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SYMPLECTIC DEFORMATIONS OF KÄHLER MANIFOLDS

Paolo de Bartolomeis

Given a compact symplectic manifold $(M,\, \kappa)$, $H^2(M,\, \mathbb{R})$ represents, in a natural sense, the tangent space of the moduli space of germs of deformations of the symplectic structure. In the case $(M,\, \kappa,\, J)$ is a compact Kähler manifold, the author provides a complete description of the subset of $H^2(M,\, \mathbb{R})$ corresponding to Kähler deformations, including the non-generic case, where (at least locally) some hyperkähler manifold factors out from M. Several examples are also discussed.

1. Introduction

The naïf deformation theory of symplectic manifolds is quite simple: let $(M,\, \kappa)$ be a compact symplectic manifold and let $\alpha \in \wedge^2(M,\, \mathbb{R})$, $d\alpha = 0$: then

$$\kappa_t := \kappa + t\alpha$$

is a (germ of) curve of symplectic structures having tangent α at 0; moreover, Moser's lemma (cf. [**2**]) ensures that $\kappa_t = \phi_t^*(\kappa)$ for a path of diffeomorphisms with $\phi_0 = id_M \iff \alpha = d\beta$ and so $H^2(M,\, \mathbb{R})$ is the tangent space of the moduli space of germs of deformations of symplectic structures and the theory is totally unobstructed (for a non-naïf version, see [**1**]).

Let $(M,\, \kappa,\, J)$ be a compact Kähler manifold: therefore, J is a κ-calibrated holomorphic structure and so $g = g_J := \kappa(J\cdot,\, \cdot)$ is a positive definite Hermitian metric; we want to investigate the subset of $H^2(M,\, \mathbb{R})$ corresponding to Kähler deformations of κ.
We have the following

Theorem 1.1. *Let $(M,\, \kappa,\, J)$ be a compact Kähler manifold; let $\mathcal{K}$ be the subset of $H^2(M,\, \mathbb{R})$ corresponding to Kähler deformations of κ;* i.e., $[\alpha] \in \mathcal{K}$ *if and only if there exists a curve of Kähler structures $(\kappa_t,\, J_t)$ with $\kappa_t = \kappa + t\alpha + o(t)$, $J_0 = J$;*

then:

$$\mathcal{K} = \mathcal{P}^{2,0+0,2} \oplus H^{1,1}(M, \mathbb{R})$$

where

$$\mathcal{P}^{2,0+0,2}(M) := \{a \in H^{2,0+0,2}(M, \mathbb{R}) \mid \nabla^M h(a) = 0\}$$

and $h(a)$ is the g-harmonic representative of a.

Note that, clearly, $\mathcal{P}^{2,0+0,2}(M)$ is generically reduced to $\{0\}$ and, if it is not the case, then (at least locally) some hyperkähler manifold factors out from M.

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2. Reduction to the $(2, 0+0, 2)$-case

We have first the following

Lemma 2.1. *Let (M, κ, J) be a compact Kähler manifold;*
then:

$$\mathcal{K} + H^{1,1}(M, \mathbb{R}) = \mathcal{K}$$

i.e., for every $a \in \mathcal{K}$, every $c \in H^{1,1}(M, \mathbb{R})$, we have $a + c \in \mathcal{K}$.

Proof. Let $\alpha \in \wedge^2(M, \mathbb{R})$, $d\alpha = 0$, such that $[\alpha] \in \mathcal{K}$.
Given $c \in H^{1,1}(M, \mathbb{R})$, let $\gamma \in \wedge^{1,1}_J(M)$ be its harmonic representative;
by assumption, there is a curve of Kähler structures (κ_t, J_t) with $\kappa_t = \kappa + t\alpha + o(t)$; by Kodaira–Spencer theory, the projection

$$P_t \colon \wedge^{1,1}_{J_t}(M) \longrightarrow \mathcal{H}^{1,1}_{g_{J_t}}(M)$$

(where, of course, $\mathcal{H}^{1,1}_{g_{J_t}}(M)$ is the space of g_{J_t}-harmonic $(1,1)$-forms on M) is smooth in t (see e.g., [**3**], p. 184).
Let

$$\check{\kappa}_t := \kappa_t + \frac{1}{2}t(\gamma + J_t\gamma),$$

i.e.,

$$\check{\kappa}(X, Y) = \kappa_t(X, Y) + \frac{1}{2}t(\gamma(X, Y) + \gamma(J_tX, J_tY))$$

and

$$\tilde{\kappa}_t := P_t(\check{\kappa}_t) = \kappa_t + \frac{1}{2}tP_t(\gamma + J_t\gamma).$$

Clearly $(\tilde{\kappa}_t, J_t)$ is a curve of Kähler structures (note: the same J_t's!) and

$$\frac{d\tilde{\kappa}_t}{d\,t}_{|t=0} = \alpha + \frac{1}{2}P_0(\gamma + J\gamma) = \alpha + \gamma.$$

□

3. The main result

Let us first recall the basic linear algebraic frame: let (T, J, g) be a Hermitian vector space, i.e., a real vector space T equipped with $J \in \mathrm{End}(T)$ satisfying $J^2 = -I$ and a positive definite scalar product g on T satisfying $g(JX, JY) = g(X, Y)$;
then

$$T^{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$$

and

$$\nu : T \longrightarrow T^{1,0}, \quad \nu(X) := \frac{1}{2}(X - iJX)$$

is a linear isomorphism such that $\nu(JX) = i\nu(X)$.
Let $V \in \mathrm{End}(T)$ with $VJ + JV = 0$;
then, we obtain

$$V : T^{0,1} \longrightarrow T^{1,0} \quad \mathbb{C} - \text{linear}$$

simply setting

$$V(X + iJX) = V(X) - iJV(X)$$

(i.e., V acts now as $\nu \circ V \circ \bar{\nu}^{-1}$); this identifies canonically $(T^*)^{0,1} \otimes T^{1,0}$ with $\{V \in \mathrm{End}(T) \,|\, VJ + JV = 0\}$.
If, moreover, $V = -{}^tV$, then, setting

$$\alpha(X, Y) := g(V(X), Y),$$

we obtain $\alpha \in \wedge^{2,0+0,2}T^*$ and

$$\alpha^{2,0}(X, Y) = \frac{1}{2}(\alpha(X, Y) - i\alpha(JX, Y)),$$

i.e., in terms of the complexified space,

$$\alpha = \gamma + \bar{\gamma},$$

with

$$\gamma \in \wedge^{2,0}T^* \quad \gamma(Z, W) = \overline{g(V(\bar{Z}), \bar{W})}.$$

Let (M, κ, J) be a compact Kähler manifold and let $(\mathcal{A}, [\,,\,], \bar{\partial}_J)$ be the DGLA governing the holomorphic deformation theory of (M, J):

$$\mathcal{A} = \bigoplus_{p \in \mathbb{Z}} \mathcal{A}_p$$

where

$$\mathcal{A}_p = \begin{cases} \wedge_J^{0,p}(M) \otimes T^{1,0}M, & \text{if } 0 \le p \le n \\ \\ 0, & \text{otherwise} \end{cases}$$

and $[\,,\,]$ is the (complex) Schouten–Nijenhuis bracket (see e.g., [**3**], p. 152);

in particular, if $U, V \in \mathcal{A}_1$ and, in terms of local holomorphic coordinates $z_1, \ldots, z_n$,

$$U = \sum_{j,k=1}^{n} a_{j,\bar{k}} d\bar{z}_k \otimes \frac{\partial}{\partial z_j} = \sum_{j=1}^{n} a^{(j)} \frac{\partial}{\partial z_j}$$

$$V = \sum_{j,k=1}^{n} b_{j,\bar{k}} d\bar{z}_k \otimes \frac{\partial}{\partial z_j} = \sum_{j=1}^{n} b^{(j)} \frac{\partial}{\partial z_j},$$

with:

$$a^{(j)} = \sum_{k=1}^{n} a_{j\bar{k}} d\bar{z}_k, \quad 1 \le j \le n$$

$$b^{(j)} = \sum_{k=1}^{n} b_{j\bar{k}} d\bar{z}_k, \quad 1 \le j \le n,$$

then:

$$[U, V] = \sum_{j,k=1}^{n} \left(a^{(j)} \wedge \frac{\partial b^{(k)}}{\partial z_j} + b^{(j)} \wedge \frac{\partial a^{(k)}}{\partial z_j} \right) \frac{\partial}{\partial z_k}$$

$$= \sum_{j,k=1}^{n} \sum_{r<s} \left(a_{j\bar{r}} \frac{\partial b_{k\bar{s}}}{\partial z_j} - b_{j\bar{s}} \frac{\partial a_{k\bar{r}}}{\partial z_j} \right) d\bar{z}_r \wedge d\bar{z}_s \otimes \frac{\partial}{\partial z_k}$$

and so

$$(1) \qquad [U, V] \left(\frac{\partial}{\partial \bar{z}_r}, \frac{\partial}{\partial \bar{z}_s} \right) = \left[U \left(\frac{\partial}{\partial \bar{z}_r} \right), V \left(\frac{\partial}{\partial \bar{z}_s} \right) \right].$$

(Of course, for general vector fields X, Y, $[U, V](X, Y) \neq [U(X), V(Y)]$!). Note that, via ν, we can put the theory in a completely real setting, where: $\mathcal{A}_p = \wedge_J^{0,p}(M) \otimes TM = \{R \in \wedge^p(M) \otimes TM \mid R(X_1, \ldots, JX_h, \ldots, X_p) = -JR(X_1, \ldots, X_h, \ldots, X_p), \quad 1 \le h \le p\}$ and, with a slight abuse of notation,

$$[R * S] = \nu^{-1}[\nu(R), \nu(S)];$$

e.g., for $p = 0$:

$$[X * Y] = \frac{1}{2}([X, Y] - [JX, JY]).$$

We shall confine to the complex form of the theory.

Let

$$\square := \bar{\partial}_J \, \bar{\partial}_J^* + \bar{\partial}_J^* \, \bar{\partial}_J \colon \mathcal{A} \longrightarrow \mathcal{A}$$

and let

$$\nabla^{TM} = \nabla' + \nabla'' \colon \mathrm{End}(TM) \longrightarrow \wedge^1(M) \otimes \mathrm{End}(TM)$$

$$\nabla^M \colon \wedge^*(M) \longrightarrow \wedge^1(M) \otimes \wedge^*(M)$$

be the exterior covariant differential operators with respect to the Levi–Civita connection (which coincides, in the Kähler case, with the Hermitian canonical connection).

Let $V \in \mathrm{End}(TM)$ such that $JV + VJ = 0$ and so, in particular

$$V \in \wedge^{0,1}(M) \otimes T^{1,0}M;$$

let $\alpha = \gamma + \bar{\gamma} \in \wedge^{2,0+0,2}(M, \mathbb{R})$ be defined by:

$$\alpha(X, Y) = \frac{1}{2} g((V - {}^t\!V)X, Y);$$

therefore, in terms of normal local holomorphic coordinates $z_1, \ldots, z_n$, we have

$$V = \sum_{j,k=1}^{n} b_{j\bar{k}} d\bar{z}_k \otimes \frac{\partial}{\partial z_j}$$

$${}^t\!V = \sum_{j,k=1}^{n} p_{j\bar{k}} d\bar{z}_k \otimes \frac{\partial}{\partial z_j}$$

with

$$p_{j\bar{k}} = \sum_{r,s}^{n} g^{r\bar{j}} b_{s\bar{r}} g_{s\bar{k}}$$

and

$$\gamma = \sum_{j<k} c_{jk} dz_j \wedge dz_k$$

with

$$\frac{1}{2}(b_{j\bar{k}} - p_{j\bar{k}}) = \sum_{r=1}^{n} g^{r\bar{j}} \overline{c_{kr}}.$$

Therefore, if $B = (b_{j\bar{k}})$, $P = (p_{j\bar{k}})$, $G = (g_{j\bar{k}})$, $C = (c_{jk})$, then:

$$P = \bar{G}^{-1\,t}BG \tag{2}$$

$$p_{j\bar{k}} = b_{k\bar{j}} + o(|z|) \tag{3}$$

$$c_{kj} = \frac{1}{2}(\bar{b}_{j\bar{k}} - \bar{b}_{k\bar{j}}) + o(|z|) \tag{4}$$

Note first that, performing the computation at the origin 0 of the system of normal holomorphic coordinates,

$$\begin{aligned}
\bar\partial_J V = 0 &\iff \frac{\partial b_{j\bar k}}{\partial \bar z_r} = \frac{\partial b_{j\bar r}}{\partial \bar z_k}, \quad 1 \le j,\, k,\, r \le n \\
&\iff \frac{\partial \overline{b_{j\bar k}}}{\partial z_r} = \frac{\partial \overline{b_{j\bar r}}}{\partial z_k}, \quad 1 \le j,\, k,\, r \le n \\
&\Longrightarrow 2\frac{\partial c_{kj}}{\partial z_r} + 2\frac{\partial c_{jr}}{\partial z_k} + 2\frac{\partial c_{rk}}{\partial z_j} = (\text{by}(4)) \\
&\frac{\partial \overline{b_{j\bar k}}}{\partial z_r} - \frac{\partial \overline{b_{k\bar j}}}{\partial z_r} + \frac{\partial \overline{b_{r\bar j}}}{\partial z_k} - \frac{\partial \overline{b_{j\bar r}}}{\partial z_k} + \frac{\partial \overline{b_{k\bar r}}}{\partial z_j} - \frac{\partial \overline{b_{r\bar k}}}{\partial z_j} = 0 \\
&1 \le j,\, k,\, r \le n \\
&\iff \partial_J \gamma = 0.
\end{aligned}$$

We have now the following.

Lemma 3.1. *Let* (M, κ, J) *be a compact Kähler manifold; let* $\alpha \in \wedge^{2,0+0,2}(M, \mathbb{R})$, $\alpha = \gamma + \bar\gamma = g(V\cdot, \cdot)$ (*and so, in particular,* $V \in \mathrm{End}(TM)$ *with* $JV + VJ = 0$ *and* $V = -{}^tV$);
then:

(1) $\Box V = 0 \iff \nabla^{TM} V = 0$
(2) $\nabla^M \gamma = 0 \iff \nabla^{TM} V = 0$
(3) $\nabla^{TM} V = 0 \Longrightarrow [V, V] = 0$

Proof. 1: we have

$$\Box V = 0 \iff \begin{cases} \bar\partial_J V = 0 \\ \bar\partial_J^* V = 0; \end{cases}$$

now, in terms of previous notations, and so, once more at 0 of our system of normal local holomorphic coordinates:

$$b_{j\bar k} = \sum_{r=1}^{n} g^{r\bar j}\overline{c_{kr}}$$

with:

$$\frac{\partial b_{j\bar k}}{\partial \bar z_r} = \frac{\partial b_{j\bar r}}{\partial \bar z_k}, \quad 1 \le j,\, k,\, r \le n \tag{5}$$

and the extra condition

$$b_{k\bar j} = -b_{j\bar k} + o(|z|) \tag{6}$$

and so, setting

$$A^r_{jk} := \frac{\partial b_{j\bar k}}{\partial \bar z_r},$$

we obtain, by (5) and (6) :

$$A^r_{jk} = -A^r_{kj} = A^k_{jr} = -A^j_{kr} = A^j_{rk} = A^k_{rj} = -A^k_{jr} = -A^r_{jk} = 0,$$

i.e.,

$$\bar\partial_J V = 0 \ \&\ V = -{}^tV \implies \nabla'' V = 0;$$

also,

$$\bar\partial^*_J V = 0 \iff \sum_{k=1}^n \frac{\partial b_{j\bar k}}{\partial z_k} = 0 \iff \partial^*_J \gamma = 0, \quad 1 \le j \le n;$$

consequently,

$$\Box V = 0 \implies \overline{\Box}_M \gamma = 0 \iff \Delta\gamma = 0;$$

finally,

$$\bar\partial\gamma = 0 \iff \frac{\partial c_{jk}}{\partial \bar z_r} = 0 \iff \frac{\partial b_{j\bar k}}{\partial z_r} = 0 \iff \nabla' V = 0$$

clearly $\nabla^{TM} V = 0 \implies \Box V = 0$ and so all the arrows can be reversed;

2: it's a general Riemannian fact that

$$(\nabla_X \alpha)(Y,\, Z) = g((\nabla_X V)Y,\, Z)$$

3: we have (cf. (1)):

$$\begin{aligned}
[V,\, V]\left(\frac{\partial}{\partial\,\bar z_j},\, \frac{\partial}{\partial\,\bar z_k}\right) &= \left[V\left(\frac{\partial}{\partial\,\bar z_j}\right),\, V\left(\frac{\partial}{\partial\,\bar z_k}\right)\right] \\
&= \sum_{r,s=1}^n \left[b_{r\bar j}\frac{\partial}{\partial z_r},\, b_{s\bar k}\frac{\partial}{\partial z_s}\right] \\
&= \sum_{r,s=1}^n \left(b_{r\bar j}\frac{\partial b_{s\bar k}}{\partial z_s} - b_{s\bar k}\frac{\partial b_{r\bar j}}{\partial z_s}\right) = 0
\end{aligned}$$

□

We have now the following

Lemma 3.2. *Let* (M, κ, J) *be a Kähler manifold; given* $L \in \wedge^{0,1}(M) \otimes T^{1,0}M$*, we have:*

$$\Box\, {}^tL = {}^t(\Box L)$$

Proof. Let $z_1, \ldots, z_n$ be local normal holomorphic coordinates; then at 0 the curvature tensor of g is given by:

$$R_{a\bar b j\bar k} = \frac{\partial^2 g_{a\bar b}}{\partial z_j\, \partial \bar z_k}$$

and

$$R_{a\bar b j\bar k} = R_{j\bar b a\bar k} = R_{j\bar k a\bar b} = R_{a\bar k j\bar b}$$

moreover, the Ricci tensor is given by

$$R_{j\bar{k}} = -\sum_{r=1}^{n} R_{j\bar{k}r\bar{r}},$$

i.e., setting $R = (R_{j\bar{k}})$, we have:

$$R = \frac{1}{2}\Delta_M G.$$

Let now

$$L = \sum_{j,k=1}^{n} l_{j\bar{k}} d\bar{z}_k \otimes \frac{\partial}{\partial z_j} = \sum_{j=1}^{n} \lambda^{(j)} \otimes \frac{\partial}{\partial z_j}$$

with, clearly, $\lambda^{(j)} = \sum_{k=1}^{n} l_{j\bar{k}} d\bar{z}_k,\ 1 \leq j \leq n$;
then we have (cf. e.g., [**3**], pp. 101–102):

$$\bar{\partial}^* L = \sum_{j=1}^{n} \left(- * \partial * \lambda^{(j)} - \sum_{r,s=1}^{n} g^{\bar{s}j} * (\partial g_{r\bar{s}} \wedge *\lambda^{(r)}) \right) \frac{\partial}{\partial z_j}$$

and so, at 0:

$$\bar{\partial}\,\bar{\partial}^* L = \sum_{j=1}^{n} \left(-\bar{\partial}(*\partial*)\lambda^{(j)} - \sum_{r=1}^{n} \bar{\partial} * (\partial g_{r\bar{\jmath}} \wedge *\lambda^{(r)}) \right) \otimes \frac{\partial}{\partial z_j},$$

while

$$\bar{\partial}^* \bar{\partial} L = \sum_{j=1}^{n} \left(- * \partial * \bar{\partial}\lambda^{(j)} \right) \otimes \frac{\partial}{\partial z_j};$$

therefore,

$$\square L = \sum_{j=1}^{n} \left(\frac{1}{2}\Delta\lambda^{(j)} - \sum_{r=1}^{n} \bar{\partial} * (\partial g_{r\bar{\jmath}} \wedge *\lambda^{(r)}) \right) \otimes \frac{\partial}{\partial z_j};$$

now:

$$\frac{1}{2}\Delta\lambda^{(j)} = \left(\frac{1}{2}\Delta l_{j\bar{k}} - \sum_{r=1}^{n} R_{r\bar{k}} l_{j\bar{r}} \right) d\bar{z}_k$$

and

$$\bar{\partial} * (\partial g_{r\bar{\jmath}} \wedge *\lambda^{(r)}) = \sum_{k,p=1}^{n} R_{r\bar{\jmath}p\bar{k}} l_{r\bar{p}} d\bar{z}_k;$$

consequently:

$$\begin{aligned}\Box L = &\sum_{j,k=1}^{n} \frac{1}{2}\Delta_M l_{j\bar{k}} d\bar{z}_k \otimes \frac{\partial}{\partial z_j} \\ &- \sum_{j,k=1}^{n}\left(\sum_{r=1}^{n} R_{r\bar{k}} l_{j\bar{r}}\right) d\bar{z}_k \otimes \frac{\partial}{\partial z_j} \\ &- \sum_{j,k=1}^{n}\left(\sum_{r,s=1}^{n} R_{r\bar{j}s\bar{k}} l_{r\bar{s}}\right) d\bar{z}_k \otimes \frac{\partial}{\partial z_j} \\ &\frac{1}{2}\Delta_M A - AR - C(A)\end{aligned}$$

where $A := (l_{j\bar{k}})$;
now:

$$^{t}L = \sum_{j,k=1}^{n} p_{j\bar{k}} d\bar{z}_k \otimes \frac{\partial}{\partial z_j}$$

with

$$p_{j\bar{k}} = \sum_{r,s=1}^{n} g^{r\bar{j}} l_{s\bar{r}} g_{s\bar{k}}.$$

Set $B := (p_{j\bar{k}})$; then:

$$B = \bar{G}^{-1\,t}AG \quad \text{and} \quad B(0) = {}^{t}A(0);$$

now (always at 0):

$$\Delta_M B = (\Delta_M \bar{G}^{-1})A + \Delta_M{}^{t}A + A\Delta_M G = \Delta_M{}^{t}A + 2^{t}AR - 2\bar{R}A;$$

consequently:

$$\begin{aligned}\Box^{t}L &= \frac{1}{2}\Delta_M{}^{t}A + {}^{t}AR - \bar{R}^{t}A - {}^{t}AR - C({}^{t}A) \\ &= \frac{1}{2}\Delta_M{}^{t}A - \bar{R}^{t}A - C({}^{t}A).\end{aligned}$$

Finally, from

- ${}^{t}R = \bar{R}$,
- $C({}^{t}A) = C(A) = {}^{t}C(A)$,

we obtain the result. □

Corollary 3.3. *If* $L \in \wedge^{0,1}(M) \otimes T^{1,0}M$ *is* $\Box$*-harmonic, so are* $1/2(L - {}^{t}L)$ *and* $1/2(L + {}^{t}L)$.

We are now in position to prove our main result, i.e., Theorem (1.1);

Proof. (a): let $\alpha \in \mathcal{P}^{2,0+0,2}$; write $\alpha = g(V\cdot, \cdot)$;
therefore, by Lemma 3.1, V satisfies

- $\nabla^{TM} V = 0$

- $[V, V] = 0$

and clearly this is also the case for $L := \frac{1}{2}VJ$;
in fact:

$$^{t}(VJ) = JV = -VJ$$

and

$$\nabla^{TM}VJ = (\nabla^{TM}V)J + V(\nabla^{TM}J) = 0:$$

then apply Lemma 3.1 (3).
Consequently, we have

$$\bar{\partial}_J tL + \frac{1}{2}t^2[L, L] = 0$$

and so

$$J_t := (id_M + tL)J(id_M + tL)^{-1}$$

is a holomorphic structure satisfying $\nabla^{TM}J_t = 0$;
consequently,

$$\kappa_t := \frac{1}{2}(\kappa + J_t\kappa)$$

is parallel, thus it is closed, and $\kappa_t = \kappa + t\alpha + o(t)$;
therefore, α is tangent to the curve of Kähler structures (κ_t, J_t);
(b): let (κ_t, J_t) be a curve of Kähler structures with

$$\kappa_t = \kappa + t\alpha + o(t), \quad \alpha \in \wedge^{2,0+0,2}(M);$$

by the basic features of holomorphic deformation theory, we can choose the $\Box$-harmonic representative of the class corresponding to the tangent endomorphism to the curve J_t;
i.e., up to diffeomorphisms, we can assume

$$J_t := (id_M + tL + o(t))J(id_M + tL + o(t))^{-1},$$

with $\Box L = 0$ and $LJ + JL = 0$;
it follows from Lemma 3.2 that, if $V = 2J(L - {}^{t}L)$, then $\Box V = 0$ and thus, by Lemma 3.1, $\nabla^{TM}V = 0$ and finally, $\nabla^{M}\alpha = 0$. $\Box$

4. Further Remarks

As we have already remarked, deformation theory of holomorphic structures ensures that, up to diffeomorphisms, the general (germ of) curve of holomorphic structures on a Hermitian manifold (M, J, g) is of the form

$$J_t := (id_M + tL + o(t))J(id_M + tL + o(t))^{-1}$$

for $L \in \mathrm{End}(TM)$ satisfying $JL + LJ = 0$, $\Box L = 0$;
but, in general, not every such an L gives rise to an actual deformation; in other words, in general, the deformation theory is obstructed.
Let $\mathcal{M}$ be the subset of $\mathrm{Ker}\,\Box$ of elements providing actual deformations.

In the Kähler case, the situation looks somehow neater;
in fact:

1) by Corollary 3.3, $\mathrm{Ker}\,\Box$ splits as

$$\mathrm{Ker}\,\Box = \mathfrak{A} \oplus \mathfrak{S}$$

where, clearly:

$$\mathfrak{A} = \{L \in \mathrm{Ker}\,\Box \,|\, L = -{}^tL\}$$
$$\mathfrak{S} = \{L \in \mathrm{Ker}\,\Box \,|\, L = {}^tL\} :$$

2) $\mathfrak{A} \subset \mathcal{M}$
and to every $L \in \mathfrak{A}$, we can associate a canonical curve of holomorphic structures:

$$J_t := (id_M + tL)J(id_M + tL)^{-1};$$

3) note that, in general, for a curve of almost symplectic structures,

$$\kappa_t = \kappa + t\alpha + o(t)$$

and a curve of κ_t-calibrated complex structures,

$$J_t := (id_M + tL + o(t))J(id_M + tL + o(t))^{-1}, \quad (LJ + JL = 0),$$

from

$$\kappa_t - J_t\kappa_t = 0,$$

by taking the t-derivative at 0, we obtain

$$\alpha^{2,0+0,2} = g((JL + {}^tLJ)\,\cdot,\,\cdot);$$

therefore, if $L \in \mathfrak{S} \cap \mathcal{M}$, and

$$J_t := (id_M + tL + o(t))J(id_M + tL + o(t))^{-1}$$

is a curve of holomorphic structures, then any curve

$$\kappa_t = \kappa + t\alpha + o(t)$$

of J_t-Kähler structures satisfies $\alpha \in \wedge^{1,1}_J(M)$;
consequently, by Lemma 2.1, it is possible to choose κ_t of the form

$$\kappa_t = \kappa + o(t)$$

therefore, the map

$$\lambda\colon \mathcal{M} \longrightarrow H^2(M, \mathbb{R}) \quad L \mapsto [g((2J(L - {}^tL)\,\cdot,\,\cdot)]$$

is a linear surjection over $\mathcal{K}/H^{1,1}(M, \mathbb{R})$, which is one-to-one when restricted to $\mathfrak{A}$.
Note that, generically,

$$\mathrm{Ker}\,\Box = \{0\},$$

but, within the exceptional range $\dim_{\mathbb{C}} \mathrm{Ker}\,\Box > 0$, then, generically,

$$\mathfrak{A} = \{0\},$$

i.e., non-trivial $L's \in \wedge^{0,1}(M) \otimes T^{1,0}M$ satisfying $\Box L = 0$ are generically symmetric: this is a sort of Ayers Rock snow flake principle.

Let us now give a closer look to the case $\mathcal{P}^{2,0+0,2}(M) \neq \{0\}$;
first of all recall that, if (M, κ, J) is a compact Kähler manifold with $\mathrm{Ric} \geq 0$, then any holomorphic form on M is parallel and thus so are harmonic forms in $H^{2,0+0,2}(M, \mathbb{R})$; consequently, for such manifolds

$$\mathcal{K} = H^2(M, \mathbb{R})$$

(recall also that $\mathrm{Ric} > 0$ at some point $\Longrightarrow H^{2,0+0,2}(M, \mathbb{R}) = 0$).
Moreover, if on a Kähler manifold (H, J, κ) there exists $\alpha \in \wedge^{2.0+0,2}(H, \mathbb{R})$, non-degenerate, satisfying $\nabla^H \alpha = 0$, then H is hyperkähler, i.e., there exists $K \in \mathfrak{C}_\kappa(H)$, satisfying $KJ + JK = 0$, $\nabla^H K = 0$ (K is nothing but the orthogonal factor of the polar decomposition of the endomorphism representing α with respect to the given Kähler metric).
Given $\alpha \in \wedge^{2,0+0,2}(H, \mathbb{R})$, with $\nabla^M \alpha = 0$, write once more $\alpha = g(V\cdot, \cdot)$, with $V = -{}^tV$, $JV + VJ = 0$, and $\nabla^{TM} V = 0$; then set:

$$E(\alpha) := \mathrm{Ker}\, V, \quad F(\alpha) := (E(\alpha))^\perp = Im\, V;$$

then:

$$X \in TM, \quad Y \in E(\alpha) \Longrightarrow \nabla^M_X Y \in E(\alpha)$$
$$X \in TM, \quad Y \in F(\alpha) \Longrightarrow \nabla^M_X Y \in F(\alpha).$$

Therefore, the distributions $E(\alpha)$ and $F(\alpha)$ are integrable, J-invariant, parallel, and totally geodesic; moreover, if $W \in \mathrm{End}(TM)$ satisfies $\nabla^{TM} W = 0$, then

$$W_{E(\alpha)} := \begin{cases} W & \text{on } E(\alpha) \\ 0 & \text{on } F(\alpha) \end{cases}$$

satisfies $\nabla^{TM} W_{E(\alpha)} = 0$;
consequently, if $\alpha = h(a)$ has maximal rank for $a \in \mathcal{P}^{2,0+0,2}(M)$, then all elements of $h(\mathcal{P}^{2,0+0,2}(M))$ vanish on $E(\alpha)$ and so $E = E(\alpha)$ is unique;
thus, passing to the universal covering $\tilde{M}$, we easily obtain that, from the Kählerian viewpoint

$$\tilde{M} = N \times H$$

where H is hyperkähler (and corresponds to F) and so:

$$M = N \times \frac{H}{\Gamma}$$

where Γ is a discrete group of holomorphic isometries of $\tilde{M}$;

summarizing:

$$\mathcal{P}^{2,0+0,2}(M) \begin{cases} = \{0\} \implies \mathcal{K} = H^{1,1}(M, \mathbb{R}) \\ \neq \{0\} \implies M = N \times \frac{H}{\Gamma}. \end{cases}$$

Note finally that both $\mathfrak{A}$ and $\mathcal{P}^{2,0+0,2}$ are complex vector spaces:

- $$\mathcal{J}_{\mathfrak{A}} L := JL;$$
- more in general, if $\alpha \in \wedge^{2,0+0,2}(M)$, define
  $$(\mathcal{J}\alpha)(X, Y) := \alpha(JX, Y) = \alpha(X, JY);$$
  then set, for $a \in \mathcal{P}^{2,0+0,2}$:
  $$\mathcal{J}a = [\mathcal{J}h(a)];$$
  from $\nabla J = 0$, it follows that $\mathcal{P}^{2,0+0,2}$ is a $\mathcal{J}$-complex space;

it is clear that $\lambda \circ \mathcal{J}_{\mathfrak{A}} = \mathcal{J} \circ \lambda$.

5. Examples

1. Let $M = \mathbb{T}^{2n}$ be the complex n-dimensional torus equipped with the standard Kähler structure (κ, J):
in particular, we have the standard global frame

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\}$$

with

$$\begin{cases} J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j} \\ \\ J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j} \end{cases} \quad 1 \leq j \leq n$$

and standard coframe

$$\{dx_1, \dots, dx_n, dy_1, \dots, dy_n\};$$

thus

$$\kappa = \sum_{h=1}^{n} dy_h \wedge dx_h;$$

therefore, if $\alpha = dx_j \wedge dx_k$, then

$$\alpha = \gamma + \bar{\gamma} + \beta$$

with

$$\gamma = \frac{1}{4} dz_j \wedge dz_k \in \wedge^{2,0}(\mathbb{T}^{2n}, \mathbb{C})$$

and

$$\beta = \frac{1}{4}(dz_j \wedge d\bar{z}_k + d\bar{z}_j \wedge dz_k) \in \wedge^{1,1}(\mathbb{T}^{2n}, \mathbb{R}).$$

Note also that

$$\alpha^{2,0+0,2} = \frac{1}{2}(dx_j \wedge dx_k - dy_j \wedge dy_k)$$

(similar formulas for $dy_j \wedge dy_k$, $dx_j \wedge dy_k$);
thus

$$\gamma = g(V\,\cdot, \,\cdot)$$

for

$$V = \frac{1}{4}\left(d\bar{z}_j \otimes \frac{\partial}{\partial\, z_k} - d\bar{z}_k \otimes \frac{\partial}{\partial\, z_j}\right);$$

or, in real terms,

$$V = \frac{1}{2}\left(dx_j \otimes \frac{\partial}{\partial\, x_k} - dy_j \otimes \frac{\partial}{\partial\, y_k} - dx_k \otimes \frac{\partial}{\partial\, x_j} + dy_k \otimes \frac{\partial}{\partial\, y_j}\right);$$

consider e.g., $n = 2$, $\alpha = 4dx_1 \wedge dx_2$;
then:

$$V = 2\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and so:

$$L = \frac{1}{2}VJ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix};$$

therefore,

$$J_t = (I + tL)J(I + tL)^{-1} = \frac{1}{1+t^2}\begin{pmatrix} 0 & 2t & t^2-1 & 0 \\ -2t & 0 & 0 & t^2-1 \\ 1-t^2 & 0 & 0 & -2t \\ 0 & 1-t^2 & 2t & 0 \end{pmatrix}$$

and

$$\kappa_t = \frac{1}{2}(\kappa + J_t\kappa) = \kappa + t\frac{1-t^2}{1+t^2}\alpha^{2,0+0,2} + \frac{1}{2}t^2(t^2-6)\kappa.$$

2. First recall that, if $\mathbb{B} := \{z \in \mathbb{C} \mid |z| < 1\}$, then there are no non-trivial parallel (1, 0)-forms on

$$\left(\mathbb{B},\ \frac{2}{(1-|z|^2)^2}dz \wedge d\bar{z}\right);$$

in fact, given $\gamma = a\,dz$, we have:

$$\nabla_{\frac{\partial}{\partial \bar{z}}}\gamma = 0 \quad \Longleftrightarrow \quad \frac{\partial a}{\partial \bar{z}} = 0$$

$$\nabla_{\frac{\partial}{\partial z}}\gamma = 0 \quad \Longleftrightarrow \quad \frac{\partial a}{\partial z} - \frac{2\bar{z}}{1-|z|^2}a = 0 \quad \Longleftrightarrow \quad a = 0;$$

consequently, if Σ is a Riemann surface covered by $\mathbb{B}$ and equipped with the constant -1 curvature Kähler metric, then there are no non-trivial parallel $(1, 0)$-forms on Σ.
Let now Σ_{g_k}, $k = 1, 2$, be compact Riemann surfaces equipped with the constant -1 curvature Kähler metric (and so $g_k \geq 2$); let $M = \Sigma_{g_1} \times \Sigma_{g_2}$; then:

- $$H^1(M, \Theta) = H^1(\Sigma_{g_1}, \Theta) \oplus H^1(\Sigma_{g_2}, \Theta)$$
  and, although $H^2(M, \Theta) \neq 0$, the holomorphic deformation theory of M is unobstructed and reduces to the deformations of Σ_{g_1} and Σ_{g_2}:
- from the previous remarks, it follows quite easily that there are no non-trivial parallel forms in $\wedge^{2,0+0,2}(M, \mathbb{R})$;

therefore, in M, we have
$$\operatorname{Ker} \Box = \mathfrak{S} \quad \text{and} \quad \mathcal{K} = H^{1,1}(M, \mathbb{R}).$$

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Dipartimento di Matematica Applicata G. Sansone
Università di Firenze
Via S. Marta 3
50139 Firenze
Italy
E-mail address: paolo.debartolomeis@unifi.it

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