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TOWARD A TOPOLOGICAL CHARACTERIZATION OF SYMPLECTIC MANIFOLDS

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A fibration-like structure called a hyperpencil is defined on a smooth, closed $2n$-manifold X, generalizing a linear system of curves on an algebraic variety. A deformation class of hyperpencils is shown to determine an isotopy class of symplectic structures on X. This provides an inverse to Donaldson's program for constructing linear systems on symplectic manifolds. In dimensions ≤ 6, work of Donaldson and Auroux provides hyperpencils on any symplectic manifold, and the author conjectures that this extends to arbitrary dimensions. In dimensions where this holds, the set of deformation classes of hyperpencils canonically maps onto the set of isotopy classes of rational symplectic forms up to positive scale, topologically determining a dense subset of all symplectic forms up to an equivalence relation on hyperpencils. In particular, the existence of a hyperpencil topologically characterizes those manifolds in dimensions ≤ 6 (and perhaps in general) that admit symplectic structures.

1. Introduction

Symplectic structures, which are closed, nondegenerate 2-forms ω on an even-dimensional manifold X, can be thought of as skew-symmetric analogs of constant curvature Riemannian metrics. The nondegeneracy condition (that each nonzero tangent vector pairs nontrivially with some vector) is the same in each case, and closure ($d\omega = 0$) corresponds to a constant curvature condition in that it is a differential equation guaranteeing that all such structures of a given dimension are locally identical. The study of constant curvature manifolds reduces, through covering space theory, to that of discrete isometry groups of Euclidean, hyperbolic and spherical space, so it is natural to ask whether symplectic structures also have some sort of topological characterization. Gromov [**Gr**], [**McS**, Theorem 7.34], showed that an open manifold X admits a symplectic structure for each choice of almost-complex structure (up to homotopy) and class $[\omega] \in H^2_{\mathrm{dR}}(X)$, reducing the existence question for symplectic structures on open manifolds to that of almost-complex structures (homotopy theory for the tangent bundle). However, the case of closed manifolds is much more difficult. For example, there exist homeomorphic pairs of smooth 4-manifolds with isomorphic tangent bundles, such that one admits symplectic structures and the other does not [**Ta**]. (See also [**GS**], [**K**].) In the present paper, we propose a solution to this problem, by introducing a topological structure called a hyperpencil, which we show determines a symplectic structure (up to isotopy). In dimensions ≤ 6, and conjecturally in general, a closed manifold admits a symplectic structure if and only if it admits a hyperpencil, and a dense subset of all symplectic structures (up to isotopy and scale) can be described as a quotient of the set of deformation classes of hyperpencils.

The prototype for hyperpencils comes from algebraic geometry (Example 2.6(a)). If $X \subset \mathbb{CP}^N$ is a smooth n-dimensional algebraic variety, we obtain a *linear k-system* $f: X - B \to \mathbb{CP}^k$ on X by intersecting X with a transverse linear subspace A of codimension $k+1$, setting $B = X \cap A$, and defining f to be the restriction of a projectivized linear surjection $\mathbb{CP}^N - A \to \mathbb{CP}^k$. Thus, any algebraic variety inherits a canonical deformation class of linear k-systems from its embedding in $\mathbb{CP}^N$. One can use the resulting local structure on X to formulate a definition of linear systems in the category of smooth manifolds. ([**G3**] studies the general case of this.) When k equals 1 and n, respectively, generic prototypes yield Lefschetz pencils and (singular) branched coverings, both of which have been extensively studied by topologists. Lefschetz used Lefschetz pencils to study the topology of algebraic varieties (e.g., [**L**]), and in recent decades these structures have also arisen in 4-manifold theory (e.g., [**GS**]). We wish to use linear systems to construct symplectic structures. To obtain the strongest theorem, we wish to use the weakest possible hypotheses. This suggests using the smallest possible value for k, since a linear k-system generates linear ℓ-systems for $\ell \leq k$ (assuming the associated almost-complex structures behave reasonably as in the algebraic case) by composition with a generic projection $\mathbb{CP}^k - A' \to \mathbb{CP}^\ell$. However, the fibers of a linear k-system have (real) dimension $2(n-k)$; when $n - k > 1$ it is already difficult to know when the fibers admit symplectic structures. Thus, the optimal case seems to be when $k = n - 1$, when generic fibers are oriented surfaces, so each has a unique symplectic form (i.e., area form in this dimension) up to isotopy and a constant scale factor. Hyperpencils (Definition 2.4) are a type of linear $(n-1)$-system derived from the algebraic prototype. We have aimed for the weakest possible hypotheses guaranteeing the existence of symplectic structures, allowing the ugliest possible local behavior. It seems likely that additional constraints should be added for other purposes; for example, it may be possible to deform any hyperpencil into a much nicer "generic" form.

Our Main Theorem 2.11 can be paraphrased as follows:

Theorem 1.1. *Let X be a smooth, closed, oriented manifold.*

(a) *For any hyperpencil on X, the space of suitably compatible almost-complex structures J is nonempty and contractible.*
(b) *Every such J is tamed by a symplectic form on X realizing a certain cohomology class determined by f, and the space of such forms is contractible (for J fixed or suitably varying).*
(c) *There is a canonical map $\Omega : \mathcal{P}(X) \to \mathcal{S}(X)$, where $\mathcal{P}(X)$ is the set of deformation classes of hyperpencils on X, and $\mathcal{S}(X)$ is the set of all isotopy classes of symplectic forms on X.*

We first consider (c). Deformations of hyperpencils are defined in Definition 2.7. Symplectic forms ω_0 and ω_1 are *isotopic* if there is a diffeomorphism $\varphi : X \to X$ isotopic to id_X with $\varphi^*\omega_1 = \omega_0$. By Moser's Theorem [**M**], this is equivalent to the existence of a deformation (smooth family of symplectic forms ω_s, $0 \le s \le 1$) for which $[\omega_s] \in H^2_{\mathrm{dR}}(X)$ is constant. Theorem 2.11(c) characterizes the symplectic forms associated by Ω to a given hyperpencil, using the intermediate structure J (see Definition 2.9 and Lemma 2.10). For example, it is easy to check that the standard Kähler form on an algebraic variety is associated in this manner to the deformation class of hyperpencils determined by its embedding in $\mathbb{CP}^N$. A more expository discussion of Theorem 2.11 (in a slightly earlier form) appears in [**G2**]. The original form of the theorem, that a 4-manifold with a Lefschetz pencil admits a symplectic structure,was first proved by the author in 1990, but remained unpublished (due to the emergence of a more direct way of constructing unusual symplectic 4-manifolds [**G1**]) until its expository appearance as [**GS**, Theorem 10.2.18 and Corollary 10.2.23].

The main motivation for Theorem 1.1 is the use of (c) in characterizing symplectic manifolds. The symplectic forms produced by the theorem are integral (i.e., with cohomology class in the image of $H^2(X;\mathbb{Z}) \to H^2_{\mathrm{dR}}(X)$). Donaldson [**D**] has proven that any integral symplectic manifold (up to scale) admits an associated Lefschetz pencil, and Auroux [**A1**] has obtained a similar result for linear 2-systems. These results imply the $n = 2, 3$ cases, respectively, of the following conjecture (which is trivially true for $n \le 1$):

Conjecture 1.2. *Let ω be any integral symplectic form on a closed $2n$-manifold X. Then for any sufficiently large integer m, the isotopy class of $m\omega$ lies in the image of Ω.*

The conjecture is still open for $n \ge 4$, motivating our attempt at the weakest possible definition of hyperpencils. However, Auroux has made some technical progress on the problem [**A2**]. One would ultimately expect hyperpencils arising from Donaldson-Auroux theory to have much nicer local properties than arbitrary hyperpencils, for example explicit holomorphic local models at the critical points. The conjecture leads to characterization of symplectic manifolds as follows. Up to scale, every rational cohomology class is integral, and the subspace $\mathcal{S}_{\mathbb{Q}}(X) \subset \mathcal{S}(X)$ of symplectic forms with $[\omega]$ rational is dense (since nondegeneracy is an open condition). Thus, we may define $\tilde{\Omega} : \mathcal{P}(X) \times \mathbb{Q}_+ \to \mathcal{S}(X)$ so that $\tilde{\Omega}(\varphi, q)$ is obtained by rescaling $\Omega(\varphi)$ to make its cohomology class q times a primitive integral class, and conclude:

Proposition 1.3. *If Conjecture* 1.2 *holds for X, then the image of the canonical map $\tilde{\Omega} : \mathcal{P}(X) \times \mathbb{Q}_+ \to \mathcal{S}(X)$ is the dense subset $\mathcal{S}_{\mathbb{Q}}(X) \subset \mathcal{S}(X)$.*

Corollary 1.4. *In dimensions where Conjecture* 1.2 *holds (e.g., dimensions ≤ 6), a closed manifold admits a symplectic structure if and only if it admits a hyperpencil. A closed 4-manifold admits a symplectic structure if and only if it admits a Lefschetz pencil with $B \neq \emptyset$.*

The last statement of the corollary follows from Theorem 1.1(c) and [**D**], if we restrict to Lefschetz pencils such that each irreducible component of each singular fiber intersects B (since these are hyperpencils); the stated version is proved directly in [**GS**, Theorem 10.2.28]. (There, the condition $B \neq \emptyset$ is contained in the definition of Lefschetz pencils.) In dimensions where Conjecture 1.2 holds, we have now topologically characterized manifolds admitting symplectic structures. From there, to topologically determine the dense subset $\mathcal{S}_{\mathbb{Q}}(X) \subset \mathcal{S}(X)$, it suffices to understand the following:

Conjecture 1.5. *The fibers of $\tilde{\Omega}$ (or equivalently, of the map $\overline{\Omega} : \mathcal{P}(X) \to \mathcal{S}_{\mathbb{Q}}(X)/\mathbb{Q}_+$ determined by Ω) are specified by a topologically defined equivalence relation on $\mathcal{P}(X)$.*

This may be easier to prove for a stronger definition of hyperpencils. The main evidence for Conjecture 1.5 is that the theorems of Donaldson and Auroux come with uniqueness statements up to a notion of stabilization, which multiplies the cohomology classes by large integers. While this stabilization comes from analytical considerations on special families of linear systems, one might hope to topologically define stabilization maps in general, $\sigma_k : \mathcal{P}(X) \to \mathcal{P}(X)$, $k \in \mathbb{Z}_+$, with $\sigma_1 = \mathrm{id}_{\mathcal{P}(X)}$, $\sigma_k \circ \sigma_\ell = \sigma_{k\ell}$ and $\Omega \circ \sigma_k = k\Omega$, and realize the equivalence relation in Conjecture 1.5 by the definition $\varphi \sim \psi$ if and only if $\sigma_k(\varphi) = \sigma_\ell(\psi)$ for some $k, \ell \in \mathbb{Z}_+$. However, these stabilizations already seem complicated in dimension 4. (For σ_2, see [**AK**].)

Our main tool for constructing symplectic structures is a method originally used by Thurston [**T**] in the context of surface bundles, to use a symplectic structure on the base to construct one on the total space. This method has been generalized to bundles with higher dimensional fibers (e.g., [**McS**, Theorem 6.3]) and to bundles with complex quadratic singularities [**GS**, Theorem 10.2.18], but we show (Theorem 3.1) that the method works for maps that may be very different from bundle projections. For example, it suffices to have a map that is J-holomorphic for suitable almost-complex structures, with suitable data in a neighborhood of each point preimage. (An *almost-complex structure* on a manifold X is a complex structure on its tangent bundle TX, or equivalently a bundle map $J : TX \to TX$ covering id_X with $J \circ J = -\mathrm{id}_{TX}$, which we should interpret as multiplication by i. A map is then *J-holomorphic* if its derivative is complex linear. Complex structures on other vector bundles may be interpreted similarly.) We apply this method to a hyperpencil $f : X - B \to \mathbb{CP}^{n-1}$, starting from the standard symplectic form ω_{std} on $\mathbb{CP}^{n-1}$. To do this, we need a suitable almost-complex structure on X, and to prove uniqueness up to isotopy we must be able to find a 1-parameter family J_s connecting any two such almost-complex structures. Thus, we need various lemmas for splicing together locally defined almost-complex structures. These are compiled into Lemma 3.2, whose proof comprises most of Section 4. To emphasize that the choice of almost-complex structure does not crucially affect the resulting symplectic forms, we define hyperpencils using only locally defined almost-complex structures and prove that the relevant space of global almost-complex structures is nonempty and contractible (Theorem 1.1(a)). For convenience, and to emphasize the topological nature of the hypotheses, we always work with C^0 almost-complex structures. Thus, our spaces of almost-complex structures will always be given the C^0-topology (or for noncompact X, its natural generalization, the compact-open topology). In contrast, we have much more flexibility in topologizing spaces of symplectic forms. For example, contractibility in Theorem 1.1(b) holds for all C^k-spaces of forms and Sobolev spaces in between. (See Theorem 2.11(b).) Our method for constructing symplectic structures has other applications besides Theorem 1.1. We study high-dimensional Lefschetz pencils and other linear systems in [**G3**], and locally holomorphic maps with 2-dimensional fibers in [**G4**].

Throughout the paper, orientations are crucial. If V is a $2n$-dimensional real vector space, any nondegenerate, skew-symmetric, bilinear form ω on V induces an orientation, since its top exterior power is a volume form. A (linear) complex structure J on V induces an orientation obtained, as usual, from any complex isomorphism $(V, J) \cong \mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ in the product orientation (where $(1, i)$ is a positively oriented real basis for $\mathbb{C}$). If V is given to be an oriented vector space, we only consider (unless otherwise specified) complex structures and nondegenerate 2-forms inducing the given orientation on V. For example, almost-complex structures and symplectic forms on oriented manifolds implicitly induce the given orientation. We let $\omega_{\rm std}$ denote the standard symplectic form on $\mathbb{CP}^k$, normalized so that $\int_{\mathbb{CP}^1}\omega_{\rm std}=1$, so $[\omega_{\rm std}]\in H^2_{\rm dR}(\mathbb{CP}^k)$ is the hyperplane class h, Poincaré dual to $[\mathbb{CP}^{k-1}]\in H_{2k-2}(\mathbb{CP}^k)$.

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2. The main theorem

To define hyperpencils, we need some preliminary definitions. We begin by generalizing some standard terminology for relating symplectic and complex structures.

Definition 2.1. Let $T:V\to W$ be a linear transformation between finite-dimensional real vector spaces, and let ω be a skew-symmetric bilinear form on W. A linear complex structure $J:V\to V$ will be called (ω,T)-*tame* if $T^*\omega(v,Jv)>0$ for all $v\in V-\ker T$. If, in addition, $T^*\omega$ is J-invariant (i.e., $T^*\omega(Jv,Jw)=T^*\omega(v,w)$ for all $v,w\in V$), we will call J (ω,T)-*compatible*. For a C^1-map $f:X\to Y$ between manifolds, with a 2-form ω on Y, an almost-complex structure J on X will be called (ω,f)-*tame* (resp. (ω,f)-*compatible*) if it is (ω,df_x)-*tame* (resp. (ω,df_x)-*compatible*) for each $x\in X$. If $T=\mathrm{id}_V$ or $f=\mathrm{id}_X$, we will shorten the terminology to ω-*tame* and ω-*compatible*.

The last sentence of the definition is standard terminology. If ω tames some J (i.e., $T=\mathrm{id}_V$ and J is ω-tame), then ω is obviously nondegenerate, so a closed, taming 2-form ω is automatically symplectic. An ω-tame J induces the same orientation as ω. (For example, homotope ω through taming forms to a compatible one.) A key advantage of ω-taming over ω-compatibility is that the former is an open condition on manifolds. In fact, the ω-taming condition is satisfied provided that it holds on the unit sphere bundle in TX, so it is preserved under ε-small perturbations of J and ω when X is compact. If J is (ω,T)-tame then $\ker T$ is a J-complex subspace of V, so $\mathrm{Im}\,T\subset W$ inherits an ω-tame complex structure T_*J making T complex linear ($T\circ J=T_*J\circ T$). This will be ω-compatible if and only if J is (ω,T)-compatible. For an (ω,f)-tame almost-complex structure, preimages of regular values of f will be J-holomorphic submanifolds (i.e., J preserves their tangent spaces), and the complex structures induced on the fibers of $\mathrm{Im}\,df\subset f^*TY\to X$ will be denoted f_*J. Both the taming and compatibility conditions are preserved under taking convex combinations $\sum t_i\omega_i$ (all $t_i\ge 0,\sum t_i=1$) for fixed f,J. An almost-complex symplectic manifold (Y,J,ω) is called *almost-Kähler* if J is ω-compatible, and *Kähler* if, in addition, (Y,J) is a complex manifold. In either case, if $f:X\to Y$ is J-holomorphic for some almost-complex structure on X, this structure is (ω,f)-compatible.

To prove uniqueness of symplectic forms induced by hyperpencils, we need a technical condition for critical points. Suppose $E,F\to X$ are real (finite dimensional) vector bundles over a metrizable topological space, and $T : E \to F$ is a (continuous) section of the bundle $\mathrm{Hom}(E, F)$. In our main application, these will be induced by a C^1-map $f : X \to Y$ between manifolds, with $T = df : TX \to f^*TY$. Motivated by this example, we call a point $x \in X$ *regular* if $T_x : E_x \to F_x$ is onto and *critical* otherwise. Let $P \subset E$ be the closure $\mathrm{cl}(\bigcup \ker T_x)$, where x varies over all the regular points of T in X, and let $P_x = P \cap E_x$. Thus, $P_x = \ker T_x$ if x is regular, and otherwise $P_x \subset \ker T_x$ consists of limits of sequences of vectors annihilated by T at regular points.

Definition 2.2. A point $x \in X$ is *wrapped* if $\mathrm{span}\, P_x$ has (real) codimension at most 2 in $\ker T_x$.

Proposition 2.3. *Suppose that in a neighborhood of a critical point $x \in X$, T is given by df, for some holomorphic map $f : U \to \mathbb{C}^{n-1}$ with U open in $\mathbb{C}^n$. If each fiber $f^{-1}(y)$ intersects the critical set K of f in at most a finite set, then x is wrapped. In fact, $P_x = \ker T_x$.*

This proposition will show that our hypothesis of wrapped critical points is broad enough to be useful. Note, however, that the proposition becomes false without the finiteness hypothesis, e.g., $n = 3$, $f(x, y, z) = (x^2, y^2)$ at $(0, 0, 0)$. (For $n = 2$, P_x equals $\ker T_x$ unless f is constant or x is a smooth point of f with multiplicity > 1; cf. [**G4**, proof of Proposition 1.3].) Similarly, P_x may not equal $\ker T_x$ if we pass from the holomorphic setting to C^∞. (The C^∞-map $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = y^3 + e^{-1/x^2} y$ has a unique critical point at $(0, 0)$, so $df_{(0,0)} = 0$, but $P_{(0,0)}$ is the x-axis.)

Proof. For $\ell \geq 2$, let $K_\ell \subset K$ be the set of $z \in U$ for which $\ker df_z$ has complex dimension $\geq \ell$. Thus, $K = K_2 \supset K_3 \supset \cdots \supset K_{n+1} = \emptyset$. We begin by showing that each K_ℓ is an analytic variety of complex codimension $\geq \ell$ in U. Analyticity follows immediately from the description of K_ℓ as the set of $z \in U$ for which every $(n - \ell + 1) \times (n - \ell + 1)$ submatrix of df_z has determinant zero. For z in the top stratum W of K_ℓ, let $Q_z = T_z W \cap \ker df_z$. If $\mathrm{codim}_{\mathbb{C}} K_\ell < \ell$, then Q_z has nonzero dimension for all $z \in W$. Choose some $z_0 \in W$ minimizing this dimension. To see that Q is a smooth distribution on W near z_0, choose a projection π of $\mathbb{C}^{n-1}$ whose restriction to $df_{z_0}(T_{z_0} W)$ is an isomorphism, and note that $\pi \circ f|W$ is a submersion at z_0 with $\ker d(\pi \circ f|W)_z$ containing $\ker d(f|W)_z = Q_z$; these latter spaces are then equal near z_0 by minimality of $\dim Q_{z_0}$. Now choose a smooth, nonzero vector field in Q near z_0. By integrating, we obtain a curve in $W \subset K$ whose image under f is a point y, contradicting finiteness of $f^{-1}(y) \cap K$.

Now observe that the subset $V = \bigcup_{z \in U} \ker df_z \subset TU = U \times \mathbb{C}^n$ is an analytic variety with complex dimension $\geq n + 1$ everywhere, since it is cut out by the system of $n - 1$ equations $df_z(v) = 0$ in (z, v). For each $\ell \geq 2$, the subset $V_\ell = V \cap ((K_\ell - K_{\ell+1}) \times \mathbb{C}^n)$ is a complex ℓ-plane bundle over $K_\ell - K_{\ell+1}$, and the latter has codimension $\geq \ell$ in U, so $\dim_{\mathbb{C}} V_\ell \leq n < \dim_{\mathbb{C}} V$. Thus, $V - K \times \mathbb{C}^n = V - \bigcup_{\ell=2}^n V_\ell$ has closure V in $U \times \mathbb{C}^n$. The proposition follows immediately. □

Definition 2.4. A *hyperpencil* on a smooth, closed, oriented, $2n$-manifold X is a (necessarily finite) subset $B \subset X$ called the *base locus* and a smooth map $f : X - B \to \mathbb{CP}^{n-1}$ such that

(1) each $b \in B$ is mapped to $0 \in \mathbb{C}^n$ by an orientation-preserving local coordinate chart in which f is given by projectivization $\mathbb{C}^n - \{0\} \to \mathbb{CP}^{n-1}$,
(2) each critical point of f is wrapped and has a neighborhood with a continuous (ω_{std}, f)-compatible almost-complex structure, and
(3) each fiber $F_y = \operatorname{cl} f^{-1}(y) \subset X$ contains only finitely many critical points of f, and each component of each $F_y - \{\text{critical points}\}$ intersects B.

Remarks 2.5. (a) The results in this paper are all trivially true for $n \leq 1$ (after Moser [**M**]), so we will assume $n \geq 2$ whenever convenient. For potential applications such as sub-hyperpencils, where it may be convenient to allow $n \leq 1$, we specify the required conventions: For $n = 0$, B equals X, and a symplectic form on a 0-manifold is its unique positive orientation. For $n = 1$, we require $B \subset X$ to be finite, and its Poincaré dual in $H^2_{\text{dR}}(X)$ is the class c_f used in Theorem 2.11.

(b) By Condition (1), the fibers F_y of a hyperpencil are complex lines near each $b \in B$. Thus, $F_y = f^{-1}(y) \cup B$ and $F_y \cap F_z = B$ for $y \neq z$, so B is closed and discrete, hence finite. Each F_y is an oriented surface (in the preimage orientation) except possibly at finitely many singularities, where F_y intersects the critical set of f in $X - B$.

(c) For $n \leq 2$ the hypothesis of wrapped critical points is trivially true. (The proof of Main Theorem 2.11 shows that the regular points of f are dense in $X - B$, so each $\operatorname{span}_{\mathbb{R}} P_x \subset T_x(X - B)$ is a nontrivial complex subspace.) For $n = 3$, this hypothesis can be eliminated if we assume that at each point in the closure of the set of unwrapped critical points, the given local almost-complex structure makes f J-holomorphic for some continuous, ω_{std}-tame local complex structure on the bundle $f^*T\mathbb{CP}^{n-1}$. (In fact, an even weaker hypothesis guarantees the Main Theorem when $n = 3$, namely $(\omega_{\text{std}}, df)$-extendability as used in Addendum 3.3 with $C = D = \emptyset$ and $E = TX$.) For arbitrary n, the hypothesis of wrapped critical points can be dropped in the presence of a global ω_{std}-compatible complex structure (standard near B) on $f^*T\mathbb{CP}^{n-1}$ making f J-holomorphic for each local almost-complex structure on $X - B$, but the resulting isotopy class of symplectic forms could then conceivably depend on the choice of this structure on $f^*T\mathbb{CP}^{n-1}$.

(d) Throughout the article, we use continuous, rather than smooth, almost-complex structures. This is both for convenience (avoiding awkward and unnecessary proofs of smoothness) and to emphasize that the purpose of the local almost-complex structures is topological rather than analytical, controlling monodromy around the critical values. For a Lefschetz pencil on a 4-manifold, for example, the monodromy consists of right-handed Dehn twists. Allowing the opposite handedness violates the hypothesis of (compatibly oriented) local almost-complex structures, and results in manifolds having no symplectic structure.

(e) It is an open question whether $(\omega_{\mathrm{std}}, f)$-compatibility can be replaced by $(\omega_{\mathrm{std}}, f)$-taming. This could be done throughout the paper if Question 4.3 had an affirmative answer, and can also be done in the situation at the end of (c) above (arbitrary n).

Examples 2.6. (a) Any smooth algebraic variety $X \subset \mathbb{CP}^N$ admits a hyperpencil. Simply pick a linear subspace $A \approx \mathbb{CP}^{N-n}$ ($n = \dim_{\mathbb{C}} X$) transverse to X in $\mathbb{CP}^N$, let $B = X \cap A$ and let f be the restriction of the holomorphic projection $\mathbb{CP}^N - A \to \mathbb{CP}^{n-1}$. Condition (1) follows from transversality. For Condition (3), note that each fiber F_y is an algebraic curve in some linear subspace of $\mathbb{CP}^N$ containing A as a codimension-1 subspace. Thus, each irreducible component of F_y intersects A, and so has a 0-dimensional intersection with the critical set (since Condition (1) locates some regular points). Now the critical points of f are wrapped by Proposition 2.3, and the obvious holomorphic structure on X completes Condition (2).
(b) On 4-manifolds, *Lefschetz pencils* are inspired by the above Example (a) with $n = 2$. Their definition is obtained from Definition 2.4 ($n = 2$) by replacing Conditions (2) and (3) with the condition that f should locally be a complex Morse function, i.e., modeled by $f(z_1, z_2) = z_1^2 + z_2^2$ at each critical point. (This holds in (a) for a generic choice of A.) See [**GS**] for a recent survey on Lefschetz pencils. Note that Condition (2) is automatically satisfied by Lefschetz pencils, as is the finiteness part of Condition (3). Lefschetz pencils need not satisfy the other part of Condition (3), however. This condition on $F_y \cap B$, which guarantees that a (4-dimensional) Lefschetz pencil is a hyperpencil, is actually not necessary for constructing symplectic structures on Lefschetz pencils. (See [**GS**, Theorem 10.2.18 and Corollary 10.2.23] for details.) It suffices to know either that $B \neq \emptyset$ or that the fibers are nontrivial in $H_2(X; \mathbb{R})$. (In fact, the only counterexamples are torus bundles, namely $L(p, 1) \times S^1 \to S^2$, and their blowups; see [**GS**, Remark 10.2.22(a)].) However, without Condition (3), one loses control of the cohomology class of the resulting form, and with it one loses uniqueness of the isotopy class.

To define an appropriate equivalence relation among hyperpencils on X, we again work by analogy with algebraic geometry. We begin by organizing hyperpencils into families over a parameter space S. Roughly, these families are given by bundles over S whose fibers have continuously varying hyperpencil structures.

Definition 2.7. A *family of hyperpencils* parametrized by a topological space S consists of a pair of fiber bundles $\pi_X : X \to S$, $\pi_Y : Y \to S$, a subset $B \subset X$ and a continuous, fiberwise smooth map $f : X - B \to Y$ covering id_S, subject to the following: The fibers X_s of X, $s \in S$, are all diffeomorphic to a fixed, closed, oriented $2n$-manifold (so the structure group consists of orientation-preserving diffeomorphisms of the fiber in the C^∞-topology), and the fibers of Y are diffeomorphic to $\mathbb{CP}^{n-1}$ with structure group $\mathbb{P}\mathrm{U}(n)$ acting in the usual way. The map $\pi_X|B : B \to S$ is a (necessarily finite) covering map. In addition:

(1) B has a neighborhood $V \subset X$ on which $\pi_X|V : V \to S$ lifts to $\tilde{\pi} : V \to B$, and $\tilde{\pi}$ is given the structure of a $\mathrm{U}(n)$-vector bundle (with zero section B and fibers oriented compatibly with those of π_X). The map $f|V - B$ is projectivization on each fiber of $\tilde{\pi}$.
(2) The map $df : T^v(X - B) \to f^*T^vY$ (where T^v denotes the bundle of tangent spaces to the fibers of π_X and π_Y) is continuous, and each critical point of df has a neighborhood in $X - B$ over which T^vX has an $(\omega_{\mathrm{std}}, df)$-compatible complex vector bundle structure.
(3) For each fiber X_s of π_X, each critical point of $f|X_s - B$ is wrapped (in X_s), and Condition (3) of Definition 2.4 is satisfied.

A *deformation* of hyperpencils is a family parametrized by $I = [0, 1]$ with $X = I \times X_0$.

It is easily verified that a family of hyperpencils parametrized by a 1-point space is the same as a hyperpencil (together with a fixed choice of the charts in Definition 2.4(1) up to $\mathrm{U}(n)$ action). If $\varphi : S \to S'$ is continuous, then a family of hyperpencils parametrized by S' pulls back to one parametrized by S. For example, any parametrized family of hyperpencils restricts to a hyperpencil on each X_s, or to a family parametrized by any subspace of S. Now, we easily obtain an equivalence relation by calling two hyperpencils on a fixed manifold *deformation equivalent* if they are realized as X_0 and X_1 for some deformation. (The only technicality is that in transitivity, the middle hyperpencil may inherit two different sets of the charts in Definition 2.4(1). However, these charts can easily be changed, by triviality of bundles over I. Closer inspection also shows that we can find a continuous family interpolating between any two such charts; cf. proof of Lemma 2.10.) For a parametrized family with S path connected, any X_s and X_t can be identified so that their hyperpencils are deformation equivalent. We can also assume the deformation is constant near each endpoint of I. If, in addition, $X = S \times X_s$ is the trivial bundle, we obtain a family of deformation equivalent hyperpencils on the fixed manifold X_s.

Examples 2.8. (a) To construct families of hyperpencils as in Example 2.6(a), let G denote the complex Grassmann manifold of codimension-n linear subspaces of $\mathbb{CP}^N$, and let $\gamma \subset G \times \mathbb{CP}^N$ be the tautological bundle whose fiber over $A \in G$ is $A \subset \mathbb{CP}^N$. If $Y \to G$ is the bundle whose fiber over A is the $(n-1)$-plane in $\mathbb{CP}^N$ with maximal distance from A, we obtain a canonical holomorphic map $f : G \times \mathbb{CP}^N - \gamma \to Y$ covering id_G, induced by linear projections on $\mathbb{C}^{N+1}$. Given an n-dimensional smooth algebraic variety X_0 in $\mathbb{CP}^N$, let $S \subset G$ be the open set of subspaces transverse to X. Restricting f to a map $S \times X_0 - \gamma \to Y|S$, we obtain a parametrized family of hyperpencils, consisting of all hyperpencils on X_0 obtained by Example 2.6(a). (The given holomorphic structure on $S \times X_0$ satisfies Condition (2) above.) For example, when $n = 2$, generic members of the family will be Lefschetz pencils, but there will typically also be parameter values for which quadratic critical points coalesce into those of higher degree. In general, the space S is path connected (since it is obtained from G by removing a subvariety of positive complex codimension), so we conclude that all hyperpencils on X_0 obtained by Example 2.6(a) are deformation equivalent (for a fixed embedding $X_0 \subset \mathbb{CP}^N$).

(b) For a family in which X is a nontrivial bundle, note that the space of all hypersurfaces of a fixed degree in $\mathbb{CP}^N$ is parametrized by some $\mathbb{CP}^M$. Let $S \subset G \times \mathbb{CP}^M$ denote the path-connected subset of pairs (A, t) such that the variety X_t is nonsingular and transverse to A. The construction of (a) above generalizes immediately to produce a parametrized family consisting of all hyperpencils as above on all nonsingular hypersurfaces of a fixed degree. It follows that any two nonsingular hypersurfaces of the same degree in $\mathbb{CP}^N$ are diffeomorphic in such a way that the canonical families of hyperpencils are deformation equivalent. We also see that the canonical deformation class of hyperpencils on a fixed hypersurface is invariant under self-diffeomorphisms induced by monodromy of the bundle of all nonsingular hypersurfaces.

We relate hyperpencils $f : X - B \to \mathbb{CP}^{n-1}$ to symplectic structures ω on X via the existence of local almost-complex structures that are simultaneously ω-tame and $(\omega_{\mathrm{std}}, f)$-compatible:

Definition 2.9. Let ω be a continuous 2-form on an oriented manifold X. A hyperpencil $f : X - B \to \mathbb{CP}^{n-1}$ is *ω-tame* if X is covered by open sets W_α with continuous, ω-tame almost-complex structures J_α such that $J_\alpha|W_\alpha - B$ is $(\omega_{\mathrm{std}}, f)$-compatible, and the structures $f_* J_\alpha$ on $\operatorname{Im} df \subset f^* T\mathbb{CP}^{n-1}|W_\alpha - B$ all agree where their domains overlap. Similarly, a family

$f : X - B \to Y$ of hyperpencils parametrized by S is *ω-tame* for a continuous family ω of 2-forms on the fibers of π_X, if X is covered by (W_α, J_α) as above. (Here W_α is open in X, J_α is a complex vector bundle structure on $T^v X|W_\alpha$, and the structures $f_* J_\alpha$ fit together on $\operatorname{Im} df \subset f^* T^v Y$.)

Lemma 2.10. *A hyperpencil f is ω-tame if and only if there is a global, continuous, ω-tame almost-complex structure J on X with $J|X-B$ $(\omega_{\mathrm{std}}, f)$-compatible. The structure J can be chosen to agree near B with the local complex structures induced by any preassigned charts as in Definition* 2.4*(1), and so that f_*J agrees with the structures f_*J_α given by Definition* 2.9 *if these are standard near B. The corresponding statements also hold for a family of hyperpencils parametrized by a metrizable space, where J is a complex vector bundle structure on T^vX agreeing on a neighborhood of B with the fiberwise complex structure determined by the vector bundle in Definition* 2.7*(1).*

In other words, the local structures J_α of Definition 2.9 can be patched together to form a global structure. The "if" direction of this lemma is obvious; the other is proved in Section 4.

Any hyperpencil $f : X - B \to \mathbb{CP}^{n-1}$ determines a class $c_f = f^*h \in H^2(X;\mathbb{Z}) \cong H^2(X-B;\mathbb{Z})$, where h is the hyperplane class, Poincaré dual to $[\mathbb{CP}^{n-2}] \in H_{2n-4}(\mathbb{CP}^{n-1};\mathbb{Z})$. This class in X is invariant under deformations of f. When $X \subset \mathbb{CP}^N$ is a smooth algebraic variety with its canonical deformation class of hyperpencils, c_f is the restriction of the hyperplane class of $\mathbb{CP}^N$, and in general for $n = 2$, c_f is Poincaré dual to any fiber F_y. We also use c_f to denote the corresponding class in $H^2_{\mathrm{dR}}(X)$; this is the cohomology class of the symplectic forms associated to f.

We are now ready to state the Main Theorem.

Theorem 2.11. *Let X be a smooth, closed, oriented $2n$-manifold.*

a) *For any hyperpencil $f : X - B \to \mathbb{CP}^{n-1}$ there is a continuous almost-complex structure J on X that is $(\omega_{\mathrm{std}}, f)$-compatible on $X - B$, and agrees near B with the complex structure given there by Definition* 2.4*(1) (for any fixed choice of charts). The space of all such structures J in the C^0-topology (for fixed f and charts near B) is contractible, as is the space of all J on X that are $(\omega_{\mathrm{std}}, f)$-compatible on $X - B$ (without the constraint near B).*
b) *For any J as in the first sentence of (a), there is a symplectic structure ω on X with $[\omega] = c_f \in H^2_{\mathrm{dR}}(X)$, taming J (so that f is ω-tame as in Definition* 2.9*). For fixed J, such structures ω form a convex subset of the space of all closed 2-forms. If only f and f_*J are fixed (otherwise allowing J to vary), the space of such (smooth) symplectic structures is still contractible in any C^k-topology, $0 \le k \le \infty$, or locally convex metric topology in between, as is the completion in such a metric.*

c) *For each deformation class φ of hyperpencils on X, there is a unique isotopy class of symplectic forms on X containing representatives ω for which some $f \in \varphi$ is ω-tame and $[\omega] = c_f$.*

There are also versions of (a) and (b) for parametrized families of hyperpencils; see Lemmas 3.2 (as applied in proving Theorem 2.11(a)) and 3.4, respectively. The completion in (b) means to complete the space of 2-forms, take the closure of the affine subspace with $[\omega] = c_f$, and restrict to an open subset using the taming condition. Examples include the C^k-spaces of taming forms, $0 \le k \le \infty$, and many Sobolev spaces. The result could be stated in even further generality; see Theorem 16 of [**P**].

Example 2.12. For a smooth algebraic variety $X \subset \mathbb{CP}^N$, the standard holomorphic structure on X is obviously $(\omega_{\mathrm{std}}|X)$-tame. Thus the isotopy class of symplectic structures determined by the canonical deformation class of hyperpencils (Example 2.6(a)) is the one containing the standard Kähler form $\omega_{\mathrm{std}}|X$.

3. Proof of the Main Theorem 2.11

Our main tool for constructing symplectic structures is the following theorem. This is based on an idea, which was used by Thurston [**T**] to construct symplectic structures on total spaces of surface bundles over symplectic manifolds, and which generalizes to bundles with symplectic fibers of arbitrary dimension (e.g., [**McS**, Theorem 6.3]). We now generalize still further, from bundle projections to singular maps suitably controlled by almost-complex structures, and work relative to a subset C of the domain. The result is general enough to apply to hyperpencils, and also has other applications [**G3**], [**G4**].

Theorem 3.1. *Let $f : X \to Y$ be a smooth map between manifolds, with $X - \operatorname{int} C$ compact for some closed subset C with a neighborhood W_C in X. Suppose that ω_Y is a symplectic form on Y, and J is a continuous, (ω_Y, f)-tame almost-complex structure on X. Fix a class $c \in H^2_{\mathrm{dR}}(X)$. Suppose that for each $y \in Y$, $f^{-1}(y)$ has a neighborhood W_y in X containing W_C, with the restriction $H^1_{\mathrm{dR}}(W_y) \to H^1_{\mathrm{dR}}(W_C)$ surjective, and with a closed 2-form η_y on W_y such that $[\eta_y] = c|W_y \in H^2_{\mathrm{dR}}(W_y)$ and such that η_y tames J on each of the complex subspaces $\ker df_x$, $x \in W_y$. Suppose that these forms η_y all agree on W_C, and that the resulting form η_C on W_C tames J on $TX|C$. Then there is a closed 2-form η on X agreeing with η_C near C with $[\eta] = c \in H^2_{\mathrm{dR}}(X)$, and such that for all sufficiently small $t > 0$ the form $\omega_t = t\eta + f^*\omega_Y$ on X tames J (and hence is symplectic).*

To show why this theorem generalizes Thurston's construction, we obtain the latter as a special case. (See [**G2**] for additional details.) Suppose f :

$X \to Y$ is a fiber bundle with X compact. Then we can take C and W_C to be empty. The required hypotheses for Thurston's construction (e.g., as given in [**McS**]) include symplectic structures on Y and the fibers $f^{-1}(y)$, which give almost-complex structures on Y and the subbundle of TX tangent to the fibers. These can easily be combined into an almost-complex structure J on X that makes f J-holomorphic and hence is (ω_Y, f)-tame. The final hypothesis for Thurston's construction guarantees the existence of a suitable class c, and the remaining hypotheses of Theorem 3.1 are now easily verified (cf. [**G2**]). The family of symplectic forms resulting from Theorem 3.1 is the same one obtained by Thurston.

The proof of Theorem 3.1 is obtained by modifying Thurston's method to exploit the almost-complex structure, allowing us to deal with a complicated critical set for f and work relative to a possibly nonempty subset C.

Proof. Fix a representative ζ of the deRham class c. For each $y \in Y$, $[\eta_y] = c|W_y$, so we can write $\eta_y = \zeta + d\alpha_y$ for some 1-form α_y on W_y. Pick some $y_0 \in Y$ and set $\alpha_C = \alpha_{y_0}|W_C$. Then for each y, $d(\alpha_C - \alpha_y) = (\eta_{y_0} - \zeta) - (\eta_y - \zeta) = 0$ on W_C, so $[\alpha_C - \alpha_y] \in H^1_{\mathrm{dR}}(W_C)$ is defined. By hypothesis, any such class extends to $H^1_{\mathrm{dR}}(W_y)$, so after adding a closed form on W_y to α_y, we can assume $\alpha_C - \alpha_y$ is exact on W_C. Choosing a function $g : W_y \to \mathbb{R}$ with $dg = \alpha_C - \alpha_y$ near C, and replacing α_y by $\alpha_y + dg$, we obtain that $\alpha_y = \alpha_C$ near C for each y. Since each $X - W_y$ is compact, each $y \in Y$ has a neighborhood disjoint from $f(X - W_y)$. Thus, we can cover Y by open sets U_i, with each $f^{-1}(U_i)$ contained in some W_y. Let $\{\rho_i\}$ be a subordinate partition of unity on Y. The corresponding partition of unity $\{\rho_i \circ f\}$ on X can be used to splice the forms α_y; let $\eta = \zeta + d\sum_i (\rho_i \circ f)\alpha_{y_i}$. Clearly, η is closed with $[\eta] = [\zeta] = c \in H^2_{\mathrm{dR}}(X)$, and $\eta = \eta_C$ near C, so it suffices to show that ω_t tames J ($t > 0$ small). In preparation, perform the differentiation to obtain $\eta = \zeta + \sum_i (\rho_i \circ f) d\alpha_{y_i} + \sum_i (d\rho_i \circ df) \wedge \alpha_{y_i}$. The last term vanishes when applied to a pair of vectors in $\ker df_x$, so on each $\ker df_x$ we have $\eta = \zeta + \sum_i (\rho_i \circ f) d\alpha_{y_i} = \sum_i (\rho_i \circ f)\eta_{y_i}$. By hypothesis, this is a convex combination of taming forms, so we conclude that $J|\ker df_x$ is η-tame for each $x \in X$.

It remains to show that there is a $t_0 > 0$ for which $\omega_t(v, Jv) > 0$ for every $t \in (0, t_0)$ and v in the unit sphere bundle $\Sigma \subset TX$ (for any convenient metric). But

$$\omega_t(v, Jv) = t\eta(v, Jv) + f^*\omega_Y(v, Jv) \ .$$

Since J is (ω_Y, f)-tame, the last term is positive for $v \notin \ker df$ and zero otherwise. Since $J|\ker df$ is η-tame, the continuous function $\eta(v, Jv)$ is positive for all v in some neighborhood U of $\ker df \cap \Sigma$ in Σ. Similarly, for $v \in \Sigma|C$, $\eta(v, Jv) = \eta_C(v, Jv) > 0$. Thus, $\omega_t(v, Jv) > 0$ for all $t > 0$ when $v \in U \cup \Sigma|C$. On the compact set $\Sigma|(X - \operatorname{int} C) - U$ containing the rest of Σ, $\eta(v, Jv)$ is bounded and the last displayed term is bounded below by a positive constant, so $\omega_t(v, Jv) > 0$ for $0 < t < t_0$ sufficiently small, as required. □

We also need some techniques for splicing together locally defined almost-complex structures. These are given by the following lemma, whose proof appears in Section 4. As in Definition 2.2 (of wrapped critical points), we let $E, F \to X$ be real vector bundles over a metrizable space, with fiber dimensions $2n$ and $2n-2$ respectively, and this time equipped with fiber orientations. We again fix a section $T : E \to F$ of $\mathrm{Hom}(E, F)$. In the applications, T will be $df : TX \to f^*TY$ for some C^1 map $f : X \to Y$, or a family of such maps, although the extra generality is useful in proving the lemma below. We wish to splice together locally defined complex structures on the bundle E, so as to extend a preassigned structure from some closed (possibly empty) subset $C \subset X$ to all of X. Our approach requires a restriction on either the topology of the critical points or the induced complex structures on the image $T(E) \subset F$. We assume the latter restriction on a closed subset D (which without loss of generality contains C) and the former restriction elsewhere. The cases we require are when D equals C or X (allowing us to set V equal to U or X below), but the general case poses no additional difficulties. A *2-form* on E or F will mean a continuously varying choice of a skew-symmetric bilinear form on each fiber of the bundle.

Lemma 3.2. *For $E, F \to X$ and T as above, let $C \subset D \subset X$ be closed subsets such that the regular points of $T|X - C$ are dense in $X - C$, and let ω_F be a nondegenerate 2-form on F (inducing the given fiber orientation). For some neighborhood U of C, let J_C be an (ω_F, T)-compatible complex structure on the (oriented) bundle $E|U$. Suppose that each $x \in X - U$ has a neighborhood W_x with an (ω_F, T)-compatible complex structure on $E|W_x$, and that for some neighborhood V of D, these can be chosen for all $x \in V - U$ so that the induced structures on $T(E|W_x) \subset F|W_x$ agree with each other and with T_*J_C wherever the domains overlap. Let J_D denote the resulting complex structure on the fibers of $T(E|V)$.*

a) *If $n \geq 3$, assume each critical point of T in $X - D$ is wrapped. Then $J_C|C$ extends to an (ω_F, T)-compatible complex structure J on E with $T_*J|D = J_D$.*
b) *Suppose $D = X$ and ω_E is a 2-form on E. If the local complex structures on E given above (including J_C) can be chosen to be ω_E-tame, then we can assume J is ω_E-tame.*
c) *Suppose that $\partial C \subset X_0 = X - \mathrm{int}\, C$ and (if $D \neq C$) $\partial D \subset X - \mathrm{int}\, D$ have disjoint collar neighborhoods compatible with the given structures on E and F. (See below.) Then in both cases (a) and (b) above, the space $\mathcal{J}$ of all complex structures J satisfying the given conclusions is weakly contractible when X_0 is locally compact and contractible when X_0 is compact.*

All of the above remains true if compatibility is replaced by taming everywhere, provided that $D = X$.

Recall that $\mathcal{J}$ is given the compact-open topology, which equals the C^0-topology when X_0 is compact. See the proof for further details. The "compatible collar" hypothesis on ∂C means that there is a subset $K \subset X_0$ homeomorphic to $I \times \partial C$ with $\{0\} \times \partial C$ mapping to ∂C in the obvious way and $\{1\} \times \partial C$ mapping onto ∂K (the boundary in X_0 in the sense of general topology), and that $E|K$ and $F|K$ can be identified with $I \times (E|\partial C)$ and $I \times (F|\partial C)$ in such a way that T, ω_F, ω_E (in case (b)), and J_D (if $D \neq C$) are constant over each $I \times \{x\} \subset I \times \partial C$.

The hypothesis of wrapped critical points can be weakened, at least when $n = 3$. For an open set $W \subset X$, we call an (ω_F, T)-compatible complex structure J on $E|W$ (ω_F, T)-*extendible* along a collection of convergent sequences of regular points in X if for each such sequence $x_i \to x$ with $x \in W$, the complex structures T_*J on each F_{x_i} (defined for all sufficiently large i) limit to an ω_F-tame complex structure on F_x (which is necessarily ω_F-compatible and an extension of T_*J on $T(E_x) \subset F_x$).

Addendum 3.3. *If $n = 3$ and there are critical points in $X - D$ that are not wrapped, fix a sequence of regular points converging to each unwrapped critical point in $X - D$, and assume that the given local complex structures on E (including J_C) are (ω_F, T)-extendible along these sequences. Then Lemma* 3.2 *still holds, where the structures J comprising $\mathcal{J}$ are all required to be (ω_F, T)-extendible along the given sequences, provided that in (c), the collar of ∂C (if $D = C$) or of ∂D (if $D \neq C$) contains no unwrapped critical points.*

Proof of Theorem 2.11. To prove (a), we wish to apply Lemma 3.2 to the hyperpencil $f : X - B \to \mathbb{CP}^{n-1}$. First, we show that the regular points of f are dense in $X - B$ by the method previously used for Proposition 2.3. If any neighborhood $W \subset X - B$ consists entirely of critical points, choose $x_0 \in W$ minimizing $\dim \ker df_x$ and note that $\ker df$ is a smooth distribution near x_0. (As before we can realize it as $\ker d(\pi \circ f)$ for a suitable projection π.) By integrating a vector field in $\ker df$, we obtain a curve of critical points in a single fiber, contradicting finiteness in Definition 2.4(3). Now we apply Lemma 3.2 to $X - B$, with $E = T(X - B)$, $F = f^*T\mathbb{CP}^{n-1}$, $T = df$ and $\omega_F = \omega_{\mathrm{std}}$ (pulled back to F). We let $C = D \subset X - B$ consist of a closed, round ball (with center deleted) about each $b \in B$ in the local charts given by Definition 2.4(1), $U = V$ be corresponding punctured open balls, and J_C be the corresponding complex structure on $E|U$. The required local complex structures over neighborhoods W_x exist by Definition 2.4(2) at critical points x, and are easy to construct at regular points. By (a) of the lemma, we obtain the required almost-complex structure J on $X - B$ (which immediately extends over X), using either the given definition of a hyperpencil or the variations of Remark 2.5(c). (For the last variation, set $D = X - B$.) Now (c) of the lemma (with the obvious radial collar of ∂C) gives contractibility of the space $\mathcal{J}_C$ of $(\omega_{\mathrm{std}}, f)$-compatible almost-complex structures on X extending J_C on C (with suitable $(\omega_{\mathrm{std}}, df)$-extendability or f_*J fixed for the above variations). Let $\mathcal{J} = \bigcup_C \mathcal{J}_C$ be the space of all such $(\omega_{\mathrm{std}}, f)$-compatible almost-complex structures on X that are standard near B (relative to fixed charts as in Definition 2.4(1), but on neighborhoods of variable size). Again fix C and U as above (with the fixed f_*J standard on U in the case of the last variation), and let $h_t : X \to X$ be a radial homotopy fixing $X - U$ with $h_0 = \mathrm{id}_X$ and h_1 collapsing C into B. Trivialize $TX|U$ in the obvious way, and extend h_t to $\tilde{h}_t : TX \to TX$ as $h_t \times \mathrm{id}_{\mathbb{C}^n}$ over U and id_{TX} elsewhere. Then pulling back induces a homotopy $H_t : \mathcal{J} \to \mathcal{J}$ with $H_0 = \mathrm{id}_{\mathcal{J}}$ and $H_1(\mathcal{J}) \subset \mathcal{J}_C$. Composing with the previous contraction of $\mathcal{J}_C$ produces the required contraction of $\mathcal{J}$. A similar argument applies to the space of all $(\omega_{\mathrm{std}}, f)$-compatible structures on X, since these must all agree on $TX|B$ by Lemma 4.4(b). The proof of (a) is now complete.

The existence part of (b) will follow from Theorem 3.1, so we begin by constructing suitable neighborhoods W_y with forms η_y. Fix $C \subset U \subset X - B$ and J on X as above, and a neighborhood W_C of C in $X - B$ with closure $\mathrm{cl}\, W_C \subset U$ a disjoint union of round balls. For each $y \in \mathbb{CP}^{n-1}$, let $K \subset F_y$ denote the (finite) subset of critical points of f lying on the fiber F_y, and let $\Delta \subset X - U$ be a disjoint union of closed balls, one centered at each point of K. Define a closed 2-form σ on $\Delta \cup U$ as follows: Choose σ to tame J on $TX|K$. For Δ sufficiently small, we can then assume J is σ-tame on Δ (by openness of the taming condition). On U, take σ to be the standard symplectic form from $\mathbb{C}^n$, in the local coordinates given by (1) of Definition 2.4, scaled so that its integral is $< 1/2$ on each complex line through 0 intersected with U. Now J is σ-tame on $\Delta \cup U$. Since J is $(\omega_{\mathrm{std}}, f)$-tame on $X - B$, $F_y - K$ is a smooth (noncompact) J-holomorphic curve in $X - K$ whose complex orientation agrees with its preimage orientation, and each component intersects B nontrivially (by Definition 2.4(3)). To allow for the (presumably unlikely) possibility of F_y being wildly knotted at K, we use the following trick: We can assume $\partial\Delta$ is transverse to F_y, so the two intersect in a finite collection of circles. Since each component of $F_y - K$ intersects B, we can connect each such circle to B by a path in $F_y - K$. Let $\Delta_0 \subset \mathrm{int}\, \Delta$ be a smaller collection of balls surrounding K, disjoint from these paths and with $\partial\Delta_0$ transverse to F_y. Then each component F_i of the compact surface $F_y - \mathrm{int}\, \Delta_0$ either lies inside $\mathrm{int}\, \Delta$ or intersects B. Let W_y be the union of $\mathrm{int}\, \Delta_0 \cup W_C$ with a tubular neighborhood rel boundary of $F_y - \mathrm{int}\, \Delta_0 \subset X - \mathrm{int}\, \Delta_0$. Extend each F_i to a closed, oriented, smooth surface $\hat{F}_i \subset W_y$ by arbitrarily attaching a surface in Δ_0. Then the classes $[\hat{F}_i] \in H_2(W_y; \mathbb{Z})$ form a basis.

We now construct the required form η_y on W_y and apply Theorem 3.1. Since F_y is J-holomorphic with J σ-tame on $\Delta \cup U$, $\sigma|F_i \cap (\Delta \cup W_C)$ is a positive area form. After rescaling σ on Δ so that $\int_{\hat{F}_i \cap \Delta} \sigma < 1/2$ for each i, we can extend σ over each F_i intersecting B as a positive area form with $\int_{\hat{F}_i} \sigma = \# F_i \cap B$, the (positive) number of points of B in F_i. Define $\pi : W_y \to W_y$ by smoothly splicing $\mathrm{id}_{W_y \cap (\Delta \cup U)}$ together with the normal bundle projection on $W_y - \Delta$ so that $\operatorname{Im} \pi \subset F_y \cup \Delta \cup U$ and $\pi | F_y \cup \Delta_0 \cup W_C$ is the identity. Then $\eta_y = \pi^* \sigma$ is a well-defined closed 2-form on W_y. On $W_y \cap (\Delta \cup W_C)$ (away from $\partial \Delta$) $\eta_y = \sigma$ tames J (hence, each $J | \ker df_x$) and η_y is standard on W_C. Similarly, $\eta_y | (F_y - K)$ tames J on each $T_x F_y = \ker df_x$, so after narrowing the tubular neighborhood defining W_y, we can assume η_y tames $J | \ker df_x$ for all $x \in W_y$ (since the taming condition is open and the set of critical points of f is closed). For each F_i intersecting B, we have $\langle \eta_y, \hat{F}_i \rangle = \langle \sigma, \hat{F}_i \rangle = \# F_i \cap B = \langle c_f, \hat{F}_i \rangle$. (The last equality follows, e.g., by computing the intersection number of $\hat{F}_i$ (pushed slightly off B) with $f^{-1}(\mathbb{CP}^{n-2}) \subset X - B$. The only intersections occur in U, and each is $+1$ by the local description of f there.) Similarly, for F_i disjoint from B we have $\langle c_f, \hat{F}_i \rangle = 0 = \langle \eta_y, \hat{F}_i \rangle$ since $\eta_y = \sigma$ is exact on the disjoint union of balls Δ containing $\hat{F}_i$. Thus $[\eta_y] = c_f | W_y \in H^2_{\mathrm{dR}}(W_y)$ since these agree on a basis of $H_2(W_y; \mathbb{Z})$. Now we can apply Theorem 3.1 to $X - B$ with $\omega_Y = \omega_{\mathrm{std}}$ on $Y = \mathbb{CP}^{n-1}$ and $c = c_f$. Note that W_C is a disjoint union of punctured open $2n$-balls with $n \geq 2$, so $H^1_{\mathrm{dR}}(W_C) = 0$. We obtain a closed 2-form η on $X - B$ that is standard on C (relative to the charts given in Definition 2.4(1)) and hence extends over X. Then $[\eta] = c_f \in H^2_{\mathrm{dR}}(X)$, and we can choose some $t > 0$ so that $\omega_t = t\eta + f^* \omega_{\mathrm{std}}$ tames J on $X - B$.

Unfortunately, the form ω_t is singular at B. We verify this with a local model, and find a way to eliminate the singularities. In the given local coordinates at $b \in B$, J is the standard complex structure on $\mathbb{C}^n$, f is projectivization, and η is the standard symplectic form. Up to a constant rescaling of the coordinates, the latter can be written in "complex spherical coordinates" (cf. [**McS**, Proposition 5.8]) as $\eta = r^2 f^* \omega_{\mathrm{std}} + \frac{1}{2\pi} d(r^2) \wedge \beta$, where r is the radial coordinate on $\mathbb{C}^n$ and β is the pull-back to $\mathbb{C}^n - \{0\} \approx S^{2n-1} \times \mathbb{R}$ of the connection 1-form on S^{2n-1} for the tautological bundle $S^{2n-1} \to \mathbb{CP}^{n-1}$, whose corresponding horizontal distribution H is orthogonal to the fibers. (To verify this formula, note that H on $\mathbb{C}^n - \{0\}$ is orthogonal to each complex line L through $0 \in \mathbb{C}^n$ under both the above η and the standard symplectic form. Both descend from S^{2n-1} to ω_{std} on $\mathbb{CP}^{n-1}$ up to a constant scale factor, and both scale by r^2 radially, so they agree up to scale on H. On each L, $d(r^2) \wedge \beta = 2r dr \wedge d\theta$ is standard up to a scale factor independent of L. The two terms of η are scaled compatibly since $d\eta = 0$ by computation, using the fact that $d\beta$ pushes down to the tautological curvature form $2\pi\omega_{\mathrm{std}}$ on $\mathbb{CP}^{n-1}$ with Chern class $-[\omega_{\mathrm{std}}] =$

$-h$.) Now $\omega_t(r) = (1 + tr^2)f^*\omega_{\text{std}} + \frac{t}{2\pi}d(r^2) \wedge \beta$ in these local coordinates. Clearly, this is singular at 0 since η is nonzero there. However, the radial change of variables $R^2 = \frac{1+tr^2}{1+t}$ (t constant) shows that $\eta(R) = \frac{1}{1+t}\omega_t(r)$, so there is a radial symplectic embedding $\varphi : (\mathbb{C}^n - \{0\}, \frac{1}{1+t}\omega_t) \to (\mathbb{C}^n, \eta)$ sending any deleted neighborhood of 0 to a deleted neighborhood of the ball $R^2 \leq \frac{1}{1+t}$. For $V \subset \mathbb{C}^n$ the image of the given coordinate chart at $b \in B$, define $\varphi_0 : V \to \mathbb{C}^n$ to be a radially symmetric diffeomorphism onto an open ball, agreeing with φ outside of a closed ball about 0 in V. Let ω be $\varphi_0^*\eta$ near each $b \in B$ and $\frac{1}{1+t}\omega_t$ elsewhere. These pieces fit together to define a symplectic form on X, since φ is a symplectic embedding. (This construction is equivalent to blowing up B, applying Theorem 3.1 with $C = \emptyset$ to the resulting singular fibration, and then blowing back down, but it avoids technical difficulties associated with taming on the blown up base locus.)

To complete the existence proof for (b), we only need to verify that the symplectic form ω on X has the required properties. Away from B, we already know that $\omega = \frac{1}{1+t}\omega_t$ tames J. Near $b \in B$, we have local coordinates with J standard and $\omega = \varphi_0^*\eta$, η standard up to a constant scale factor. Since φ_0 is radially symmetric, it preserves the horizontal distribution H on $\mathbb{C}^n - \{0\}$ and the form $\eta|H$ up to (nonconstant) scale. Also, φ_0 preserves each complex line through 0, and $\varphi_0^*\eta$ is a positive area form on each. Since these complex lines and H are η-orthogonal and J-holomorphic, J is ω-tame near B and hence everywhere on X. To compute the cohomology class $[\omega] \in H^2_{\text{dR}}(X)$, it suffices to work outside C. Then $[\omega] = \frac{1}{1+t}[\omega_t] = \frac{1}{1+t}(tc_f + f^*[\omega_{\text{std}}]) = c_f$ since $[\omega_{\text{std}}] = h \in H^2_{\text{dR}}(\mathbb{CP}^{n-1})$. Thus, the constructed form ω has the required properties.

For fixed J, the space of taming ω with $[\omega]$ fixed is obviously convex; the rest of the theorem depends on the following lemma. This lemma allows us to extend a family of taming forms to a parametrized family of hyperpencils, starting from any subfamily with reasonable local topology.

Lemma 3.4. *Let $f : X - B \to Y$ be a family of hyperpencils parametrized by a metrizable space S, and suppose the restriction of f to some closed subset $S_0 \subset S$ is ω_0-tame for some continuous family ω_0 of continuous 2-forms ω_s on the fibers X_s, $s \in S_0$. Suppose S_0 has a neighborhood $U_0 \subset S$ with a C^0 retraction $r : U_0 \to S_0$ covered by retractions $r_X : \pi_X^{-1}(U_0) \to \pi_X^{-1}(S_0)$ and $r_Y : \pi_Y^{-1}(U_0) \to \pi_Y^{-1}(S_0)$ with fiberwise derivatives dr_X and dr_Y continuous over U_0, r_Y preserving ω_{std}, r_X a fiberwise diffeomorphism preserving B and complex linear near B with respect to the vector bundle structure, and $f \circ r_X = r_Y \circ f$. Then ω_0 extends to a family ω parametrized by S, taming f. If ω_0 consists of C^k-forms varying continuously in the C^k-topology,* ($0 \leq k \leq \infty$)*, then ω inherits this property (provided dr_X is C^k-continuous),* *and if the family ω_0 is C^k (for $\pi_X : X \to S$ a C^{k+1}-bundle map of C^{k+1}-manifolds) then so is ω. Similarly, closure of each ω_s and the condition $[\omega_s] = c_f \in H^2_{\mathrm{dR}}(X_s)$ are inherited by ω. We can also assume that ω tames a preassigned J on T^vX as specified in Lemma* 2.10 *(provided that ω_0 tames J over S_0), or that ω tames some J for which f_*J on $\operatorname{Im} df \subset f^*T^vY$ is preassigned, r_Y invariant, standard near B, ω_{std}-compatible, and induced from local complex structures on T^vX as in Definition 2.9 (over S_0) and Definition 2.7(2) (over S).*

Proof. If J was not preassigned, construct it as follows: Find an ω_0-tame J_0 over S_0 by Lemma 2.10, and pull it back over U_0 by r_X. Then J_0 is continuous and $(\omega_{\mathrm{std}}, f)$-compatible over U_0, standard near B there, and if f_*J is preassigned, it agrees with f_*J_0 there. Now $J_0|\pi_X^{-1}(S_0)$ extends to J on T^vX as specified in Lemma 2.10 (where ω_0-tameness of J only applies over S_0), by Lemma 3.2(a) applied to $X - B$ with $E = T^v(X - B)$, $F = f^*T^vY$, $C = \pi_X^{-1}(S_0)$ union the closure of a suitable neighborhood of B, and D equal to C or X, depending on whether f_*J was given. (Note that X is metrizable since it is locally metrizable and paracompact Hausdorff, e.g., [**Mu**, §42], [**P**].) By the existence part of (b) of Theorem 2.11, each X_s, $s \in S - S_0$, has a smooth symplectic form ω_s taming $J|X_s$, with $[\omega_s] = c_f$. By openness of the taming condition and compactness of each fiber X_s, each ω_s extends over some neighborhood W_s of $s \in S - S_0$ (e.g., via local triviality of π_X) so that it tames $J|\pi_X^{-1}(W_s)$. Similarly, each $s_0 \in S_0$ has a neighborhood W_{s_0} in S over which ω_0 extends as a taming form, preserving any additional conditions. (If only a continuous family is required, pull back by r_X. For a C^k-family of closed forms, $k \geq 1$, locally trivialize, C^k-extend the map $[\omega_s] : S_0 \to H^2_{\mathrm{dR}}(X_s)$ over some W_{s_0} (possibly with $[\omega_s] = c_f$ by hypothesis), find a C^k-family of representatives, and correct by $d\alpha$ for a suitably extended family of 1-forms α to recover the original subfamily ω_0.) Using a partition of unity on S subordinate to the cover $\{W_s | s \in S\}$, splice together the local families ω_s into a global family ω extending ω_0. The taming and closure conditions are preserved since each $\omega|X_s$ is a convex combination (with constant coefficients) of such forms. The lemma follows immediately. □

To complete the proof of Theorem 2.11, first let ω_s, $s = 0, 1$, be symplectic forms on X associated to hyperpencils f_s in a deformation class φ as in (c). Then there is a deformation agreeing with f_0 and f_1, respectively, on neighborhoods of $0, 1$ in I. The above lemma, with $S = I$ and $S_0 = \{0, 1\}$, gives a smooth family of symplectic forms ω_s, $0 \leq s \leq 1$, interpolating between ω_0 and ω_1, with $[\omega_s] = c_{f_0} = c_{f_1}$ for each s. By Moser's Theorem [**M**], any deformation of cohomologous symplectic forms is realized by an isotopy, proving (c). A similar argument proves the last sentence of (b), completing the proof of the theorem: For fixed k, $1 \leq k \leq \infty$, let $\mathcal{C}$ denote the C^k-space of all C^k-symplectic forms on X as in (b) (for fixed f and f_*J). Any continuous map $\varphi : \partial D^{m+1} \to \mathcal{C}$ (D^{m+1} an $(m+1)$-disk) can be interpreted as a C^k-continuous family ω_0 of symplectic forms on X parametrized by ∂D^{m+1}. Applying the above lemma to the constant family of hyperpencils on X with $S = D^{m+1}$, $S_0 = \partial D^{m+1}$, and the given f_*J (which ensures that f over S_0 is ω_0-tame), we extend to a family parametrized by D^{m+1}, or equivalently a continuous map $D^{m+1} \to \mathcal{C}$ extending φ. This shows that $\mathcal{C}$ is weakly contractible ($\pi_m(\mathcal{C}) = 0$ for all m). But $\mathcal{C}$ is an open subset of the affine subspace of 2-forms determining $c_f \in H^2_{\mathrm{dR}}(X)$, so it is a metrizable manifold of infinite dimension. In particular, it is an ANR, so weak contractibility implies contractibility [**P**]. Since the C^∞-forms are dense in the C^0-space of forms, and $\mathcal{C}$ extends to an open subset of a closed affine subspace of the latter, the required assertion follows from Theorem 16 of [**P**]. □

4. Proofs of Lemmas 2.10 and 3.2

The proofs of Lemmas 2.10 and 3.2 require a canonical method for interpolating between almost-complex structures. Our approach is a generalization of that of [**ABKLR**], Proposition 6.2. This depends on the $r = -\frac{1}{2}$ case of the following proposition, a similar form of which is stated without proof on p.100 of [**ABKLR**]. The proof below follows a suggestion of L. Sadun.

Proposition 4.1. *For $r \in \mathbb{R}$, let $\rho_0 : \mathbb{C} - (-\infty, 0] \to \mathbb{C}$ be the branch of z^r with $\rho_0(1) = 1$. Let $\mathcal{A} \subset \mathrm{GL}(m, \mathbb{C})$ be the open subset of matrices with no eigenvalues in $(-\infty, 0]$. Then there is a unique holomorphic map $\rho : \mathcal{A} \to \mathrm{GL}(m, \mathbb{C})$, which will be denoted by $\rho(A) = A^r$, such that each λ-eigenvector of A is a $\rho_0(\lambda)$-eigenvector of A^r. This map has the following properties:*

a) *For $n \in \mathbb{Z}, A^n$ agrees with its usual meaning.*
b) *For $|r| \le 1$ or $s \in \mathbb{Z}, (A^r)^s = A^{rs}$.*
c) *If $T : \mathbb{C}^m \to \mathbb{C}^k$ is a linear transformation with $TA = BT$, then $TA^r = B^r T$ (whenever both sides are defined).*
d) *If A is real, then so is A^r.*
e) *If A is real and self-adjoint with respect to a given inner product g on $\mathbb{R}^m$, then so is A^r.*
f) *For $n \in \mathbb{Z} - \{0\}$, $A^{1/n}$ is the unique solution to the equation $X^n = A$ for which all eigenvalues of X lie in $\operatorname{Im} \rho_0$.*

Note that (d) and (c) imply that ρ is canonically defined for any finite-dimensional real vector space.

Proof. For any $A \in \mathcal{A}$, the Jordan form of A splits $\mathbb{C}^m$ as the direct sum of the generalized eigenspaces $V_\lambda = \ker(A - \lambda I)^m$ (λ ranging over the eigenvalues of A). On each V_λ, A has the form $\lambda(I + N)$ for the nilpotent transformation $N = \frac{1}{\lambda}A - I$. Set $\rho(A) = \rho_0(\lambda)p(N)$ on V_λ, where $p(z)$ is the power series expansion about 0 of the function $\rho_0(1+z) = (1+z)^r$. (Note that $p(N)$ is a polynomial in the nilpotent transformation N.) On each λ - eigenspace, $N = 0$ and $A^r = \rho_0(\lambda)I$ as required. Properties (a) and (b) follow immediately from the corresponding properties for ρ_0 and p. Property (c) is also immediate, once we observe that the condition $TA = BT$ implies T preserves or annihilates each generalized λ-eigenspace. For (d), take A real and note that $\mathbb{R}^m$ is the span of the vectors $v + \bar{v}, v \in V_\lambda$ (where λ ranges over all eigenvalues of A). Since $\bar{v} \in V_{\bar{\lambda}}$, $A^r(v+\bar{v}) = A^r v + A^r \bar{v} = A^r v + \overline{A^r v}$, so this is real as required. Now (e) follows immediately from g-orthogonal diagonalizability. For (f), take any X as given and write it as $\lambda(I+N)$ on each of its generalized eigenspaces V_λ. Then $A = X^n = \lambda^n(I+N)^n$ on each V_λ, and the last factor has the form $I +$ (nilpotent). Thus $A^{1/n} = \rho(X^n) = X$, since λ is the unique n^{th} root of λ^n in $\operatorname{Im}\rho_0$ and the power series for $\rho_0(1+z)$ inverts the exponentiation of $(I+N)^n$.

It remains to prove holomorphicity of ρ, from which uniqueness follows by density of diagonalizable matrices in $\mathcal{A}$. First note that the equation $\det(A - \lambda I) = 0$ in (A, λ) exhibits the eigenvalues of matrices in $\mathcal{A}$ as an algebraic variety in $\mathcal{A} \times \mathbb{C}$. The subset $\mathcal{S} \subset \mathcal{A}$ of matrices failing to have m distinct eigenvalues is then also a variety, and over $\mathcal{A} - \mathcal{S}$ the m eigenvalues vary holomorphically. One can now locally construct holomorphically varying bases of eigenvectors over $\mathcal{A} - \mathcal{S}$, and in these bases, ρ is easily seen to be holomorphic as required. To show that ρ extends from $\mathcal{A} - \mathcal{S}$ to some holomorphic map $\hat{\rho} : \mathcal{A} \to \mathrm{GL}(m, \mathbb{C})$, note that nondiagonalizable matrices with $m-1$ distinct eigenvalues form a dense open subset of $\mathcal{S}$. Given such a matrix A, restrict it to its 2-dimensional generalized eigenspace V_λ, and note that in a suitable basis we obtain the matrix A_0 in the family

$$A_z = \begin{pmatrix} \lambda + z & 0 \\ 1 & \lambda \end{pmatrix}, \text{ with } \rho(A_z) = \begin{pmatrix} \rho_0(\lambda+z) & 0 \\ \dfrac{\rho_0(\lambda+z) - \rho_0(\lambda)}{z} & \rho_0(\lambda) \end{pmatrix}$$

for small $z \neq 0$. (The latter equality is easy to verify using the eigenvectors $(0,1)$ and $(z,1)$ for A_z.) Clearly, $\rho(A_z)$ extends holomorphically over $z = 0$. The Cauchy Integral Formula now provides the required holomorphic extension $\hat{\rho}$ on $\mathcal{A}$ (since the remaining subset of $\mathcal{S}$ has higher codimension in $\mathcal{A}$; cf. Hartogs' Theorem [**GH**]).

Finally, we show $\hat{\rho} = \rho$. First, we assume $r = 1/n, n \in \mathbb{Z} - \{0\}$. For $A \in \mathcal{S}$, consider a sequence (A_i) in $\mathcal{A} - \mathcal{S}$ converging to it. Then $\rho(A_i) = \hat{\rho}(A_i) \to \hat{\rho}(A)$. By (a) and (b), $(\rho(A_i))^n = A_i$; taking the limit shows that $(\hat{\rho}(A))^n = A$. Similarly, this sequence allows us to write each eigenvalue λ of $\hat{\rho}(A)$ as $\lim \lambda_i$ for λ_i an eigenvalue of $\rho(A_i)$. By the definition of ρ, $\lambda_i \in \operatorname{Im}\rho_0$, so $\lambda_i = \rho_0(\lambda_i^n)$ and $\lambda = \rho_0(\lambda^n)$. Now $\hat{\rho}(A) = \rho(A)$ by (f) as required. If $r = p/q \in \mathbb{Q}$, consider $A_i \to A$ as before. Then $\hat{\rho}(A_i) = (A_i^{1/q})^p$, so $\hat{\rho}(A) = (A^{1/q})^p = \rho(A)$ (by the previous case). The case of irrational r now follows by continuity of ρ and $\hat{\rho}$ with respect to r. □

Corollary 4.2. *For a real, finite-dimensional vector space V, let $\mathcal{B} \subset \mathrm{Aut}(V)$ be the open set of linear operators with no real eigenvalues, and let $\mathcal{J} \subset \mathcal{B}$ denote the set of complex structures on V (for both orientations of V). Then there is a canonical real-analytic retraction $j : \mathcal{B} \to \mathcal{J}$. For any linear transformation $T : V \to W$ with $TA = BT$, we have $Tj(A) = j(B)T$ (whenever both sides are defined).*

Proof. We generalize [**ABKLR**]. For $B \in \mathcal{B}$, note that $-B^2$ has no real eigenvalues ≤ 0. (If $-\lambda^2$ were an eigenvalue of $-B^2$ with $\lambda \in \mathbb{R}$ then $\pm\lambda$ would be an eigenvalue of B, since $0 = \det(-B^2 + \lambda^2 I) = \pm \det(B + \lambda I)\det(B - \lambda I)$.) Thus, we can define $j(B)$ to be $B(-B^2)^{-1/2}$. The two factors commute by Proposition 4.1(c), so $(j(B))^2 = -I$ as required. For $J \in \mathcal{J}$, $j(J) = J$. The rest of the corollary also follows immediately. □

Now fix $J_1, \ldots, J_k \in \mathcal{J}$ and let $t = (t_1, \ldots, t_k)$ vary over the simplex $\sum t_i = 1$, each $t_i \geq 0$. Suppose each $B_t = \sum t_i J_i$ has no real eigenvalues. For example, this is guaranteed if $J_1, \ldots, J_k$ are all ω-tame for a fixed ω on V (since for $v \neq 0$, $\omega(v, B_t v) = \sum t_i \omega(v, J_i v) > 0$ but $\omega(v, v) = 0$). Then we obtain an analytic simplex $j(B_t)$ of complex structures (complex simplex?) with vertices $J_1, \ldots, J_k$. We show that if the vertices are ω-compatible for a fixed ω on V, then so is each $j(B_t)$ (cf. [**ABKLR**]): Compatibility implies that the bilinear form $g_i(v, w) = \omega(v, J_i w)$ is positive definite and symmetric for each i, as is the form $g = \sum t_i g_i$. Since $g(v, w) = \omega(v, B_t w)$, we have $g(B_t v, w) = \omega(B_t v, B_t w) = -\omega(B_t w, B_t v) = -g(B_t w, v) = -g(v, B_t w)$. Thus B_t is skew-adjoint with respect to the inner product g, so $-B_t^2$ and $(-B_t^2)^{-1/2}$ are self-adjoint (the latter by Proposition 4.1(e)). Hence $j(B_t)$ is skew-adjoint, implying that $\omega(j(B_t)v, j(B_t)w) = g(j(B_t)v, B_t^{-1} j(B_t) w) = g(v, B_t^{-1} w) = \omega(v, w)$ as required. Furthermore, $\omega(v, j(B_t)v) = \omega(v, B_t(-B_t^2)^{-1/2} v) = g(v, (-B_t^2)^{-1/2} v) > 0$ for $v \neq 0$, since $-B_t^2$ and hence $(-B_t^2)^{-1/2}$ are g-orthogonally diagonalizable with all eigenvalues positive.

Question 4.3. *If $J_1, \ldots, J_k$ are only given to be ω-tame (for a fixed ω), is each $j(B_t)$ ω-tame?*

An affirmative answer would allow us to replace compatibility by taming throughout the paper, define ω-tame hyperpencils in a purely local way, remove $f_* J$ from the statement of the Main Theorem 2.11(b), and significantly simplify some of the proofs. For example, (b) of Lemma 3.2 with $D = C$ would follow immediately from the proof of (a), rendering the case of the lemma with $D \neq C$ unnecessary, along with the nonlocal condition on $f_* J_\alpha$ in the definition of ω-tame hyperpencils, and the fixed $f_* J$ in Theorem 2.11(b).

Proof of Lemma 2.10. The main difficulty is that Lemma 3.2(b) does not apply directly at the base locus. For an ω-tame hyperpencil $f : X - B \to \mathbb{CP}^{n-1}$ and $b \in B$, Definition 2.9 gives a neighborhood W_b of b in X with an ω-tame J_b on W_b for which $J_b|W_b - \{b\}$ is $(\omega_{\mathrm{std}}, f)$-compatible. If f_*J_b is not standard near b, we must correct this by a perturbation. First note that $J_b|T_bX$ must agree with the standard structure i, by (b) of the following lemma, which is proved below.

Lemma 4.4. (a) *A linear complex structure J on $\mathbb{R}^{2n}$, $n \neq 1$, is determined by its 1-dimensional (oriented) complex subspaces.*

(b) *If $f : \mathbb{C}^n - \{0\} \to \mathbb{CP}^{n-1}$ denotes projectivization, $n \geq 2$, and J is a continuous (positively oriented) almost-complex structure on a neighborhood W of 0 in $\mathbb{C}^n$, with $J|W - \{0\}$ $(\omega_{\mathrm{std}}, f)$-tame, then $J|T_0\mathbb{C}^n$ is the standard complex structure.*

Thus there is a neighborhood W of b in W_b for which the operators $A_t = (1-t)J_b + ti$, $0 \leq t \leq 1$, have no real eigenvalues, and for which the resulting complex structures $j(A_t)$ from Corollary 4.2 are ω-tame. Since $f_*j(A_t) = j((1-t)f_*J_b + tf_*i)$ by the naturality statement in Corollary 4.2, the paragraph following the proof of that corollary shows that each $j(A_t)$ on $W - \{b\}$ is $(\omega_{\mathrm{std}}, f)$-compatible. Let $\rho : X \to [0,1]$ be a continuous function with support in W and identically 1 near b. Then we can add $(W, j(A_\rho))$ to the open cover $\{W_\alpha\}$ and replace each W_α by $W_\alpha - \operatorname{supp}\rho$, recovering the hypotheses of the lemma, with the unique structure near b standard. After applying this procedure when necessary to each $b \in B$, we invoke Lemma 3.2(b), with $E = T(X - B)$, $F = f^*T\mathbb{CP}^{n-1}$, $T = df$, and C equal to the closure of a sufficiently small neighborhood of B so that we can let J_C be the standard structure i near each $b \in B$. The resulting structure on $X - B$ extends in the obvious way over B, completing the proof for a single hyperpencil.

A similar argument applies to families over metrizable parameter spaces. Cover $B \subset X$ by neighborhoods W as before, and choose $\rho : X \to [0,1]$ supported in their union. The resulting forms $f_*j(A_\rho)$ fit together as required, and Lemma 3.2(b) again completes the proof. □

Proof of Lemma 4.4. To reduce (b) to (a), note that each complex line through 0 (with respect to the standard complex structure i on $\mathbb{C}^n$) intersects W in a J-complex curve (with the same orientation), since its tangent spaces away from 0 are given by $\ker df$ and J is $(\omega_{\mathrm{std}}, f)$-tame. ($J$-complexity extends over 0 by continuity.) Thus, the i-complex lines through 0 in $T_0\mathbb{C}^n$ are also J-complex lines. To prove (a), assume each 1-dimensional i-linear subspace of $\mathbb{C}^n = \mathbb{R}^{2n}$ is J-linear, pick two i-linearly independent vectors $v, w \in \mathbb{C}^n$ and let $W = \operatorname{span}_i w$ and $V_\lambda = \operatorname{span}_i(v + \lambda w)$ for $\lambda \in \mathbb{C}$. These subspaces are both i-complex and J-complex lines. Thus, the projections of $V_\lambda \subset V_0 \oplus W$ to V_0 and W determine a map $\varphi_\lambda : V_0 \to W$ (whose graph is

V_λ) that is both i-linear and J-linear, and is determined by the condition $\varphi_\lambda(v) = \lambda w$. Now $\psi = \varphi_1^{-1} \circ \varphi_i : V_0 \to V_0$ is both i- and J-linear and given by $\psi(v) = iv$, so ψ is multiplication by i. It follows that J commutes with i on the complex line V_0, so that these two complex structures on V_0 agree. But V_0 was chosen arbitrarily, so $J = i$ everywhere. □

Proof of Lemma 3.2 *and Addendum* 3.3. We begin proving (a) with a partition of unity argument that works outside of $D - C$. Cover the space X by open sets W_α with complex structures J_α on $E|W_\alpha$ as in the lemma. We can assume that U is the unique W_α intersecting C (with corresponding $J_\alpha = J_C$), and that any W_α intersecting D lies in V, so that $T_* J_\alpha = J_D|W_\alpha$ there. Since X is metrizable, it is paracompact by Stone's Theorem (e.g., [**MS**], [**Mu**, Theorem 41.4]), so there is a partition of unity $\{\rho_\alpha\}$ subordinate to the covering $\{W_\alpha\}$. Let $A = \sum \rho_\alpha J_\alpha : E \to E$. To make this into a complex structure by the retraction in Corollary 4.2, we must investigate where A could have real eigenvalues. First note that the map $B = \sum \rho_\alpha T_* J_\alpha : T(E) \to T(E)$ has no real eigenvalues on any fiber since each $T_* J_\alpha$ is ω_F-tame. Since $TA = BT$, each λ-eigenvector of A is mapped by T to 0 or a λ-eigenvector of B, so any real eigenvector of A must lie in $\ker T$.

To rule out real eigenvectors in $\ker T$, recall the subset $P \subset \ker T \subset E$ introduced for Definition 2.2 (of wrapped points). For each $x \in X$, any $v \in P_x$ can be written as $\lim v_i$ with $v_i \in \ker T_{x_i}$ for some sequence (x_i) of regular points converging to x. After passing to a subsequence, we can assume the oriented 2-planes $\ker T_{x_i}$ (in the preimage orientation) converge to an oriented 2-plane $\Pi \subset P_x$ containing v. For each J_α defined on E_x, each $\ker T_{x_i}$ (i large) will be J_α-complex (compatibly oriented) as will the limit Π. It is now easy to construct a decomposition $\operatorname{span}_{\mathbb{R}} P_x = \bigoplus \Pi_j$, where each oriented real 2-plane Π_j is a J_α-complex line for each J_α defined at x. Clearly, the quotient $Q_x = \ker T_x / \operatorname{span}_{\mathbb{R}} P_x$ inherits a complex structure $\bar{J}_\alpha$ from each such J_α, and these are all compatible with the same orientation on Q_x (inherited via $\ker T_x$ from $\omega_F|T(E_x)$). If x is wrapped, then $\dim_{\mathbb{C}} Q_x \leq 1$, so each $\bar{J}_\alpha$ is ω-tame for some fixed ω on Q_x. It follows as before that $\sum \rho_\alpha(x) \bar{J}_\alpha$ has no real eigenvalues on Q_x, so any real eigenvector of A_x lies in $\operatorname{span}_{\mathbb{R}} P_x = \bigoplus \Pi_j$. But a direct sum ω on this space tames each J_α, so we conclude that A_x has no real eigenvalues when x is wrapped. If $x \in X - D$ is not wrapped (Addendum 3.3), then by hypothesis $n = 3$, and $\dim_{\mathbb{C}} Q_x \geq 2$, so $T_x = 0$ (since density of regular points implies $\dim_{\mathbb{C}} \operatorname{span}_{\mathbb{R}} P_x \geq 1$). We are also given a sequence $x_i \to x$ of regular points along which each J_α is (ω_F, T)-extendible, so each relevant $T_* J_\alpha$ extends continuously and ω_F-compatibly to $F|\{x_i\} \cup \{x\}$. If we pass to a suitable subsequence, $\frac{T_{x_i}}{\|T_{x_i}\|}$ will converge to a nonzero transformation that is complex linear for each relevant α. Then over the subspace $\{x_i\} \cup \{x\}$, $\frac{T}{\|T\|}$ will have a wrapped point at x, and each relevant J_α will be $(\omega_F, \frac{T}{\|T\|})$-compatible, so the previous argument again shows that A_x has no real eigenvalues. Thus, A can have real eigenvalues only at unwrapped critical points in D.

Corollary 4.2 provides a complex structure $j(A)$ that is well-defined and continuous except where A has real eigenvalues. By construction, $j(A) = J_C$ on some neighborhood of C, and $T_* j(A) = j(B)$, which is defined everywhere and equals J_D on a neighborhood V' of D. Furthermore $j(A)$ is (ω_F, T)-compatible (in particular, (ω_F, T)-tame) since each $T_* J_\alpha$, and hence $j(B)$, is ω_F-compatible. (This is the only place where we require compatibility instead of taming; cf. Question 4.3. Our use of compatibility in proving Lemma 2.10 can be avoided using openness of taming, cf. [**G3**, second paragraph following Addendum 2.6].) Similarly, in the case of Addendum 3.3, $j(A)|X - D$ is (ω_F, T)-extendible along the given sequences, as is any structure mapping to $j(B)$. This proves (a) of Lemma 3.2 (and its addendum) when $D = C$. The general case reduces to the case $D = X$ of the lemma after we intersect each W_α with V', add a new $W_\alpha = X - D$ with $J_\alpha = j(A)|X - D$, and extend J_D to $j(B) : T(E) \to T(E)$.

We now complete the proof of (a) in the remaining case $D = X$, while simultaneously preparing for (b). Recall that we just showed each $v \in P$ lies in an oriented 2-plane $\Pi = \lim \ker T_{x_i} \subset P_x$, for some sequence (x_i) of regular points converging to x, and that Π is a complex line for each J_α (including J_C) defined on E_x. Each such J_α now induces a complex structure J_Π on the quotient E_x/Π, and this structure is the limit of the corresponding structures on $E_{x_i}/\ker T_{x_i}$. Since these latter structures are also determined by J_D on $T(E_{x_i})$ via the isomorphism induced by T_{x_i}, they are independent of α, as is J_Π. If $Z \subset X$ denotes the subset of X for which $\dim_{\mathbb{C}} \operatorname{span}_{\mathbb{R}} P_x \geq 2$, then each E_x with $x \in Z$ contains at least two such distinct planes Π_1 and Π_2, and the J_α-complex monomorphism $E_x \hookrightarrow E_x/\Pi_1 \oplus E_x/\Pi_2$ shows that the structures J_α all agree on E_x. Thus, over $C' = C \cup \operatorname{cl} Z$, the structures $J_\alpha|W_\alpha - C$ and J_C all fit together into a continuous (ω_F, T)-compatible structure $J_{C'}$ on $E|C'$ with $T_* J_{C'} = J_D|C'$, depending only on J_C and J_D. We can now construct a $J_{C'}$-Hermitian fiber metric on $E|C'$ whose real part extends to a fiber metric g on E (using partitions of unity and the Tietze Extension Theorem). Over $X - C'$, P is an oriented (real) 2-dimensional subbundle of E. (To verify that $P|X - C'$ is a continuous section of the Grassmann bundle, note that for $x \in X - C'$ and any neighborhood V_Π of the plane $\Pi = P_x$, there is a neighborhood U_x of x in X on which all regular points map into V_Π; otherwise one could construct a sequence $x_i \to x$ with $\ker T_{x_i}$ converging to a plane $\neq \Pi$. Now $U_x - C'$ maps into the closure of V_Π.) Using the given orientation of $P|X - C'$ and the metric g, define $J|X - C'$ to be counterclockwise $\frac{\pi}{2}$-rotation on P and the structure determined by J_{P_x} on each $P_x^\perp \cong E_x/P_x$. (Note that J_{P_x} is continuous in x since it is locally induced by the structures J_α.) For each $x \in X - C'$, T_x factors through

E_x/P_x on which J and J_α agree, so $J|X-C'$ is (ω_F,T)-compatible, with $T_*J|X-C'=J_D|X-C'$. Thus the proof of (a) is completed by showing that $J|X-C'$ and $J_{C'}$ fit together continuously at each $x\in C'$. If this fails, then there is a sequence $x_i\to x$ for which each J_{x_i} lies outside a fixed neighborhood of J_x in the bundle $\mathrm{End}(E)$. By continuity of $J_{C'}$, we can pass to a subsequence (x_i) lying in $X-C'$, and then further assume the oriented 2-planes P_{x_i} converge to some Π at x by compactness of the Grassmann manifold. Since g is continuous and Hermitian at x, it is now routine to obtain the contradiction that $J_{x_i}\to J_x$. (Compare with a fixed J_α on the given orthogonal summands.)

To prove (b) of the lemma, we introduce a 2-form ω_E on E that is assumed to tame each J_α (including J_C). On $X-C'$, let Q denote the ω_E-orthogonal complement of the 2-dimensional (real) oriented subbundle $P\subset E|X-C'$. Since each P_x is a complex line for some ω_E-tame J_α on E_x, ω_E is nondegenerate (and positively oriented) on P. Thus $E|X-C'=Q\oplus P$ is an ω_E-orthogonal direct sum splitting. Now any subbundle of $E|X-C'$ complementary to P can be written as $\operatorname{graph}\psi$ for some continuous section ψ of $\mathrm{Hom}(Q,P)$. For J as constructed in the previous paragraph, let J_ψ denote the complex structure on $E|X-C'$ given by J on P and by J_{P_x} on each $\operatorname{graph}\psi_x\cong E_x/P_x$. To express any J_ψ in terms of J_0 (which is given by J_{P_x} on each Q_x), note that these agree on the quotient E_x/P_x, so for any $(q,\psi(q))\in\operatorname{graph}\psi\subset Q\oplus P$ we have $J_\psi(q,\psi(q))=(J_0q,\psi(J_0q))$. Then for any $(q,p)\in Q\oplus P$ we have $J_\psi(q,p)=J_\psi(q,\psi(q))+J_\psi(0,-\psi(q)+p)=(J_0q,(\psi J_0-J\psi)q+Jp)$. In particular,

$$(4.1)\quad \omega_E((q,p),J_\psi(q,p))=\omega_E(q,J_0q)+\omega_E(p,Jp)+\omega_E(p,(\psi J_0-J\psi)q).$$

Note that J is ω_E-tame on $P|X-C'$ since the fibers are correctly oriented J-complex lines. In the next paragraph, we will show that J_0 is ω_E-tame on Q, so the first two terms on the right side of Equation (4.1) have positive sum whenever $(q,p)\neq(0,0)$. Now choose ψ so that $\operatorname{graph}\psi$ is a J-complex subbundle of $E|X-C'$. Then $J_\psi=J|X-C'$. At each $x\in C'$, J agrees with some J_α, so it is ω_E-tame on E_x. By openness of the taming condition, we conclude that J is ω_E-tame over some neighborhood U' of C'. Thus, the left side of (4.1) is positive for $x\in U'-C',(q,p)\neq(0,0)$. Replacing ψ by $\rho\psi$ in (4.1) for any $\rho:U'-C'\to[0,1]$, we obtain a convex combination of two positive quantities, showing that $J_{\rho\psi}$ is ω_E-tame on $E|U'-C'$. If we choose $\rho:X-C'\to[0,1]$ to be 1 near C' and 0 outside U', then $J_{\rho\psi}$ extends over C' as J, providing the required ω_E-tame complex structure on E.

We finish the proof of (b) by showing that $J_0|Q_x$ is ω_E-tame for all $x\in X-C'$. If g denotes the J_0-invariant, symmetric bilinear form on Q_x given by $g(v,w)=\frac{1}{2}(\omega_E(v,J_0w)+\omega_E(w,J_0v))$, then $g(q,q)=\omega_E(q,J_0q)$, so it suffices to show g is positive definite. For a fixed J_0-invariant inner product on Q_x, let $Q_-\subset Q_x$ be the span of all nonpositive eigenvectors of

g. By J_0-invariance of g and the background inner product, Q_- is a J_0-complex subspace of Q_x. Let $\psi : Q_x \to P_x$ be a (real) linear transformation whose graph is a J_α-complex subspace of $E_x = Q_x \oplus P_x$ for some α. Then $J_\psi | \operatorname{graph} \psi = J_\alpha | \operatorname{graph} \psi$ is ω_E-tame. Thus (4.1) is positive for all nonzero (q, p) with $p = \psi(q)$. The latter condition cancels two terms, so we obtain $\omega_E(q, J_0 q) + \omega_E(p, \psi(J_0 q)) > 0$. Since the first term is $g(q, q)$, it is nonpositive on Q_-, implying that $p = \psi(q)$ cannot vanish on Q_- unless q does. Hence, $\psi | Q_-$ is a monomorphism and $\dim_{\mathbb{C}} Q_- \leq \dim_{\mathbb{C}} P_x = 1$. It now suffices to rule out the case $\dim_{\mathbb{C}} Q_- = 1$. In this case, note that J_α and ω_E induce the same orientation on E_x and on P_x, hence, on $Q_x \cong E_x / P_x$ (by ω_E-orthogonality of the splitting $E_x = Q_x \oplus P_x$). If $\dim_{\mathbb{C}} Q_x = 1$, this implies $J_0 | Q_x$ is ω_E-tame, so it suffices to assume $\dim_{\mathbb{C}} Q_x > 1$. Then the function $q \mapsto g(q, q)$ realizes both positive and nonpositive values on $Q_x - \{0\}$, so by connectedness there is a nonzero $q \in Q_x$ with $g(q, q) = 0$. It follows that ω_E vanishes on the J_0-complex line $Q_0 \subset Q_x$ containing q, so it is degenerate on the ω_E-orthogonal sum $Q_0 \oplus P_x \subset E_x$. But this latter subspace is J_α-complex since it projects to a J_{P_x}-complex line in E_x / P_x, contradicting the hypothesis that J_α is ω_E-tame.

To prove (c), assume $X_0 = X - \operatorname{int} C$ is locally compact, and let $\mathcal{C}(\xi)$ be the space of (continuous) sections of the bundle $\xi = \operatorname{End}(E | X_0)$ of endomorphisms $E_x \to E_x$ ($x \in X_0$) in the compact-open topology. A subbasis for this topology is given by all subsets of the form $S(B, W) = \{\sigma \in \mathcal{C}(\xi) | \sigma(B) \subset W\} \subset \mathcal{C}(\xi)$, for $B \subset X_0$ compact and $W \subset \xi$ open. (See, e.g., [**Mu**] for the space of *all* maps $X_0 \to \xi$, then restrict to the subspace of sections $\mathcal{C}(\xi)$.) We need three basic facts about this topology: (1) The evaluation map $e : X_0 \times \mathcal{C}(\xi) \to \xi$ (by $e(x, \sigma) = \sigma(x)$) is continuous. (This follows easily from local compactness.) (2) If S is any topological space and $\pi_{X_0} : X_0 \times S \to X_0$ is projection, then any section of the bundle $\pi^*_{X_0} \xi = \xi \times S \to X_0 \times S$ has the form (τ, π_S) where $\tau : X_0 \times S \to \xi$ can be interpreted as a map $S \to \mathcal{C}(\xi)$. This correspondence gives a bijection between continuous sections of $\pi^*_{X_0} \xi$ and continuous maps $S \to \mathcal{C}(\xi)$ (cf. [**Mu**]). (3) If X_0 is compact, then the compact-open topology on $\mathcal{C}(\xi)$ equals the C^0-topology induced by any fiber metric on ξ. (This is clear once we observe that subtracting a section of ξ induces a fiberwise-isometric homeomorphism of ξ, so it suffices to compare neighborhoods of the 0-section.)

In either case (a) or (b), let $\mathcal{J} \subset \mathcal{C}(\xi)$ be the subspace consisting of all complex structures on $E | X_0$ satisfying the conclusion of the lemma — that is, (ω_F, T)-compatible extensions J of $J_C | \partial C$ with $T_* J = J_D$ on $D \cap X_0$, and with J ω_E-tame in case (b) or (for the addendum) (ω_F, T)-extendible along the given sequences. For X_0 compact, the first step in showing the contractibility of $\mathcal{J}$ is to apply the lemma to the metrizable space $\tilde{X} = I \times X_0 \times \mathcal{J}$, with the structures $\tilde{E}$, $\tilde{F}$, $\tilde{T}$, $\omega_{\tilde{F}}$ (and $\omega_{\tilde{E}}$ in case (b)) pulled back in the obvious way by projection $\pi_{X_0} : \tilde{X} \to X_0$. We set $\tilde{C} = (\{0,1\} \times X_0 \cup I \times \partial C) \times \mathcal{J}$ and $\tilde{D} = (\{0,1\} \times X_0 \cup I \times (D \cap X_0)) \times \mathcal{J}$. To define $J_{\tilde{C}}$ on a neighborhood $\tilde{U}$ of $\tilde{C}$, first consider the tautological complex structure $J_{\mathrm{taut}} = (e, \pi_{\mathcal{J}}) : X_0 \times \mathcal{J} \to \xi \times \mathcal{J} = \pi^*_{X_0}\xi = \mathrm{End}(\tilde{E}|\{0\} \times X_0 \times \mathcal{J})$. Let $h : [0, \frac{1}{2}] \times I \to I$ be continuous with $h(0,t) = t$, $h(s,1) = 1$ and $h(s,t) = 0$ for $s \geq t$. Extend to a map $[0, \frac{1}{2}] \times K \to K$, for the collar $K \approx I \times \partial C$ in X_0, by $(h \circ (\mathrm{id}_{[0,1/2]} \times \pi_I), \pi_{\partial C})$ (so the ∂C factor is carried along trivially), and let $H : [0, \frac{1}{2}] \times X_0 \to X_0$ be the resulting extension by π_{X_0} outside of K. Compatibility of the collar neighborhood shows that $(H \times \mathrm{id}_{\mathcal{J}})^*(\tilde{E}|\{0\} \times X_0 \times \mathcal{J})$ can be identified with $\tilde{E}|([0, \frac{1}{2}] \times X_0 \times \mathcal{J})$, and similarly for $\tilde{F}$, $\omega_{\tilde{F}}$, $\tilde{T}$ (and $\omega_{\tilde{E}}$ and $J_{\tilde{D}} = \pi^*_{X_0}(J_D|D)$ when relevant), so $(H \times \mathrm{id}_{\mathcal{J}})^* J_{\mathrm{taut}}$ can be considered an $(\omega_{\tilde{F}}, \tilde{T})$-compatible complex structure on $\tilde{E}|([0, \frac{1}{2}] \times X_0 \times \mathcal{J})$. Now fix some $J \in \mathcal{J}$ and pull it back to $\tilde{E}|([\frac{1}{2}, 1] \times X_0 \times \mathcal{J})$ by the same method, using the function $h'(s,t) = h(1-s,t)$. Then the two complex structures agree over $\{\frac{1}{2}\} \times [0, \frac{1}{2}) \times \partial C \times \mathcal{J}$ (where they are both pulled back from $J_C|\partial C$), so together they define a complex structure $J_{\tilde{C}}$ on a neighborhood $\tilde{U}$ of $\tilde{C}$ as required. Construct the required local complex structures elsewhere on $\tilde{X}$ by (for example) pulling back the given structures on X_0 by π_{X_0}. The hypotheses of the lemma are now satisfied on $\tilde{X}$ if D equals C or X. For the general case, extend $J_{\tilde{D}}$ over a neighborhood $\tilde{V}$ of $\tilde{D}$ by a procedure similar to the one above, pulling back each local structure on $\tilde{E}$ (including $J_{\tilde{C}}$) along the product lines of the given collar of ∂D. The lemma gives a section $\tilde{J} : \tilde{X} = I \times X_0 \times \mathcal{J} \to \mathrm{End}(\tilde{E})$, and hence a continuous map $\varphi : I \times \mathcal{J} \to \mathcal{C}(\xi)$ with image in $\mathcal{J}$. Since $\varphi|\{0\} \times \mathcal{J} = \mathrm{id}_{\mathcal{J}}$ and $\varphi|\{1\} \times \mathcal{J} = J$ (as a constant map into $\mathcal{J}$), $\varphi : I \times \mathcal{J} \to \mathcal{J}$ is the required contraction.

If X_0 is only locally compact, we wish to show that every map $S^m \to \mathcal{J}$ is homotopic to a constant map. But such a map can be interpreted as a section of the bundle $\pi^*_{X_0}\xi \to X_0 \times S^m$ that is in fact a complex structure on $\pi^*_{X_0}(E|X_0) \to X_0 \times S^m$. The previous argument with S^m in place of $\mathcal{J}$ provides the required nullhomotopy. □

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