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THE GUILLEMIN FORMULA AND KÄHLER METRICS ON TORIC SYMPLECTIC MANIFOLDS

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We discuss the construction of toric Kähler metrics on symplectic $2n$-manifolds with a hamiltonian n-torus action and present a simple derivation of the Guillemin formula for a distinguished Kähler metric on any such manifold. The results also apply to orbifolds.

Introduction

There is a one-to-one correspondence, established by T. Delzant in [**5**], between the class of compact connected symplectic manifolds of dimension $2n$ with an effective hamiltonian action of the n-dimensional torus T^n, known as compact *toric* symplectic manifolds, and a class a convex polytopes in $\mathbb{R}^n$, known as *Delzant polytopes.* More precisely, given a compact toric symplectic manifold, the image of the momentum map is a Delzant polytope; conversely, to each Delzant polytope Δ is *canonically* associated a compact toric symplectic manifold M_Δ, in such a way that Δ may be identified with the image of the corresponding momentum map. By extending the class of Delzant polytopes, a similar correspondence can be established for orbifolds [**9**].

It turns out that the symplectic manifold M_Δ canonically associated to Δ comes naturally equipped with a Kähler structure (M_Δ, g, J) and a holomorphic action of the complexified torus $T^n_{\mathbb{C}}$, which makes the complex manifold (M_Δ, J) into a toric variety; moreover, on the dense open orbit of $T^n_{\mathbb{C}}$, the Kähler structure admits a T^n-invariant globally defined Kähler potential, which has been computed by V. Guillemin in [**6**] (see also [**7**] and [**1**]).

The aim of this note is to provide a simple derivation of the Guillemin formula. The idea is to systematically exploit the symplectic geometry of the Delzant construction, in which the symplectic manifold M_Δ is obtained as a symplectic quotient of $\mathbb{R}^{2d}$ (equipped with its standard hamiltonian action of T^d) by a $(d-n)$-subtorus of T^d (here d is the number of codimension one faces of the Delzant polytope). The canonical Kähler metric is obtained from the identification of $\mathbb{R}^{2d}$ with $\mathbb{C}^d$ induced by the action of T^d: the flat metric on $\mathbb{C}^d$ then descends to a Kähler metric on M_Δ, the Guillemin metric.

The key observation is that a Kähler metric on a toric symplectic manifold determines and—modulo the choice of angular coordinates—is determined by a *reduced* metric on the interior of the image of its momentum map, and that these reduced metrics behave in a straightforward way with respect to symplectic quotients. In particular, in the Delzant construction, the dual $\mathbb{R}^{n*}$ of the Lie algebra of the quotient torus T^n acting on M_Δ is naturally an affine subspace of the dual $\mathbb{R}^{d*}$ of the Lie algebra of T^d, and the reduced metric on the interior Δ_0 of Δ is simply the pullback of the reduced metric induced (on a quadrant of $\mathbb{R}^{d*}$) by the flat metric on $\mathbb{C}^d$. To carry out this pullback, one simply has to write the flat metric on $\mathbb{C}^d$ in *momentum coordinates*, i.e.,

$$g_0 = \sum_{j=1}^{d} \frac{d\mu_j{}^2}{2\mu_j} + \sum_{j=1}^{d} 2\mu_j dt_j^2. \tag{1}$$

(Here the first sum gives the reduced metric, the second the metric on the d-torus fibres.)

This procedure leads naturally to the Guillemin metric on M_Δ in momentum coordinates, which are the derivatives of a Kähler potential with respect to the more traditional *holomorphic coordinates* associated to the complexified torus action. The Legendre transform G of the Kähler potential F gives a *dual* or *symplectic* potential for the toric Kähler metric in momentum coordinates [**6**, **7**]. The usefulness of dual potentials and momentum coordinates in toric symplectic geometry has been emphasised by M. Abreu [**1**, **2**]. In particular, dual potentials, like reduced metrics, behave straightforwardly under symplectic quotients, and so the dual potential of M_Δ is easy to compute. Conversely, inverting the Legendre transform, we can reobtain the holomorphic coordinates and the Kähler potential F, yielding the Guillemin formula.

Although this indirect approach is very simple, it seems not to have been described in detail before. However, it is closely related to the construction of toric hyperkähler metrics in [**3**] by R. Bielawski and A. Dancer, using the method of [**8**, Section 2(C)]. As they point out, the hyperkähler approach also gives the Guillemin formula by setting to zero half of the coordinates. Nonetheless, we hope it will be of use to present directly the Kähler story, with complete arguments, here.

We begin by reviewing the Delzant–Lerman–Tolman theory of toric symplectic manifolds and orbifolds. Then we introduce toric Kähler metrics and their description in momentum coordinates. We prove that under symplectic quotient, the reduced metrics are related by pullback, then apply this to the flat metric on $\mathbb{C}^d$.

Finally, in an appendix, we explain how the Guillemin formula for the Kähler potential is also a direct consequence of a general formula appearing in [**4**] (cf. also [**8**, Section 3 (E)]).

1. Hamiltonian group actions

Let (M, ω) be a symplectic manifold with an effective action of a Lie group G. For simplicity, we assume that G is compact. Then we say the action is hamiltonian if there is a momentum map $\mu\colon M \to \mathfrak{g}^*$ for the action, i.e., μ is G-equivariant and $\iota_{X_\xi}\omega = -\langle d\mu, \xi\rangle$ for any $\xi \in \mathfrak{g}$, where X_ξ is the corresponding vector field on M (generating the action of a 1-parameter subgroup of G).

If c is a regular value of μ invariant under the coadjoint action of G, then $\mu^{-1}(c)$ is a G-invariant submanifold of M and the action of G on $\mu^{-1}(c)$ is locally free, with finite isotropy groups. If this action is free, then $M/\!/_c G := \mu^{-1}(c)/G$ is a symplectic manifold, the *symplectic quotient* of M by G. In general, however, $M/\!/_c G$ is a symplectic orbifold.

Symplectic reduction can also be used to construct symplectic manifolds and orbifolds with hamiltonian group actions, by taking a symplectic quotient by a normal subgroup of the symmetry group.

Lemma 1. *Let G be a (compact) Lie group with closed Lie subgroup N, and suppose that $(\tilde{M}, \tilde{\omega})$ has a hamiltonian G-action with momentum map $\mu\colon \tilde{M} \to \mathfrak{g}^*$, so that $\mu_N = i^* \circ \mu\colon \tilde{M} \to \mathfrak{n}^*$ is the momentum map for the induced hamiltonian N-action (here i^* is adjoint to the inclusion of $\mathfrak{n}$ in $\mathfrak{g}$). Suppose further that N is normal in G, that*

$\tilde{c} \in \mathfrak{g}^*$ is invariant under the coadjoint action of G, and that $c = i^*(\tilde{c})$ is a regular value of μ_N such that the action of N on $\mu_N^{-1}(c)$ is locally free.

Then $M = \tilde{M}/\!/_c N$ has a hamiltonian action of G/N whose momentum map, viewed as an N-invariant function on $\mu_N^{-1}(c)$, is the restriction of $\mu - \tilde{c}$ to $\mu_N^{-1}(c)$.

Proof. The action of $[g] \in G/N$ on $[z] \in M = \mu_N^{-1}(c)/N$ is given by $[g] \cdot [z] = [g \cdot z]$. To see that this is well defined, note that the elements z of $\mu_N^{-1}(c)$ are characterized by $\langle \mu(z) - \tilde{c}, \zeta \rangle = 0$ for all $\zeta \in \mathfrak{n}$; now since $\tilde{c}$ is G-invariant, for any z in $\mu_N^{-1}(c)$ and any g in G, we then have $\langle \mu(g \cdot z) - \tilde{c}, \zeta \rangle = \langle \mu(z) - \tilde{c}, \mathrm{Ad}_{g^{-1}} \zeta \rangle$. Thus, since N is normal in G, $g \cdot z$ is in $\mu_N^{-1}(c)$ and $[g \cdot z]$ depends only on $[g]$ and $[z]$.

The fact that $\langle \mu(z) - \tilde{c}, \zeta \rangle = 0$ for any $z \in \mu_N^{-1}(c)$ means that the restriction of $\mu - \tilde{c}$ to $\mu_N^{-1}(c)$, say ν, has its values in $(\mathfrak{g}/\mathfrak{n})^*$, identified with the annihilator $\mathfrak{n}^0$ of $\mathfrak{n}$ in $\mathfrak{g}^*$. Moreover, ν is N-invariant: since N is normal, Ad_g^* acts by the identity on $\mathfrak{n}^0$ for any g in N.

Now ν gives a momentum map for the action of G/N on M, since for any $\xi \in \mathfrak{g}$, $\langle d\nu, [\xi] \rangle$ pulls back to $\langle d\mu, \xi \rangle = -\iota_{X_\xi} \tilde{\omega}$ on $\mu_N^{-1}(c)$. $\square$

We remark that if $u^* \colon (\mathfrak{g}/\mathfrak{n})^* \to \mathfrak{g}^*$ is the natural inclusion (adjoint to $u \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$) then the affine embedding $\ell = u^* + \tilde{c}$ gives a commutative diagram

$$
\begin{array}{ccc}
\mu_N^{-1}(c) & \xrightarrow{\mu} & (i^*)^{-1}(c) \subset \mathfrak{g}^* \\
{\scriptstyle q}\big\downarrow & & \big\uparrow{\scriptstyle \ell} \\
M & \xrightarrow{\nu} & (\mathfrak{g}/\mathfrak{n})^*
\end{array}
\tag{2}
$$

where q is the natural projection. In particular ℓ identifies the image of ν with the intersection of the affine subspace $(i^*)^{-1}(c)$ and the image of $\mu \colon M \to \mathfrak{g}^*$.

2. Toric symplectic manifolds

A *hamiltonian m-torus action* on a symplectic manifold (M, ω) of dimension $2n$ is an effective torus action generated by an m-dimensional family of hamiltonian vector fields $X \in C^\infty(M, TM) \otimes \mathbb{R}^{m*}$ with a momentum map $\mu \colon M \to \mathbb{R}^{m*}$ whose components Poisson commute, i.e., $\iota_X \omega = -d\mu$ and $\omega(X, X) = 0$. (This ensures that the components of X commute, and that μ is T^m-equivariant.)

Observe that $\mathbb{R}^m$ is here the Lie algebra of the torus, and so contains a lattice $\mathbb{Z}^m$ such that the torus is $T^m = \mathbb{R}^m/2\pi\mathbb{Z}^m$.

Since the orbits are isotropic with respect to ω, it follows that $m \leq n$—if equality holds, we say that (M, ω, μ) is a *toric* symplectic manifold.

In this section we summarize the classification of *compact* toric symplectic manifolds, established by Delzant [**5**], and the extension of this theory to orbifolds [**9**]. It turns out that there is a one to one correspondence between compact connected toric symplectic manifolds (or orbifolds) and a class of compact convex polytopes, called *Delzant polytopes*. More precisely, given a compact connected toric symplectic manifold (M^{2n}, ω, μ), the image of the momentum map μ is a compact convex polytope Δ in $\mathbb{R}^{n*}$ by the Atiyah–Guillemin–Sternberg convexity theorem. The same holds for toric orbifolds; the polytopes obtained belong to the following class.

Definition 1. Let $\mathbb{R}^n$ be an n-dimensional vector space with a lattice $\mathbb{Z}^n$, and consider a compact convex polytope Δ in $\mathbb{R}^{n*}$ defined by the equations

$$\langle x, u_j \rangle \geq \lambda_j, \tag{3}$$

($u_j \in \mathbb{R}^n$, $\lambda_j \in \mathbb{R}$) for $j = 1, \ldots d$, where $d > n$ is the number of $(n-1)$-faces (simply called *faces* in the sequel) of Δ.

Then Δ is said to be a *rational Delzant polytope* if the u_j belong to the lattice $\mathbb{Z}^n$ and the elements corresponding to faces meeting any given vertex of the polytope form a basis of $\mathbb{R}^n$. (It follows that the polytope is n-valent, i.e., n edges meet each vertex—the dual basis of $\mathbb{R}^{n*}$ gives the directions of these edges.)

The polytope is said to be *integral* or simply a *Delzant polytope* if the u_j corresponding to faces meeting any vertex form a ($\mathbb{Z}$-)basis of the lattice $\mathbb{Z}^n$.

Remark 1. Note that in this definition we consider the vectors u_j and the lattice $\mathbb{Z}^n$ as part of the data. If Δ is integral, then in fact the u_j are uniquely determined by Δ and the lattice, since they must be primitive; conversely the u_j generate the lattice in this case.

More generally, the u_j are determined by Δ, $\mathbb{Z}^n$ and a positive integer labelling of the faces of Δ, since we may write $u_j = m_j y_j$ with y_j primitive and $m_j \in \mathbb{Z}^+$—because of this description, rational Delzant polytopes are called *labelled polytopes* in [**9**]. Now, conversely, the u_j in general only generate a (finite index) sublattice $\hat{\mathbb{Z}}^n$ of $\mathbb{Z}^n$. However, there is not much generality lost by replacing $\mathbb{Z}^n$ with this sublattice—see Remark 3 below.

The image of the momentum map of a symplectic toric orbifold (manifold) is a rational (integral) Delzant polytope. Conversely, to each rational Delzant polytope is canonically associated a compact connected toric symplectic orbifold $(M_\Delta, \omega_\Delta, \mu_\Delta)$ such that Δ is the image of the momentum map μ_Δ. A toric symplectic orbifold is uniquely determined by its polytope up to equivariant symplectomorphism, so the two operations are mutually inverse.

The construction of M_Δ will be important in the sequel, so we summarize it here, following [**7**, **9**]. The idea is to obtain M_Δ as a symplectic quotient of a $2d$-dimensional symplectic vector space by the natural action of a $(d-n)$-dimensional subgroup N of $T^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$.

Here, $\mathbb{Z}^d$ may be thought of (canonically) as the free abelian group generated by the d faces σ_j of Δ, and $\mathbb{R}^d$ is the corresponding real vector space $\mathbb{Z}^d \otimes_{\mathbb{Z}} \mathbb{R}$. We take the symplectic vector space to be $\mathbb{R}^{2d} = \mathbb{Z}^d \otimes_{\mathbb{Z}} \mathbb{R}^2$, where $\mathbb{R}^2$ is equipped with its standard area form and circle action. In terms of polar coordinates on $\mathbb{R}^2 \cong \mathbb{C}$, we then have a natural action of the torus T^d on $\mathbb{R}^{2d}$:

$$[a_1, \dots, a_d] \cdot \sum_{j=1}^{d} \sigma_j \otimes (r_j, \theta_j) = \sum_{j=1}^{d} \sigma_j \otimes (r_j, \theta_j + a_j) \tag{4}$$

where $[a_1, \dots, a_d]$ denotes the class of $(a_1, \dots, a_d) \in \mathbb{R}^d$ modulo $2\pi\mathbb{Z}^d$. This action is hamiltonian with respect to the standard symplectic form, which in polar coordinates is $\sum_{j=1}^{d} r_j dr_j \wedge d\theta_j$. It follows that the momentum map μ of the T^d action is given (up to translation) by

$$\mu(v) = \frac{1}{2}(r_1^2, \dots r_d^2) = \frac{1}{2} \sum_{j=1}^{d} r_j^2 \sigma_j^* \tag{5}$$

where σ_j^* is the basis of $\mathbb{R}^{d*}$ dual to σ_j, and $v = \sum_{j=1}^{d} \sigma_j \otimes (r_j, \theta_j)$.

Let $u\colon \mathbb{Z}^d \to \mathbb{Z}^n$ be the group homomorphism determined by $\sigma_j \mapsto u_j$. We denote also by u the corresponding homomorphisms from $\mathbb{R}^d$ to $\mathbb{R}^n$ and from T^d to T^n, which are surjective, and we denote the kernels by $\mathfrak{n}$ and N respectively.

Remark 2. $u\colon \mathbb{Z}^d \to \mathbb{Z}^n$ is surjective if and only if the u_j generate $\mathbb{Z}^n$, in which case N is the quotient of its (abelian) Lie algebra $\mathfrak{n}$ by the kernel of $u\colon 2\pi\mathbb{Z}^d \to 2\pi\mathbb{Z}^n$, and so is a subtorus of T^d.

More generally if the u_j generate a sublattice $\hat{\mathbb{Z}}^n$, then the quotient of N by the connected component of the identity is the finite group $\mathbb{Z}^n/\hat{\mathbb{Z}}^n$.

We now consider the restriction of the action of T^d on $\mathbb{R}^{2d}$ to the subgroup N. The action of N is still hamiltonian with momentum map $\mu_N = i^* \circ \mu$, where $i^*\colon \mathbb{R}^{d*} \to \mathfrak{n}^*$ is adjoint to the inclusion $i\colon \mathfrak{n} \to \mathbb{R}^d$. We let $c = i^*(-\lambda)$, where $\lambda = (\lambda_1, \dots \lambda_d)$ is naturally the element $\sum_{j=1}^d \lambda_j \sigma_j^*$ of $\mathbb{R}^{d*}$, and consider the momentum level set $\mu_N^{-1}(c)$.

The annihilator in $\mathbb{R}^{d*}$ of $\mathfrak{n}$ is the image of the adjoint u^* of u. It follows that $\mu_N^{-1}(c)$ is the set of elements v of $\mathbb{R}^{2d}$ for which there exists x in $\mathbb{R}^n$ with $\mu(v) + \lambda = u^*(x)$.

In other words, since $u^*(x) = \sum_{j=1}^d \langle x, u_j\rangle\, \sigma_j^*$, $v = \sum_{j=1}^d \sigma_j \otimes (r_j, \theta_j)$ belongs to $\mu_N^{-1}(c)$ if and only if there exists x in $\mathbb{R}^{n*}$ such that

$$\ell_j(x) := \langle x, u_j\rangle - \lambda_j = \frac{r_j^2}{2}, \tag{6}$$

for $j = 1, \dots, d$.

We observe that $(\ell_1, \dots \ell_d)$ are the components of the affine embedding $\ell = u^* - \lambda$ of $\mathbb{R}^{n*}$ into $\mathbb{R}^{d*}$. Hence v belongs to $\mu_N^{-1}(c)$ if and only if $\mu(v)$ belongs to the image of ℓ, and then x appearing in (6) is uniquely determined by v and belongs to Δ. This defines a map from $\mu_N^{-1}(c)$ to Δ, which, following Lemma 1, we denote by ν. We thus have:

$$\ell(\nu(v)) = \mu(v) \tag{7}$$

for all $v = \sum \sigma_j \otimes (r_j, \theta_j)$ in $\mu_N^{-1}(c)$; ν is clearly surjective and for each x in Δ and $\nu^{-1}(x)$ coincides with the orbit of any of its elements under the action of T^d. In particular, $\mu_N^{-1}(c)$ is compact.

Notice that $\nu(v)$ belongs to the interior of Δ (the complement of the union of its faces) if and only if all the r_j are non-zero; the isotropy group of v in T^d is then zero.

It turns out that for *any* v in $\mu_N^{-1}(c)$, the isotropy group $N_v = T_v \cap N$ of v in N is finite, and is zero if the polytope is integral. Indeed, if $\nu(v)$ belongs to the intersection of k faces $\sigma_{j_1}, \dots \sigma_{j_k}$, then $r_{j_1} = \dots = r_{j_k} = 0$ and the isotropy group T_v of v in T^d is the k-dimensional subtorus of T^d whose Lie algebra $\mathfrak{t}_v$ is the set of $(a_1, \dots a_d)$ with $a_j = 0$ for $j \neq j_1, \dots, j_k$. The intersection of T_v with N is thus the set of $[a_1, \dots a_d] \in T_v$ with $\sum_{j=1}^d a_j u_j \in 2\pi\mathbb{Z}^d$. However, the u_j with $a_j \neq 0 \mod 2\pi\mathbb{Z}$, are part of a basis for a (finite index) sublattice of $\mathbb{Z}^d$ (choose a vertex in $\sigma_{j_1} \cap \dots \cap \sigma_{j_k}$). This means that the u_j with $a_j \neq 0 \mod 2\pi\mathbb{Z}$

form a basis for a finite index sublattice of $\mathfrak{t}_v \cap \mathbb{Z}^d$; N_v is the quotient of $\mathfrak{t}_v \cap \mathbb{Z}^d$ by this sublattice.

Therefore N acts locally freely on $\mu_N^{-1}(c)$, so c is a regular value of μ_N, $\mu_N^{-1}(c)$ is a closed submanifold of $\mathbb{R}^{2d}$ and the quotient $M := \mu_N^{-1}(c)/N$ is a compact symplectic orbifold, of (real) dimension $2d-2(d-n) = 2n$. This is the symplectic quotient, at momentum level c, of $\mathbb{R}^{2d}$ by N.

By Lemma 1, ν induces a momentum map with image Δ for the natural action of the quotient torus $T^n = T^d/N$ on $M = \mu_N^{-1}(c)/N$.

Remark 3. If the u_j do not generate $\mathbb{Z}^n$, then we can instead work with the sublattice $\hat{\mathbb{Z}}^n$ generated by the u_j: $\hat{N}$ is then the connected component of the identity in N, so $\hat{M}$ is a finite cover of M, with a hamiltonian action of the cover $\hat{T}^n = \mathbb{R}^n/2\pi\hat{\mathbb{Z}}^n$ of $T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. Thus M is the quotient of $\hat{M}$ by the finite group $\mathbb{Z}^n/\hat{\mathbb{Z}}^n$. It is usual in the theory of orbifolds to pass to such global covers if they exist, as this will (at least partially) resolve some of the orbifold singularities. This explains why there is not much generality lost in assuming that the u_j generate $\mathbb{Z}^n$.

3. Kähler quotients

In general, let (M, g, J, ω) be a Kähler manifold with an effective action of a group G by isometries, which is hamiltonian with momentum map $\mu \colon M \to \mathfrak{g}^*$. Let $M/\!/_c G$ be a symplectic quotient of M by G; then since G acts by isometries on $\mu^{-1}(c)$ (and we are supposing the action is locally free), $M/\!/_c G$ is a Kähler orbifold, the *Kähler quotient* of M by G.

We now consider the situation described by Lemma 1, where G acts on $(\tilde{M}, \tilde{\omega})$ and N is a normal subgroup of G. We assume additionally that G acts freely on $\tilde{M}$ and that the dimension of G is equal to half the dimension of $\tilde{M}$ so that $\mu \colon \tilde{M} \to \mathfrak{g}^*$ is a G-principal bundle over its image. Then if $\tilde{g}$ is a G-invariant compatible metric on $\tilde{M}$, we obtain a uniquely defined riemannian metric, say $\tilde{g}_{\mathrm{red}}$ on $\operatorname{im}\mu$ such that μ is a riemannian submersion from $\tilde{g}$ to $\tilde{g}_{\mathrm{red}}$.

Similarly, the quotient group G/N acts freely on the Kähler quotient $M = \tilde{M}/\!/_c N$ and this makes $\nu \colon M \to (\mathfrak{g}/n)^*$ into a G/N-principal bundle over its image, and ν is a riemannian submersion for a uniquely defined riemannian metric g_{red} on $\operatorname{im}\nu$, when M is equipped with the metric g induced by $\tilde{g}$.

We denote by $\ell = u^* + \tilde{c}$ the affine map from $\operatorname{im}\nu$ to $\operatorname{im}\mu$ induced by the natural inclusion $u^*\colon (\mathfrak{g}/\mathfrak{n})^* \to \mathfrak{g}^*$.

Lemma 2. *The reduced metrics are related by $\ell^*\tilde{g}_{\mathrm{red}} = g_{\mathrm{red}}$.*

Proof. Let $z \in \mu_N^{-1}(c)$ and let $\tilde{X} \in T_z\tilde{M}$ be orthogonal to the G-orbit through z. Then the (isometric) projection X of $\tilde{X}$ onto T_xM is orthogonal to the G/N-orbit through $x = q(z)$. We thus have

$$|\nu_*(X)|_{g_{\mathrm{red}}} = |X|_g = |\tilde{X}|_{\tilde{g}} = |\mu_*(\tilde{X}))|_{\tilde{g}_{\mathrm{red}}}.$$

Since $\mu(z) = \ell(\nu(x))$ by Lemma 1, this is what we want. $\square$

We wish to emphasise at this point that the Kähler quotient is a construction in symplectic geometry—the riemannian metric on M (and hence the complex structure) is just a passenger; that is, *any* G-invariant Kähler metric on M with Kähler form ω descends to a compatible Kähler metric on $M/\!/_cG$.

In general it is difficult to parameterize G-invariant Kähler metrics with a fixed Kähler form; instead one usually fixes the complex structure, so that compatible Kähler metrics are parameterized (locally) by *Kähler potentials* K (i.e., $dd^cK = \omega$, where $d^c = J \circ d$). In the case of a compact toric symplectic manifolds however, compatible toric Kähler metrics on the open set $M_0 = \mu^{-1}(\Delta_0)$ determine and are determined by a T^m-invariant lagrangian foliation (the integral leaves of the distribution generated by vector fields $JK_1, \dots, JK_m$) and by the induced riemannian metric on Δ_0, which is of hessian type, i.e. defined in terms of a *dual potential*, which is related to the Kähler potential by Legendre transform (more details in the next section).

4. Dual potentials for toric Kähler metrics

Suppose that (M^{2n}, ω) is a toric symplectic manifold with momentum map $x\colon M \to \mathbb{R}^{n*}$ and that (g, J) is a compatible toric Kähler structure. Let $X \in C^\infty(M, TM) \otimes \mathbb{R}^{n*}$ be the corresponding family of hamiltonian Killing vector fields with $\iota_X\omega = -dx$.

We wish to study the geometry of M locally, so we assume that the torus action is free. Then JX is a family of holomorphic vector fields and the components of X and JX together form a frame of commuting vector fields. The closed 1-forms in the dual frame are naturally the components of $\mathbb{R}^n$-valued 1-forms α and $J\alpha = -\alpha \circ J$ with $\alpha(X)$ equal to the identity of $\mathbb{R}^n$. If we locally write $\alpha = dt$ and $J\alpha = -dy$ with

$y, t \colon M \to \mathbb{R}^n$, then $y + it \colon M \to \mathbb{C}^n$ is a local holomorphic chart on M.

The components of $J\alpha$ and $\iota_X\omega$ span the same rank n-subbundle of T^*M, the annihilator of tangent space to the orbits of the torus action. Hence there must be mutually inverse functions $\boldsymbol{F} \colon M \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{n*})$ and $\boldsymbol{G} \colon M \to \mathrm{Hom}(\mathbb{R}^{n*}, \mathbb{R}^n)$ such that

$$\iota_X\omega = \langle \boldsymbol{F}, J\alpha \rangle \qquad \text{and} \qquad J\alpha = \langle \boldsymbol{G}, \iota_X\omega \rangle.$$

Since $g(X, X) = \omega(X, JX) = \boldsymbol{F}$, we deduce that $\boldsymbol{F}$ and $\boldsymbol{G}$ are, at each point of M, symmetric and positive definite elements of $\mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$ and $\mathbb{R}^n \otimes \mathbb{R}^n$.

Concretely, with respect to a basis of the Lie algebra $\mathbb{R}^n$, we have

$$dx_r = \sum_s F_{rs} dy_s \qquad \text{and} \qquad dy_r = \sum_s G_{rs} dx_s.$$

Now $\sum_s G_{rs} dx_s$ is closed if and only if G_{rs} is the hessian with respect to x of a function G. Indeed $\sum_s G_{rs} dx_s = dy_r$ if and only if $G_{rs} = \partial y_r / \partial x_s$, in which case $y_r = \partial G / \partial x_r$ for some function G, since G_{rs} is symmetric. Similarly $\sum_s F_{rs} dy_s$ is closed if and only if F_{rs} is the hessian with respect to y of a function F.

We therefore have functions F, G on M such that $x_r = \partial F / \partial y_r$ and $y_r = \partial G / \partial x_r$; without loss of generality, we may also assume that the constant $F + G - \langle x, y \rangle$ is zero. Thus the coordinate systems x and y, and the functions F and G, are related by Legendre transform:

$$F + G = \langle x, y \rangle = \sum_r x_r \frac{\partial G}{\partial x_r} = \sum_r y_r \frac{\partial F}{\partial y_r}.$$

F is a Kähler potential, since $\omega = \sum_{r,s} F_{rs} dy_s \wedge dt_r = dd^c F$. On the other hand G is the dual potential we need to parameterize toric Kähler metrics with fixed symplectic form.

Proposition 1. [**6**, **1**] *Let G be a function on $\mathbb{R}^{n*}$ whose hessian G_{rs} is positive definite with inverse F_{rs}. Then*

$$\sum_{r,s} (G_{rs} dx_r dx_s + F_{rs} dt_r dt_s)$$

is a toric Kähler metric with Kähler form

$$\omega = \sum_r dx_r \wedge dt_r$$

and momentum map x.

Observe that g_{red} is the metric $\sum_{r,s} G_{rs} dx_r dx_s$ on the image of the momentum map x. The following simple observation explains why dual potentials are convenient when taking Kähler quotients of toric Kähler manifolds.

Proposition 2. *Let $(\tilde{M}, \tilde{g})$ be a toric Kähler manifold of real dimension $2d$. Denote by $\tilde{M}_0$ the open set on which the T^d action is free, so that the momentum map $\mu\colon \tilde{M}_0 \to U \subset \mathbb{R}^{d*}$ is a principal T^d-bundle.*

Let N be a subgroup of T^d of codimension n, with momentum map, $\mu_N = i^* \circ \mu$, where $i^*\colon \mathbb{R}^{d*} \to \mathfrak{n}^*$ is the natural projection. Choose $\tilde{c} \in \mathbb{R}^{n*}$ such that $c = i^*(\tilde{c})$ is a regular value of μ_N and the action of N on $\mu_N^{-1}(c)$ is locally free. Let $M = \mu_N^{-1}(c)/N$ be the Kähler quotient with metric g and $q\colon \mu_N^{-1}(c) \to M$ the natural projection.

The action of T^d on $\tilde{M}$ preserves $\mu_N^{-1}(c)$, hence induces a hamiltonian action of the quotient torus $T^n := T^d/N$ on M with momentum map ν defined by

$$\nu \circ q = \mu - \tilde{c}. \tag{8}$$

Let Δ_0 denote the image of ν restricted to M_0, the quotient of $\mu_N^{-1}(c) \cap \tilde{M}_0$ by N.

Then the reduced metric g_{red} on Δ_0 is the pullback by ℓ of the reduced metric $\tilde{g}_{\mathrm{red}}$ on U and so if $\tilde{G}\colon U \to \mathbb{R}$ is a dual potential for $\tilde{g}$ on $\tilde{M}_0$, then $G = \tilde{G} \circ \ell\colon \Delta_0 \to \mathbb{R}$ is a dual potential for g on M_0.

Proof. Most of the statements in this proposition are immediate from Lemma 1, since the coadjoint actions are trivial and N is necessarily a normal subgroup of T^d. It remains to establish the conclusion that $G = \ell^* \tilde{G}$ is a dual potential for the quotient. This follows from Lemma 2, since ℓ is an affine map with respect to the flat affine connections D and $\tilde{D}$ on Δ_0 and U. Indeed, for the dual potentials we have

$$Dd(\ell^* \tilde{G}) = \ell^* \tilde{D} d\tilde{G} = \ell^* \tilde{g}_{\mathrm{red}} = g_{\mathrm{red}}.$$

□

5. The Guillemin formula

In section 2 we explained how any toric symplectic orbifold M could be obtained as a symplectic quotient of a symplectic vector space $\mathbb{R}^{2d}$. In canonical terms $\mathbb{R}^{2d} = \mathbb{Z}^d \otimes_{\mathbb{Z}} \mathbb{R}^2$, where $\mathbb{Z}^d$ denotes the free abelian group generated by the faces $\sigma_1, \dots \sigma_d$ of the (rational) Delzant polytope Δ, and $\mathbb{R}^2$ is the standard symplectic vector space with a (linear) circle action. To emphasize the symplectic nature of the construction we deliberately suppressed the obvious identification of $\mathbb{R}^2$ with $\mathbb{C}$, and hence of $\mathbb{R}^{2d}$ with $\mathbb{C}^d$. In these terms the torus action (4) on $\mathbb{C}^d$ is given by

$$[a_1, \dots, a_d] \cdot (z_1, \dots z_d) = (e^{ia_1} z_1, \dots e^{ia_1} z_d) \tag{9}$$

where $[a_1, \dots a_d] \in \mathbb{R}^d/2\pi\mathbb{Z}^d$ and $z_j = r_j e^{i\theta_j}$.

Thus $\mathbb{R}^{2d}$ is in fact equipped with a canonical flat Kähler metric compatible with the symplectic form $\frac{i}{2} \sum dz_j \wedge d\overline{z}_j$, and this induces a canonical Kähler metric on M, called the *Guillemin metric*.

In this section, we obtain an explicit expression, the *Guillemin formula*, for the reduced metric on the interior Δ_0 of the (rational) Delzant polytope Δ. We also obtain formulae for the Kähler potential and dual potential. These results are due to Guillemin [**6**, **7**] (see also Abreu [**1**, **2**]), but we obtain a simple new proof using Proposition 2.

Recall that the toric symplectic manifold (M, ω, ν) fits into the commutative diagram

$$\begin{array}{ccc} \mu_N^{-1}(c) & \xrightarrow{\mu} & \mathbb{R}^{d*} \\ {\scriptstyle q}\big\downarrow & & \big\uparrow{\scriptstyle \ell} \\ M & \xrightarrow{\nu} & \Delta \subset \mathbb{R}^{n*} \end{array} \tag{10}$$

where $\ell \colon \mathbb{R}^{n*} \to \mathbb{R}^{d*}$ is the affine map

$$\ell(x) = (\ell_1(x), \dots, \ell_d(x)) \tag{11}$$

with

$$\ell_j(x) = \langle u_j, x \rangle - \lambda_j. \tag{12}$$

Now we restrict attention to the open subset of $\mathbb{C}^d$ where the T^d action is free, and the corresponding open subset M_0 of M. These are principal T^d and T^n bundles over $U \subset \mathbb{R}^{d*}$ and $\Delta_0 \subset \mathbb{R}^{n*}$ respectively, where U is the positive quadrant of $\mathbb{R}^{d*}$.

It remains only to compute the reduced metric $\tilde{g}_{\mathrm{red}}$ and dual potential $\tilde{G}$ for the flat Kähler structure on $\mathbb{C}^d$ and pull these back by ℓ. The Kähler potential can then be found by Legendre transform.

The explicit derivation of $\tilde{g}_{\mathrm{red}}$ and $\tilde{G}$ is a straightforward coordinate transformation: $z_j = r_j e^{i\theta_j}$ and the components of the momentum map μ are given by $\mu_j = r_j^2/2$. Thus the flat metric on $\mathbb{C}^d$ is

$$\sum_{j=1}^d dz_j \, d\overline{z}_j = \sum_{j=1}^d \bigl(dr_j{}^2 + r_j^2 \, d\theta_j{}^2 \bigr) = \sum_{j=1}^d \Bigl(\frac{d\mu_j{}^2}{2\mu_j} + 2\mu_j \, d\theta_j{}^2 \Bigr). \tag{13}$$

Hence

$$\tilde{g}_{\mathrm{red}} = \frac{1}{2}\sum_{j=1}^{d} \frac{d\mu_j{}^2}{\mu_j} \tag{14}$$

so that a dual potential $\tilde{G}$ is given by

$$\tilde{G} = \frac{1}{2}\sum_{j=1}^{d} \mu_j \log \mu_j. \tag{15}$$

Theorem 1. *Let (M^{2n}, ω) be a toric symplectic orbifold with momentum map $\nu\colon M \to \Delta \subset \mathbb{R}^{n*}$, where the (rational) Delzant polytope Δ is given by*

$$\{x \in \mathbb{R}^{n*} : \ell_j(x) \geq 0, \quad j = 1, \dots d\}, \tag{16}$$

where $\ell_j(x) = \langle u_j, x\rangle - \lambda_j$ for $u_j \in \mathbb{Z}^n$ and $\lambda_j \in \mathbb{R}$.

Equip M with the Guillemin metric g. Then the reduced metric on Δ_0 is

$$g_{\mathrm{red}} = \frac{1}{2}\sum_{j=1}^{d} \frac{d\ell_j{}^2}{\ell_j} \tag{17}$$

and a dual potential and Kähler potential are given by

$$G = \frac{1}{2}\sum_{j=1}^{d} \ell_j(x) \log \ell_j(x) \tag{18}$$

$$F = \frac{1}{2}\sum_{j=1}^{d} \big(\lambda_j \log \ell_j(x) + \ell_j(x)\big) \tag{19}$$

Proof. Using Proposition 2, the reduced metric and dual potential follow immediately from (14) and (15) by writing $\mu_j = \ell_j(x)$. It remains to compute the Legendre transform. The dual coordinates y to the momentum map x are given by

$$y = \frac{\partial G}{\partial x} = \frac{1}{2}\sum_{j=1}^{d} u_j \log \ell_j(x) + u_j, \tag{20}$$

Thus a Kähler potential is given by

$$
\begin{aligned}
-G(x) + \langle x, y\rangle &= \frac{1}{2}\sum_{j=1}^{d}\big((\langle u_j, x\rangle - \ell_j(x))\log \ell_j(x) + \langle u_j, x\rangle\big) \\
&= \frac{1}{2}\sum_{j=1}^{d}\big(\ell_j(x) + \lambda_j + \lambda_j \log \ell_j(x)\big).
\end{aligned} \tag{21}
$$

This differs from F by an additive constant. □

Appendix: An alternative derivation of the Kähler potential

In this Appendix, we show that the expression of the Kähler potential F appearing in Theorem 1 can also be easily derived from a general expression of the Kähler potential of a Kähler reduction appearing in [**4**].

We again consider the construction of M_Δ as a Kähler reduction of $\mathbb{R}^{2d} \simeq \mathbb{C}^d$ with respect to the standard action of a subgroup N of T^d, cf. section 2. Notations are the same as in the body of the paper. We denote by $N_{\mathbb{C}}$, $T^d_{\mathbb{C}}$, $T^n_{\mathbb{C}}$ the complexifications of N, T^d, T^n; the complex torus $T^d_{\mathbb{C}} \simeq \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ (d times) acts holomorphically on $\mathbb{R}^{2d} \cong \mathbb{C}^d$ by $(\zeta_1, \cdots, \zeta_d) \cdot (z_1, \ldots, z_d) = (\zeta_1 z_1, \ldots, \zeta_d z_d)$, where, for each $j = 1, \ldots, d$, $\zeta_j = t_j e^{i a_j}$, $t_j > 0$, is a non-zero complex number; the restriction of this action to the compact part T^d coincides with the action already considered. The restriction of this action to $N_{\mathbb{C}}$ is again a holomorphic extension of the action of N.

Let $\mathbb{C}^d_s$ be the set of points of $\mathbb{C}^d$ whose $N_{\mathbb{C}}$-orbit cuts $\mu_N^{-1}(c)$ along a non-empty orbit of N ($\mathbb{C}^d_s$ is called the *stable* part of $\mathbb{C}^d$ in [**7**]). Then, as a complex manifold, the Kähler reduction M coincides with the (ordinary) quotient of $\mathbb{C}^d_s$ by $N_{\mathbb{C}}$.

Moreover, for any z in $\mathbb{C}^d_s$, there exists a *unique* element $t_z = (t_{z,1}, \ldots, t_{z,d})$ in $\exp(iV) \subset N_{\mathbb{C}} \subset T^d_{\mathbb{C}}$ such that $t_z \cdot z$ belongs to $\mu_N^{-1}(c)$.

We denote by p the natural projection from $\mathbb{C}^d_s$ onto $M = \mathbb{C}^d_s / N_{\mathbb{C}}$; we then have:

$$
p(z) = [t_z \cdot z] = [t_{z,1} z_1, \ldots, t_{z,d} z_d], \tag{22}
$$

for any z in $\mathbb{C}^d_s$, where $[t_z \cdot z]$ denotes the class mod N of $t_z \cdot z$ in $\mu_N^{-1}(c)$.

We temporarily assume that the $\lambda_j \in 2\pi\mathbb{Z}$ for all j, so that λ can be identified with the character χ_λ of T^d defined by $\chi_\lambda \colon [a_1, \ldots, a_d] \to e^{i(\sum_{j=1}^{d} \lambda_j a_j)}$, whereas $-c$ is identified with the restriction, χ_{-c}, of χ_λ to

N. We still denote χ_{-c} the natural extension of χ_{-c} to $N_{\mathbb{C}}$, with values in $\mathbb{C}^*$.

Proposition 3. *The pull-back $p^*\omega$ admits a globally defined N invariant potential, $\hat{K}$, on $\mathbb{C}^d_s$, i.e.*

$$p^*\omega = dd^c\hat{K}, \tag{23}$$

where $\hat{K}$ is given by

$$\hat{K}(z) = \frac{1}{2}\sum_{j=1}^{d}(\ell_j(x) + \lambda_j \log t^2_{z,j}). \tag{24}$$

Here, $x = (\nu \circ p)(z)$ is the point of Δ corresponding to $p(z)$ and $\ell_j(x) = (x, u_j) - \lambda_j$.

Proof. According to Theorem 3.1 in [**4**], we have that

$$\hat{K}(z) = K_0(t_z \cdot z) + \frac{1}{2}\log|\chi_{-c}(t_z)|^2, \tag{25}$$

where $K_0 = \frac{r^2}{4}$ is the natural Kähler potential of the flat metric of $\mathbb{C}^d_S$. By (6), we then have: $K_0(t_z \cdot z) = \frac{1}{4}|t_z \cdot z|^2 = \frac{1}{2}\sum_{j=1}^d \ell_j(x)$, where $x = (\nu \circ p)(z)$, whereas $|\chi_{-c}(t_z)|^2 = t^{2\lambda_1}_{z,1} \ldots t^{2\lambda_d}_{z,d}$. □

We define $\mathbb{C}^d_0$ as the set of z in $\mathbb{C}^d_s$ such that $z_j \neq 0$ for all j. Its image by p is the open subset, M_0, of M which maps to the interior, Δ_0, of Δ by ν.

As an easy consequence of Proposition 3, we get an alternative derivation of the Guillemin formula for the Kähler potential [**7**]:

Proposition 4. *The restriction of ω to M_0 has a globally defined, T^d/N-invariant potential K, i.e.,*

$$\omega|_{M_0} = dd^cK, \tag{26}$$

with $K = F \circ \nu$, and

$$F(x) = \frac{1}{2}\sum_{j=1}^{d}(\ell_j(x) + \lambda_j \log \ell_j(x)), \tag{27}$$

for any x in Δ_0.

Proof. Let z be an element of $\mathbb{C}^d_0$, so that $z_j \neq 0$ for all $j = 1, \ldots, d$. The element t_z, viewed as an element of $\mathbb{R}^+ \times \ldots \times \mathbb{R}^+$ (d times), is then written as $t_z = (t_{z,1}, \ldots, t_{z,d})$, where, by (6), each positive real number $t_{z,j}$ is *determined* by: $t_{z,j}^2 = \frac{2\ell_j(x)}{|z_j|^2}$. We then have:

$$\begin{aligned} \log \chi_{-c}(|t_z|^2) &= \sum_{j=1}^{d} \log t_{z,j}^{2\lambda_j} = \sum_{j=1}^{d} \lambda_j \log t_{z,j}^2 \\ &= \sum_{j=1}^{d} \lambda_j (\log \ell_j - \log |z_j|^2 + \log 2). \end{aligned} \tag{28}$$

Since the terms in $\log |z_j|^2$ and the constant term $\log 2$ have no effect in the computation of $\omega|_{M_0}$, we directly derive (27) from (24). $\square$

Remark 4. In this approach, we assumed the λ_j were integral. However, an easy rescaling argument yields the same result when the λ_j are rational, hence finally, by a continuity argument, in the general case where the λ_j are real, i.e., for *any* Delzant polytope, as we showed in the body of the paper.

Bibliography

[1] M. Abreu, *Kähler geometry of toric varieties and extremal metrics*, Int. J. Math. **9** (1998), 641–651.

[2] M. Abreu, *Kähler metrics on toric orbifolds*, J. Diff. Geom. **58** (2001), no. 1, 151–187

[3] R. Bielawski and A. Dancer, *The geometry and topology of toric hyperkähler manifolds*, Commun. Anal. Geom. **8**, No.4 (2000), 727-760.

[4] O. Biquard and P. Gauduchon, *Hyperkähler metrics on cotangent bundles of hermitian symmetric spaces*, in *Geometry and Physics*, eds.: J. E. Andersen, J. Dupont, H. Pedersen, A. Swann, Lecture Notes in Pure and applied mathematics **184**, Dekker (1997).

[5] T. Delzant, *Hamiltoniens périodiques et image convexe de l'application moment*, Bull. Soc. Math. France **116** (1988), 315–339.

[6] V. Guillemin, *Kähler structures on toric varieties*, J. Diff. Geom. (1994), 285–309.

[7] V. Guillemin, *Moment Maps and Combinatorial Invariants of Hamiltonian T^n-spaces*, Progress in Mathematics **122**, Birkhäuser (1994).

[8] N. J. Hitchin, A. Karlhede, U. Lindström, M. Roček, *Hyperkähler metrics and supersymmetry*, Comm. Math. Phys. **108** (1987), 535–559.

[9] E. Lerman and S. Tolman, *Hamiltonian torus actions on symplectic orbifolds and symplectic varieties*, Trans. Amer. Math. Soc. **349** (1997), 4201–4230.

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