopf algebras.

Proof. The map $\gamma_h : Y_h(2n) \rightarrow Y(2n)$ is given by $\tilde{t}_{ij}^{(M)} \mapsto t_{ij}^{(M)} h^M$, extended linearly. \square

In order to describe the classical limit of $Y(2n)$, we need first some terminology. Let U denote a formal neighborhood of ∞ in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. If u is the usual coordinate on \mathbb{C} , then u^{-1} is a coordinate on a neighborhood of ∞ . By a “formal” neighborhood, we mean that the space of functions on U is the space of formal power series in the local coordinate u^{-1} . Let $U \times \mathbb{C}^{2n}$ be a trivial vector bundle over the formal neighborhood of ∞ . A gauge transformation of this vector bundle is given by an element $F(u) \in \text{Maps}(U, GL(2n, \mathbb{C}))$. Here, $F(u)$ is a *formal* power series with coefficients in $\mathfrak{gl}(2n, \mathbb{C})$, with the additional restriction that it is invertible as a formal power series. Multiplication in the gauge group is given pointwise. The *pointed* gauge group $\text{Maps}_1(U, GL(2n, \mathbb{C}))$ is the subgroup such that the point $\infty \in U$ maps to the identity element in $GL(2n, \mathbb{C})$. Thus an element $F(u) \in \text{Maps}_1(U, GL(2n, \mathbb{C}))$ is of the form

$$F(u) = \mathbb{1} + A_1 u^{-1} + A_2 u^{-2} + \dots,$$

where $A_i \in \mathfrak{gl}(2n, \mathbb{C})$. Note that since the 0-th coefficient is the identity matrix, such formal power series are invertible for any choice of $A_i \in \mathfrak{gl}(2n, \mathbb{C})$.

Theorem 4.2. *The classical limit of the Yangian $Y(2n)$ is the (infinite-dimensional) pointed gauge group $\mathcal{G}_{2n} := \text{Maps}_1(U, GL(2n, \mathbb{C}))$ of a trivial \mathbb{C}^{2n} -bundle over a formal neighborhood U of $\infty \in \mathbb{P}^1$. Moreover, \mathcal{G}_{2n} has a Poisson structure compatible with the product structure in the gauge group, making it a Poisson–Lie group.*

Proof. We denote by $z_{ij}^{(M)}$ the coordinate function on $\text{Maps}_1(U, GL(2n, \mathbb{C}))$ which reads off the (i, j) -th entry of the coefficient of u^{-M} of an element $A(u) = \sum_{M=0}^{\infty} A_M u^{-M} \in \text{Maps}_1(U, GL(2n, \mathbb{C}))$. Note that $A_0 = \mathbb{1}$, so $z_{ij}^{(0)} \equiv \delta_{ij}$. It is straightforward to check that the Hopf algebra structure on the space of functions on $\text{Maps}_1(U, GL(2n, \mathbb{C}))$ (coming from the group structure on the gauge group), generated by the $z_{ij}^{(M)}$, is the same as that on $Y_0(2n)$. (The identification sends the element $z_{ij}^{(M)}$ to the generator $t_{ij}^{(M)}$ of $Y_0(2n)$.)

The Poisson structure on \mathcal{G}_{2n} is given by the first-order term in h in the deformation of the algebra structure in $Y_h(2n)$, so a glance at (4.1) yields the Poisson structure

$$\{z_{ij}^{(M)}, z_{kl}^{(N)}\} = \sum_{r=0}^{\min(M, L)-1} \left(z_{kj}^{(r)} z_{il}^{(M+L-1-r)} - z_{kj}^{(M+L-1-r)} z_{il}^{(r)} \right). \quad (4.2)$$

It remains to check that the product and Poisson structures on \mathcal{G}_{2n} are compatible, i.e., the multiplication map $\mathcal{G}_{2n} \times \mathcal{G}_{2n} \rightarrow \mathcal{G}_{2n}$ is Poisson, where

$\mathcal{G}_{2n} \times \mathcal{G}_{2n}$ has the product Poisson structure. This translates to the condition that for f_1, f_2 functions on \mathcal{G}_{2n} and $g, g' \in \mathcal{G}_{2n}$,

$$\{f_1, f_2\}(gg') = \{L_g^* f_1, L_g^* f_2\}(g') + \{R_{g'}^* f_1, R_{g'}^* f_2\}(g). \quad (4.3)$$

This follows from the compatibility of the coalgebra and algebra structures on $Y_h(2n)$ (B.7). In the case of $Y_h(2n)$, we have defined $\Delta_1 \equiv 0$, so we get the simplified compatibility equation

$$\Delta(\mu_1(a_1 \otimes a_2)) = (\mu \otimes \mu_1 + \mu_1 \otimes \mu)(\Delta^{13}(a_1)\Delta^{24}(a_2)). \quad (4.4)$$

Here, μ indicates the commutative (pointwise) multiplication and μ_1 , being the first-order term of the deformation of μ , is the Poisson bracket. Finally, $\Delta^{13}\Delta^{24}$ corresponds to the pullback induced by the multiplication map

$$(m_{13}, m_{24}) : G \times G \times G \times G \rightarrow G \times G \\ (g_1, g_2, g_3, g_4) \mapsto (g_1 g_3, g_2 g_4).$$

From this, it follows that the compatibility (4.3) is a consequence of (4.4). \square

We now describe the classical-limit of the twisted Yangian, which turns out to be a Poisson homogeneous space associated to the Poisson–Lie group \mathcal{G}_{2n} . We first define an involution σ on \mathcal{G}_{2n} as follows. Let $A(u) \in \mathcal{G}_{2n}$. Then we define

$$\sigma : A(u) \mapsto Q^{-1}(A(-u)^t)^{-1}Q,$$

where Q is the matrix defining the standard symplectic form (A.3). Note that this is just a pointwise version of the standard involution on $GL(2n, \mathbb{C})$ whose fixed point set is $Sp(2n, \mathbb{C})$. We then define \mathcal{H}_{2n} to be the fixed point set \mathcal{G}_{2n}^σ under this involution.

Theorem 4.3. *The classical limit of the twisted Yangian is $\text{Fun}(\mathcal{G}_{2n}/\mathcal{H}_{2n})$.*

Proof. We will show directly that the degeneration of the twisted Yangian, which we denote by \mathcal{A} , consists exactly of functions f on \mathcal{G}_{2n} with the property that for $g, g' \in \mathcal{G}_{2n}$,

$$g' = g \cdot h, \quad h \in \mathcal{H}_{2n} \Rightarrow f(g) = f(g'). \quad (4.5)$$

The generators of the twisted Yangian are given by the $s_{ij}^{(M)}$ in (B.5). This is written collectively in matrix form as $S(u) = T(u)T(-u)^\tau$ as in (B.4). Here τ is the symplectic transpose defined in (A.4). Since the classical limit of $Y^-(2n)$ is generated by these $s_{ij}^{(M)}$, it suffices to check the relation (4.5) for these generators.

Let $A(u) \in \mathcal{G}_{2n}$. We have

$$S(A(u)) := \left(\sum_M s_{ij}^{(M)}(A(u)) \cdot u^{-M} \right).$$

It is straightforward to see that

$$S(A(u)) = A(u)A(-u)^\tau.$$

We first show that the functions $S(u)$ are invariant under the action of \mathcal{H}_{2n} . Let $B(u), C(u) \in \mathcal{G}_{2n}$, where $B(u) = C(u)A(u)$ for an element $A(u) \in \mathcal{H}_{2n}$. We want to show that

$$S(B(u)) = S(C(u)).$$

By the above, this is equivalent to showing that

$$B(u)B(-u)^\tau = C(u)C(-u)^\tau.$$

Since $B(u) = C(u)A(u)$ and the symplectic transpose τ is an antihomomorphism, this is equivalent to showing that, for $A(u) \in \mathcal{H}_{2n}$,

$$A(u)A(-u)^\tau = \mathbb{1}.$$

By definition of \mathcal{H}_{2n} , we have

$$A(-u)^t = QA(u)^{-1}Q^{-1}.$$

Therefore

$$\begin{aligned} A(u)A(-u)^\tau &= A(u)Q^{-1}A(-u)^tQ \\ &= A(u)Q^{-1}(QA(u)^{-1}Q^{-1})Q \\ &= A(u)A(u)^{-1} \\ &= \mathbb{1}, \end{aligned}$$

and we are done. This argument is reversible, i.e., if $B(u) = C(u)A(u)$, and $S(B(u)) = S(C(u))$, then $A(u) \in \mathcal{H}_{2n}$. Therefore, the functions $S(u) = (s_{ij}(u))$ are precisely the functions on $\mathcal{G}_{2n}/\mathcal{H}_{2n}$. \square

Remark 4.4. Heuristically, the fact that $Y^-(2n)$ has classical limit, a homogeneous space of \mathcal{G}_{2n} may be motivated as follows. The important observation is that $Y^-(2n)$, being a Hopf coideal of $Y(2n)$, has a classical limit which is a Hopf coideal of $Y_0(2n)$. Suppose now that G is any Poisson–Lie group and H is a subgroup. The multiplication map $m : G \times G \rightarrow G$ induces a map

$$\bar{m} : G \times \frac{G}{H} \rightarrow \frac{G}{H}$$

since G/H is a quotient by H on the right. This map then dualizes to

$$\text{Fun}\left(\frac{G}{H}\right) \rightarrow \text{Fun}(G) \otimes \text{Fun}\left(\frac{G}{H}\right),$$

which is simply the coproduct $\Delta : \text{Fun}(G) \rightarrow \text{Fun}(G) \otimes \text{Fun}(G)$ restricted to the subalgebra $\text{Fun}(G/H)$. Hence, $\Delta(\text{Fun}(G/H)) \in \text{Fun}(G) \otimes \text{Fun}(G/H)$, and $\text{Fun}(G/H)$ is a Hopf coideal in $\text{Fun}(G)$.

Just as there is a geometric interpretation of the classical limit of the Yangian, there is also a geometric interpretation of the classical limit of the Yangian, namely as a space of sections on which \mathcal{G}_{2n} acts transitively with stabilizer \mathcal{H}_{2n} . Let $E = U \times \mathbb{C}^{2n}$ denote the total space of the trivial \mathbb{C}^{2n} -bundle over the formal neighborhood U . Let $\alpha : U \rightarrow U$ be the involution on U given by

$$\alpha : u \mapsto -u.$$

Let α^*E denote the pullback by α of the bundle E , and let $(E \otimes \alpha^*E)^*$ be the bundle over U whose fiber over a point u is the space of \mathbb{C} -bilinear pairings of E_u with $(\alpha^*E)_u = E_{-u}$. We will show below that the homogeneous space $\mathcal{G}_{2n}/\mathcal{H}_{2n}$ can be identified with the subset of the space of sections $\Phi(u)$ of $(E \otimes \alpha^*E)^*$ satisfying the ‘‘skew’’ condition

$$\Phi(u)^t = -\Phi(-u) \tag{4.6}$$

and the condition $\Phi(\infty) = Q$. Here, we have used the triviality of E to identify all fibers with a single \mathbb{C}^{2n} and express $\Phi(u) \in (E \otimes \alpha^*E)^*$ as a complex $2n \times 2n$ matrix.

Theorem 4.5. *The classical limit of the twisted Yangian can be identified with sections of a vector bundle over the formal neighborhood U satisfying (4.6) and $\Phi(\infty) = Q$. Thus*

$$\begin{aligned} \frac{\mathcal{G}_{2n}}{\mathcal{H}_{2n}} &\cong \Gamma((E \otimes \alpha^*E)^*)^{skew} := \{\Phi : \Phi(u)^t = -\Phi(-u), \Phi(\infty) = Q\} \\ &\subseteq \Gamma((E \otimes \alpha^*E)^*). \end{aligned}$$

Proof. We need to show that \mathcal{G}_{2n} acts transitively on $\Gamma((E \otimes \alpha^*E)^*)^{skew}$, where some point has stabilizer \mathcal{H}_{2n} . Let $\Phi \in \Gamma((E \otimes \alpha^*E)^*)$ be a bilinear pairing as above. The action of $B \in \mathcal{G}_{2n}$ is given by

$$(B \cdot \Phi)(u) = (B(u)^{-1})^t \Phi(u) B(-u)^{-1}.$$

We first show that there is an element of \mathcal{G}_{2n} fixed by \mathcal{H}_{2n} . Consider the canonical section Ω of $(E \otimes \alpha^*E)^*$ given by the standard symplectic pairing Q in (A.3), so $\Omega(u) \equiv Q$. Now suppose $A(u) \in \mathcal{G}_{2n}$ and $A \cdot \Omega(u) = \Omega(u)$. This holds if and only if

$$(A(u)^{-1})^t \Omega(u) A(-u)^{-1} = \Omega(u)$$

as a formal power series in u^{-1} . This means

$$(A(u)^{-1})^t Q A(-u)^{-1} = Q.$$

By properties of Q and the definition of the symplectic transpose τ , the above is equivalent to the equation

$$A(u) A(-u)^\tau = \mathbb{1},$$

i.e., $A(u) \in \mathcal{H}_{2n}$. Thus, the stabilizer of the ‘‘constant’’ pairing $\Omega(u) := Q$ is precisely the subgroup \mathcal{H}_{2n} .

We now show that the action of \mathcal{G}_{2n} is transitive. Let Φ be a bilinear pairing satisfying the conditions

$$\Phi(u)^t = -\Phi(-u), \quad \Phi(\infty) = Q.$$

We wish to show that there exists $B(u) \in \mathcal{G}_{2n}$ such that

$$\Phi(u) = (B \cdot \Omega)(u) := (B(u)^{-1})^t Q B(-u)^{-1}.$$

Equivalently, we wish to find $C(u) = B(u)^{-1}$ such that

$$\Phi(u) = C(u)^t Q C(u),$$

as formal power series in u^{-1} .

Let $\Phi(u) = \sum_{k=0}^{\infty} \Phi_k u^{-k}$ and set $C(u) = \sum_{k=0}^{\infty} C_k u^{-k} \in \mathcal{G}_{2n}$. By the conditions on Φ , we must have that $\Phi_0 = Q$, $\Phi_k^t = \Phi_k$ for k odd, and $\Phi_k^t = -\Phi_k$ for k even. Expanding in powers of u^{-1} , we have

$$C(u)^t Q C(-u) = \sum_{m=0}^{\infty} \left(\sum_{\substack{k+l=m \\ k, l \geq 0}} (-1)^l C_k^t Q C_l \right) u^{-m}.$$

Thus, for each $m \geq 0$, we wish to solve for C_m in the equation

$$\Phi_m = \sum_{\substack{k+l=m, \\ k, l \geq 0}} (-1)^l C_k^t Q C_l.$$

For $m = 0$, this is automatic since $\Phi_0 = Q, C_0 = \mathbb{1}$. The equations for $m \geq 1$ are solved for C_m in a straight-forward manner by induction and by using the fact that Φ_k is symmetric for k odd and antisymmetric for k even. \square

4.2. The classical limit of the map Ψ . We now turn our attention to the crucial algebra map Ψ in equation (2.8). In order to take the classical limit of Ψ , it is convenient to first decompose Ψ into pieces. We have the following:

$$\begin{array}{c}
 \Psi \\
 \curvearrowright \\
 Y^-(2) \xrightarrow{\iota} Y^-(2n) \xrightarrow{\psi} Y^-(2n) \xrightarrow{\phi} U(\mathfrak{sp}(2n, \mathbb{C})), \quad (4.7)
 \end{array}$$

where $\Psi := \phi \circ \psi \circ \iota$. It turns out that the image of Ψ is in $U(\mathfrak{sp}(2n, \mathbb{C}))^{\mathfrak{sp}(2(n-1), \mathbb{C})}$ [13]. The two maps ι and ϕ are relatively simple to describe. The technical heart of this section lies in the analysis of the middle algebra map ψ .

We begin with the map ι . The twisted Yangian $Y^-(2n)$ contains the sub-algebra $Y^-(2)$, generated by $s_{i,j}(u), i, j \in \{-n, n\}$. We denote the inclusion map by

$$\iota : Y^-(2) \hookrightarrow Y^-(2n). \quad (4.8)$$

We have chosen indices so that the subalgebra $Y^-(2)$ sit in the “corner entries” of the matrix $S(u)$.

In order to write down the maps ϕ and ψ , it is convenient to first set some notation. Let \mathcal{A} be an algebra with generators $a_{ij}^{(M)}$. Let

$$A(u) = \sum_{M=0}^{\infty} A_M u^{-M}$$

be a formal power series with matrix coefficients, where $A_M := (a_{ij}^{(M)})$. Then we may specify a map f between \mathcal{A} and an algebra \mathcal{B} by setting

$$f(A(u)) = B(u),$$

where f is understood to be linear over u^{-1} and in the matrix entries, so that

$$f(a_{ij}^{(M)}) = b_{ij}^{(M)},$$

for some $b_{ij}^{(M)} \in \mathcal{B}$. We use this notation below.

Let $F = (F_{ij})$ be the $2n \times 2n$ matrix whose (i, j) -th entry is given by the generator F_{ij} in $\mathfrak{sp}(2n, \mathbb{C})$ (A.7). Then ϕ is defined by the formula

$$\phi : S(u) \mapsto 1 + \frac{F}{u - (1/2)}. \quad (4.9)$$

This is a well-defined algebra map [14].

We now come to the middle automorphism ψ of $Y^-(2n)$. It turns out that this is the restriction to $Y^-(2n)$ of an algebra automorphism $\widehat{\psi}$ of $Y(2n)$ [13, 15], so we have the commutative diagram

$$\begin{array}{ccc} Y(2n) & \xrightarrow{\widehat{\psi}} & Y(2n) \\ \uparrow & & \uparrow \\ Y^-(2n) & \xrightarrow{\psi} & Y^-(2n) \end{array}$$

and in order to take the classical limit of ψ , it is convenient to first analyze that of $\widehat{\psi}$.

We will describe $\widehat{\psi}$ as a composition of well-known “basic” algebra automorphisms of $Y(2n)$ [14]. These are

1. $m_{g(u)} : T(u) \mapsto g(u)T(u)$, where $g(u)$ is a formal power series of the form

$$g(u) := 1 + g_1 u^{-1} + g_2 u^{-2} + \dots, \quad g_i \in \mathbb{C}.$$

2. $\tau_a : T(u) \mapsto T(u + a)$, $a \in \mathbb{C}$,

- 3. $\text{inv} : T(u) \mapsto T(-u)^{-1}$,
- 4. $\bar{\tau} : T(u) \mapsto T(-u)^\tau$,

where τ is the symplectic transpose defined in (A.4).

Using these “basic” automorphisms, the map $\hat{\psi}$ is given by

$$\hat{\psi} = \tau_n \circ \bar{\tau} \circ \text{inv} \circ m_{g(u)}, \tag{4.10}$$

where $g(u) = 1 + g_1 u^{-1} + g_2 u^{-2} + \dots$ is an element of $Z(Y(2n))[[u^{-1}]]$ and $Z(Y(2n))$ is the center of $Y(2n)$ [13].

Remark 4.6. The proof in [14] showing that $m_{g(u)}$ is an algebra automorphism for $g(u)$ with coefficients in \mathbb{C} also shows that $m_{g(u)}$ is also an automorphism for any $g(u)$ with coefficients in the center of the Yangian. This is because $g(u)T(u)$ still satisfies the ternary relation (B.2).

We will now take the classical limits of these algebra maps ϕ, ψ, ι . We first concentrate on the middle algebra map ϕ . Since we have decomposed ϕ as a composition of the basic automorphisms, it suffices to calculate the classical limit of each of these four basic types of algebra automorphisms.

The theorem below is the technical heart of this Section. The magic that occurs here is that two of the basic automorphisms degenerate to the *trivial* automorphism in the classical limit, while the other two basic automorphisms remain the *same*. This “all-or-nothing” phenomenon accounts for the amazing simplifications that occur in the classical limit and in large part explains the simplicity of the formulæ for the functions integrating \mathcal{O}_λ in Theorem 3.2.

Theorem 4.7. *The automorphisms $m_{g(u)}, \tau_a$ degenerate to the identity at the level of the classical limit. The classical limits of the automorphisms $\text{inv}, \bar{\tau}$ are expressed by the same formulæ as for the original automorphisms.*

Proof. We will show that the multiplication map goes to the identity and that the inverse map remains the same. The calculations for the other two types of automorphisms are similar.

We first consider the multiplication map $m_{g(u)} : T(u) \mapsto g(u)T(u)$. In order to take the classical limit, we need to find an algebra automorphism $(m_{g(u)})_h$ of $Y_h(2n)$ such that the following diagram commutes for any $h \neq 0$:

$$\begin{array}{ccc} Y_h(2n) & \xrightarrow{(m_{g(u)})_h} & Y_h(2n) \\ \gamma_h \downarrow & & \downarrow \gamma_h \\ Y(2n) & \xrightarrow{m_{g(u)}} & Y(2n) \end{array} \tag{4.11}$$

Here, γ_h is the isomorphism between $Y_h(2n)$ and $Y(2n)$ used in the proof of Lemma 4.1. We first observe that the original automorphism $m_{g(u)}$ acts as

follows on the generators:

$$m_{g(u)} : t_{ij}^{(M)} \longmapsto \sum_{\substack{K+L=M \\ K,L \geq 0}} g_K t_{ij}^{(L)}.$$

Using this explicit form and the definition of γ_h in the proof of Lemma 4.1, one may immediately compute that the map $m_{g(u)} \circ \gamma_h$ takes

$$m_{g(u)} \circ \gamma_h : \tilde{t}_{ij}^{(M)} \longmapsto h^M \left(\sum_{\substack{K+L=M \\ K,L \geq 0}} g_K t_{ij}^{(L)} \right).$$

Thus, in order to have the diagram (4.11) commute, we are forced to define $(m_{g(u)})_h$ as follows:

$$(m_{g(u)})_h : \tilde{t}_{ij}^{(M)} \longmapsto \sum_{\substack{K+L=M \\ K,L \geq 0}} g_K \tilde{t}_{ij}^{(L)} h^K.$$

Thus, when we set $h = 0$, this degenerates to the map

$$(m_{g(u)})_0 : \tilde{t}_{ij}^{(M)} \longmapsto \tilde{t}_{ij}^{(M)},$$

i.e., the identity map.

Now we consider the automorphism $\text{inv} : T(u) \mapsto T(-u)^{-1}$. We wish to make a diagram similar to (4.11) commute, using now inv instead of $m_{g(u)}$. We first observe that the automorphism inv behaves as follows on the generators, $M \geq 1$:

$$\text{inv} : t_{ij}^{(M)} \longmapsto \sum_{s=1}^M (-1)^{M+s} \left(\sum_{\substack{m_1+\dots+m_s=M \\ m_i \geq 1}} \sum_{\substack{a_1, \dots, a_{s-1} \\ a_i \in \mathcal{I}}} t_{i, a_1}^{(m_1)} t_{a_1, a_2}^{(m_2)} \dots t_{a_{s-1}, j}^{(m_s)} \right).$$

The generators with $M = 0$ are sent to themselves. Then we immediately compute that for the map $\text{inv} \circ \gamma_h$, we have for $M \geq 1$

$$\text{inv} \circ \gamma_h : \tilde{t}_{ij}^{(M)} \longmapsto h^M \left[\sum_{s=1}^M (-1)^{M+s} \left(\sum_{\substack{m_1+\dots+m_s=M \\ m_i \geq 1}} \sum_{\substack{a_1, \dots, a_{s-1} \\ a_i \in \mathcal{I}}} t_{i, a_1}^{(m_1)} \dots t_{a_{s-1}, j}^{(m_s)} \right) \right].$$

Thus, we see that the map inv_h on $Y_h(2n)$ is *independent* of h , and thus the classical limit inv_0 is given by the same formula as for the map inv on $Y(2n)$. \square

The computation of the classical limits of the basic automorphisms immediately leads us to a quick computation of the classical limit of both $\widehat{\psi}$ and ψ .

Corollary 4.8. *The map $\widehat{\psi}$ degenerates to $\bar{\tau} \circ \text{inv}$ in the classical limit and sends*

$$T(u) \mapsto (T(u)^\tau)^{-1}.$$

Proof. The map $\widehat{\psi}$ is decomposed as $\tau_n \circ \bar{\tau} \circ \text{inv} \circ m_{g(u)}$. Both $m_{g(u)}$ and τ_n degenerate to the identity, and inv and $\bar{\tau}$ remain the same. Thus, the limit is simply $\bar{\tau} \circ \text{inv}$, as desired. \square

Remark 4.9. On the quantum level, the map $T(u) \mapsto (T(u)^\tau)^{-1}$ is also a coalgebra automorphism [14]. On the classical level, one therefore expects it to degenerate to a group automorphism of \mathcal{G}_{2n} . This is indeed the case, as may be checked directly.

The classical limit $\widehat{\psi}_0$ of $\widehat{\psi}$ preserves the subgroup \mathcal{H}_{2n} and therefore induces a map on the homogeneous space $\mathcal{G}_{2n}/\mathcal{H}_{2n}$, giving us the classical limit of ψ . We now give the explicit formula for this map, written in terms of the coordinates $S(u) = (s_{ij}(u))$.

Corollary 4.10. *The classical limit ψ_0 of the map ψ is given by $S(u) \mapsto (S(-u))^{-1}$.*

Proof. Since $\widehat{\psi}_0$ takes $T(u) \mapsto (T(u)^\tau)^{-1}$, we see that $T(-u)^\tau \mapsto ((T(-u)^\tau)^{-1})^\tau$. Hence the coordinates $S(u) = T(u)T(-u)^\tau$ are mapped to $(T(u)^\tau)^{-1}((T(-u)^\tau)^{-1})^\tau$. Since τ is an involutory automorphism, we have that $(T(-u)^\tau)^{-1} = (T(-u)^{-1})^\tau$, and thus

$$S(u) = T(u)T(-u)^\tau \mapsto (T(u)^\tau)^{-1}T(-u)^{-1} = S(-u)^{-1}.$$

Thus the map on the homogeneous space is given by $S(u) \mapsto S(-u)^{-1}$. \square

Now that we have calculated the limit ψ_0 of ψ , we now focus on the two algebra maps ϕ and ι in (4.7). As advertised previously, the calculations here are more straightforward than those for ψ .

Proposition 4.11. *The classical limit ϕ_0 of the map ϕ is given by*

$$\phi_0 : \begin{cases} \tilde{s}_{ij}^{(0)} := \delta_{ij} \mapsto \delta_{ij}, \\ \tilde{s}_{ij}^{(1)} \mapsto F_{ij}, \\ \tilde{s}_{ij}^{(M)} \mapsto 0, \quad \text{for } M \geq 2. \end{cases}$$

Proof. In order to compute ϕ_0 , we need to find algebra maps ϕ_h such that a diagram similar to (4.11) commutes. Using the expansion

$$\frac{1}{u - 1/2} = \sum_{r=0}^{\infty} (-1)^r \left(\frac{1}{2}\right)^r u^{-r-1},$$

we see that an explicit formula for ϕ is given by

$$s_{ij}^{(M)} (-1)^{M-1} \left(\frac{1}{2}\right)^{M-1} \mapsto F_{ij}.$$

It is then a straightforward computation to see that ϕ_h is given by

$$\tilde{s}_{ij}^{(M)} \mapsto F_{ij} \left(\frac{1}{2}\right)^{M-1} (-1)^{M-1} h^{M-1}.$$

In particular, for $h = 0$, the only non-zero images are those of $\tilde{s}_{ij}^{(1)}$, and we have the formulæ as desired. \square

In order to give a geometric description of ϕ_0 , we first recall that the classical limit of $U_h(\mathfrak{sp}(2n, \mathbb{C}))$ is well known to be $\text{Sym}(\mathfrak{sp}(2n, \mathbb{C})^*)$ (see e.g., [1]), the space of polynomial functions on the dual of the Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$. The underlying Poisson space is $\mathfrak{sp}(2n, \mathbb{C})^*$. With this in hand, it is immediate from Proposition 4.11 that the geometric map corresponding to ϕ_0 is given by

$$X \in \mathfrak{sp}(2n, \mathbb{C})^* \mapsto \mathbb{1} + Xu^{-1}, \quad (4.12)$$

where the image is interpreted as representing an element in $\mathcal{G}_{2n}/\mathcal{H}_{2n}$.

Finally, we calculate the classical limit of the inclusion map ι .

Proposition 4.12. *The classical limit ι_0 of the map ι is given by*

$$\tilde{s}_{\pm n, \pm n}^{(M)} \in Y^-(2) \longrightarrow \tilde{s}_{\pm n, \pm n}^{(M)} \in Y^-(2n).$$

Proof. The classical limit ι_0 is given by the same formula as for ι since the inclusion map preserves degrees (where $\deg(s_{ij}^{(M)}) = M$). \square

Interpreted as a geometrical map from $\mathcal{G}_{2n}/\mathcal{H}_{2n}$ to $\mathcal{G}_2/\mathcal{H}_2$, it is given in coordinates by

$$S(u) \mapsto \begin{bmatrix} s_{-n, -n}(u) & s_{-n, n}(u) \\ s_{n, -n}(u) & s_{n, n}(u) \end{bmatrix},$$

i.e., it is the quotient map given by “taking the corner entries” of $S(u)$. For an element $A(u) \in \mathcal{G}_{2n}/\mathcal{H}_{2n}$, let $A(u)_{\pm n, \pm n}$ denote the element in $\mathcal{G}_2/\mathcal{H}_2$ gotten by taking the corner entries as above.

We now give the formula for the classical limit of Ψ , interpreted as a geometric map on the underlying Poisson spaces. Recall that the Poisson space which is the classical limit of $U(\mathfrak{sp}(2n, \mathbb{C}))$ is $\mathfrak{sp}(2n, \mathbb{C})^*$ [1], and thus the classical limit of the centralizer $U(\mathfrak{sp}(2n, \mathbb{C}))^{\mathfrak{sp}(2(n-1), \mathbb{C})}$ is the Poisson quotient of the Lie algebra dual by a subgroup, $\mathfrak{sp}(2n, \mathbb{C})^*/Sp(2(n-1), \mathbb{C})$.

Theorem 4.13. *The classical limit Ψ_0 of Ψ is given by the composition $\iota_0 \circ \psi_0 \circ \phi_0$. As a map on the underlying geometric objects, Ψ_0 is given in*

coordinates as follows. Let $X \in \mathfrak{sp}(2n, \mathbb{C})^*$.

$$\Psi_0 : X \mapsto \left(\sum_{M=0}^{\infty} X^M u^{-M} \right)_{\pm n, \pm n}$$

This is invariant under the action of $Sp(2(n-1), \mathbb{C})$, so it is well defined on $\mathfrak{sp}(2n, \mathbb{C})^*/Sp(2(n-1), \mathbb{C})$.

Proof. The first map ϕ_0 sends X to $\mathbb{1} + Xu^{-1}$. The map ψ_0 sends

$$\mathbb{1} + Xu^{-1} \mapsto (\mathbb{1} - Xu^{-1})^{-1} = \mathbb{1} + Xu^{-1} + X^2u^{-2} + \dots,$$

and then ι_0 takes the corner entries, as desired. Since the $(\pm n, \pm n)$ entries are untouched by the action of $Sp(2(n-1), \mathbb{C})$, the map is invariant under this action. \square

4.3. The derivation of the integrable system. So far, in taking the geometric classical limit of Molev's constructions in [13], we have followed his conventions in using algebras over \mathbb{C} . In so doing, we have obtained, in Theorem 4.13, a map on the quotient of the complex Lie algebra dual $\mathfrak{sp}(2n, \mathbb{C})^*$. In this section, we take the compact analog of Ψ_0 in order to obtain the symplectic-geometric picture, with \mathbb{R} -valued functions on (a quotient of) the compact form $\mathfrak{u}(n, \mathbb{H})^*$.

Recall that $\mathfrak{gl}(n, \mathbb{H})$ is naturally a subalgebra of $\mathfrak{gl}(2n, \mathbb{C})$ by a restriction of scalars. Given an element $A(u) \in \mathfrak{gl}(n, \mathbb{H})[[u^{-1}]]$, we denote by $A(u)_{nn}$ the element of $\mathfrak{gl}(1, \mathbb{H})[[u^{-1}]] \cong \mathbb{H}[[u^{-1}]]$ obtained by taking the (n, n) -th matrix entry of $A(u)$.

Definition 4.14. For $X \in \mathfrak{u}(n, \mathbb{H}) \cong \mathfrak{u}(n, \mathbb{H})^*$, we define the compact form $\Psi_{\mathbb{H}}$ of Ψ_0 as

$$\Psi_{\mathbb{H}} : X \mapsto (1 + Xu^{-1} + X^2u^{-2} + \dots)_{nn}. \quad (4.13)$$

This map is well defined on the quotient $\mathfrak{u}(n, \mathbb{H})^*/U(n-1, \mathbb{H})$ since it is invariant under the action of $U(n-1, \mathbb{H})$.

As we saw in Section 2.1, the key ingredient for the construction of a Gel'fand–Cetlin basis for the case of $U(n, \mathbb{C})$ is the presence of a large family of commuting operators on $V(\lambda)$, the classical limit of which gave a large family of Poisson-commuting functions on \mathcal{O}_λ . Following this example, we look now for commuting elements of $Y^-(2)$, which via Molev's map Ψ are commuting operators on an irreducible representation $V(\lambda)$ of $U(n, \mathbb{H})$. We are in luck: Molev observes [13] that the coefficients of $\text{tr}(S(u))$ generate a commuting subalgebra of $Y^-(2)$. Thus, in the classical limit, the functions $s_{-n, -n}^{(M)} + s_{n, n}^{(M)}$ for $M \geq 1$ Poisson-commute on $\mathcal{G}_2/\mathcal{H}_2$, and hence (since $\Psi_{\mathbb{H}}$ is a Poisson map) their pullbacks Poisson-commute on $\mathfrak{u}(n, \mathbb{H})^*/U(n-1, \mathbb{H})$. Note that as functions on $\mathcal{G}_2/\mathcal{H}_2$, the $s_{-n, -n}^{(M)} + s_{n, n}^{(M)}$ are a priori \mathbb{C} -valued, since they simply read off certain matrix entries in $\mathfrak{gl}(2n, \mathbb{C})$. However, the

image of $\Psi_{\mathbb{H}}$ is by definition contained in the intersection $\mathfrak{gl}(1, \mathbb{H})[[u^{-1}]] \cap \mathcal{G}_2/\mathcal{H}_2 \cong \mathbb{H}[[u^{-1}]] \cap \mathcal{G}_2/\mathcal{H}_2$. Restricted to this subset, the functions are in fact \mathbb{R} -valued. We record the following calculation.

Lemma 4.15. *The functions $\left(s_{-n, -n}^{(M)} + s_{n, n}^{(M)}\right)$ are \mathbb{R} -valued when restricted to $\mathbb{H}[[u^{-1}]] \cap \mathcal{G}_2/\mathcal{H}_2$. In particular, for $A(u) \in \mathcal{G}_2/\mathcal{H}_2$,*

$$\left(s_{-n, -n}^{(M)} + s_{n, n}^{(M)}\right)(A(u)) = 2 \cdot \operatorname{Re}(A_M),$$

where $A_M \in \mathbb{H} \subset \mathfrak{gl}(2, \mathbb{C})$, and $\operatorname{Re}(A_M)$ denotes the real part in the quaternionic sense.

Proof. An element in $\mathbb{H}[[u^{-1}]] \cap \mathcal{G}_2/\mathcal{H}_2$ is a formal series

$$A(u) = \mathbb{1} + A_1 u^{-1} + A_2 u^{-2} + \cdots \in \mathbb{1} + u^{-1} \mathfrak{gl}(2, \mathbb{C})[[u^{-1}]],$$

where $A(u) = B(u)B(-u)^\tau$, for $B(u) \in \mathcal{G}_2$. Since $A(u) \in \mathbb{H}[[u^{-1}]]$, each coefficient A_M is an element in \mathbb{H} , where these are considered as element of $\mathfrak{gl}(2, \mathbb{C})$ by the standard inclusion $\mathbb{H} \hookrightarrow \mathfrak{gl}(2, \mathbb{C})$:

$$A_M = \alpha_M + j\beta_M \mapsto \begin{bmatrix} \bar{\alpha}_M & \beta_M \\ -\bar{\beta}_M & \alpha_M \end{bmatrix}.$$

Then, it is immediate that

$$\left(s_{-1, -1}^{(M)} + s_{1, 1}^{(M)}\right)(A(u)) = \bar{\alpha}_M + \alpha_M = 2 \cdot \operatorname{Re}(\alpha)$$

and in particular is \mathbb{R} -valued. Here, $\operatorname{Re}(\alpha)$ denotes the real part in the \mathbb{H} sense. \square

We are now prepared to obtain the formulæ for the functions $f_{n, m}$ used in Section 3.1.

Theorem 4.16. *Let \mathcal{O}_λ be a generic coadjoint orbit of $U(n, \mathbb{H})$. Let $\Psi_{\mathbb{H}} : \mathfrak{u}(n, \mathbb{H})/U(n-1, \mathbb{H}) \rightarrow \mathcal{G}_2/\mathcal{H}_2$ denote the compact form of Ψ_0 . Let $X \in \mathcal{O}_\lambda$, so $X = AD_\lambda A^*$ for some $A \in U(n, \mathbb{H})$. Then*

$$(\Psi_{\mathbb{H}})^* \left(s_{-1, -1}^{(M)} + s_{1, 1}^{(M)}\right)(X) = \operatorname{rtr}((AD_\lambda A^*)^M E_{nn}), \quad (4.14)$$

where the $a_{n, \ell}$ denote the entries in the bottom row of $A = (a_{ij})$. In particular, for M odd, the pullback functions are identically 0. For all M , the pullback functions are invariant under $U(n-1, \mathbb{H})$ and hence well defined on the quotient $\mathfrak{u}(n, \mathbb{H})^*/U(n-1, \mathbb{H})$.

Proof. Given the diagonalization $X = AD_\lambda A^*$, any X^M is of the form $X^M = (AD_\lambda A^*)^M = AD_\lambda^M A^*$. Since D_λ is diagonal, its power D_λ^M is a diagonal matrix with diagonals given by

$$((i\lambda_1)^M, \dots, (i\lambda_n)^M).$$

The (n, n) -th entry in X^M is therefore given by

$$(X^M)_{(n,n)} = \sum_{\ell=1}^n a_{n,\ell} (i\lambda_\ell)^m \overline{a_{n,\ell}},$$

where $A = (a_{ij})$. Since the functions $s_{-1,-1}^{(M)} + s_{1,1}^{(M)}$ read off the real part of M -th coefficient, as in Lemma 4.15,

$$(\Psi_{\mathbb{H}})^* \left(s_{-1,-1}^{(M)} + s_{1,1}^{(M)} \right) (X) = 2 \cdot \sum_{\ell=1}^n \operatorname{Re} \left(a_{n,\ell} (i\lambda_\ell)^M \overline{a_{n,\ell}} \right).$$

Using the reduced trace (A.10), this can be rewritten as

$$(\Psi_{\mathbb{H}})^* \left(s_{-1,-1}^{(M)} + s_{1,1}^{(M)} \right) (X) = \operatorname{rtr}((AD_\lambda A^*)^M E_{nn}). \quad (4.15)$$

Notice that for M odd, each $(i\lambda_\ell)^M$ is pure imaginary, and hence each term in the sum is pure imaginary in \mathbb{H} . Hence its real part is 0. Therefore, for M odd, this pullback is identically 0 as a function on \mathcal{O}_λ . Hence we only get non-trivial functions for $M = 2m$ even. Moreover, since the function reads off only the (n, n) -th entry, it is invariant under $U(n-1, \mathbb{H})$. \square

5. Appendix A: on quaternionic linear algebra and $U(n, \mathbb{H})$

The quaternions \mathbb{H} are defined as the set of quadruples

$$q = a + ib + jc + kd,$$

where $a, b, c, d \in \mathbb{R}$ and the i, j, k satisfy the relations $i^2 = j^2 = k^2 = -1$, $ij = k$. The quaternions are *not* commutative, since $ij = -ji = k$. We define conjugation in \mathbb{H} as

$$\bar{q} = a - ib - jc - kd,$$

for q as above. We define $\operatorname{Re}(q) := a$ and $\operatorname{Im}(q) := ib + jc + kd$ for q as above. We define the norm of an element q to be $\|q\| = \sqrt{q\bar{q}} \in \mathbb{R}$.

We now describe our conventions for linear algebra over \mathbb{H} . Let \mathbb{H}^n be the n -dimensional quaternionic vector space of n -tuples in \mathbb{H} , equipped with scalar multiplication by \mathbb{H} on the *right*. Elements of \mathbb{H}^n will be represented by column vectors, and \mathbb{H} -linear transformations will then be represented by matrix multiplication on the *left*. We denote by $\mathfrak{gl}(n, \mathbb{H})$ the algebra of $n \times n$ matrices with entries in \mathbb{H} . The standard basis vectors are the $e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)^t$, where the 1 is in the i -th place. We define the conjugate \bar{v} of a vector $v \in \mathbb{H}^n$ componentwise, and the norm $\|v\|$ also as usual, by a sum of norms of the components. For any $m \times n$ matrix $A = (a_{ij})$ with entries in \mathbb{H} , we define the conjugate transpose as the $n \times m$ matrix

$$A^* := (\bar{A})^t,$$

where conjugation is quaternionic conjugation, and the transpose t is as usual.

Given two vectors $v = (v_1, \dots, v_n)^t, w = (w_1, \dots, w_n)^t \in \mathbb{H}^n$, the standard quaternionic hermitian form is defined by

$$\langle v, w \rangle := \sum_{l=1}^n \bar{v}_l w_l.$$

More compactly,

$$\langle v, w \rangle := v^* w,$$

where the conjugate transpose is defined above. We define the *compact symplectic group* to be the subset of \mathbb{H} -linear transformations preserving the quaternionic hermitian form.

$$U(n, \mathbb{H}) := \{A \in \mathfrak{gl}(n, \mathbb{H}) : \langle Av, Aw \rangle = \langle v, w \rangle, \forall v, w \in \mathbb{H}^n\}.$$

Again, more compactly,

$$U(n, \mathbb{H}) = \{A \in \mathfrak{gl}(n, \mathbb{H}) : A^* A = \mathbf{1}\}.$$

The Lie algebra is

$$\mathfrak{u}(n, \mathbb{H}) = \{X \in \mathfrak{gl}(n, \mathbb{H}) : X^* + X = 0\}.$$

Observe that $\mathbb{H} \cong \mathbb{C} \oplus j\mathbb{C}$, where the \mathbb{C} is thought of as $\mathbb{R} \oplus i\mathbb{R}$. Similarly, $\mathbb{H}^n \cong \mathbb{C}^n \oplus j\mathbb{C}^n$. By a restriction of scalars from \mathbb{H} to \mathbb{C} , \mathbb{H}^n can be thought of as a \mathbb{C} -vector space of dimension $2n$, with ordered basis

$$\{e_{-n}, e_{-(n-1)}, \dots, e_{-2}, e_{-1}, e_1, e_2, \dots, e_{n-1}, e_n\}, \quad (\text{A.1})$$

where $e_{-i} := e_i \cdot j \in \mathbb{H}^n \cong \mathbb{C}^{2n}$. With this basis of \mathbb{C}^{2n} in mind, we take the indexing set to be

$$\mathcal{I} := \{-n, -n+1, \dots, -2, -1, 1, 2, \dots, n-1, n\}. \quad (\text{A.2})$$

Note that the index 0 is skipped. Since any \mathbb{H} -linear map is also \mathbb{C} -linear, there is a natural inclusion $\mathfrak{gl}(n, \mathbb{H})$ into $\mathfrak{gl}(2n, \mathbb{C})$, the space of $2n \times 2n$ matrices with \mathbb{C} entries.

Using the decomposition $\mathbb{H} \cong \mathbb{C} \oplus j\mathbb{C}$, we may also write the quaternionic hermitian form as a sum

$$\langle v, w \rangle = H(v, w) + jQ(v, w) \in \mathbb{C} \oplus j\mathbb{C},$$

where H is the standard hermitian form on \mathbb{C}^{2n} , and Q is a *complex symplectic form* on \mathbb{C}^{2n} . Written with respect to the ordered basis of \mathbb{C}^{2n} above, the symplectic form is given by

$$Q(v, w) := v^t Q w.$$

Here, Q is the $2n \times 2n$ matrix

$$Q = \begin{bmatrix} \mathbf{0} & \tilde{I}_n \\ -\tilde{I}_n & \mathbf{0} \end{bmatrix}, \quad (\text{A.3})$$

where \tilde{I}_n is the $n \times n$ matrix with ones along the *antidiagonal*, i.e., $\tilde{I}_n = (a_{ij})$ where $a_{ij} = \delta_{i, (n+1)-j}$.

Using the symplectic form Q above, we define the *symplectic transpose* τ as the involution on $\mathfrak{gl}(2n, \mathbb{C})$ which satisfies

$$Q(Av, w) = Q(v, A^\tau w)$$

for all $v, w \in \mathbb{C}^{2n}$, $A \in \mathfrak{gl}(2n, \mathbb{C})$. An explicit formula for the symplectic transpose is given by

$$\tau : A \mapsto Q^{-1} A^t Q, \quad (\text{A.4})$$

where the matrix Q is given in A.3 and the t is the usual transpose.

The subgroup of $\mathfrak{gl}(2n, \mathbb{C})$ preserving the complex symplectic form Q is defined to be the *complex symplectic group* $Sp(2n, \mathbb{C})$. We have

$$Sp(2n, \mathbb{C}) := \{A \in \mathfrak{gl}(2n, \mathbb{C}) : Q(Av, Aw) = Q(v, w) \quad \forall v, w \in \mathbb{C}^{2n}\}, \quad (\text{A.5})$$

which again can be rewritten as

$$Sp(2n, \mathbb{C}) = \{A \in \mathfrak{gl}(2n, \mathbb{C}) : A^t Q A = Q\} \quad (\text{A.6})$$

for the matrix Q defined above. Since the compact symplectic group is precisely the subgroup preserving both the usual hermitian form on \mathbb{C}^{2n} the symplectic form Q , we have

$$U(n, \mathbb{H}) = U(2n, \mathbb{C}) \cap Sp(2n, \mathbb{C}).$$

We may conclude from this fact, plus a dimension count, that $U(n, \mathbb{H})$ is the compact form of $Sp(2n, \mathbb{C})$. This justifies the terminology.

The Lie algebra of $Sp(2n, \mathbb{C})$ is given by

$$\mathfrak{sp}(2n, \mathbb{C}) = \{X \in \mathfrak{gl}(2n, \mathbb{C}) : X^t Q + QX = 0\}.$$

A basis of $\mathfrak{sp}(2n, \mathbb{C})$ is given by the elements

$$F_{i,j} = E_{i,j} + \text{sgn}(i) \cdot \text{sgn}(j) E_{-j,-i}, \quad (\text{A.7})$$

where the index set for basis elements of $\mathfrak{gl}(2n, \mathbb{C})$ is $\mathcal{I} := \{-n, -(n-1), \dots, -1, 1, 2, \dots, n\}$ (note that we skip the index 0).

We now collect some standard Lie-group-theoretic facts about the compact symplectic group $U(n, \mathbb{H})$. A further discussion can be found in [2].

As a real manifold, $U(n, \mathbb{H})$ has dimension $2n^2 + n$. The dimension of a maximal torus of $U(n, \mathbb{H})$ is n . We will always take as choice of maximal torus the diagonal subgroup $T^n \cong (S^1)^n$, where $S^1 = \{e^{i\theta}\}$. This is the S^1 sitting in the *first* factor of $\mathbb{H} \cong \mathbb{C} \oplus j\mathbb{C}$. Any element X of the Lie algebra $\mathfrak{u}(n, \mathbb{H})$ can be conjugated by an element A of $U(n, \mathbb{H})$ to a diagonal matrix:

$$AXA^* = D_\lambda.$$

Since the Weyl group of $U(n, \mathbb{H})$ is the group of *signed* permutations, we may choose D_λ to be of the form

$$D_\lambda := \begin{bmatrix} i\lambda_1 & & & \\ & i\lambda_2 & & \\ & & \ddots & \\ & & & i\lambda_n \end{bmatrix}, \quad \text{where } 0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n. \quad (\text{A.8})$$

We denote by $\lambda = (\lambda_1, \dots, \lambda_n)$ the n -tuple of “eigenvalues.”

The Killing form on $\mathfrak{u}(n, \mathbb{H})$ is the restriction of the usual Killing form on $\mathfrak{u}(2n, \mathbb{C})$ restricted to $\mathfrak{u}(n, \mathbb{H})$. This is called the *reduced trace* pairing, where the reduced trace of a quaternionic matrix A is defined by

$$\text{rtr}(A) := \text{tr}(\iota(A)), \quad (\text{A.9})$$

and ι is the inclusion $\mathfrak{u}(n, \mathbb{H}) \hookrightarrow \mathfrak{u}(2n, \mathbb{C})$. It will be convenient to express the reduced trace purely in quaternionic terms. For an element $A = (a_{ij}) \in \mathfrak{gl}(n, \mathbb{H})$, the reduced trace is given by

$$\text{rtr}(A) = 2 \cdot \text{Re} \left(\sum_{i=1}^n a_{ii} \right), \quad (\text{A.10})$$

i.e., it is twice the real part of the sum of the diagonals.

6. Appendix B: the Yangian and twisted Yangian

We now briefly recall essential facts about the Yangian and twisted Yangian. See [14] for details.

The Yangian $Y(2n) = Y(\mathfrak{gl}(2n))$ is defined² as a complex associative unital algebra with countably many generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \dots$, where the indices i, j are in the indexing set \mathcal{I} as in Appendix A. In particular, the Yangian is infinite-dimensional. The generators satisfy the following defining commutation relations

$$\left[t_{ij}^{(M)}, t_{kl}^{(L)} \right] = \sum_{r=0}^{\min(M,L)-1} \left(t_{kj}^{(r)} t_{il}^{(M+L-1-r)} - t_{kj}^{(M+L-1-r)} t_{il}^{(r)} \right). \quad (\text{B.1})$$

Moreover, the Yangian is a Hopf algebra with coproduct defined as

$$\Delta(t_{ij}(u)) := \sum_{a=-n}^n t_{ia}(u) \otimes t_{aj}(u).$$

It turns out to be useful to write these relations more compactly. We first set some definitions and notation. Define the formal power series

$$t_{ij}(u) := \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in Y(2n)[[u^{-1}]],$$

²The Yangian $Y(N)$ can be defined for any N , but we are only interested in the case $N = 2n$ even.

and assemble them in a single “ T -matrix” as

$$T(u) := \sum_{i,j} t_{ij}(u) \otimes E_{ij} \in Y(2n)[[u^{-1}]] \otimes \text{End}(\mathbb{C}^{2n}).$$

Moreover, for an operator $X \in \text{End}(\mathbb{C}^{2n})$, we set

$$X_1 := X \otimes \mathbb{1}, \quad X_2 := \mathbb{1} \otimes X,$$

in $\text{End}(\mathbb{C}^{2n} \otimes \mathbb{C}^{2n})$. Then we may define

$$T_1(u) := \sum_{i,j} t_{ij}(u) \otimes (E_{ij})_1, \quad T_2(v) := \sum_{i,j} t_{ij}(v) \otimes (E_{ij})_2.$$

In a similar spirit, we define

$$T_{[1]}(u) := \sum_{i,j=1}^{2n} t_{ij}(u) \otimes 1 \otimes E_{ij} \in Y(2n)[[u^{-1}]]^{\otimes 2} \otimes \text{End}(\mathbb{C}^{2n}),$$

and

$$T_{[2]}(u) := \sum_{i,j=1}^{2n} 1 \otimes t_{ij}(u) \otimes E_{ij} \in Y(2n)[[u^{-1}]]^{\otimes 2} \otimes \text{End}(\mathbb{C}^{2n}).$$

With the notation above, the relations in equation (B.1) can be written more compactly as the following single relation for the T -matrix:

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v), \quad (\text{B.2})$$

where $R(u) := 1 - \frac{P}{u}$, and P is the permutation operator on $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$.³

The coproduct structure may similarly be described by the single matrix equation

$$\Delta(T(u)) = T_{[1]}(u)T_{[2]}(u) \in Y(2n)[[u^{-1}]]^{\otimes 2} \otimes \text{End}(\mathbb{C}^{2n}). \quad (\text{B.3})$$

The antipode map is defined by $S : T(u) \mapsto T^{-1}(u)$ and the counit is $\epsilon(T(u)) := 1$. This makes the Yangian into a Hopf algebra [14].

The twisted Yangian $Y^-(2n)$ is defined as the subalgebra of $Y(2n)$ with generators the entries in the matrix

$$S(u) := T(u)T(-u)^\tau, \quad (\text{B.4})$$

where τ was defined in (A.4). In terms of the matrix entries

$$s_{ij}(u) = \sum_{a=-n}^n \theta_{aj} t_{ia}(u) t_{-j,-a}(-u) = \sum_{M=0}^{\infty} s_{ij}^{(M)} u^{-M},$$

³The object $R(u) \in \text{End}(\mathbb{C}^{2n}) \otimes \text{End}(\mathbb{C}^{2n}) \otimes \mathbb{C}(u)$ is called the *Yang–Baxter R -matrix*. The relation above is called the *ternary relation*.

we obtain the formula for the generators $s_{ij}^{(M)}$ in terms of the generators of $Y(2n)$.

$$s_{ij}^{(M)} = \sum_{a=-n}^n \sum_{K+L=M, K, L \geq 0} (-1)^L \theta_{aj} t_{ia}^{(K)} t_{-j, -a}^{(L)}. \quad (\text{B.5})$$

It can be shown [14, Proposition 4.17] that the twisted Yangian is a left Hopf coideal of the Yangian, i.e.

$$\Delta(Y^-(2n)) \subset Y(2n) \otimes Y^-(2n).$$

In particular,

$$\Delta(s_{ij}(u)) = \sum_{k,l} \theta_{lj} t_{ik}(u) t_{-j, -l}(-u) \otimes s_{kl}(u),$$

where $i, j \in \{-n, -(n-1), \dots, -1, 1, \dots, n-1, n\}$.

Finally, we briefly recall the definitions of a topological Hopf algebra and a deformation of a Hopf algebra. See [1] for details.

Definition B.1. *A topological Hopf algebra over $\mathbb{C}[[h]]$ is a complete $\mathbb{C}[[h]]$ -module A_h equipped with $\mathbb{C}[[h]]$ -linear maps $\iota_h, \mu_h, \epsilon_h, \Delta_h, S_h$ satisfying the Hopf algebra axioms, but with algebraic tensor products replaced by the completions in the h -adic topology.*

Definition B.2. *A deformation of a Hopf algebra $(A, \iota, \mu, \epsilon, \Delta, S)$ over \mathbb{C} is a topological Hopf algebra A_h over $\mathbb{C}[[h]]$ such that*

1. A_h is isomorphic to $A[[h]]$ as a $\mathbb{C}[[h]]$ -module,
2. $\mu_h \equiv \mu \pmod{h}$, $\Delta_h \equiv \Delta \pmod{h}$.

In particular, the compatibility condition that Δ_h is an algebra homomorphism from A_h to $A_h \otimes A_h$ is expressed by

$$\Delta_h(\mu_h(a_1 \otimes a_2)) = (\mu_h \otimes \mu_h) \Delta_h^{13}(a_1) \Delta_h^{24}(a_2). \quad (\text{B.6})$$

Here, for Δ any coalgebra structure, we define Δ^{13} and Δ^{24} as follows. If $\Delta(a) = \sum a_i \otimes a'_i$, $\Delta(b) = \sum b_j \otimes b'_j$, then

$$\Delta^{13}(a) \Delta^{24}(b) := \sum a_i \otimes b_j \otimes a'_i \otimes b'_j.$$

We are interested in the first-order terms in h . In particular, the first-order term in h of equation (B.6) for $\Delta_h = \sum_{k=0}^{\infty} \Delta_k h^k$ and $\mu_h = \sum_{k=0}^{\infty} \mu_k h^k$ on $A[[h]]$ is

$$\begin{aligned} \Delta_1(\mu_1(a_1 \otimes a_2)) + \Delta_1(\mu_0(a_1 \otimes a_2)) &= (\mu_0 \otimes \mu_1 + \mu_1 \otimes \mu_0) \Delta^{13}(a_1) \Delta^{24}(a_2) \\ &\quad + \Delta_1(a_1) \Delta_0(a_2) + \Delta_0(a_1) \Delta_1(a_2). \end{aligned} \quad (\text{B.7})$$

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