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# DISTINGUISHING THE CHAMBERS OF THE MOMENT POLYTOPE

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**Let $M$ be a compact manifold with a Hamiltonian $T$ action and moment map $\Phi$. The restriction map in rational equivariant cohomology from $M$ to a level set $\Phi^{-1}(p)$ is a surjection, and we denote the kernel by $I_p$. When $T$ has isolated fixed points, we show that $I_p$ distinguishes the chambers of the moment polytope for $M$. In particular, counting the number of distinct ideals $I_p$ as $p$ varies over different chambers is equivalent to counting the number of chambers.**

## 1. Introduction

Let $(M, \omega)$ be a compact symplectic manifold with a Hamiltonian torus action by $T$. Then $M$ has a moment map

$$\Phi : M \longrightarrow \mathfrak{t}^*$$

from $M$ to the dual of the Lie algebra of $T$ which has the property that for every $\xi \in \mathfrak{t}$, $d\langle \Phi(p), \xi \rangle = \omega(X_\xi, \cdot)$, where $X_\xi$ is the vector field on $M$ generated by $\xi$. The torus action preserves level sets of $\Phi$. If $p$ is a regular value of the moment map, then $T$ acts locally freely on $\Phi^{-1}(p)$ and the *symplectic reduced space* $M_p := \Phi^{-1}(p)/T$ has at worst orbifold singularities. In particular, the equivariant cohomology of the level set is (rationally) isomorphic to the ordinary cohomology of the quotient $\Phi^{-1}(p)/T$. The *Kirwan map*

$$\kappa_p : H_T^*(M) \longrightarrow H^*(M_p)$$

is the restriction to the level set in equivariant cohomology followed by this isomorphism. Kirwan [**Ki**] showed that $\kappa_p$ is a surjection for all regular values $p$. Let

$$I_p := \ker \kappa_p$$

be the kernel of this map. The main aim of this article is to observe that this ideal distinguishes connected components of the set of regular values of $\Phi$, when the fixed point set consists of isolated points.

In general, this ideal may be hard to compute. S. Tolman and J. Weitsman described a set of generators of $I_p$ in terms of their restriction properties to the fixed point set of the $T$ action on $M$ [**TW**]. The first author (of this article) subsequently described these classes in terms of the moment map polytope $\Phi(M)$[**Go2**]. Preceding these techniques, the third author described the ideal indirectly by using cohomology pairings and the residue formula [**JK1**], a technique we exploit further here.

Let $\Delta := \Phi(M)$ be the image of the moment map. For $M$ compact, $\Delta$ is a convex polytope which we call the *moment polytope* [**At**], [**GS1**]. The connected components of the set of regular values of $\Phi$ form *chambers* of the polytope. These chambers are bounded by the (non-regular) values $\bigcup \Phi(M^H)$, where the union is over all one-dimensional subtori $H \subset T$ and $M^H$ denotes the fixed point set of $H$. For any particular $H \subset T$, we call the connected components of $\Phi(M^H)$ *walls* of $\Delta$, and note that a wall will be codimension-1 in $\mathfrak{t}^*$ if $H$ is one-dimensional and the quotient $T/H$ acts effectively on $M^H$. While there have been numerous attempts to describe the geometry as a regular value $p$ crosses a codimension-1 wall (most notably [**GK**], [**Ma**], [**GS2**]), there has so far been little evidence that global observations about $\Delta$ can be made using the techniques of equivariant cohomology. In [**GLS2**] the authors are able to use explicit calculations to find the number of chambers in the image of certain degenerate $SU(4)$ coadjoint orbits, and E. Rassart [**Ra**] has since extended these methods to count the number for any coadjoint orbit of $SU(4)$. The result we present below suggest that an algebraic approach may have merit in answering these questions.

**Theorem 1.1.** *Let $M$ be a compact connected symplectic manifold with a Hamiltonian $T$ action. Suppose the action has isolated fixed points, and denote the moment map by $\Phi$. Let $p$ and $q$ be two regular values of $\Phi$. Then $p$ and $q$ in are in distinct chambers of the moment polytope if and only if $I_p \neq I_q$.*

One of the difficulties in proving such a theorem is that, for $p$ and $q$ in distinct chambers, it is not sufficient to find different generating sets for the two ideals. These two sets could generate the same ideal. For this reason, our task is to find specific classes which are in one ideal but not in the other.

**Remark 1.2.** The quotient of $H_T^*(M)$ by each of these ideals may result in isomorphic rings. For example, the reduced space of any generic $SU(3)$ coadjoint orbit is a 2-sphere for any regular value. Thus the reduced spaces have isomorphic cohomology even though the points of reduction may not be in the same chamber.

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## 2. Background

The kernel of the Kirwan map can be described in two equivalent ways: in terms of cohomology classes and in terms of cohomology pairings. The Tolman-Weitsman theorem ([**TW**], Theorem 3) describes a generating set of the classes in the kernel of the Kirwan map, while the Jeffrey-Kirwan theorem describes how to evaluate pairings (or integrals) on the reduced space. Because the reduced space is finite-dimensional, and in particular the Poincaré pairing is non-degenerate, the kernel $I_p$ of the Kirwan map $\kappa_p$ consists of those classes $\beta$ for which $\int_{M_p} \kappa_p(\alpha\beta) = 0$ for all $\alpha$. Our interest is to show that certain classes have non-zero integral and are therefore *not* in the kernel of the Kirwan map.

Our main tool for describing any class is its restriction to the fixed point set $M^T$.

**Theorem 2.1** (Kirwan)**.** *Let $M$ be a compact symplectic manifold with Hamiltonian $T$ action and fixed point set $M^T$. Let $M^T_{cc}$ indicate the connected components of the fixed point set. The restriction to the fixed point set*

$$\iota^* : H_T^*(M) \longrightarrow H_T^*(M^T) = \oplus_{F \in M^T_{cc}} H_T^*(F) \tag{2.1}$$

*is an injection.*

In particular, a class is determined by its behavior on the fixed point set. For this reason, we refer to the set of connected components of the fixed point set where the class is nonzero as the *support* of a class. We denote the restriction of $\alpha$ to $F \in M^T_{cc}$ by $\iota_F^*(\alpha)$ or $\alpha|_F$.

**Definition 2.2.** *For $\alpha \in H_T^*(M)$, the* support *of $\alpha$ is*

$$supp\ \alpha = \{F \in M^T_{cc} :\ \iota_F^*(\alpha) \neq 0\},$$

*the set of fixed point components $F$ on which $\alpha$ restricts to a non-zero class.*

**2.1. The Jeffrey-Kirwan residue formula.** The residue formula expresses cohomology pairings on reduced spaces in terms of a multi-dimensional residue of certain rational holomorphic functions on $\mathfrak{t}$ [**JK1**]. The functions are obtained by restriction of equivariant classes to fixed points, and dividing by equivariant Euler classes. While the theorem applies to compact $M$ reduced by any compact group $K$, we need (and present) only the Abelian case below.

The computation of terms in the residue formula depends on a cone $\Lambda$ in $\mathfrak{t}$, even though the result of the formula is independent of this choice. Let $\gamma_1, \dots, \gamma_k$ be the set of all weights that occur by the $T$ action at any of the fixed point components. Choose some $\xi \in \mathfrak{t}$ such that $\gamma_i(\xi) \neq 0$ for all $i$. Let $\beta_i = \gamma_i$ if $\gamma_i(\xi) > 0$ and $\beta_i = -\gamma_i$ if $\gamma_i(\xi) < 0$. Thus $\beta_i(\xi) > 0$ for all $i$. The cone $\Lambda$ is the set of all vectors in $\mathfrak{t}$ which behave like $\xi$:

$$\Lambda = \{X \in \mathfrak{t} : \ \beta_i(X) > 0, \text{ for all } i\}.$$

**Theorem 2.3** (Jeffrey-Kirwan)**.** *Let $(M, \omega)$ be a compact symplectic manifold with a Hamiltonian $T$ action and moment map $\Phi$, where $T$ is a compact torus. Denote by $M^T_{cc}$ the connected components of the fixed point set of $T$ on $M$. Let $p$ be a regular value of $\Phi$ and $\omega_p$ the Marsden-Weinstein reduced symplectic form on $M_p$. Then for $\beta \in H^*_T(M)$ and $\kappa_p : H^*_T(M) \to H^*(M_p)$ we have*

$$\int_{M_p} \kappa_p(\beta) e^{\omega_p} = c \cdot \mathrm{Res}^{\Lambda} \left( \sum_{F \in M^T_{cc}} e^{i(\Phi(F)-p)(X)} \int_F \frac{\iota^*_F(\beta(X)e^{\omega})}{e_F(X)} [dX] \right)$$

*where $c$ is a non-zero constant, $X$ is a variable in $\mathfrak{t} \otimes \mathbb{C}$, and $e_F(X)$ is the equivariant Euler class of the normal bundle to $F$ in $M$. The multi-dimensional residue* $\mathrm{Res}^{\Lambda}$ *is defined below.*

It is often more useful to refer to the dual cone $\Lambda^* \subset \mathfrak{t}^*$, the convex cone formed by the positive span of the vectors $\beta_1, \dots, \beta_k$, or the set of vectors $\beta$ which can be written $\beta = \sum_{i=1}^k s_i \beta_i$, with $s_i \geq 0$, perhaps not uniquely. Assume $\dim T = l$, let $J = (j_1, \dots, j_l)$ be a multi-index, and $X^J = X_1^{j_1} \cdots X_l^{j_l}$. The operator $\mathrm{Res}^{\Lambda}$ (defined in [**JK1**]) is defined by linearity and the following properties ([**JK2**], Proposition 3.2):

1) Let $\alpha_1, \dots, \alpha_v \in \Lambda^*$ be vectors in the dual cone. Suppose that $\lambda$ is not in any cone of dimension $l-1$ or less spanned by a subset

of the $\{\alpha_i\}$. Then

$$\mathrm{Res}^{\Lambda}\left(\frac{X^J e^{i\lambda(X)}[dX]}{\prod_{i=1}^{v}\alpha_i(X)}\right) = 0$$

unless all of the following properties are satisfied:
a) $\{\alpha_i\}_{i=1}^{v}$ span $\mathfrak{t}^*$ as a vector space,
b) $v - (j_1 + \cdots + j_l) \geq l$,
c) $\lambda \in \langle \alpha_1, \ldots, \alpha_v \rangle^+$, the positive span of the vectors $\{\alpha_i\}$.

2) If properties (1)(a)-(c) above are satisfied, then

$$\mathrm{Res}^{\Lambda}\left(\frac{X^J e^{i\lambda(X)}[dX]}{\prod_{i=1}^{v}\alpha_i(X)}\right) = \sum_{m\geq 0}\lim_{s\to 0^+}\mathrm{Res}^{\Lambda}\left(\frac{X^J (i\lambda(X))^m e^{is\lambda(X)}[dX]}{m!\prod_{i=1}^{v}\alpha_i(X)}\right),$$

and all but one term in this sum are 0 (the non-vanishing term being that with $m = v - (j_1 + \cdots + j_l) - l$).

3) The residue is not identically 0. If properties (1) $(a)-(c)$ are satisfied with $\alpha_1, \ldots, \alpha_l$ linearly independent in $\mathfrak{t}^*$, then

$$\mathrm{Res}^{\Lambda}\left(\frac{e^{i\lambda(X)}[dX]}{\prod_{i=1}^{l}\alpha_i(X)}\right) = \frac{1}{\det\overline{\alpha}},$$

where $\overline{\alpha}$ is the nonsingular matrix whose columns are the coordinates of $\alpha_1, \ldots, \alpha_l$ with respect to any orthonormal basis of $\mathfrak{t}$ defining the same orientation.

4) When $\lambda$ is of the form $\sum_{j=1}^{k} s_j\beta_j$ where fewer than $l$ of the $s_j$ are nonzero, then we define the residue

$$\mathrm{Res}^{\Lambda}\left(\frac{e^{i\lambda(X)}[dX]}{\prod_{i=k_1}^{k_l}\beta_i(X)}\right)$$

as the limit of the residues at $\lambda + s\rho$ where $s$ is a small positive real parameter and $\rho \in \mathfrak{t}^*$ is chosen so that $\lambda + s\rho$ does not lie in any cone of dimension $l-1$ or less spanned by a subset of the $\{\beta_j\}$.

**Remark 2.4.** Suppose $X^J = 1$, so all $j_m = 0$. Then the residue is equal to

$$\frac{i^{k-l}H_\beta(\lambda)}{(2\pi)^l}$$

where $H_\beta$ is the pushforward of Lebesgue measure from $\mathbb{C}^k$ to $\mathfrak{t}^*$ (in other words, the Duistermaat-Heckman function for the linear action of $T$ on $\mathbb{C}^k$ via the collection of weights $\beta_1, \dots, \beta_k$). (See [**JK1**] Proposition 8.11 (ii) and [**GLS1**].)

**Remark 2.5.** By property (4), the residue equals 0 when $\lambda = 0$ provided that $k > l$, because the functions $H_\beta$ are piecewise polynomial functions of degree $k - l$ and so must approach 0 at $s\rho$ as $s \to 0^+$, when $\rho$ is a point in the cone $\Lambda^*$. Likewise, when $k = l$ the value of the residue at $\lambda = 0$ equals the value given in (3).

**Remark 2.6.** It turns out that it is legitimate to expand the exponential in the left hand side of (2) using a Taylor series and then to compute the right hand side in property (2) using linearity and the properties listed in (1) and (3). In particular, $(i\lambda(X))^m$ is a polynomial in X and can be distributed. Then for a multi-index $I = (i_1, \dots i_l)$,

$$\lim_{s\to 0^+} \operatorname{Res}^\Lambda\big(\frac{X^J X^I e^{is\lambda(X)}[dX]}{\prod_{i=1}^v \alpha_i(X)}\big) = 0$$

unless $v - (j_1 + \dots j_l) - (i_1 + \dots i_l) = l$.

Suppose $M^T$ consists of isolated fixed points. Consider the case that $\beta$ is of homogeneous degree, and that $\alpha \in H_T^*(M)$ is homogeneous and complementary in degree in the sense that

$$\deg \alpha = \dim M_p - \deg \beta.$$

Then the residue theorem states

$$\int_{M_p} \kappa_p(\alpha\beta) = c \cdot \operatorname{Res}^\Lambda \left( \sum_{F \in M^T} e^{i(\Phi(F)-p)(X)} \frac{\iota_F^*(\alpha\beta)}{e_F}[dX] \right). \tag{2.2}$$

Alternatively (see [**JK2**] Proposition 3.4) the residue can be reformulated as follows. For $f$ a meromorphic function of one complex variable $z$ which is of the form $f(z) = g(z)e^{i\lambda z}$ where $g$ is a rational function, we define

$$\operatorname{Res}_z^+ f(z)dz = \sum_{b \in \mathbb{C}} \operatorname{Res}(g(z)e^{i\lambda z}; z = b).$$

We extend this definition by linearity to linear combinations of functions of this form.

Viewing $f$ as a meromorphic function on the Riemann sphere and observing that the sum of all the residues of a meromorphic 1-form on the Riemann sphere is 0, we observe that

$$\operatorname{Res}_z^+ (f(z)dz) = -\operatorname{Res}_{z=\infty} (f(z)dz).$$

If $X \in \mathfrak{t}$, define

$$h(X) = \frac{q(X)e^{i\lambda(X)}}{\prod_{j=1}^{k} \beta_j(X)}$$

for some polynomial function $q(X)$ of $X$ and some $\lambda, \beta_1, \ldots, \beta_k \in \mathfrak{t}^*$. Suppose that $\lambda$ is not in any proper subspace of $\mathfrak{t}^*$ spanned by a subset of $\{\beta_1, \ldots, \beta_k\}$. Let $\Lambda$ be any nonempty open cone in $\mathfrak{t}$ contained in some connected component of

$$\{X \in \mathfrak{t} : \beta_j(X) \neq 0, 1 \leq j \leq k\}.$$

Then for a generic choice of coordinate system $X = (X_1, \ldots, X_l)$ on $\mathfrak{t}$ for which $(0, \ldots, 0, 1) \in \Lambda$ we have

$$\text{Res}^{\Lambda}(h(X)[dX]) = \triangle \text{Res}^{+}_{X_1} \circ \ldots \circ \text{Res}^{+}_{X_l} (h(X) dX_1 \ldots dX_l) \tag{2.3}$$

where the variables $X_1, \ldots, X_{m-1}$ are held constant while calculating $\text{Res}^{+}_{X_m}$, and $\triangle$ is the determinant of any $l \times l$ matrix whose columns are the coordinates of an orthonormal basis of $\mathfrak{t}$ defining the same orientation as the chosen coordinate system. We assume that if $(X_1, \ldots, X_l)$ is a coordinate system for $X \in \mathfrak{t}$, then $(0, 0, \ldots, 1) \in \Lambda$. We also require an additional technical condition on the coordinate systems, which is valid for almost any choice of coordinate system (see Remark 3.5 (1) from [**JK2**]).

One proof of (2.3) involves checking that the object on the right hand side of (2.3) satisfies a subset of the properties (1)-(3) above which characterize the residue uniquely (see [**JK2**], Proposition 3.4).

**2.2. Equivariant Morse theory.** The goal of this section is to use the restriction properties of $\alpha \in H_T^*(M)$ to find level sets on which $\alpha$ restricts to 0 under the corresponding Kirwan map. In fact we show something stronger, namely that in the case of isolated fixed points there is a basis of $H_T^*(M)$ (as an $H_T^*(pt)$-module) given by certain classes $\{\alpha_F\}_{F \in M_{cc}^T}$ indexed by the fixed point set (although the choice of these classes may not be unique). The classes $\alpha_F$ are in $I_p$ for a very specific (and generally rather large) set of regular values $p$, as we describe below.

First consider the case that $p$ is not in the convex hull of the image under $\Phi$ of *supp* $\alpha$. One can prove that $\alpha \in I_p$ using equivariant Morse theory. In particular, if $p$ is not in this convex hull, then there is some $\xi \in \mathfrak{t}$ whose corresponding moment map component $\Phi^\xi := \langle \Phi, \xi \rangle$ is such that $\Phi^\xi(\Phi^{-1}(p)) < \Phi^\xi(F)$ for every $F \in$ *supp* $\alpha$. One may choose $\xi$ so that $\Phi^\xi$ has distinct values for distinct components of the fixed point set. Denote these components by $C_0, C_1, \dots, C_k$ and order them so that $i < j$ if and only if $\Phi^\xi(C_i) < \Phi^\xi(C_j)$. We note that $\Phi^\xi$ is a Morse-Bott function. Following [**AB**] one defines

$$M_i^+ := (\Phi^\xi)^{-1}(-\infty, \Phi^\xi(C_i) + \epsilon_i) \text{ and} \tag{2.4}$$

$$M_i^- := (\Phi^\xi)^{-1}(-\infty, \Phi^\xi(C_i) - \epsilon_i) \tag{2.5}$$

where $\epsilon_i > 0$ is small enough that $C_i$ is the only critical set in the interval $(\Phi^\xi)^{-1}(\Phi^\xi(C_i) - \epsilon_i, \Phi^\xi(C_i) + \epsilon_i)$. For each $i$, consider the long exact sequence in equivariant cohomology

$$\cdots \to H_T^*(M_i^+, M_i^-) \to H_T^*(M_i^+) \to H_T^*(M_i^-) \to H_T^{*+1}(M_i^+, M_i^-) \to \cdots . \tag{2.6}$$

Using the Thom isomorphism, we may identify

$$H_T^*(M_i^+, M_i^-) \cong H_T^{*-\lambda_i}(C_i),$$

where $\lambda_i$ is the index of the negative normal bundle to $C_i$ (under $\Phi^\xi$). Atiyah and Bott observed that the map $H_T^{*-\lambda_i}(C_i) \to H_T^*(M_i^+)$ is an *injection* and hence the sequence splits

$$0 \to H_G^{*-\lambda_i}(C_i) \to H_G^*(M_i^+) \to H_G^*(M_i^-) \to 0. \tag{2.7}$$

It follows by induction on $i$ that if $\alpha$ is 0 on $C_1$,..., $C_i$, then $\alpha$ is 0 on $M_i^+$ and therefore $\alpha$ restricts to 0 on $\Phi^{-1}(p) \subset (\Phi^\xi)^{-1}(p)$ if $\Phi^\xi(p) < \Phi^\xi(C_{i+1})$.

When $M$ is not Kähler, we may not have the familiar convexity properties. The convex hull of $\Phi(supp\ \alpha)$ may contain regular values that are not in the image of the gradient flow-outs of *supp* $\alpha$: see the non-Kähler example in Section 4.3. For these regular values, it is not immediately clear that the class $\alpha$ restricts to zero at these regular values. We use some additional Morse theory to prove that $\alpha$ does indeed restrict to 0 on these level sets. We first define the *extended stable set* of a critical set $C$.

**Definition 2.7.** *Let $f$ be a Morse-Bott function on $M$, and order the critical sets $C_0, C_1, \dots, C_k$ so that $i < j$ if and only if $f(C_i) < f(C_j)$. Let grad $f$ be the gradient flow of $f$ with respect to a compatible Riemannian metric. The* extended stable set *of $C$ is the set of points $x$ in $M$ such that there is a sequence of critical sets $C_{i_1}, C_{i_2}, \dots, C_{i_m} = C$ in which $x$ converges to $C_{i_1}$ under the flow of $-grad\ f$, and there exist points in the negative normal bundle of $C_{i_j}$ that converge to $C_{i_{j+1}}$ under the flow of $-grad\ f$.*

The extended stable set is also called the extended flow-up (where "up" means that the value of $f$ increases under the flow of *grad* $f$ rather than $-$*grad* $f$).

**Remark 2.8.** Note that the extended stable set is *not* the same as the closure of the stable flow-outs of the critical set $C$. A manifold with a Morse-Bott function for which these two are not the same is the Hirzebruch surface $F_2$ (see Figure 1).

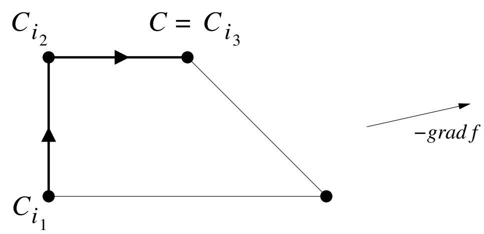


**Figure 1.** The bold lines are the image of the *extended stable set* of $C$ under $\Phi$. The closure of the *stable flow-out* of $C$ is just the (top) horizontal bold line.

A slight generalization of these methods (see [**Go2**][1] ) yields

**Theorem 2.9.** *For every fixed point set component $F$ there exists a homogeneous class $\alpha_F$ which has support on the extended stable set of $F$ and such that $\iota_F^*(\alpha_F) = e_T(\nu^- F)$, the equivariant Euler class of the negative normal bundle to $F$. We call such a class a* Morse-Thom class *associated to $F$.*

It now follows easily that, in the case of isolated fixed points, any class $\alpha \in H_T^*(M)$ can be written as an element of the ideal generated by the classes $\alpha_F$ as $F$ ranges over the fixed point set. Let $i$ be the smallest number such that $\iota_{C_i}^*(\alpha) \neq 0$. Then $\iota_{C_i}^*(\alpha)$ is a multiple $m$ of the equivariant Euler class of the negative normal bundle to $C_i$. Let $\alpha_i$ be a Morse-Thom class associated to $C_i$. Then $\alpha - m\alpha_i$ restricts to 0 on all classes $C_j$ where $j \leq i$. Continue inductively to express $\alpha$ in terms of the $\alpha_i, i = 1, \ldots, k$.

**Remark 2.10.** In the case that some component $f$ of the moment map is *Palais-Smale*, i.e. the (ordinary, non-extended) unstable and stable

[1]in [**Go2**] the extended stable set was termed "extended stable manifold," a misnomer since these sets are not in general manifolds

manifolds intersect transversally, there is a unique Morse-Thom class for each component of the fixed point set: Suppose $F$ and $G$ are two isolated fixed points and that the unstable manifold of $F$ intersects the stable manifold of $G$ transversally and non-trivially. If $F \neq G$, then the intersection is not zero-dimensional and $\dim \nu_G^+ + \dim \nu_F^- - \dim M > 0$, where $\nu_G^+$ is the positive normal bundle of $G$ in $M$ and $\nu_F^-$ is the negative normal bundle of $F$ in $M$, both with respect to $f$. Since $\dim \nu_G^+ = \dim M - \dim \nu_G^-$, it follows that $\dim \nu_F^- > \dim \nu_G^-$. The degree of any Morse-Thom class out of $G$ is equal to $\dim \nu_G^-$, and thus homogeneity assures that the class associated to $G$ is unique. This is the case for coadjoint orbits of complex reductive groups. A similar argument works in the case that $F$ and $G$ are not isolated.

Let $p$ be a regular value of $\Phi$, $f$ a Morse-Bott function on $M$, and $C_0, \ldots, C_k$ the connected components of the critical set. We choose $f$ generically enough so that we may order the critical sets by $f(C_i) < f(C_j)$ for $i < j$. Suppose that $\alpha \in H_T^*(M)$ is an equivariant Euler class associated to the connected component $C_i$ of the critical set. Suppose that $\Phi^{-1}(p)$ does not intersect the extended stable set associated to $F$. Then $\alpha|_{\Phi^{-1}(p)} = 0$, for $\Phi^{-1}(p)$ is a subset of a space equivariantly homotopic to $M_i^-$ (see [**Go2**]), and $\alpha|_{M_i^-} = 0$ by assumption that it is an Euler class associated to $C_i$. Thus $\alpha$ restricted to a level set $\Phi^{-1}(p)$ is non-zero only if this level set is in the extended stable set of a fixed point component $F \in supp\ \alpha$. We have proven

**Theorem 2.11.** *Let $\alpha \in H_T^*(M)$, where $M$ is a compact Hamiltonian $T$-space with moment map $\Phi$. Suppose $p$ is a regular value of $\Phi$, and that for some component $\Phi^\xi$ of $\Phi$, $\Phi^{-1}(p)$ does not intersect the extended stable set with respect to $\Phi^\xi$ of any of the points in the support of $\alpha$. Then $\alpha|_{\Phi^{-1}(p)} = 0$.*

We will need this theorem to prove Theorem 1.1 in the general case, in which we are not guaranteed that the supports of Morse-Thom classes are convex. We note, however, that we have proven

**Corollary 2.12.** *Let $H \subset \mathfrak{t}^*$ be a rational hyperplane passing through $p$. We orient $H$ by realizing it as a level set of a vector $\xi \in \mathfrak{t}$ and denoting by $H^+$ the set of all points $q \in \mathfrak{t}^*$ such that $\xi(q) > \xi(H)$, and by $H^-$ the set of all points $q \in \mathfrak{t}^*$ such that $\xi(q) < \xi(H)$. If $\Phi(supp\ \alpha) \subset H^+$ or $\Phi(supp\ \alpha) \subset H^-$, then $\alpha \in I_p$.*

A much harder fact to prove is that the set of these classes, each of which has this vanishing property for some hyperplane $H$, generate the kernel of the Kirwan map. This is the content of the Tolman-Weitsman theorem [**TW**].

Notice that regular values are never on walls of the moment polytope. If $p$ is regular, then $\Phi^{-1}(p)$ has a locally free $T$ action, and in particular points in the level set can be fixed by at most 0-dimensional subsets of $T$.

## 3. The Main Theorem and its Proof

We begin with a proposition about the behavior of $I_p$ as $p$ moves within a chamber.

**Proposition 3.1.** *Let $M$ be a compact Hamiltonian $T$ space with moment map $\Phi$. If $p$ and $q$ are two regular values of $\Phi$ in the same chamber of the moment polytope, then $I_p = I_q$.*

*Proof.* Since any two points $p$ and $q$ in the chamber $C$ may be connected by a piecewise linear path lying entirely in $C$, the Proposition follows from the following theorem of Guillemin-Kalkman [**GK**] and Martin [**Ma**].

**Theorem 3.2** ([**GK**],[**Ma**]). *Let there be a piecewise linear path of regular values between two regular values $p$ and $q$ of the moment map. Let $\eta \in H_T^*(M)$. Then for any $\zeta \in H_T^*(M)$ we have*

$$\int_{M_p} \kappa_p(\zeta)\kappa_p(\eta) = \int_{M_q} \kappa_q(\zeta)\kappa_q(\eta).$$

Since $\kappa_p$ and $\kappa_q$ are surjective, it follows that $\kappa_p(\eta) = 0$ if and only if $\kappa_q(\eta) = 0$ (using Poincaré duality). □

We now assume that $T$ acts on $M$ with isolated fixed points, and prove the harder part of Theorem 1.1, namely

**Proposition 3.3.** *Let $M$ be a compact Hamiltonian $T$ space with isolated fixed points and moment map $\Phi$. If $p$ and $q$ are two regular values of $\Phi$ in distinct chambers of the moment polytope, then $I_p \neq I_q$.*

We prove this by construction. For any $p$ and $q$ in distinct chambers $C_p$ and $C_q$ respectively, we find a class $\gamma$ with certain restriction properties that make computations easy. We then show that the Jeffrey-Kirwan localization theorem implies that $\gamma \in I_p$ but $\gamma \notin I_q$.

We note that any wall can be oriented as follows. Suppose $W \subset \mathfrak{t}^*$ is a codimension-1 wall of $\Delta$. Choose $\xi \in \mathfrak{t}$ perpendicular to $W$ in the sense that $W$ lies in the intersection of $\Delta$ with a level set of $\xi$. We write

$W^+$ for $\{x \in \mathfrak{t}^* \mid \xi(x) > \xi(W)\}$ and $W^-$ for $\{x \in \mathfrak{t}^* \mid \xi(x) < \xi(W)\}$. Thus $W^+$ and $W^-$ are open half spaces in $\mathfrak{t}^*$. We denote by $H$ the hyperplane of $\mathfrak{t}^*$ containing $W$. Define $H^+ = W^+$ and $H^- = W^-$.

Let $p$ and $q$ be regular values of the moment map in distinct connected components of $\Delta$. Choose $W$ a codimension-1 wall of the chamber of $p$, and orient $W$ so that $p \in W^+$ and $q \in W^-$. Note that $W$ need not equal $C_p \cap W$. For every fixed point $F$, let $\{\beta_i^F\}$ denote the set of weights of the $T$ action on $T_F M$.

We begin by finding a special subset $P$ of the fixed point set $M^T$. For every $F$, let $J_F$ be the index set with $i \in J_F$ if and only if $\Phi(F) + \beta_i^F \in H^+ \cup H$. The set $P$ consists of the fixed points $F$ that satisfy two conditions:

1) $\Phi(F) \in W$, and
2) $p$ is in the positive linear span of $\{\Phi(F) + \beta_i^F\}_{i \in J_F}$. We will write $p \in \Phi(F) + \langle \beta_i^F \rangle_{i \in J_F}^+$.

**Lemma 3.4.** *$P$ is non-empty.*

*Proof.* Let $W$ be a wall adjacent to the chamber $C_p$ containing $p$, and $H$ the hyperplane containing $W$. The wall $W \subseteq H$ is contained in the image under $\Phi$ of a connected component $M_0$ of the fixed point set $M^S$ of a circle $S \subseteq T$. This component $M_0$ is itself a compact symplectic manifold with Hamiltonian $T/S$ action, and must therefore have fixed points. Therefore, there are fixed points that satisfy condition (1).

Since $W$ is a wall on the chamber of $p$, we may assume that $p$ is very close to $W$. As above, $M_0$ is a connected component of $M^S$ such that $W \subseteq \Phi(M_0) \subset H$. Consider the orthogonal projection $\pi : \mathfrak{t}^* \to H$. Since $p$ is close to $W$, we may assume that $\pi(p) \in W$.

We note that the maximal value of $\xi \circ \Phi$ is not attained on $M_0$ since $W$ is necessarily an internal wall. Therefore, at every fixed point $F \in M_0^T$, the action of $T$ on $T_F M_0$ must have at least one weight pointing into $H^+ = W^+$.

Now consider the weights pointing along $H$. The $T/S$ action on $M_0$ is effective, and thus the image of the moment map for this action is a convex set in $Lie(T/S)^*$, identified with a subset of $H$ containing $W$. It follows that, for any point in $W$ (and in particular, for $\pi(p)$), there is a fixed point $F \in M_0^T$ whose weights have positive span containing this point. Choosing a fixed point $F$ with $\pi(p)$ contained in the positive span of the weights $\{\beta_i^F\}$ at $F$ pointing into $H$, we may move $p$ close enough to $W$ so that $p$ is contained in the positive span of $\{\Phi(F)+\beta_i^F\}$.

This proves that there is an $F \in M^T$ satisfying the two conditions in the definition of $P$. $\square$

We now choose an appropriate dual cone $\Lambda^*$ in $\mathfrak{t}^*$. Consider the set $\{\beta_i^F\}_{i,F}$ of all weights that occur at any fixed point. Choose any $F_0 \in P$. Since $p \in \Phi(F_0) + \langle \beta_i^{F_0} \rangle^+_{i \in J_{F_0}}$, there exists a subset of weights

$$\{\beta_i^{F_0}\}_{i \in I_{F_0}} \subset \{\beta_i^{F_0}\}_{i \in J_{F_0}}$$

indexed by $I_{F_0} \subset J_{F_0}$ such that

(a) $\langle \beta_i^{F_0} \rangle^+_{i \in I_{F_0}}$ does not contain a line, and
(b) $p \in \Phi(F_0) + \langle \beta_i^{F_0} \rangle^+_{i \in I_{F_0}}$.

Choose any $X \in \mathfrak{t}$ with $\beta_i^F(X) \neq 0$ for all $i, F$ and satisfying:

1) $\beta_i^{F_0}(X) < 0$ for all $i \in I_{F_0}$, and
2) If $\Phi(F) \in H$ and $\Phi(F) + \beta_i^F \in H^-$, then $\beta_i^F(X) > 0$.

The vector $X$ defines a polarization of the weights. Let $\gamma_i^F = \beta_i^F$ if $\beta_i^F(X) > 0$ and $\gamma_i^F = -\beta_i^F$ if $\beta_i^F(X) < 0$. Finally, let $\Lambda^*$ be the cone generated by the span of the vectors $\{\gamma_i^F\}_{i,F}$.

Armed with $\Lambda^*$, we are now prepared to find a class $\gamma \in I_q$ with $\gamma \notin I_p$. We first show that the choice of $\Lambda^*$ ensures that certain fixed points (those whose image under $\Phi$ lie in $H^+$) will not contribute to the residue for any $\beta \in H_T^*(M)$. By Proposition 3.1, we may assume that $p$ is close to $H$. The choice of polarization implies that (for $p$ close enough to $H$) if $\Phi(F) \in H^+$, then $\Phi(F) \notin p + \Lambda^*$. In particular by Property (1)(c) of the residue,

$$\operatorname{Res}^\Lambda \frac{e^{i(\Phi(F)-p)}}{\prod_{i=1}^l \alpha_i} = 0$$

for *any* $\alpha_1, \ldots, \alpha_l \in \Lambda^*$, whenever $\Phi(F) \in H^+$.

We now show that there is a certain set of classes $\alpha \in H_T^*(M)$ which restrict to 0 on fixed points $F$ with $\Phi(F) \in H^-$. Recall that $H$ is level set of $\xi \in \mathfrak{t}$. Consider $\eta \in \mathfrak{t}$, a small perturbation of $\xi$ with the properties that

1) For $F_1, F_2 \in M^T$ with $\Phi(F_1), \Phi(F_2) \in H$ but $\Phi(F_1) \neq \Phi(F_2)$, we have $\eta(\Phi(F_1)) \neq \eta(\Phi(F_2))$.
2) For any $F \in M^T$ with $\Phi(F) \in H^+$ and any $F_1 \in M^T$ with $\Phi(F_1) \in H$, we have $\eta(\Phi(F_1)) < \eta(p) < \eta(\Phi(F))$.

Consider $\eta \circ \Phi$ restricted to $P$. Let $F_1 \in P$ be the fixed point where $\eta \circ \Phi|_P$ is maximized. Recall that by construction, $\Phi(F_1) \in H$. By Theorem 2.9 there exists a Morse-Thom class $\alpha = \alpha_{F_1}$ which has support (in the sense of definition 2.2) on the extended flow-up out of $F_1$ along *grad* $\eta$. In particular,

$$\alpha|_F = 0 \quad \text{if} \quad \Phi(F) \in H^-.$$

It now follows immediately that

**Lemma 3.5.** *Let $\alpha = \alpha_{F_1}$ as above. Then $\beta\alpha \in I_q$ for any $\beta \in H_T^*(M)$.*

*Proof.* Using the cone $\Lambda^*$, the only terms that contribute to the integral over $M_q$ are fixed points $F$ such that $\Phi(F) - q \in \Lambda^*$, or $q \in \Phi(F) - \Lambda^*$. Since $q \in H^-$, it follows that the only possibly contributing $F$ are those whose images under the moment map are in $H^-$. However, $\beta\alpha$ restricts to 0 on all these points (as $\alpha$ does), which implies the residue contributions are 0. □

It is left to show there exists $\beta \in H_T^*(M)$ such that $\int_{M_p} \kappa_p(\beta\alpha) \neq 0$. We first simplify the integral by showing that there is only one possibly nonzero term in the residue formula (2.2).

**Claim 3.6.** *Let $\alpha$ be a Morse-Thom class of $F_1$ and $\beta \in H_T^*(M)$ be any class. Then*

$$\int_{M_p} \kappa_p(\beta\alpha) = c \cdot \mathrm{Res}^\Lambda e^{i(\Phi(F_1)-p)(X)} \frac{\iota_{F_1}^*(\beta\alpha)}{e_{F_1}}[dX]. \tag{3.1}$$

*where $e_{F_1}$ is the equivariant Euler class at $F_1$.*

*Proof.* Let $F$ be a fixed point with $\Phi(F) \in H^+$. Then we showed that $p \notin \Phi(F) - \Lambda^*$, and the corresponding residue term in Equation (2.2) is therefore 0.

Suppose instead that $F$ is a fixed point with $\Phi(F) \in H^-$. Then $\alpha|_F = 0$ implies that any corresponding residue term is 0.

Finally, suppose that $F$ is a fixed point with $\Phi(F) \in H$. If $F \notin P$, then by construction $p \notin \Phi(F) - \Lambda^*$ and therefore the corresponding residue is 0. On the other hand, if $F \in P$ and $F \neq F_1$, then $\alpha|_F = 0$ since $\alpha$ is supported on the flow-up out of $F_1$ along *grad* $\eta$ and $F_1$ is where $\eta \circ \Phi$ attains its maximum among points in $P$. Therefore the only remaining possible contribution to the integral (3.1) is from $F_1$. □

The final step is to prove our original claim, namely that

**Lemma 3.7.** *There exists $\beta \in H_T^*(M)$ such that $\beta\alpha \notin I_p$, where $\alpha$ is a Morse-Thom class associated to $F_1$, as above.*

*Proof.* We show that, for an appropriate choice of $\beta$, the single term in (3.1) is nonzero.

Let $R$ be the set of all weights of the $T$ action at $F_1$. Let $R^+ = \Lambda^* \cap R$ and $R^- = R \backslash R^+$. Since $F_1$ is isolated, the normal bundle is topologically trivial (although not equivariantly) and so the Euler class of the normal bundle and that of the negative normal bundle with respect to $\eta \circ \Phi$ are products of weights. In particular

$$e_{F_1} = \prod_{w \in R} w$$

and

$$\iota_{F_1}^*(\alpha) = \alpha|_{F_1} = \prod_{w \in R^-} w.$$

We simplify

$$\frac{\iota_{F_1}^*(\beta\alpha)}{e_{F_1}} = \frac{\iota_{F_1}^*(\beta) \prod_{w \in R-} w}{\prod_{w \in R} w} = \frac{\iota_{F_1}^*(\beta)}{\prod_{w \in R^+} w} \tag{3.2}$$

We choose $\beta$ carefully. Let $\gamma_1, \ldots, \gamma_l$ be any basis of $\mathfrak{t}^*$ chosen among elements of $R^+$. Let

$$\beta = \prod_{w \in R^+ - \{\gamma_1, \ldots, \gamma_l\}} w.$$

As $\beta$ is in the image of $H_T^*(pt)$, $\iota_F^* \beta = \beta$ for all fixed points $F$. Thus the formula (3.2) becomes

$$\frac{\iota_{F_1}^*(\beta)}{\prod_{w \in R^+} w} = \frac{1}{\prod_{i=1}^l \gamma_i}.$$

By Property 3 of the residue formula, the residue

$$c \cdot \mathrm{Res}^\Lambda \left( e^{i(\Phi(F)-p)(X)} \frac{1}{\prod_l \gamma_i} [dX] \right) = c \cdot \frac{1}{\det \overline{\gamma}}$$

where $\overline{\gamma}$ is a matrix whose columns are $\gamma_i$ in an appropriate basis. Since the $\gamma_i$ are linearly independent and $c$ is non-zero, the residue is non-zero. This is the only contribution to the sum (2.2), so we have shown the integral of $\beta\alpha$ over the reduced space at $p$ is non-zero. In other words, $\gamma = \beta\alpha \notin I_p$. In particular, this implies that $\alpha \notin I_p$, as well as $\beta \notin I_p$. This completes the proof of the lemma. □

## 4. Applications of the main theorem

**4.1. Schubert classes and counting chambers.** In the case that $M$ is a coadjoint orbit of any complex reductive group, a specific basis for $I_p$ has been described in terms of the choice of symplectic structure on the coadjoint orbit and the reduction point $p$ ([**Go1**], [**GM**]). The generators are *permuted Schubert classes* defined by their duality properties with certain subvarieties of $M$. They can also be uniquely described by their restriction properties due to Kirwan injectivity. We present the latter description here. Theorem 4.1 can be taken as a definition for readers unfamiliar with Schubert classes.

Let $G$ be a compact semi-simple Lie group with Borel subgroup $B$ and maximal torus $T$. Let $M$ be a coadjoint orbit of $G$ through $\lambda \in \mathfrak{t}^*$. We identify the Weyl group $W := N(T)/T$ with the fixed point set by associating $\lambda$ to the trivial coset, and the remaining points can be identified by the transitive action of $W$ on the fixed points. We write $\lambda_\sigma$ to indicate the fixed point corresponding to $\sigma$. Let $f$ be a moment map for a generic $S^1$ action on $M$, such that $f|_{M^T}$ is minimized at $\lambda$.

**Theorem 4.1.** *For every $\sigma \in W$ the associated Schubert class $\beta_\sigma \in H_T^*(M)$ is the unique homogeneous class defined by the following restriction properties.*

1) *$\beta_\sigma|_{\lambda_\sigma} = e(\nu_f^- \lambda_\sigma)$, the equivariant Euler class of the negative normal bundle (with respect to $f$) of the fixed point associated to $\sigma$.*
2) *$\beta_\sigma|_{\lambda_\tau} = 0$ unless $\tau \geq \sigma$ in the Bruhat order of $W$.*

Schubert classes form a basis for $H_T^*(M)$ as a module over $H_T^*(pt)$. If we use the same construction while allowing $f$ to be minimized at other fixed points, we obtain the *permuted Schubert classes*

$$\beta_\sigma^\tau := \tau \cdot \beta_{\tau^{-1}\sigma},$$

whose restriction properties can be described in a similar fashion.

Permuted Schubert classes are precisely the classes predicted by Theorem 2.9. In the case of coadjoint orbits the *extended* flow-up is precisely the closure of the flow-up from any fixed point. They form (permuted) Schubert varieties.

There is a very simple description of ideals $I_p$ for coadjoint orbits of complex reductive groups [**Go1**], [**GM**]. For any regular value $p$, the ideal $I_p$ is generated by the permuted Schubert classes $\beta_\sigma^\tau$ such that $p$ is not in the convex hull of *supp* $\beta_\sigma^\tau$. Since the number of chambers in $\Delta$ is determined by the number of distinct ideals $I_p$, this simple description offers the possibility of an algebraic method for ascertaining the number of chambers in the moment polytope for coadjoint orbits $SU(n)$, for example. Of course, these numbers depend on the non-zero value $\lambda \in \mathfrak{t}^*_+$ in the positive Weyl chamber, whose coadjoint orbit we are considering. For $SU(3)$, the number of chambers is six or seven, and does not offer much of a challenge. E. Rassart has computed this number for $SU(4)$ orbits [**Ra**] using geometric arguments, and has found numbers in the hundreds (depending on the value of $\lambda$), but the $SU(5)$ case is already out of reach. It would be interesting to find a bound on the number of chambers for general $n$.

**4.2. Relation to a wall-crossing formula.** There are several places in the literature in which the behavior of the reduced space, or integrals over these spaces, is studied as the regular value moves over exactly one codimension-1 wall of the moment map. Most notable, perhaps, is the formula due to Guillemin-Kalkman [**GK**], and independently S. Martin [**Ma**]. We present here a short exposition on this result, and then show that Theorem 1.1 allows us to gain some information about the reduction at singular values of the moment map in the case that the fixed points are isolated.

Let $L$ be a hyperplane of critical values of $\Phi$ such that the line segment between two regular values $p$ and $q$ intersects $L$ but intersects no other hyperplanes of critical values of $\Phi$. Let $N$ be a component of the fixed point set of a circle $S \subset T$ such that $\Phi(N) \subset L$. Then Guillemin-Kalkman give

$$\int_{M_p} \kappa_p(\zeta\eta) - \int_{M_q} \kappa_q(\zeta\eta) = \int_{N_{\mathrm{red}}} \kappa_{T/S}\left(\mathrm{Res}\left[\frac{\iota_N^*(\zeta\eta)}{e_N}\right]\right).$$

Here, $\Phi_S(M_p) > \Phi_S(M_q)$, where $\Phi_S$ is the component of $\Phi$ corresponding to $S$. This requires the choice of a basis element in the Lie algebra of $S$, and the same basis element must be used to define the residue. Note that $N_{\mathrm{red}}$ is the reduced space of $N$ with respect to the action of $T/S$, and $e_N$ is the equivariant Euler class of the normal bundle of $N$ in $M$; $\kappa_{T/S}$ is the Kirwan map for the action of $T/S$ on $N$.

The operation Res is defined as follows. Introduce a basis element $X$ for $\mathrm{Lie}(S)$ and basis elements $Y_1, \ldots, Y_n$ for $\mathrm{Lie}(H)$, where $H$ is a codimension-1 subtorus transverse to $S$, so $H \cong T/S$. The normal bundle $\nu$ to $N$ can be assumed to be a sum of line bundles:

$$\nu = L_1 \oplus \cdots \oplus L_l.$$

We put

$$c_1^T(L_i) = m_i X + \mu_i + \sum_j f_i^j Y_j$$

where $m_i \in \mathbf{Z}$ is the weight of the representation of $S$ on $L_i$ and $\mu_i + \sum_j f_i^j Y_j$ is a Cartan representative of $c_1^H(L_i)$ (so $\mu_i \in \Omega^2(N)$ and $f_i^j \in \Omega^0(N)$). Write

$$\frac{1}{c_1^T(L_i)} = \frac{1}{m_i X(1 + \sum_j f_i^j Y_j/X + \mu_i/X)}$$

and expand

$$\frac{1}{1+A} = 1 - A + A^2 - \dots$$

using $A = (\sum_j f_i^j Y_j + \mu_i)/X$. By expanding into a product of power series in $X$ we can rewrite $\iota_N^*(\zeta\eta)/e_N$ as a sum $\sum_{j=-\infty}^{m_0} \beta_j X^j$ where each $\beta_j$ is a polynomial in $Y_1, \dots, Y_n$ with coefficients in $\Omega^*(N)$. In particular, $\beta_j \in \Omega^*_{T/S}(N)$. We define

$$\mathrm{Res}(\iota_N^*(\zeta\eta)/e_N) = \beta_{-1}.$$

It is shown in [**GK**] that $\beta_{-1}$ is well defined independent of the choice of the variables $X$ and $Y_j$.

**Remark 4.2.** This is the residue at $X = \infty$ of the meromorphic function of $X$ defined by $\iota_N^*(\zeta\eta)/e_N$. Thus it equals minus the sum of the residues at finite values of $X$ (using the fact that the sum of the residues of a meromorphic 1-form over the Riemann sphere is zero). Here the meromorphic 1-form depends on one complex variable $X$ in the Riemann sphere, but takes values in $\Omega^*_{T/S}(N)$

The technique for crossing one wall can of course be inductively applied to more than one wall, obtaining a formula for the difference in the integral over reduced spaces at *any* two regular values. However the resulting formula does not imply that the chambers are distinguished by the ideals $I_p$, and indeed it may be a lengthy computation using this method to obtain that this difference is non-zero for some class.

We finish this section with a small result on the behavior of the classes that distinguish ideals. Let $p$ and $q$ be regular values with a line between them crossing exactly one singular value $q_1$ of $\Phi$. By Theorem 1.1, there is a class $\alpha$ such that $\alpha \in I_q$ but $\alpha \notin I_p$. We may choose a class with the additional property that $\deg \alpha = \dim M_p$. By the wall-crossing formula,

$$\int_{M_p} \kappa_p(\alpha) - 0 = \int_{N_{\mathrm{red}}} \kappa_{T/S} \left( \mathrm{Res} \left[ \frac{\iota_N^*(\alpha)}{e_N} \right] \right) \neq 0. \tag{4.1}$$

Choose a $T$-invariant Riemannian metric on $M$. Then Equation (4.1) implies that the image of $\alpha$ under the restriction

$$H_T^*(M) \xrightarrow{\;i^*\;} H_T^*(N) \xrightarrow{\;p^*\;} H_T^*(S(\nu N))$$

$$\xrightarrow{\;\cong\;} H_{T/S}^*(S(\nu N)/S) \xrightarrow{\;\pi_*\;} H_{T/S}^{*-k}(N) \xrightarrow{\;\kappa_{T/S}\;} H^{*-k}(N_{red})$$

is nonzero. The above composition is the "localization map" defined by [**Ma**], Definition 5.1. Here $S(\nu N)$ is the unit sphere bundle (with respect to the invariant metric) of the normal bundle to $N$, and $N_{red}$ is the reduced space at the singular value $q_1$. The maps $i^*$ and $p^*$ are induced by the inclusion $i : N \hookrightarrow M$ and the projection $p : S(\nu N) \to N$, respectively. The isomorphism on the curved arrow follows from the fact that $S$ acts locally freely on the sphere bundle (and the cohomology is in rational coefficients). Lastly, $\pi_*$ is "integration over the fibers", and reduces degree by $k$, the dimension of the fibers of the (weighted) projective bundle $S(\nu N)/S \to N$, according to the induced orientation defined in [**Ma**]. In particular, not only are the fundamental classes described by Theorem 2.9 nonzero when restricted to level sets near $C$ and in the flow-up from $C$, but these classes are nonzero when restricted to the *singular walls* on the boundary (near $C$) of the cone out of $C$.

It is shown in Section 4 of [**GK**] how to use the formula (4.1) inductively to give a formula for $\int_{M_p} \kappa_p(\alpha)$ in terms of the restrictions of $\alpha$ to the components of the fixed point set of $T$. This formula is obtained by drawing a line $l$ from $p$ to the boundary of $\Phi(M)$ which avoids all codimension-2 walls. Next, for each intersection $p_j$ of $l$ with a codimension-1 wall $W_j$, one draws a line $l_j$ within $W_j$ from $p_j$ to the boundary of $\Phi(M) \cap W_j$. This process gives rise to a graph (the term "dendrite" is used in [**GK**]) originating at $p$ and terminating in points $\Phi(F)$ where $F$ is a fixed point of $T$ on $M$. The quantity $\int_{M_p} \kappa_p(\alpha)$ may thus be expressed in terms of the data $\alpha|_F$ for a distinguished collection of fixed points $F$.

One can easily deduce from the description of the residue given in (2.3) that Guillemin and Kalkman's iterated residue is equivalent to

Jeffrey and Kirwan's.[2] See also Kalkman's proof of their equivalence [**Ka**].

**4.3. A non-Kähler example.** We now explore the main theorem in the context of a non-Kähler manifold $M$, introduced by Sue Tolman [**To**]. In this non-Kähler setting, the extended stable sets are not manifolds, or even complex varieties. We find Morse-Thom classes on an unusual extended flow, whose image under $\Phi$ is not convex.

We first sketch Tolman's construction of $M$. It is obtained from two manifolds, each diffeomorphic to $N = \mathbb{C}P^2 \times \mathbb{C}P^1$. This manifold $N$ is a toric variety with an effective Hamiltonian $T^3$ action. We restrict our attention to two different $T^2$ actions on $N$, whose moment polytopes are given in Figure 2 (a) and (b). We then take a symplectic cut of each of these manifolds, along the dotted line shown in the figure. Notice that the symplectic slices, whose moment images are the intersection of the dotted line with the moment polytopes, are equivariantly symplectomorphic. We use the technique of symplectic gluing to glue the bottom half of Figure 2 (a) and the top half of Figure 2 (b) to construct $M$, whose moment polytope is shown in Figure 2 (c).

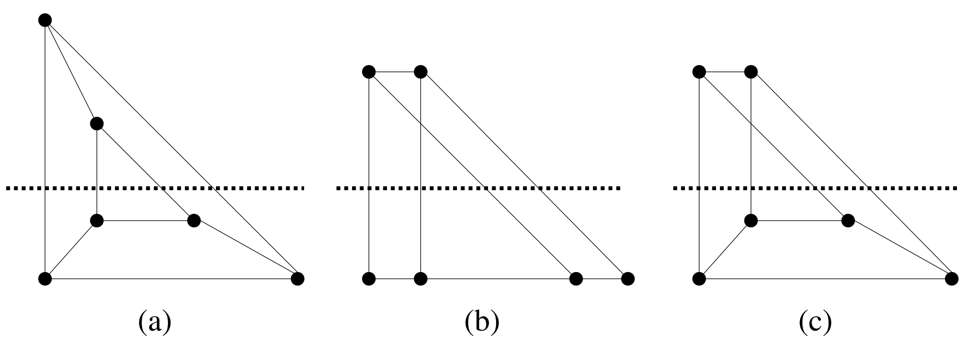


**Figure 2.** The non-Kähler manifold $M$ with moment polytope in (c) is achieved as a symplectic gluing of the bottom half of (a) to the top half of (b).

There is a Hamiltonian $T^2$ action on $M$, with isolated fixed points. The moment polytope is given in Figure 2 (c), and the vertices of

[2]Both the Jeffrey-Kirwan residue and the Guillemin-Kalkman residue depend on a set of choices: for Jeffrey-Kirwan one chooses the cone $\Lambda$, while for Guillemin-Kalkman one chooses a "dendrite".

this graph correspond to the fixed points. The restriction of a cohomology class $\alpha \in H_T^*(M)$ to a fixed point $F \in M^T$ is an element of $H_T^*(pt) \cong \mathbb{C}[x, y]$ (see Theorem 2.1). Thus, we may describe a class in the equivariant cohomology of $M$ by giving a polynomial in two variables at each vertex of the moment polytope.

In Figure 3, we label two points $p$ and $q$ which are regular values of the moment map for $T$ acting on $M$. They are in different chambers of the moment polytope, and hence by the main theorem, the kernel of the Kirwan map is different for each of these values. We may apply the techniques from the proof of the main theorem to construct equivariant cohomology classes distinguishing these chambers. In Figure 3 (a), we demonstrate a class $\alpha$ in $I_q$ but not in $I_p$, and in (b), we show a class $\beta$ in $I_p$, not in $I_q$. These classes distinguish these two ideals.

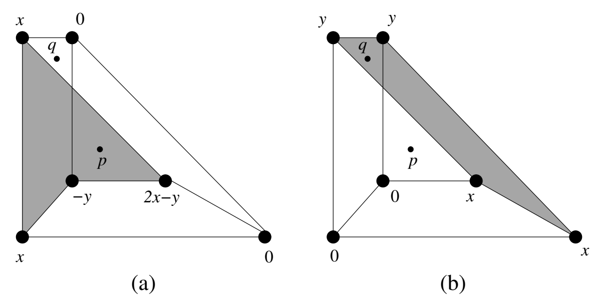


**Figure 3.** Figure (a) shows a class $\alpha \in I_q$ such that $\alpha \notin I_p$, and (b) shows a class in $\beta \in I_p$ such that $\beta \notin I_q$.

The shaded region represents the image under the moment map of the flow-out of the support of each class, for a given choice of $f$. The fact that the region in (a) is non-convex reflects the fact that the manifold $M$ is non-Kählerizable. We may contrast this to the setting given in Section 4.1, where the classes constructed that distinguish among ideals are supported on permuted Schubert varieties. While these varieties may not be manifolds, they are complex, and hence have convex moment image.

One important point about these extended flows is the following. Suppose $p$ and $q$ are in distinct chambers, and a wall $W$ adjacent to the chamber $C_p$ separates them, such that $p \in H^+$ and $q \in H^-$, where $H$ is the hyperplane containing $W$. Let the class $\alpha$ be a Morse-Thom class (associated to a point on the wall $W$) constructed by Lemma 3.5, so that $\alpha \in I_q$ but not in $I_p$. When the extended stable set is convex, then Corollary 2.12 implies that $\alpha \in I_r$ for any $r \in H^-$ (the negative half space). However, without this convexity requirement, $\alpha$ is not necessarily in $I_r$ for all $r \in H^-$. Let $p, q$ and $r$ be as indicated in Figure 4. Let $W$ be the wall slightly to the left of $q$, as indicated. The class $\alpha$ supported on the shaded region will be in $I_q$ and not in $I_p$, by the argument given in Figure 3. However, it is clear that $r \in H^-$ and yet $\alpha$ is not in $I_r$ since indeed $\alpha$ is also an equivariant Euler class associated with a wall close to $r$. This example demonstrates the need for the extended flows introduced in Section 2.2.

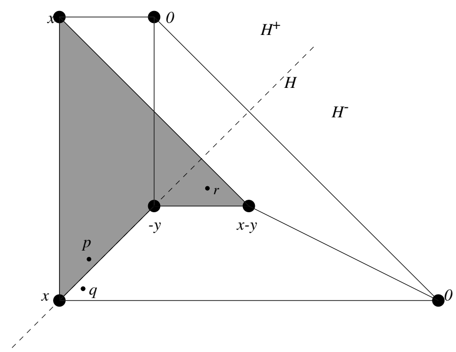


**Figure 4**

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