

MOMENT MAPS, SYMPLECTOMORPHISM GROUPS AND COMPATIBLE COMPLEX STRUCTURES

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In this paper, we apply Donaldson's general moment map framework for the action of a symplectomorphism group on the corresponding space of compatible (almost) complex structures to the case of rational ruled surfaces. This gives a new approach to understanding the topology of their symplectomorphism groups, based on a result of independent interest: the space of compatible integrable complex structures on any symplectic rational ruled surface is (weakly) contractible. We also explain how in general, under this condition, there is a direct relationship between the topology of a symplectomorphism group, the deformation theory of compatible complex structures and the groups of complex automorphisms of these complex structures.

1. Introduction

The known results regarding the topology of symplectomorphism groups, for two-dimensional surfaces and four-dimensional rational ruled surfaces, indicate a direct relation with the topology of the corresponding groups of complex automorphisms. The goal of this paper is to formulate this empirical relation more precisely, within a general framework involving infinite dimensional groups, manifolds and moment maps. This general framework goes back to Atiyah and Bott [5], was made rigorous and precise in finite dimensions by Kirwan [18] and was shown to apply to the action of a symplectomorphism group on the corresponding space of compatible (almost) complex structures by Donaldson [10].

It is usually quite difficult to make this general framework rigorous and precise in infinite dimensions, and we will make no attempt at that here. However, its formal application is often very useful as a guide to the results one should expect and to the approach one might take in proving them. We will present here two results, Theorems 3.5 and 3.6, that are an example of this. Their rigorous proofs will appear in [2].

The paper is organized as follows. In Section 2, we describe the general moment map framework and how it applies to the action of a symplectomorphism group on the corresponding space of compatible (almost) complex structures. In Section 3, we discuss the particular case of rational ruled surfaces, where the geometric picture suggested by the moment map framework is quite accurate and gives a new approach to understanding the topology of their symplectomorphism groups. This approach is based on the fact that the space of compatible integrable complex structures on any symplectic rational ruled surface is (weakly) contractible (Theorem 3.5). Any time this condition holds, there should be a direct relationship between the problem of understanding the topology of the symplectomorphism group and the following two problems: understanding the deformation theory of compatible complex structures and understanding the topology of the groups of complex automorphisms of these complex structures. This general relation is explained in Section 4.

2. Moment map geometry

In this section, following [10], we recall a general moment map framework and how it applies to the action of a symplectomorphism group on the corresponding space of compatible almost-complex structures.

General framework. Let G be a Lie group, \mathcal{G} its Lie algebra, $\langle \cdot, \cdot \rangle$ an inner product on \mathcal{G} invariant under the adjoint action, \mathcal{G}^* the dual Lie algebra naturally identified with \mathcal{G} via $\langle \cdot, \cdot \rangle$ and $G^{\mathbb{C}}$ a complexification of G .

Let (X, J, Ω) be a Kähler manifold equipped with an action of G by Kähler isometries, i.e., a homomorphism

$$\rho : G \longrightarrow \text{Iso}(X, J, \Omega) = \text{Hol}(X, J) \cap \text{Symp}(X, \Omega).$$

Suppose this action satisfies the following two conditions:

- (i) the holomorphic action of G on (X, J) extends to a holomorphic action of $G^{\mathbb{C}}$ on (X, J) ;
- (ii) the symplectic action of G on (X, Ω) admits a suitably normalized equivariant moment map $\mu : X \rightarrow \mathcal{G}^*$.

Then we have the following two general principles.

General Principle I. The complex and symplectic quotients of X by G are naturally identified. More precisely,

$$\frac{\mu^{-1}(0)}{G} = \frac{X^s}{G^{\mathbb{C}}}$$

where $X^s \subset X$ is an open, $G^{\mathbb{C}}$ -invariant, subset of “stable points.” We will not define here this notion of stability, the important point being that it should only depend on the holomorphic geometry of the situation. The content of this principle is that on each stable $G^{\mathbb{C}}$ -orbit, there is a point $p \in \mu^{-1}(0)$, unique up to the action of G .

General Principle II. The map $\|\mu\|^2 \equiv \langle \mu, \mu \rangle : X \rightarrow \mathbb{R}$ behaves like a G -invariant Morse–Bott function, whose critical manifolds compute the equivariant cohomology $H_G^*(X) \equiv H^*(X \times_G EG)$ (over \mathbb{Q} and, in good special cases, also over \mathbb{Z}).

Combining these two general principles, one gets the following geometric picture for the action of G on X :

- The gradient flow of $-\|\mu\|^2$ induces an invariant stratification

$$X = V_0 \sqcup V_1 \sqcup V_2 \sqcup \dots,$$

where each V_k is the stable manifold of some critical set C_k of $\|\mu\|^2$.

- Let \mathcal{O}_k denote the coadjoint orbit $G \cdot \xi_k \subset \mathcal{G}^*$, where $\xi_k = \mu(p_k)$ for some $p_k \in C_k$. Then

$$\frac{V_k}{G^{\mathbb{C}}} \simeq \frac{\mu^{-1}(\mathcal{O}_k)}{G}.$$

If $C_0 = \mu^{-1}(0)$, then $\mathcal{O}_0 = \{0\}$ and $V_0 = X^s$.

- The equivariant cohomology $H_G^*(X)$ can be computed from $H_G^*(V_k)$, $k = 0, 1, 2, \dots$ (over \mathbb{Q} and, in good special cases, also over \mathbb{Z}).

Symplectomorphism groups and compatible complex structures.

Consider a compact symplectic manifold (M, ω) , of dimension $2n$ and assume that $H^1(M, \mathbb{R}) = 0$. Let $G \equiv \text{Symp}(M, \omega)$ be the symplectomorphism group of (M, ω) . This is an infinite dimensional Lie group whose Lie algebra \mathcal{G} can be identified with the space of functions on M with integral zero:

$$\mathcal{G} = C_0^\infty(M) \equiv \left\{ f : M \rightarrow \mathbb{R} : \int_M f \frac{\omega^n}{n!} = 0 \right\}.$$

\mathcal{G} has a natural invariant inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle \equiv \int_M f \cdot g \frac{\omega^n}{n!},$$

which will be used to identify \mathcal{G}^* with \mathcal{G} .

Consider now the space $\mathcal{J}(M, \omega)$ of almost complex structures J on M , which are compatible with ω , i.e., for which the bilinear form

$$g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$$

is a Riemannian metric on M . This is the space of sections of a bundle over M with fiber the contractible symmetric Kähler manifold $Sp(2n, \mathbb{R})/U(n) \equiv$ Siegel upper half space [24]. This fiberwise symmetric Kähler structure, together with the volume form induced by ω on M , turns $\mathcal{J}(M, \omega)$ into an infinite dimensional (contractible) Kähler manifold.

The symplectomorphism group G acts naturally on $\mathcal{J}(M, \omega)$ by Kähler isometries:

$$\phi \cdot J \equiv \phi_*(J) = d\phi \circ J \circ d\phi^{-1}, \quad \forall \phi \in G, \quad J \in \mathcal{J}(M, \omega).$$

To fit the previous general framework, this action should satisfy conditions (i) and (ii).

The first (holomorphic) condition poses an immediate problem since there is no complexification $G^{\mathbb{C}}$ of the symplectomorphism group G . However, we can certainly complexify the Lie algebra \mathcal{G} to

$$\mathcal{G}^{\mathbb{C}} \equiv \left\{ f : M \rightarrow \mathbb{C} : \int_M f \frac{\omega^n}{n!} = 0 \right\}$$

and the infinitesimal action of \mathcal{G} on $\mathcal{J}(M, \omega)$ extends to an action of $\mathcal{G}^{\mathbb{C}}$, since the complex structure on $\mathcal{J}(M, \omega)$ is integrable. This gives rise to an integrable complex distribution on $\mathcal{J}(M, \omega)$ whose leaves play the role of “connected components of orbits of the group $G^{\mathbb{C}}$.”

In the holomorphic side of General Principles I and II that we want to apply, $G^{\mathbb{C}}$ is not that important when compared with the role played by its orbits. The geometric meaning of these “ $G^{\mathbb{C}}$ -orbits” becomes quite clear if one restricts the actions under consideration to the invariant Kähler submanifold X of compatible integrable complex structures

$$X \equiv \mathcal{J}^{\text{int}}(M, \omega) \subset \mathcal{J}(M, \omega),$$

determined by the vanishing of the Nijenhuis tensor. Here, it follows from Donaldson’s analysis in [10] that

$J, J' \in X$ belong to the same “ $G^{\mathbb{C}}$ -orbit” iff there exists $\varphi \in \text{Diff}(M)$ such that

$$[\varphi^*(\omega)] = [\omega] \in H^2(M, \mathbb{R}) \quad \text{and} \quad \varphi_*(J) = J'.$$

This explicit description of a “ $G^{\mathbb{C}}$ -orbit” is good enough to consider that the holomorphic action of $G = \text{Symp}(M, \omega)$ on $X = \mathcal{J}^{\text{int}}(M, \omega)$ satisfies condition (i).

Regarding condition (ii), Donaldson [10] shows that there always exists an equivariant and suitably normalized moment map

$$\mu : \mathcal{J}(M, \omega) \rightarrow \mathcal{G}^* \cong C_0^\infty(M)$$

for the symplectic action of G on $\mathcal{J}(M, \omega)$ given by

$$\mu(J) = (\text{Hermitian scalar curvature } S(J) \text{ of the metric } g_J) - d,$$

where d is the constant defined by

$$d \cdot \int_M \frac{\omega^n}{n!} \equiv 2\pi c_1(M) \wedge [\omega]^{n-1}(M) = \int_M S(J) \frac{\omega^n}{n!}.$$

Note that on $X \subset \mathcal{J}(M, \omega)$, i.e., for integrable J , the Hermitian scalar curvature $S(J)$ coincides with the usual scalar curvature of the Riemannian metric g_J .

We have concluded that the Kähler action of G on X satisfies conditions (i) and (ii) of the general framework, and so General Principles I and II should apply. What do they say in this context?

General Principle I says that each stable compatible complex structure is diffeomorphic to one in $\mu^{-1}(0)$, unique up to the action of G . Since

$$J \in \mu^{-1}(0) \iff \mu(J) = 0 \iff S(J) = d = \text{constant},$$

this says that on each diffeomorphism class of stable compatible complex structures there should exist a unique $\text{Symp}(M, \omega)$ -orbit whose corresponding Kähler metric has constant scalar curvature. (See the work of Donaldson [11–13], exploring this consequence of general principle I.)

General Principle II says that the critical points of

$$\|\mu\|^2 : X = \mathcal{J}^{\text{int}}(M, \omega) \longrightarrow \mathbb{R}, \quad \|\mu\|^2(J) = \int_M S^2(J) \frac{\omega^n}{n!} + \text{constant},$$

determine the equivariant cohomology $H_G^*(X)$.

These critical points are, in particular, extremal Kähler metrics in the sense of Calabi [6, 7]. When extremal Kähler metrics exist, they minimize $\|\mu\|^2$ on the corresponding “ $G^{\mathbb{C}}$ -orbit” [16] and are conjecturally unique up to the action of G [8].

In the concrete examples, we will discuss (Riemann surfaces and rational ruled surfaces) these general principles do hold. Whenever this is the case, one gets the following geometric picture for the action of $G = \text{Symp}(M, \omega)$ on $X = \mathcal{J}^{\text{int}}(M, \omega)$:

- There is a stratification of X of the form

$$X = V_0 \sqcup V_1 \sqcup V_2 \sqcup \dots .$$

In this stratification, each V_k contains the set of compatible complex structures which are diffeomorphic to an extremal one with normalized scalar curvature in

$$\mathcal{O}_k \equiv \text{coadjoint orbit } G \cdot (S(J_k) - d) \text{ in } C_0^\infty(M) \cong \mathcal{G}^*,$$

where J_k is some critical compatible complex structure in V_k .

- “ $V_k/G^{\mathbb{C}}$ ” $\equiv \{“G^{\mathbb{C}}$ -orbits” in $V_k\} \simeq \mu^{-1}(\mathcal{O}_k)/G$ is some moduli space of complex structures that one might try to understand using methods from complex geometry (deformation theory).
- Let \mathcal{O}_{J_k} denote the “ $G^{\mathbb{C}}$ -orbit” through some extremal $J_k \in V_k$ and let $K_k \equiv \text{Iso}(M, \omega, J_k) \subset G$. Then, if the group $\text{Hol}_{[\omega]}(M, J_k)$ of holomorphic automorphisms which preserve the cohomology class of ω is the complexification of K_k (this is always the case if the groups are connected by [7]), then the inclusion

$$\frac{G}{K_k} \cong G \cdot J_k \hookrightarrow \mathcal{O}_{J_k} = \frac{“G^{\mathbb{C}}”}{\text{Hol}_{[\omega]}(M, J_k)}$$

is a homotopy equivalence.

- The equivariant cohomology $H_G^*(X)$ can be computed from $H_G^*(V_k)$, $k = 0, 1, 2, \dots$. From the previous two points, each $H_G^*(V_k)$ should be determined from finite dimensional considerations involving moduli spaces of complex structures and subgroups of isometries in $G = \text{Symp}(M, \omega)$. Recall that $\mathcal{J}(M, \omega)$ is always contractible. If $X = \mathcal{J}^{\text{int}}(M, \omega) \subset \mathcal{J}(M, \omega)$ is also contractible, then

$$H_G^*(X) = H^*(BG).$$

3. Rational ruled surfaces

In this section, we discuss the particular case of rational ruled surfaces, formulating the precise results suggested by the framework of Section 2.

Symplectic structures. As smooth 4-manifolds, rational ruled surfaces are S^2 -bundles over S^2 . Since $\pi_2(B \text{Diff}^+(S^2)) \cong \pi_2(BSO(3)) \cong \mathbb{Z}/2$, there are only two diffeomorphism types classified by the second Stiefel–Whitney class of the bundle (the mod 2 reduction of the Euler class): a trivial S^2 -bundle over S^2 and a non-trivial S^2 -bundle over S^2 . Since the story for each of these is analogous, we will concentrate here on the trivial bundle, i.e.,

$$M = S^2 \times S^2.$$

Symplectic structures on $S^2 \times S^2$ are classified by the following theorem [19].

Theorem 3.1. *If ω is a symplectic form on $S^2 \times S^2$, then it is diffeomorphic to $\lambda\sigma \oplus \mu\sigma$ for some real $\lambda, \mu > 0$, where σ denotes the standard area form on S^2 with $\int_{S^2} \sigma = 1$.*

Since the symplectomorphism group and the space of compatible almost complex structures are not affected by positive scalings of the symplectic form and we can switch the two S^2 -factors, it will suffice to consider symplectic structures of the form

$$\omega_\lambda = \lambda\sigma \oplus \sigma \quad \text{with} \quad 1 \leq \lambda \in \mathbb{R}.$$

From now on, we will use the following notation: $M_\lambda = (S^2 \times S^2, \omega_\lambda)$, $1 \leq \lambda \in \mathbb{R}$; $G_\lambda = \text{Symp}(M_\lambda) = \text{symplectomorphisms of } M_\lambda$; $\mathcal{J}_\lambda = \mathcal{J}(M_\lambda) = \text{compatible almost complex structures}$; $X_\lambda = \mathcal{J}^{\text{int}}(M_\lambda) = \text{compatible integrable complex structures}$.

We will also use the following obvious isomorphism:

$$\begin{aligned} H_2(S^2 \times S^2, \mathbb{Z}) &\xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z} \\ m[S^2 \times \text{pt}] + n[\text{pt} \times S^2] &\longmapsto (m, n). \end{aligned}$$

Compatible integrable complex structures. As a complex manifold, a rational ruled surface is a holomorphic \mathbb{P}^1 -bundle over \mathbb{P}^1 . These are the well-known Hirzebruch surfaces

$$H_k = P(\mathcal{O} \oplus \mathcal{O}(-k)) \quad \text{for some } k \in \mathbb{N}_0,$$

where we write $\mathcal{O}(-1)$ for the tautological line bundle over \mathbb{P}^1 and $P(E)$ for the projectivization of a vector bundle E .

Any complex structure J on $S^2 \times S^2$ is isomorphic to H_{2k} for some $k \in \mathbb{N}_0$, while the “odd” Hirzebruch surfaces are diffeomorphic to the non-trivial S^2 -bundle over S^2 [22]. When $(S^2 \times S^2, J)$ has two embedded \mathbb{P}^1 ’s with self-intersection 0 and themselves intersecting at one point, then $(S^2 \times S^2, J) \cong H_0$. When $(S^2 \times S^2, J)$ has an embedded \mathbb{P}^1 with self-intersection $-2k < 0$, then $(S^2 \times S^2, J) \cong H_{2k}$.

To understand which of these complex structures J can be made compatible with a symplectic form ω_λ , for some $1 \leq \lambda \in \mathbb{R}$, it is important to note that the compatibility condition implies that the symplectic form evaluates positively on any J -holomorphic curve. Hence, for a compatible $J \in X_\lambda$, a homology class $(m, n) \in H_2(S^2 \times S^2; \mathbb{Z})$ can only be represented by a J -holomorphic curve if $\lambda m + n > 0$. This rules out embedded curves with self-intersection less than -2ℓ , where $\ell \in \mathbb{N}_0$ is such that $\ell < \lambda \leq \ell + 1$. In particular, the class $(1, -k) \in H_2(S^2 \times S^2; \mathbb{Z})$, with self-intersection $-2k$, can only be represented by a J -holomorphic curve for some $J \in X_\lambda$ if $\lambda - k > 0$.

This turns out to be the only relevant condition. In fact, we have the following theorem.

Theorem 3.2. *Given $1 \leq \lambda \in \mathbb{R}$, there is a stratification of X_λ of the form*

$$X_\lambda = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_\ell,$$

with $\ell \in \mathbb{N}_0$ such that $\ell < \lambda \leq \ell + 1$ and where:

(i)

$$\begin{aligned} V_k &\equiv \{J \in X_\lambda : (S^2 \times S^2) \cong H_{2k}\} \\ &= \{J \in X_\lambda : (1, -k) \in H_2(M, \mathbb{Z}) \text{ is represented} \\ &\quad \text{by a } J\text{-holomorphic sphere}\}. \end{aligned}$$

(ii) V_0 is open and dense in X_λ . For $k \geq 1$, V_k has codimension $4k - 2$ in X_λ .

(iii) $\overline{V_k} = V_k \sqcup V_{k+1} \sqcup \cdots \sqcup V_\ell$.

(iv) For each $k \in \mathbb{N}_0$, there is a complex structure $J_k \in V_k$, unique up to the action of G_λ , for which $g_{\lambda,k} \equiv \omega_\lambda(\cdot, J_k \cdot)$ is an extremal Kähler metric.

(v) Denoting by K_k the Kähler isometry group of $(S^2 \times S^2, \omega_\lambda, J_k)$, we have that

$$K_k \cong \begin{cases} \mathbb{Z}/2 \times (SO(3) \times SO(3)), & \text{if } k = 0; \\ S^1 \times SO(3), & \text{if } k \geq 1. \end{cases}$$

(vi) Given $J \in V_k$, there exists $\varphi \in \text{Diff}(S^2 \times S^2)$ such that

$$[\varphi^*(\omega_\lambda)] = [\omega_\lambda] \in H^2(S^2 \times S^2; \mathbb{R}) \quad \text{and} \quad \varphi_*(J_k) = J,$$

so each stratum V_k consists of a unique “ $G_\lambda^{\mathbb{C}}$ ”-orbit, the orbit through J_k . Moreover, the inclusion

$$\begin{aligned} \left(\frac{G_\lambda}{K_k} \right) &\longrightarrow V_k = \mathcal{O}_{J_k} \\ [\psi] &\longmapsto \psi_*(J_k) \end{aligned}$$

is a weak homotopy equivalence.

(vii) Each V_k has a tubular neighborhood $NV_k \subset X_\lambda$, with normal slice given by

$$H^1(H_{2k}, \Theta) \cong \mathbb{C}^{2k-1},$$

where $\Theta =$ sheaf of holomorphic vector fields on H_{2k} .

(viii) For $k \geq 1$, the representation of $K_k \cong S^1 \times SO(3)$ on the normal slice \mathbb{C}^{2k-1} at $J_k \in V_k$ is the following: S^1 acts diagonally and $SO(3)$ acts irreducibly with highest weight $2(k - 1)$.

Proof. Points (i), (ii), (iii), (v) and (vii) follow from standard complex geometry and deformation theory applied to complex structures on $S^2 \times S^2$ [9, I.6]. One needs to check that standard deformation theory can in fact be used here in the context of compatible complex structures. This, together with points (vi) and (viii), will be proved in [2]. Point (iv) is proved in [6]. \square

This theorem shows that the geometric picture suggested by the moment map framework of Section 2 is quite accurate for rational ruled surfaces. It implies, by standard equivariant cohomology theory [2], the following corollary.

Corollary 3.3. *Given $\ell \in \mathbb{N}_0$ and $\lambda \in]\ell, \ell + 1]$, we have that*

$$H_{G_\lambda}^*(X_\lambda; \mathbb{Z}) \cong H^*(BSO(3) \times BSO(3); \mathbb{Z}) \oplus \bigoplus_{k=1}^\ell \Sigma^{4k-2} H^*(BS^1 \times BSO(3); \mathbb{Z}),$$

where \cong indicates a group isomorphism.

Compatible almost complex structures. We will now look at the contractible space \mathcal{J}_λ of compatible almost complex structures on $M_\lambda = (S^2 \times S^2, \omega_\lambda)$ for a given $1 \leq \lambda \in \mathbb{R}$. Gromov [15] was the first to study the topology of the symplectomorphism group G_λ by analyzing its natural action on this space \mathcal{J}_λ . He used pseudo-holomorphic curves techniques and this approach turned out to be quite fruitful [1, 3]. In fact, the action of G_λ is compatible with a natural geometric stratification of \mathcal{J}_λ , analogous to the one presented in Theorem 3.2 for $X_\lambda \subset \mathcal{J}_\lambda$. Replacing holomorphic spheres by pseudo-holomorphic ones, and complex deformation theory by gluing techniques for pseudo-holomorphic spheres, one can prove the following theorem [21].

Theorem 3.4. *Given $1 \leq \lambda \in \mathbb{R}$, there is a stratification of the contractible space \mathcal{J}_λ of the form*

$$\mathcal{J}_\lambda = U_0 \sqcup U_1 \sqcup \cdots \sqcup U_\ell,$$

with $\ell \in \mathbb{N}_0$ such that $\ell < \lambda \leq \ell + 1$ and where:

(i)

$U_k \equiv \{J \in \mathcal{J}_\lambda : (1, -k) \in H_2(M; \mathbb{Z}) \text{ is represented by a } J\text{-holomorphic sphere}\}.$

In particular, $V_k = U_k \cap X_\lambda \subset U_k$.

(ii) U_0 is open and dense in \mathcal{J}_λ . For $k \geq 1$, U_k has codimension $4k - 2$ in \mathcal{J}_λ .

(iii) $\overline{U}_k = U_k \sqcup U_{k+1} \sqcup \cdots \sqcup U_\ell$.

(iv) The inclusion

$$\begin{aligned} \left(\begin{array}{c} G_\lambda \\ K_k \end{array} \right) &\longrightarrow U_k \\ [\psi] &\longmapsto \psi_*(J_k) \end{aligned}$$

is a weak homotopy equivalence, where $J_k \in V_k \subset U_k$ and $K_k = \text{Iso}(S^2 \times S^2, \omega_\lambda, J_k)$ were characterized in Theorem 3.2. In particular, the inclusion

$$V_k \longrightarrow U_k$$

is also a weak homotopy equivalence.

(v) Each U_k has a tubular neighborhood $NU_k \subset \mathcal{J}_\lambda$ which fibers over U_k as a ball bundle.

Contractibility of X_λ . Given $\ell \in \mathbb{N}_0$ and $1 \leq \lambda \in]\ell, \ell + 1]$, we can combine the results of Theorems 3.2 and 3.4 to obtain a finite family of diagrams, one for each $0 \leq k \leq \ell$, of the form

$$\begin{array}{ccccc} F_k & \longrightarrow & NV_k & \longrightarrow & V_k \subset X_\lambda \\ \downarrow & & \downarrow & & \downarrow \\ F_k & \longrightarrow & NU_k & \longrightarrow & U_k \subset \mathcal{J}_\lambda \end{array}$$

where the vertical arrows are inclusions, the one on the left representing the identity between the fibers of the tubular neighborhoods over $V_k \subset U_k$. These diagrams are G_λ -equivariant in a suitable sense. Given that \mathcal{J}_λ is contractible and V_k is weakly homotopy equivalent to U_k , one can use this finite family of diagrams to prove the following theorem [2].

Theorem 3.5. *Given $1 \leq \lambda \in \mathbb{R}$, the space X_λ of compatible integrable complex structures on $(S^2 \times S^2, \omega_\lambda)$ is weakly contractible.*

As far as we know, these are the first known examples of dimension greater than two where the topology of the space of compatible integrable complex structures is understood.

Cohomology of BG_λ . Theorem 3.5 implies that

$$H_{G_\lambda}^*(X_\lambda; \mathbb{Z}) \cong H^*(BG_\lambda; \mathbb{Z}).$$

Combining this isomorphism with Corollary 3.3, we get the following theorem.

Theorem 3.6. *Given $\ell \in \mathbb{N}_0$ and $\lambda \in]\ell, \ell + 1]$, we have that*

$$H^*(BG_\lambda; \mathbb{Z}) \cong H^*(BSO(3) \times BSO(3); \mathbb{Z}) \oplus \bigoplus_{k=1}^{\ell} \Sigma^{4k-2} H^*(BS^1 \times BSO(3); \mathbb{Z}),$$

where \cong indicates a group isomorphism.

Although we used a new point of view, this theorem is not the first result regarding the topology of BG_λ :

- A formula for the rational cohomology ring of BG_λ was determined in [3] for any $1 \leq \lambda \in \mathbb{R}$. That formula is incomplete since the relation that determines it is only correct up to “higher order terms” that have to be taken into account. See [2] for details regarding this issue.
- The integral cohomology ring of BG_λ was determined for $\lambda = 1$ in [15] (where Gromov proves that G_1 is homotopy equivalent to $SO(3) \times SO(3)$) and for $1 < \lambda \leq 2$ in [4].

We hope to include in [2] a description of the ring structure of $H^*(BG_\lambda; \mathbb{Z})$ for any $1 \leq \lambda \in \mathbb{R}$.

4. Symplectomorphisms and complex automorphisms

Let (M, ω) be a symplectic manifold and $\mathcal{J}^{\text{int}}(M, \omega)$ the corresponding space of compatible integrable complex structures. The aim of this section is to explain how the condition

$$(1) \quad \mathcal{J}^{\text{int}}(M, \omega) \sim *$$

that the space of compatible complex structures be contractible, essentially equates the problem of understanding the topology of $\text{Symp}(M, \omega)$ with the following two problems.

- Understanding the (large-scale) deformation theory of complex structures compatible with ω .
- Understanding the topology of the groups of complex automorphisms of each of these complex structures.

The same is true if one replaces compatible complex structures by tame complex structures in which case the deformation theory in question is the usual deformation theory of Kodaira and Spencer (as the tameness condition is open).

We should point out straight away that we only know condition (1) to hold in a few very simple examples: Riemann surfaces (where the condition holds

because all almost complex structures are integrable) and simple rational complex surfaces such as \mathbb{P}^2 and rational ruled surfaces. The work of Lalonde and Pinsonnault [20] suggests that condition (1) also holds for blow-ups of \mathbb{P}^2 at two points. It would be very interesting to have some understanding of the generality of (1).

The reason why (1) equates the problems above is that it establishes a sort of weak equivalence between two moduli problems—one in symplectic geometry and another in complex geometry. To explain this, we must first recall some basic facts regarding topological groupoids. The reader is referred to [17, Section I] for more explanation and details.

Topological groupoids. A topological groupoid is a small topological category in which every morphism is invertible. Thus, a topological groupoid consists of a pair of spaces (O, M) together with some structure maps. O is the space of objects and M the space of isomorphisms between the objects in O . The structure maps are $\iota : O \rightarrow M$ assigning the identity isomorphism to each object, two maps $d_0, d_1 : M \rightarrow O$ assigning to a morphism its domain and range, a composition map $\mu : M \times_O M \rightarrow M$ and an inverse map $c : M \rightarrow M$ satisfying the obvious identities.

To a groupoid one can associate the space of isomorphism classes O/I , which we will call the *coarse moduli space*, defined as the quotient of O by the equivalence relation generated by $d_0(\alpha) \sim d_1(\alpha)$. Of course, the groupoid carries much more information than the quotient space.

In many cases, including all those that will concern us, moduli problems can be described in terms of topological groupoids of the following special form:

Example 4.1. Let G be a topological group acting on a space X . We define a groupoid $\Gamma(X, G)$ with objects $O = X$ and $M = G \times X$. The structure maps are $\iota(x) = (1, x)$, $d_0(g, x) = x$, $d_1(g, x) = g \cdot x$, $\mu(g, x, h, g \cdot x) = (hg, x)$ and $c(g, x) = (g^{-1}, g \cdot x)$. The associated quotient space is the space of orbits X/G .

Note that $\Gamma(*, G)$ can be naturally identified with the topological group G .

Let $\Gamma = (O, M)$ be a topological groupoid, B be a space and $\mathcal{U} = \{U_i\}$ be an open cover of B . A 1-cocycle over \mathcal{U} with values in Γ is the assignment to each pair i, j of a map

$$g_{ij} : U_i \cap U_j \longrightarrow M$$

satisfying the cocycle condition $\mu(g_{ij}, g_{jk}) = g_{ik}$ on triple intersections. In particular, $g_{ii} : U_i \rightarrow M$ sends each point of U_i to an identity morphism and hence amounts to an object selection map $f_i : U_i \rightarrow O$. The cocycle condition then says that these selections are compatible so that the 1-cocycle represents a continuous family of objects over B . One defines equivalence

between 1-cocycles in the obvious way. An equivalence class of 1-cocycles is called a Γ -structure on X .

Two Γ -structures σ_i , $i = 0, 1$, on Y are said to be *concordant*¹ if there is a Γ -structure σ on $Y \times [0, 1]$ such that $\sigma|_{Y \times 0} = \sigma_0$ and $\sigma|_{Y \times 1} = \sigma_1$. If $\Gamma = \Gamma(*, G)$ is a topological group, concordance is the same as equality, but this is generally not the case.

Any topological groupoid Γ has a classifying space $B\Gamma$ which is a homotopy invariant version of the coarse moduli space. It is determined by the equation

$$[Y, B\Gamma] = \{\text{concordance classes of } \Gamma\text{-structures on } Y\}$$

In the case of Example 4.1, the classifying space of $\Gamma(X, G)$ is the Borel construction on the G -space X . That is, if we write EG for a contractible free G -space, we have

$$B\Gamma(X, G) \simeq EG \times_G X$$

In particular, if the G -space X consists of a single orbit and H denotes the isotropy subgroup of one of the points of X , we have

$$(2) \quad B\Gamma \simeq \frac{EG \times_G G}{H} = \frac{EG}{H} \simeq BH$$

and so the classifying space of the groupoid is the same as the classifying space of the isotropy group (of any object). If there is more than one orbit, then the map $EG \rightarrow *$ still induces a map

$$\pi : B\Gamma \longrightarrow \frac{X}{G}$$

with fibers

$$\pi^{-1}(Gx) \simeq B \text{Aut}(x),$$

where we have written $\text{Aut}(x)$ for the isotropy group of x . Intuitively, this says that the space $B\Gamma$ is obtained by gluing the classifying spaces of the automorphism groups $B \text{Aut}(x)$ via the topology of the moduli space X/G , which one can write

$$(3) \quad B\Gamma \simeq \int_{X/G} B \text{Aut}(x).$$

In good situations, this is a precise statement [14, Proposition, p. 183]. Even if it is not, it still provides a useful guide to understanding the topology of $B\Gamma$. We will see how this plays out in Examples 4.4 and 4.5 below.

We will say that a map $\phi : \Gamma_1 \rightarrow \Gamma_2$ of topological groupoids is a *weak equivalence* if the induced map of classifying spaces

$$B\phi : B\Gamma_1 \rightarrow B\Gamma_2$$

¹The term used by Haefliger is homotopic.

is a weak homotopy equivalence. In the examples we are concerned with, the source of weak equivalences is the following immediate consequence of the homotopy invariance of the Borel construction:

Lemma 4.2. *If X and Y are G -spaces and $f : X \rightarrow Y$ is a G -equivariant map which is a weak equivalence, then the induced map of groupoids $\Gamma f : \Gamma(X, G) \rightarrow \Gamma(Y, G)$ is a weak equivalence.*

Examples of moduli problems. Let M be a compact manifold and $G = \text{Diff}^+(M)$ the group of orientation preserving diffeomorphisms of M with the C^∞ topology. Consider the following G -spaces:

- (i) The space $\mathcal{J}^{\text{int}}(M)$ of complex structures on M compatible with some (unspecified) symplectic form on M .
- (ii) The space $\Omega^{\text{int}}(M)$ of symplectic forms on M .
- (iii) The space $K^{\text{int}}(M) = \{(J, \omega) \in \mathcal{J}^{\text{int}}(M) \times \Omega^{\text{int}}(M) : J \text{ compatible with } \omega\}$ of Kähler structures on M .

We will write $\Gamma\mathcal{J}$, $\Gamma\Omega$ and ΓK for the associated topological groupoids. The projections

$$\mathcal{J}^{\text{int}}(M) \xleftarrow{\pi_1} K^{\text{int}}(M) \xrightarrow{\pi_2} \Omega^{\text{int}}(M)$$

are G -equivariant. The fibers of π_1 are always convex, so π_1 is a weak equivalence. Lemma 4.2 then says that the groupoids $\Gamma\mathcal{J}$ and ΓK are weakly equivalent. Condition (1) is precisely the condition that the fibers of π_2 be contractible. If in addition we know that π_2 is surjective, i.e., that all symplectic structures on M are Kähler, then Lemma 4.2 says that $\Gamma\Omega$ is also weakly equivalent to $\Gamma\mathcal{J}$ and ΓK .

Remark 4.3. *We can also consider the non-integrable moduli problems derived from the space $\mathcal{J}(M)$ of almost complex structures compatible with some symplectic form on M and $K(M)$ of almost-Kähler structures. These groupoids are always weakly equivalent to $\Gamma\Omega$. In this light, condition (1) can be seen as the requirement that the inclusions $\mathcal{J}^{\text{int}}(M) \rightarrow \mathcal{J}(M)$ or $K^{\text{int}}(M) \rightarrow K(M)$ induce weak equivalences of topological groupoids.*

Now consider the following variation on the previous situation. Fix a symplectic form ω on M and consider the orbit

$$X_0 = \text{Diff}^+(M) \cdot \omega \subset \Omega^{\text{int}}(M).$$

Let $X_1 = \pi_1^{-1}(X_0) \subset K^{\text{int}}(M)$ and $X_2 = \pi_2(X_1) \subset \mathcal{J}^{\text{int}}(M)$ so that we have G -equivariant maps

$$X_0 \xleftarrow{\pi_1} X_1 \xrightarrow{\pi_2} X_2$$

If (1) holds for the symplectic form ω , then as before all three groupoids are weakly equivalent and using (2) and (3) we have

$$(4) \quad B \text{ Symp}(M, \omega) \simeq B\Gamma(X_0, G) \simeq B\Gamma(X_2, G) \simeq \int_{[J] \in Z} B \text{ Aut}(J)$$

where $Z = X_2/\text{Diff}^+(M)$ is the coarse moduli space of complex structures compatible with ω , and $\text{Aut}(J)$ is the group of biholomorphisms of the complex manifold (M, J) .

Example 4.4 Elliptic curves. *Let $M = S^1 \times S^1$ and let ω be the standard symplectic form. The topology of the diffeomorphism groups of surfaces is well understood. The inclusion of the affine orientation preserving automorphisms of $S^1 \times S^1$ in $\text{Symp}(M, \omega)$ is a weak equivalence, i.e.,*

$$SL_2(\mathbb{Z}) \ltimes (S^1 \times S^1) \simeq \text{Symp}(M, \omega).$$

The complex structures on a torus (which are all compatible with any given area form) are parametrized by points in the upper half plane up to the action of $SL(2, \mathbb{Z})$ by fractional linear transformations. Thus the space Z in (4) is obtained from

$$\Omega = \{z \in \mathbb{C} : -\frac{1}{2} \leq \text{Re}(z) \leq \frac{1}{2}, |z| \geq 1, \text{Im}(z) > 0\}$$

by identifying the boundary points through the maps $z \mapsto z+1$ and $z \mapsto -1/z$. If E_z denotes the elliptic curve parametrized by $z \in \Omega$, we have

$$\text{Aut}_{\mathbb{C}}(E_z) = \text{Aut}(\langle 1, z \rangle) \ltimes E_z$$

where $\text{Aut}(\langle 1, z \rangle)$ denotes the automorphisms of the lattice generated by $\{1, z\}$. Most lattices have multiplication by -1 as their only non-trivial automorphism. The exceptions are $\langle 1, i \rangle$ which has $\mathbb{Z}/4$ as automorphism group and $\langle 1, e^{\pi i/3} \rangle$ which has $\mathbb{Z}/6$ as automorphism group.

Thus, in this case, (4) says that $B \text{Symp}(M, \omega)$ can be expressed as the double mapping cylinder (or homotopy pushout) of the maps

$$B\left(\frac{\mathbb{Z}}{4} \ltimes (S^1 \times S^1)\right) \longleftarrow B\left(\frac{\mathbb{Z}}{2} \ltimes (S^1 \times S^1)\right) \longrightarrow B\left(\frac{\mathbb{Z}}{6} \ltimes (S^1 \times S^1)\right)$$

which amounts to the familiar decomposition of $SL(2, \mathbb{Z})$ as the amalgam of $\mathbb{Z}/4$ and $\mathbb{Z}/6$ over $\mathbb{Z}/2$ [23, 1.5.3].

Example 4.5. $S^2 \times S^2$. Let us resume Section 3 from the point of view of this section. A theorem of Qin [22] says that any complex structure on $S^2 \times S^2$ is isomorphic to one of the even Hirzebruch surfaces H_{2k} , with $k \geq 0$. Theorem 3.1 (Lalonde-McDuff) says that any symplectic form on $S^2 \times S^2$ is, up to scale, diffeomorphic to $\omega_\lambda = \lambda\sigma \oplus \sigma$ for some $\lambda \geq 1$. The complex structures which are compatible with ω_λ are the H_{2k} with $k < \lambda$. So in this example, the coarse moduli space Z is a finite set of points (with a non-Hausdorff topology) and (4) expresses $B \text{Symp}(M, \omega_\lambda)$ as a union of spaces with the homotopy type of $B \text{Aut}_{\mathbb{C}}(H_{2k})$ for the allowable values of k .

The way the different spaces $B \text{Aut}_{\mathbb{C}}(H_{2k})$ fit together is controlled by the deformation theory of the complex structures, which in this case is the usual Kodaira–Spencer theory.

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