Japan J. Indust. Appl. Math., 25 (2008), 117–147 Area ⟨1⟩

Dynamics of Front Solutions in a Specific Reaction-Diffusion System in One Dimension

Shin-Ichiro Ei*, Hideo Ikeda† and Takeyuki Kawana‡

**Faculty of Mathematics, Kyushu University*
Fukuoka 810–8560, Japan
E-mail: ichiro@math.kyushu-u.ac.jp
†Department of Mathematics, University of Toyama
Toyama 930–8555, Japan
E-mail: ikeda@sci.u-toyama.ac.jp
‡Graduate School of Integrated Science
Yokohama City University
Yokohama 236–0027, Japan

Received April 25, 2007
Revised October 15, 2007

In this paper, two component reaction-diffusion systems with a specific bistable nonlinearity are concerned. The systems have the bifurcation structure of pitch-fork type of traveling front solutions with opposite velocities, which is rigorously proved and the ordinary differential equations describing the dynamics of such traveling front solutions are also derived explicitly. It enables us to know rigorously precise information on the dynamics of traveling front solutions. As an application of this result, the imperfection structure under small perturbations and the dynamics of traveling front solutions on heterogeneous media are discussed.

Key words: reaction-diffusion systems, traveling fronts, pitch-fork bifurcation, heterogeneity

1. Introduction

Reaction-diffusion systems have been widely treated to describe various phenomena such as chemical reaction and pattern formation in biology. The typical examples of them in 1-dimensional problem are the two-components activator-inhibitor systems of the form

$$\begin{cases} \varepsilon\tau u_t = \varepsilon^2 u_{xx} + f(u,v), \\ v_t = Dv_{xx} + g(u,v), \end{cases} \quad t > 0, \quad x \in \boldsymbol{R}, \tag{1.1}$$

where ε, τ, D are all positive, and $f(u,v) = u(1-u)(u-a) - v$ and $g(u,v) = u - \gamma_1 v + \gamma_2$ for constants $0 < a < \frac{1}{2}$, $\gamma_1 > 0$ and γ_2. The type of this equation has been extensively studied to consider localized patterns such as pulse and front solutions ([3], [12]). When the nonlinearity of (1.1) is mono-stable with suitable conditions, (1.1) has traveling pulse solutions (e.g. [10]). On the other hand, when the nonlinearity is bistable, traveling front solutions exist in (1.1).

Here we assume (1.1) is bistable and odd symmetric with respect to a middle unstable equilibrium, that is, f and g may be replaced by

$$f(u,v) = u - u^3 - v, \quad g(u,v) = u - \gamma v \tag{1.2}$$

with a positive constant $\gamma > \gamma_0 \equiv \frac{2}{3\sqrt{3}}$. For small $\varepsilon > 0$, there exist a stable stationary front solution for large τ and two stable traveling front solutions with opposite velocities for small τ. Thus, we can observe bifurcation phenomena with respect to τ ([6], [13] and [18]). In [11], Kokubu et al. show the bifurcation of the stationary front into traveling fronts is supercritical as τ decreases (see Fig. 5.2) when γ is sufficiently large by using the method of Lyapunov–Schmidt. This situation corresponds to the case that v is almost flat, that is, (1.1) with (1.2) will be close to a single equation. And they do not discuss the dynamics of these solutions. After that a similar problem is studied by several authors. In particular, Hagberg and Meron in the series of papers [7], [8] and [9] discuss a front bifurcation as stated above using matched asymptotic expansions when D is sufficiently small or γ is sufficiently large and get the same results in [11]. But their analysis is very formal and not quite rigorous.

In this paper, we consider the two-component system of (1.1) with

$$f(u,v) = \left(u+\frac{1}{2}\right)\left(\frac{1}{2}-u\right)\left(u-\frac{1}{2}v\right), \quad g(u,v) = u - v. \tag{1.3}$$

The system (1.1) with (1.3) has two stable equilibria $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$ and one unstable one $(0,0)$. Since the system (1.1) with (1.3) is symmetric with respect to $(0,0)$, this has a front solution with the velocity 0 connecting two stable equilibria $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$. Here we use $g(u,v) = u - v$ in place of $g(u,v) = u - \gamma v$ $\left(\gamma > \frac{1}{2}\right)$ simply, because its analysis is the same as the case $g(u,v) = u - v$. For concrete examples of (1.2) and (1.3), the reader refers [15] and [18]. The construction of the front solution will be stated in Section 2. Let the front solution $S(x)$. Here we note that $S(x)$ exists for any $\tau > 0$ because the stationary solution $S(x)$ is independent of τ in (1.1). We consider $S(x)$ as the trivial solution. Then we can show the bifurcation point is $\tau(\varepsilon) = \tau_0 + O(\varepsilon) = \frac{1}{4\sqrt{2D}} + O(\varepsilon)$ such that traveling front solutions, say $S_\pm(x-ct)$, with velocities c of opposite directions appear and $S(x)$ is unstable for $\tau < \tau(\varepsilon)$. On the other hand, for $\tau > \tau(\varepsilon)$, $S(x)$ is a stable stationary solution and there are no traveling solutions in the neighborhood of $S(x)$. Thus, we find that the bifurcation structure is a super-critical pitchfork type, which is completely determined by eigenfunctions and the adjoint eigenfunctions of the linearized operator with respect to $S(x)$. Furthermore we can give more precise information about the dynamics of front solutions, which means that for $\tau < \tau(\varepsilon)$, the unstable stationary front solution $S(x)$ plays the role of a separator between stable traveling front solutions $S_\pm(x-ct)$.

For the system (1.1) with (1.2) or more general one, we can construct eigenfunctions and adjoint eigenfunctions rigorously (see Section 4). But unfortunately we can not determine the sign of M_1 in Section 2 explicitly, which gives the direction of the bifurcation. But for the system (1.1) with (1.3) we can construct all necessary eigenfunctions and find the signs of M_1 and M_2 (see Theorem 3.3). Then we obtain the complete bifurcation diagram explicitly and front dynamics is also rigorously analyzed according to the method stated in [1]. By virtue of this, the dynamics of traveling front solutions on inhomogeneous media is analytically shown as an application (cf. [5], [14], [16] and [17]). Thus, this paper provides a nice example in which we can calculate all necessary things analytically and explicitly in order to study the bifurcation diagram and the dynamics of solutions.

This paper is organized as follows: First, we construct a stationary front solution in Section 2. In Section 3, some general theories to study the bifurcation structure and the dynamics are explained by according to the method in [1]. We shall construct eigenfunctions and adjoint ones for the system (1.1) with (1.3) in Section 4. In Section 5, important constants M_1 and M_2 which will determine the type of bifurcation structure are calculated. In Section 6, by using the information obtained in Section 5, we analyze the imperfection structure under small perturbations and the dynamics of traveling front solutions on inhomogeneous media.

The results of this paper are available to the study of interaction of front solutions. We will show repulsive interaction and the occurrence of oscillatory dynamics of interacting two fronts in the forthcoming paper [2].

2. Construction of a stationary front solution

In this section, we shall construct a stationary front solution satisfying

$$\begin{cases} 0 = \varepsilon^2 u_{xx} + f(u,v), & x \in \boldsymbol{R}, \\ 0 = D v_{xx} + g(u,v), & \\ (u,v)(\pm\infty) = \left(\mp\dfrac{1}{2}, \mp\dfrac{1}{2}\right), & \end{cases} \tag{2.1}$$

where

$$f(u,v) = \left(u + \frac{1}{2}\right)\left(\frac{1}{2} - u\right)\left(u - \frac{1}{2}v\right), \quad g(u,v) = u - v.$$

We impose $u(0) = 0$ because (2.1) is translation free. Since $f(u,v) = -f(-u,-v)$, $g(u,v) = -g(-u,-v)$ hold, the solution of (2.1) is necessarily an odd function. Hence, it suffices to consider (2.1) only on $\boldsymbol{R}_+ := (0,\infty)$. Thus the problem (2.1) is reduced to

$$\begin{cases} 0 = \varepsilon^2 u_{xx} + f(u,v), & x \in \boldsymbol{R}_+, \\ 0 = D v_{xx} + g(u,v), & \\ (u,v)(0) = (0,0), \quad (u,v)(\infty) = \left(-\dfrac{1}{2}, -\dfrac{1}{2}\right). & \end{cases} \tag{2.2}$$

By using the singular perturbation technique ([4], [6]), we construct the solution of (2.2) as the asymptotic expansions with respect to ε.

First, consider the outer approximations. Let us expand as $u = U_0(x) + \varepsilon U_1(x) + \cdots$, $v = V_0(x) + \varepsilon V_1(x) + \cdots$ and substitute them into (2.2). Then we first obtain from the terms ε^0

$$\begin{cases} f(U_0, V_0) = 0, & x \in \boldsymbol{R}_+, \\ DV_0'' + g(U_0, V_0) = 0, & \\ V_0(0) = 0, \quad V_0(\infty) = -\dfrac{1}{2}. \end{cases} \tag{2.3}$$

Then they are solved as $U_0(x) = -\frac{1}{2}$ and $V_0(x) = -\frac{1}{2}\big(1 - e^{-x/\sqrt{D}}\big)$.

The terms ε^1 lead

$$\begin{cases} -\dfrac{1}{2}(1 + V_0(x))U_1 = 0, & x \in \boldsymbol{R}_+, \\ DV_1'' + U_1 - V_1 = 0, & \\ V_1(0) = V_1(\infty) = 0. \end{cases} \tag{2.4}$$

(2.4) leads $U_1(x) = V_1(x) = 0$.

Next, we consider the inner expansions in the neighborhood of $x = 0$. Let expand by using the following form

$$\begin{cases} u(x) = U_0(x) + \varepsilon U_1(x) + \cdots + W_0(\xi) + \varepsilon W_1(\xi) + \cdots, \\ v(x) = V_0(x) + \varepsilon V_1(x) + \cdots + \varepsilon^2 Y_0(\xi) + \varepsilon^3 Y_1(\xi) + \cdots, \end{cases} \tag{2.5}$$

where $\xi := x/\varepsilon$ and substitute (2.5) into (2.2). Equating the terms of ε^0, we have

$$\begin{cases} \ddot{W}_0 + f(U_0(0) + W_0, 0) = 0, & \xi \in \boldsymbol{R}_+, \\ D\ddot{Y}_0 + W_0 = 0, & \\ W_0(0) = \dfrac{1}{2}, \quad W_0(\infty) = 0, & \\ Y_0(\infty) = \dot{Y}_0(\infty) = 0 \end{cases} \tag{2.6}$$

and the solutions of (2.6)

$$W_0(\xi) = \frac{1}{2}\left(1 - \tanh\frac{\xi}{2\sqrt{2}}\right), \quad Y_0(\xi) = -\frac{1}{D}\int_\xi^\infty \int_z^\infty W_0(s)\,ds\,dz,$$

where $\dot{W}$ means $\frac{dW}{d\xi}$ and so on.

From the terms of ε^1, we have

$$\begin{cases} \ddot{W}_1 + f_u(U_0(0) + W_0, 0)W_1 = -f_v(U_0(0) + W_0)V_0'(0)\xi, & \xi \in \boldsymbol{R}_+, \\ D\ddot{Y}_1 + W_1 = 0, & \\ W_1(0) = W_1(\infty) = 0, & \\ Y_1(\infty) = \dot{Y}_1(\infty) = 0. \end{cases} \tag{2.7}$$

Here we imposed the boundary condition $\dot{Y}_1(\infty) = 0$ in order that a solution $Y_1(\xi)$ must decay to 0 as $\xi \to \infty$. Note that $V_0(x) = V_0(\varepsilon\xi) = \varepsilon V_0'(0)\xi + O(\varepsilon^2)$ holds. Then we have

$$
\begin{aligned}
W_1(\xi) &= -\frac{1}{2}V_0'(0)\dot{W}_0(\xi)\int_0^\xi (\dot{W}_0(z))^{-2}\int_z^\infty \dot{W}_0(s)sW_0(s)(1-W_0(s))\,ds\,dz,\\
Y_1(\xi) &= -\frac{1}{D}\int_\xi^\infty\int_z^\infty W_1(s)\,ds\,dz.
\end{aligned}
$$

Thus, the approximate solution of (2.2), say $(u_0(x;\varepsilon), v_0(x;\varepsilon))$, is given by

$$
\begin{aligned}
u_0(x;\varepsilon) &:= -\frac{1}{2} + W_0(x/\varepsilon) + \varepsilon W_1(x/\varepsilon),\\
v_0(x;\varepsilon) &:= V_0(x) + \varepsilon^2 Y_0(x/\varepsilon) + \varepsilon^3 Y_1(x/\varepsilon).
\end{aligned}
$$

The existence of an exact solution of (2.2) is guaranteed by the singular perturbation technique ([6]).

Theorem 2.1. *For a small positive ε, there exist solutions of (2.2), say $(u^+(x;\varepsilon), v^+(x;\varepsilon))$, such that*

$$
\|u^+(\,\cdot\,;\varepsilon) - u_0(\,\cdot\,;\varepsilon)\|_{X^1_{\rho,\varepsilon}(\boldsymbol{R}_+)} + \|v^+(\,\cdot\,;\varepsilon) - v_0(\,\cdot\,;\varepsilon)\|_{X^1_{\rho,1}(\boldsymbol{R}_+)} = o(\varepsilon)
$$

as $\varepsilon \downarrow 0$, where

$$
X^k_{\rho,\varepsilon}(\boldsymbol{R}_+) := \left\{ u \in C^k(\boldsymbol{R}_+);\ \|u\|_{X^k_{\rho,\varepsilon}(\boldsymbol{R}_+)} := \sum_{j=0}^k \sup_{x\in\boldsymbol{R}_+}\left| e^{\rho x}\left(\varepsilon\frac{d}{dx}\right)^j u(x)\right| < \infty \right\}
$$

($k = 0, 1, 2, \dots$) and $\rho > 0$ is a sufficiently small constant independent of ε.

In [6], we construct traveling front solutions with the velocity c on the whole interval $\boldsymbol{R}$. There the transversal condition is important to match solutions smoothly at the origin. Here we assume f and g being odd symmetric and consider stationary front solutions (with the velocity 0), which implies these solutions are also odd symmetric and there is no necessity to match them smoothly at the origin. As we have seen in the above, the solutions $(u^+(x;\varepsilon), v^+(x;\varepsilon))$ are described by the two different scales x and x/ε. Note that the solutions have a sharp transition layer in a small neighborhood of $x = 0$. Apart from the neighborhood of $x = 0$, outer approximations, which are functions of x, are good ones. But the outer ones do not approximate the solution in the neighborhood of $x = 0$ because the solutions have a sharp transition layer at $x = 0$. Indeed, the slope of $u^+(x;\varepsilon)$ at $x = 0$ becomes $O(1/\varepsilon)$ as $\varepsilon \to 0$. Then we must modify these outer approximations by adding functions of x/ε, which describe a sharp boundary layer at $x = 0$. In the form of the approximate solutions $(u_0(x;\varepsilon), v_0(x;\varepsilon))$, $-1/2$, $V_0(x)$ are outer approximations, and $W_0(x/\varepsilon)$, $W_1(x/\varepsilon)$, $Y_0(x/\varepsilon)$, $Y_1(x/\varepsilon)$ are modified functions.

More precisely, we have the following expression to the solutions $(u^+(x;\varepsilon), v^+(x;\varepsilon))$:

$$\begin{cases} u^+(x;\varepsilon) = -\dfrac{1}{2} + \varepsilon^2 E_1(x) + W_0(x/\varepsilon) + \varepsilon W_1(x/\varepsilon) + \varepsilon^2 E_2(x/\varepsilon), \\ v^+(x;\varepsilon) = V_0(x) + \varepsilon^2 E_3(x) + \varepsilon^2 Y_0(x/\varepsilon) + \varepsilon^3 Y_1(x/\varepsilon) + \varepsilon^4 E_4(x/\varepsilon), \end{cases} \quad x \in [0,\infty),$$

where $E_j(x) = E_j(x;\varepsilon) \in X^2_{\rho_1,1}(\boldsymbol{R}_+)$ uniformly in a small positive ε for some positive constant ρ_1 $(j = 1,2,3,4)$. These expressions are very convenient to calculate L^2-inner product of these functions (see Sections 4 and 5).

Finally, we note that a solution of (2.1) is an odd function. Then we find a solution of (2.1) is represented as

$$(u(x;\varepsilon), v(x;\varepsilon)) = \begin{cases} (u^+(x;\varepsilon), v^+(x;\varepsilon)), & x \in [0,\infty), \\ (-u^+(-x;\varepsilon), -v^+(-x;\varepsilon)), & x \in (-\infty, 0]. \end{cases}$$

3. General theory on the dynamics of front solutions and the bifurcation diagram

In this section, we shall give general theory in order to analyze the dynamics of a single front solution of (1.1) in the neighborhood of a bifurcation point according to [1].

We write (1.1) in the form

$$\boldsymbol{u}_t = \mathcal{L}(\boldsymbol{u};\tau), \tag{3.1}$$

where $\boldsymbol{u} := (u,v)$ and

$$\mathcal{L}(\boldsymbol{u};\tau) := \begin{pmatrix} \dfrac{\varepsilon}{\tau} u_{xx} + \dfrac{1}{\varepsilon\tau} f(u,v) \\ D v_{xx} + g(u,v) \end{pmatrix}.$$

As shown in the previous section, (3.1) has a stationary front solution, say $S(x)$ for any $\tau > 0$. Let $L(\tau)$ be the linearized operator with respect to $S(x)$, that is, $L(\tau) := \mathcal{L}'(S(x);\tau)$. Note that 0 is necessarily the eigenvalue of $L(\tau)$ associated with the eigenfunction S_x. We assume the following:

(H1) There exists $\tau(\varepsilon) > 0$ such that $L := L(\varepsilon) := L(\tau(\varepsilon))$ is degenerate in Jordan block type. Namely, there exists a function $\Psi(x)$ satisfying $L\Psi = -S_x$ and the generalized eigenspace with respect to 0 is spanned by S_x and Ψ.

(H2) Let Σ_0 be the spectrum of L. Then Σ_0 is given by $\Sigma_0 = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1 := \{0\}$ and $\Sigma_2 \subset \{z \in \boldsymbol{C};\ \mathrm{Re}(z) < -\gamma_0\}$ for a constant $\gamma_0 > 0$.
Let $X := L^2(\boldsymbol{R})$ and Q, R be the projections corresponding to the spectral sets Σ_1 and Σ_2, respectively.

(H3) $X_0 := QX$ is spanned by S_x and Ψ, that is, $X_0 = \mathrm{span}\{S_x, \Psi\}$.

(H1) will be shown in Section 4.1, and (H2) is given in [13] for more general nonlinearities f and g. Here we note that (H1) also holds for more general nonlinearities f and g as in [13]. (H3) emphasizes that the dimension of the generalized eigenspace associated with 0 eigenvalue is just 2, which holds for our problems by solving eigenvalue problems with respect to $L(\tau(\varepsilon))$ in Section 4.1. Let L^* be the adjoint operator of L. L^* has also the same properties. That is, there exist Φ^* and Ψ^* satisfying $L^*\Phi^* = 0$ and $L^*\Psi^* = -\Phi^*$. These functions Φ^* and Ψ^* are constructed in Sections 4.2 and 4.3, respectively.

Proposition 3.1. *These eigenfunctions are uniquely determined by the normalization* $\langle \Psi, S_x\rangle_2 = 0$, $\langle S_x, \Psi^*\rangle_2 = 1$ *and* $\langle \Psi, \Psi^*\rangle_2 = 0$, *where* $\langle\,\cdot\,,\,\cdot\,\rangle_2$ *denotes the inner product in* X. *Note that* $\langle \Psi, \Phi^*\rangle_2 = 1$ *and* $\langle S_x, \Phi^*\rangle_2 = 0$ *hold.*

This is shown in Proposition 4.1 in [1]. In the problem (1.1) with (1.3), we note that $\Psi(x)$, $\Phi^*(x)$ and $\Psi^*(x)$ can be taken as even functions by the symmetry of f and g. The normalization stated in Proposition 3.1 will be done in Section 4.4.

We consider (3.1) in the neighborhood of $\tau = \tau(\varepsilon)$ and put $\tau = \tau(\varepsilon) + \eta$ for small η. Then (3.1) is written as

$$\boldsymbol{u}_t = \mathcal{L}(\boldsymbol{u}) + \eta G(\boldsymbol{u};\eta), \tag{3.2}$$

where $\mathcal{L}(\boldsymbol{u}) := \mathcal{L}(\boldsymbol{u}, \tau(\varepsilon))$ and $\eta G(\boldsymbol{u};\eta) := \mathcal{L}(\boldsymbol{u};\tau(\varepsilon)+\eta) - \mathcal{L}(\boldsymbol{u})$.

Define $\Xi(x;r) := S(x) + r\Psi(x)$ and $(\kappa(l)\boldsymbol{u})(x) := \boldsymbol{u}(x-l)$ and $M(r^*) := \{\kappa(l)\Xi(\,\cdot\,;r);\ l \in \boldsymbol{R},\ |r| < r^*\}$.

Theorem 3.2. *For a small positive* ε, *there exist positive constants* C_0, r^*, η^* *and a neighborhood* U_N *of* $M(r^*)$ *such that if the initial data* $\boldsymbol{u}(0) \in U_N$, *then there exist functions* $l(t)$, $r(t)$ *such that the solution* $\boldsymbol{u}$ *of* (3.1) *satisfies*

$$\|\boldsymbol{u}(t) - \kappa(l(t))\Xi(\,\cdot\,;r(t))\|_\infty \le C_0(|r(t)|^2 + |\eta|)$$

for $|\eta| < \eta^*$ *as long as* $|r(t)| < r^*$. $l(t)$ *and* $r(t)$ *are estimated as*

$$\dot{l} = r + O(|r|^2 + |\eta|^2), \quad \dot{r} = O(|r|^2 + |\eta|^2).$$

Theorem 3.2 indicates that the movement of a front like solution $\boldsymbol{u}$ is essentially governed by $l(t)$, $r(t)$ and $r(t)$ gives the velocity. Here we note that the Landau symbol O means to be estimated by constants independent of small r and η, which may depend on ε. We take in this paper η to be sufficiently small by fixing ε small enough.

The explicit form of the equations of $l(t)$ and $r(t)$ is given as follows. Define functions $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ in $X_1 := RX$ by

$$L\boldsymbol{\zeta}_1 + \frac{1}{2}F''(S(x))\Psi\cdot\Psi + \Psi_x = 0, \quad L\boldsymbol{\zeta}_2 + G_0(S(x)) = 0$$

and constants

$$\begin{cases} M_1 := \langle \partial_x \boldsymbol{\zeta}_1, \Phi^* \rangle_2 + \langle F''(S)\Psi \cdot \boldsymbol{\zeta}_1, \Phi^* \rangle_2 + \dfrac{1}{6}\langle F'''(S)\Psi^3, \Phi^* \rangle_2, \\ M_2 := \langle \partial_x \boldsymbol{\zeta}_2, \Phi^* \rangle_2 + \langle F''(S)\Psi \cdot \boldsymbol{\zeta}_2, \Phi^* \rangle_2 + \langle G_0'(S)\Psi, \Phi^* \rangle_2, \end{cases} \tag{3.3}$$

where

$$F(\boldsymbol{u}) = \begin{pmatrix} \dfrac{1}{\varepsilon\tau(\varepsilon)} f(u(x), v(x)) \\ g(u(x), v(x)) \end{pmatrix},$$

$$G(\boldsymbol{u}; \eta) = \begin{pmatrix} -\dfrac{1}{\varepsilon(\tau(\varepsilon)+\theta\eta)^2}\{\varepsilon^2 u_{xx} + f(u(x), v(x))\} \\ 0 \end{pmatrix}$$

for $0 < \theta < 1$ and

$$G_0(\boldsymbol{u}) = G(\boldsymbol{u}; 0) = \begin{pmatrix} -\dfrac{1}{\varepsilon\tau^2(\varepsilon)}\{\varepsilon^2 u_{xx} + f(u(x), v(x))\} \\ 0 \end{pmatrix}$$

for $\boldsymbol{u} = \boldsymbol{u}(x) = (u(x), v(x))$.

Theorem 3.3. *Let $l(t)$ and $r(t)$ be the functions stated in* Theorem 3.2. *Then for $|\eta| < \eta^*$,*

$$\begin{aligned} \dot{l} &= r + O\big(|r|^3 + |\eta|^{3/2}\big), \\ \dot{r} &= K(r; \eta) + O(|r|^4 + |\eta|^2) \end{aligned}$$

hold as long as $|r(t)| < r^*$, where $K(r; \eta) := M_1 r^3 + M_2 \eta r$.

4. Construction of eigenfunctions

In this section, we construct explicitly eigenfunctions Ψ, Φ^* and Ψ^* stated in the previous section for the system (1.1) with (1.3) together with the bifurcation point $\tau(\varepsilon)$.

Let $S(x) := {}^t(u(x;\varepsilon), v(x;\varepsilon))$ be the stationary front solution constructed in Section 2. Then L and L^* are

$$\begin{aligned} L &= \begin{pmatrix} \dfrac{\varepsilon}{\tau(\varepsilon)}\dfrac{d^2}{dx^2} + \dfrac{1}{\varepsilon\tau(\varepsilon)} f_u(u, v) & \dfrac{1}{\varepsilon\tau(\varepsilon)} f_v(u, v) \\ g_u(u, v) & D\dfrac{d^2}{dx^2} + g_v(u, v) \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{\varepsilon}{\tau(\varepsilon)}\dfrac{d^2}{dx^2} + \dfrac{1}{\varepsilon\tau(\varepsilon)}\left(-3u^2 + uv + \dfrac{1}{4}\right) & \dfrac{1}{\varepsilon\tau(\varepsilon)}\left(\dfrac{1}{2}u^2 - \dfrac{1}{8}\right) \\ 1 & D\dfrac{d^2}{dx^2} - 1 \end{pmatrix} \end{aligned}$$

and

$$L^* = \begin{pmatrix} \dfrac{\varepsilon}{\tau(\varepsilon)}\dfrac{d^2}{dx^2} + \dfrac{1}{\varepsilon\tau(\varepsilon)}\Big(-3u^2 + uv + \dfrac{1}{4}\Big) & 1 \\ \dfrac{1}{\varepsilon\tau(\varepsilon)}\Big(\dfrac{1}{2}u^2 - \dfrac{1}{8}\Big) & D\dfrac{d^2}{dx^2} - 1 \end{pmatrix},$$

where $u = u(x;\varepsilon)$ and $v = v(x;\varepsilon)$.

In the following subsections, eigenfunctions will be constructed by asymptotic expansions. As eigenfunctions are not unique, we construct them with the same order of the trivial eigenfunction $S_x(x)$, say $O\big(\frac{1}{\varepsilon}\big)$. Though they are all approximations, the existence of rigorous eigenfunctions with respect to the constructed approximate functions is shown by the standard implicit function theorem. As we stated in Section 3, Ψ, Φ^* and Ψ^* can be taken as even functions. Then we construct them only on the half interval $\boldsymbol{R}_+$.

4.1. Construction of $\boldsymbol{\Psi}$

We first construct the eigenfunction Ψ satisfying $L\Psi = -S_x$. Let $\Psi(x) = {}^t(\psi_1(x), \psi_2(x))$. Then ψ_1 and ψ_2 satisfy

$$\begin{cases} \varepsilon^2\dfrac{d^2}{dx^2}\psi_1 + \Big(-3u^2 + uv + \dfrac{1}{4}\Big)\psi_1 + \Big(\dfrac{1}{2}u^2 - \dfrac{1}{8}\Big)\psi_2 = -\varepsilon\tau(\varepsilon)u_x, \\ D\dfrac{d^2}{dx^2}\psi_2 + \psi_1 - \psi_2 = -v_x \end{cases} \tag{4.1}$$

where $u = u(x;\varepsilon)$ and $v = v(x;\varepsilon)$. Since $\Psi(x)$ is an even function, it suffices to consider (4.1) in $\boldsymbol{R}_+$ with the boundary conditions

$$\frac{d\psi_1}{dx}(0) = 0, \quad \psi_1(\infty) = 0, \quad \frac{d\psi_2}{dx}(0) = 0, \quad \psi_2(\infty) = 0. \tag{4.2}$$

First, let us consider outer solutions. Expanding $\psi_j(x) = \psi_j^0(x) + \varepsilon\psi_j^1(x) + \cdots$ $(j = 1, 2)$, $\tau(\varepsilon) = \tau_0 + O(\varepsilon)$ and substituting them into (4.1) with (4.2), we have

$$\begin{cases} -\dfrac{1}{2}(1 + V_0(x))\psi_1^0 = 0, \\ D\dfrac{d^2\psi_2^0}{dx^2} + \psi_1^0 - \psi_2^0 = \dfrac{1}{2\sqrt{D}}e^{-\frac{x}{\sqrt{D}}}, \quad x \in \boldsymbol{R}_+, \\ \psi_2^0(0) = A_0, \quad \psi_2^0(\infty) = 0, \end{cases}$$

with an unknown constant A_0 from the coefficients of ε^0. A_0 is determined later. Here we note that the front solution $S(x)$ is approximated by $(u_0(x;\varepsilon), v_0(x;\varepsilon))$ as in Theorem 2.1. Solving the above equations, we have

$$\psi_1^0(x) = 0, \quad \psi_2^0(x) = \Big(A_0 - \frac{x}{4D}\Big)e^{-\frac{x}{\sqrt{D}}}. \tag{4.3}$$

Similarly, we have the equation of $\psi_j^1(x)$ as

$$\begin{cases} -\dfrac{1}{2}(1+V_0(x))\psi_1^1 = 0, \\ D\dfrac{d^2\psi_2^0}{dx^2} + \psi_1^1 - \psi_2^1 = 0, \quad x \in \boldsymbol{R}_+, \\ \psi_2^1(0) = A_1, \quad \psi_2^1(\infty) = 0 \end{cases}$$

with an unknown constant A_1 and the solution

$$\psi_1^1(x) = 0, \quad \psi_2^1(x) = A_1 e^{-\frac{x}{\sqrt{D}}}. \tag{4.4}$$

Next, we consider inner approximations. Expand

$$\begin{cases} \psi_1(x) = \psi_1^0(x) + \varepsilon\psi_1^1(x) + \cdots + \dfrac{1}{\varepsilon}p_0\Big(\dfrac{x}{\varepsilon}\Big) + p_1\Big(\dfrac{x}{\varepsilon}\Big) + \cdots, \\ \psi_2(x) = \psi_2^0(x) + \varepsilon\psi_2^1(x) + \cdots + \varepsilon q_0\Big(\dfrac{x}{\varepsilon}\Big) + \varepsilon^2 q_1\Big(\dfrac{x}{\varepsilon}\Big) + \cdots, \end{cases}$$

and substitute into (4.1) with (4.2). Then we get the equations of p_j and q_j $(j = 1, 2)$ as follows $(\xi = x/\varepsilon)$.

The equation of p_0, q_0 is derived from the coefficients of ε^{-1} as

$$\begin{cases} \ddot{p}_0 + \Big(-3W_0^2 + 3W_0 - \dfrac{1}{2}\Big)p_0 = 0, \qquad \xi \in \boldsymbol{R}_+, \\ D\ddot{q}_0 + p_0 = 0, \\ \dot{p}_0(0) = 0, \quad p_0(\infty) = 0, \\ q_0(\infty) = 0, \quad \dot{q}_0(\infty) = 0. \end{cases}$$

One solution of the above equation is

$$p_0(\xi) = \dot{W}_0(\xi), \quad q_0(\xi) = -\frac{1}{D}\int_\xi^\infty \int_z^\infty p_0(s)\,ds\,dz. \tag{4.5}$$

The boundary condition at $x = 0$ stated in (4.2) requires

$$\frac{d\psi_2^0}{dx}(0) + \dot{q}_0(0) = 0$$

and then A_0 is determined by $A_0 = -\frac{3}{4\sqrt{D}}$ because $\dot{q}(0) = \frac{1}{D}\int_0^\infty p_0(s)\,ds = \frac{1}{D}\int_0^\infty \dot{W}_0(s)\,ds = -\frac{1}{2D}$.

We have the equation from the terms of ε^0 as

$$\begin{cases} \ddot{p}_1 + \left(-3W_0^2 + 3W_0 - \dfrac{1}{2}\right)p_1 = r(\xi;\tau_0), & \xi \in \boldsymbol{R}_+, \\ D\ddot{q}_1 + p_1 = 0, & \\ \dot{p}_1(0) = 0, \quad p_1(\infty) = 0, & \\ q_1(\infty) = 0, \quad \dot{q}_1(\infty) = 0, & \end{cases} \tag{4.6}$$

where $r(\xi;\tau_0) := -\tau_0\dot{W}_0 + \frac{3}{8\sqrt{D}}W_0(W_0-1) + \frac{\xi}{2\sqrt{D}}\dot{W}_0\big(-\frac{1}{2}+W_0\big) + 6W_1\big(W_0-\frac{1}{2}\big)\dot{W}_0$.

The existence of the solution p_1 in (4.6) requires the solvability condition

$$\int_0^\infty \dot{W}_0 r(\xi;\tau_0)\,d\xi = 0. \tag{4.7}$$

The lowest order τ_0 of the bifurcation point $\tau(\varepsilon)$ is determined by (4.7). We first obtain the value of τ_0. From (2.7), W_1 satisfies the equation

$$\begin{cases} \ddot{W}_1 + \left(-3W_0^2 + 3W_0 - \dfrac{1}{2}\right)W_1 = -\dfrac{\xi}{4\sqrt{D}}W_0(1-W_0), \quad \xi \in \boldsymbol{R}_+, \\ W_1(0) = W_1(\infty) = 0. \end{cases} \tag{4.8}$$

Differentiating (4.8) with respect to ξ and putting $q := \dot{W}_1$, we have

$$\begin{cases} \ddot{q} + \left(-3W_0^2 + 3W_0 - \dfrac{1}{2}\right)q = r_1(\xi), \quad \xi \in \boldsymbol{R}_+, \\ \dot{q}(0) = q(\infty) = 0. \end{cases} \tag{4.9}$$

where $r_1(\xi) := 6W_1\big(W_0-\frac{1}{2}\big)\dot{W}_0 - \frac{1}{4\sqrt{D}}W_0(1-W_0) + \frac{\xi}{2\sqrt{D}}\big(W_0-\frac{1}{2}\big)\dot{W}_0$. Since $\dot{W}_0$ is a solution of the homogeneous equation of (4.9), the solvability condition

$$\int_0^\infty r_1(\xi)\dot{W}_0\,d\xi = 0 \tag{4.10}$$

has to hold for the existence of q in (4.9). (4.10) leads to

$$\begin{aligned} 6\int_0^\infty W_1\left(W_0 - \frac{1}{2}\right)(\dot{W}_0)^2\,d\xi &= \frac{1}{4\sqrt{D}}\int_0^\infty W_0(1-W_0)\dot{W}_0\,d\xi \\ &\quad - \frac{1}{2\sqrt{D}}\int_0^\infty \xi\left(W_0 - \frac{1}{2}\right)(\dot{W}_0)^2\,d\xi \\ &= \frac{1}{4\sqrt{D}} \times \left(-\frac{1}{12}\right) - \frac{1}{2\sqrt{D}} \times \left(-\frac{1}{48}\right) \\ &= -\frac{1}{96\sqrt{D}} \end{aligned}$$

because $W_0(\xi) = \frac{1}{1+e^{\xi/\sqrt{2}}}$. By using this calculation for (4.7), we have

$$\begin{aligned}
0 &= -\tau_0 \int_0^\infty (\dot{W}_0)^2 \, d\xi + \frac{3}{8\sqrt{D}} \int_0^\infty W_0(W_0-1)\dot{W}_0 \, d\xi \\
&\quad + \frac{1}{2\sqrt{D}} \int_0^\infty \xi \Big(W_0 - \frac{1}{2} \Big) (\dot{W}_0)^2 \, d\xi + 6 \int_0^\infty W_1 \Big(W_0 - \frac{1}{2} \Big) (\dot{W}_0)^2 \, d\xi \\
&= -\frac{\sqrt{2}}{24}\tau_0 + \frac{3}{8\sqrt{D}} \times \frac{1}{12} + \frac{1}{2\sqrt{D}} \Big(-\frac{1}{48} \Big) - \frac{1}{96\sqrt{D}} \\
&= -\frac{\sqrt{2}}{24}\tau_0 + \frac{1}{96\sqrt{D}},
\end{aligned}$$

which leads to

$$\tau_0 = \frac{1}{4\sqrt{2D}}. \tag{4.11}$$

Note that $\sqrt{2}\dot{W}_0 = W_0(W_0-1)$. We find that $r(\xi;\tau_0) = r_1(\xi)$ for $\xi \in \boldsymbol{R}_+$. Compare (4.6) with (4.9), we can take $p_1(\xi) = q(\xi) = \dot{W}_1(\xi)$. Then q_1 is given by p_1 as

$$q_1(\xi) = -\frac{1}{D} \int_\xi^\infty \int_z^\infty p_1(s) \, ds \, dz = \frac{1}{D} \int_\xi^\infty W_1(z) \, dz.$$

Similarly $\frac{d\psi_2^1}{dx}(0) + \dot{q}_1(0) = 0$ has to hold by the boundary condition (4.2). Then the constant A_1 is determined by $A_1 = \frac{1}{\sqrt{D}} \int_0^\infty p_1(s) \, ds = -\frac{1}{\sqrt{D}} W_1(0) = 0$.

Thus, we can construct eigenfunctions of (4.1), (4.2) in $\boldsymbol{R}_+$ together with $\tau(\varepsilon)$ as

$$\begin{cases}
\tau(\varepsilon) = \tau_0 + O(\varepsilon), \\
\psi_1(x;\varepsilon) = \varepsilon^2 E_1(x) + \dfrac{1}{\varepsilon} p_0\Big(\dfrac{x}{\varepsilon}\Big) + p_1\Big(\dfrac{x}{\varepsilon}\Big) + \varepsilon E_2(x/\varepsilon), \\
\psi_2(x;\varepsilon) = \psi_2^0(x) + \varepsilon^2 E_3(x) + \varepsilon q_0\Big(\dfrac{x}{\varepsilon}\Big) + \varepsilon^2 q_1\Big(\dfrac{x}{\varepsilon}\Big) + \varepsilon^3 E_4(x/\varepsilon),
\end{cases}$$

for $x \in [0,\infty)$ which are rigorously proved by the implicit function theorem as in Theorem 2.1, where $E_j(x) = E_j(x;\varepsilon) \in X^2_{\rho_1,1}(\boldsymbol{R}_+)$ uniformly in a small positive ε for some positive constant ρ_1 $(j = 1,2,3,4)$. (4.1) is an eigenvalue problem with unknown value $\tau(\varepsilon)$. First, we construct solutions (ψ_1, ψ_2) for arbitrarily fixed τ and then using the relation $\langle \varPsi, S_x \rangle_2 = 0$ in Proposition 3.1 we can determine the eigenvalue $\tau(\varepsilon)$ uniquely (refer the relation (4.7) to determine τ_0). Since the function $(\psi_1(x;\varepsilon), \psi_2(x;\varepsilon))$ is an even function, we can easily extend this into the whole interval $\boldsymbol{R}$. This shows (H1) in Section 3.

4.2. Construction of Φ^*

In this subsection, we construct an eigenfunction Φ^* satisfying $L^*\Phi^* = 0$. Since Φ^* is an even function, the problem becomes

$$\begin{cases} \varepsilon^2 \dfrac{d^2}{dx^2}\phi_1^* + \left(-3u^2 + uv + \dfrac{1}{4}\right)\phi_1^* + \varepsilon\tau(\varepsilon)\phi_2^* = 0, \\ \varepsilon\tau(\varepsilon) D \dfrac{d^2}{dx^2}\phi_2^* + \left(\dfrac{1}{2}u^2 - \dfrac{1}{8}\right)\phi_1^* - \varepsilon\tau(\varepsilon)\phi_2^* = 0, \\ \dfrac{d\phi_1^*}{dx}(0) = 0, \quad \phi_1^*(\infty) = 0, \\ \dfrac{d\phi_2^*}{dx}(0) = 0, \quad \phi_2^*(\infty) = 0, \end{cases} \tag{4.12}$$

where $u = u(x;\varepsilon)$, $v = v(x;\varepsilon)$ and $\Phi^*(x) = {}^t(\phi_1^*(x), \phi_2^*(x))$.

Similarly to the previous subsection, we first expand $\Phi^*(x)$ as outer solutions

$$\begin{cases} \phi_1^*(x) = \phi_1^{*,0}(x) + \varepsilon\phi_1^{*,1}(x) + \cdots, \\ \phi_2^*(x) = \phi_2^{*,0}(x) + \varepsilon\phi_2^{*,1}(x) + \cdots \end{cases}$$

and substitute them into (4.12). Then we have

$$\phi_1^{*,0}(x) = 0, \quad \phi_2^{*,0}(x) = A_0^* e^{-\frac{x}{\sqrt{D}}}, \quad \phi_1^{*,1}(x) = 2\tau_0\phi_2^{*,0}(x)/(1 + V_0(x)), \tag{4.13}$$

where A_0^* is an arbitrary constant to be fixed later. $\phi_2^{*,1}(x)$ is also determined similarly.

Next we expand $\Phi^*(x)$ as inner solutions

$$\begin{cases} \phi_1^*(x) = \varepsilon\phi_1^{*,1}(x) + \cdots + \dfrac{1}{\varepsilon}p_0^*\left(\dfrac{x}{\varepsilon}\right) + p_1^*\left(\dfrac{x}{\varepsilon}\right) + \cdots, \\ \phi_2^*(x) = \phi_2^{*,0}(x) + \varepsilon\phi_2^{*,1}(x) + \cdots + q_0^*\left(\dfrac{x}{\varepsilon}\right) + \varepsilon q_1^*\left(\dfrac{x}{\varepsilon}\right) + \cdots \end{cases}$$

and substitute them into (4.12). Then we have from the coefficients of ε^{-1} terms

$$\begin{cases} \ddot{p}_0^* + \left(-3W_0^2 + 3W_0 - \dfrac{1}{2}\right)p_0^* = 0, \\ \tau_0 D\ddot{q}_0^* + \left(\dfrac{1}{2}\left(W_0 - \dfrac{1}{2}\right)^2 - \dfrac{1}{8}\right)p_0^* = 0, \\ \dot{p}_0^*(0) = p_0^*(\infty) = 0, \\ \dot{q}_0^*(0) = q_0^*(\infty) = 0, \end{cases} \quad \xi \in \boldsymbol{R}_+, \tag{4.14}$$

($\xi = x/\varepsilon$) and the solution $p_0^*(\xi) = B_0^*\dot{W}_0(\xi)$ for an arbitrary constant B_0^*. From the equation of q of (4.14), we have $B_0^* = 0$, which leads $q_0^*(\xi) = 0$ and hence $p_0^*(\xi) = 0$.

The equations from terms of ε^0 are

$$\begin{cases} \ddot{p}_1^* + \left(-3W_0^2 + 3W_0 - \dfrac{1}{2}\right)p_1^* = 0, \\ \tau_0 D\ddot{q}_1^* + \left(\dfrac{1}{2}\left(W_0 - \dfrac{1}{2}\right)^2 - \dfrac{1}{8}\right)p_1^* = 0, \\ \dot{p}_1^*(0) = p_1^*(\infty) = 0, \\ q_1^*(\infty) = \dot{q}_1^*(\infty) = 0. \end{cases} \quad \xi \in \boldsymbol{R}_+, \tag{4.15}$$

p_1^* is given by $p_1^*(\xi) = B_1^*\dot{W}_0(\xi)$ with an arbitrary constant B_1^*. We take $B_1^* = 1$. Then q_1^* is solved as

$$q_1^*(\xi) = \frac{1}{2\tau_0 D}\int_\xi^\infty \left(\frac{W_0^3(s)}{3} - \frac{W_0^2(s)}{2}\right) ds.$$

The condition $\frac{d\phi_2^{*,0}}{dx}(0) + \dot{q}_1^*(0) = 0$ leads $A_0^* = \frac{1}{24\tau_0\sqrt{D}}$.

Thus, we have Φ^* as

$$\begin{cases} \phi_1^*(x;\varepsilon) = \varepsilon\phi_1^{*,1}(x) + \varepsilon^2 E_1(x) + p_1^*\left(\dfrac{x}{\varepsilon}\right) + \varepsilon E_2(x/\varepsilon), \\ \phi_2^*(x;\varepsilon) = \phi_2^{*,0}(x) + \varepsilon\phi_2^{*,1}(x) + \varepsilon^2 E_3(x) + \varepsilon q_1^*\left(\dfrac{x}{\varepsilon}\right) + \varepsilon^2 E_4(x/\varepsilon), \end{cases} \quad x \in [0,\infty),$$

where $E_j(x) = E_j(x;\varepsilon) \in X^2_{\rho_1,1}(\boldsymbol{R}_+)$ uniformly in a small positive ε for some positive constant ρ_1 $(j = 1,2,3,4)$. We can easily extend them into the whole interval $\boldsymbol{R}$.

4.3. Construction of Ψ^*

Let us obtain the eigenfunction $\Psi^* = {}^t(\psi_1^*, \psi_2^*)$ satisfying $L^*\Psi^* = -\Phi^*$, which is the solution of

$$\begin{cases} \varepsilon^2\dfrac{d^2\psi_1^*}{dx^2} + \left(-3u^2 + uv + \dfrac{1}{4}\right)\psi_1^* + \varepsilon\tau(\varepsilon)\psi_2^* = -\varepsilon\tau(\varepsilon)\phi_1^*, \\ \varepsilon\tau(\varepsilon)D\dfrac{d^2\psi_2^*}{dx^2} + \left(\dfrac{1}{2}u^2 - \dfrac{1}{8}\right)\psi_1^* - \varepsilon\tau(\varepsilon)\psi_2^* = -\varepsilon\tau(\varepsilon)\phi_2^*. \end{cases} \tag{4.16}$$

Since Ψ^* is an even function, we impose the boundary condition

$$\frac{d\psi_1^*}{dx}(0) = \psi_1^*(\infty) = 0, \quad \frac{d\psi_2^*}{dx}(0) = \psi_2^*(\infty) = 0 \tag{4.17}$$

and consider (4.16) on $\boldsymbol{R}_+$.

First, we construct outer approximations. Let expand

$$\begin{cases} \psi_1^*(x) = \psi_1^{*,0}(x) + \varepsilon\psi_1^{*,1}(x) + \cdots, \\ \psi_2^*(x) = \psi_2^{*,0}(x) + \varepsilon\psi_2^{*,1}(x) + \cdots \end{cases}$$

and plug them into (4.16).

The equation with respect to ε^0 is

$$-\frac{1}{2}(1+V_0(x))\psi_1^{*,0}=0,$$

which leads to $\psi_1^{*,0}(x)=0$. Then the equation with respect to ε^1 is

$$\begin{cases} -\dfrac{1}{2}(1+V_0(x))\psi_1^{*,1}+\tau_0\psi_2^{*,0}=0, & \\ D\dfrac{d^2\psi_2^{*,0}}{dx^2}-\psi_2^{*,0}=-\phi_2^{*,0}, & x\in \boldsymbol{R}_+, \\ \psi^{*,0}(0)=A_1^*, \quad \psi^{*,0}(\infty)=0. & \end{cases} \tag{4.18}$$

A_1^* is determined later. We have

$$\psi_2^{*,0}(x)=\left\{A_1^*+\frac{x}{48\tau_0 D}\right\}e^{-\frac{x}{\sqrt{D}}}, \quad \psi_1^{*,1}(x)=\frac{2\tau_0\psi_2^{*,0}(x)}{1+V_0(x)}$$

from (4.18) and $\phi_2^{*,0}(x)=-\frac{1}{24\tau_0\sqrt{D}}e^{-x/\sqrt{D}}$. $\psi_2^{*,1}(x)$ is also determined similarly.

Next we consider inner expansions by

$$\begin{cases} \psi_1^*(x)=\varepsilon\psi_1^{*,1}(x)+\cdots+\dfrac{1}{\varepsilon}\hat{p}_0\left(\dfrac{x}{\varepsilon}\right)+\hat{p}_1\left(\dfrac{x}{\varepsilon}\right)+\cdots, \\ \psi_2^*(x)=\psi_2^{*,0}(x)+\varepsilon\psi_2^{*,1}(x)+\cdots+\hat{q}_0\left(\dfrac{x}{\varepsilon}\right)+\varepsilon\hat{q}_1\left(\dfrac{x}{\varepsilon}\right)+\cdots \end{cases}$$

and plug them into (4.16). The coefficients of ε^{-1} leads to

$$\begin{cases} \ddot{\hat{p}}_0+\left(-3W_0^2+3W_0-\dfrac{1}{2}\right)\hat{p}_0=0, & \\ \tau_0 D\ddot{\hat{q}}_0+\left(\dfrac{1}{2}\left(W_0-\dfrac{1}{2}\right)^2-\dfrac{1}{8}\right)\hat{p}_0=0, & \xi\in\boldsymbol{R}_+ \end{cases}$$

with $\dot{\hat{p}}_0(0)=\hat{p}_0(\infty)=0$, $\dot{\hat{q}}_0(0)=\hat{q}_0(\infty)=0$. Then $\hat{p}_0(\xi)=B_2^*\dot{W}_0(\xi)$ is obtained with an arbitrary constant B_2^* and the equation of $\hat{q}_0$ leads to $B_2^*=0$ ($\hat{p}_0(\xi)=0$) and $\hat{q}_0(\xi)=0$.

The equation with respect to ε^0 is

$$\begin{cases} \ddot{\hat{p}}_1+\left(-3W_0^2+3W_0-\dfrac{1}{2}\right)\hat{p}_1=0, & \\ \tau_0 D\ddot{\hat{q}}_1+\left(\dfrac{1}{2}\left(W_0-\dfrac{1}{2}\right)^2-\dfrac{1}{8}\right)\hat{p}_1=0, & \xi\in\boldsymbol{R}_+ \end{cases}$$

with $\dot{\hat{p}}_1(0) = \hat{p}_1(\infty) = 0$, $\hat{q}_1(\infty) = \dot{\hat{q}}_1(\infty) = 0$. $\hat{p}_1$ is solved as $\hat{p}_1(\xi) = B_3^* \dot{W}_0(\xi)$ with an arbitrary constant B_3^*. We take $B_3^* = 1$. Then from the equation of $\hat{q}_1$, we have

$$\hat{q}_1(\xi) = \frac{1}{2\tau_0 D} \int_\xi^\infty \left(\frac{1}{3} W_0^3(s) - \frac{1}{2} W_0^2(s) \right) ds.$$

A_1^* is determined such that $\frac{d\psi_2^{*,0}}{dx}(0) + \dot{\hat{q}}_1(0) = 0$, which leads to $A_1^* = \frac{1}{16\tau_0\sqrt{D}}$ and $\psi_2^{*,0}(x) = \{ \frac{1}{16\tau_0\sqrt{D}} + \frac{x}{48\tau_0 D} \} e^{-x/\sqrt{D}}$. Thus, Ψ^* is constructed as

$$\begin{cases} \psi_1^*(x;\varepsilon) = \varepsilon\psi_1^{*,1}(x) + \varepsilon^2 E_1(x) + \hat{p}_1\left(\dfrac{x}{\varepsilon}\right) + \varepsilon E_2(x/\varepsilon), \\ \psi_2^*(x;\varepsilon) = \psi_2^{*,0}(x) + \varepsilon\psi_2^{*,1}(x) + \varepsilon^2 E_3(x) + \varepsilon \hat{q}_1\left(\dfrac{x}{\varepsilon}\right) + \varepsilon^2 E_4(x/\varepsilon), \end{cases} \quad x \in [0,\infty),$$

where $E_j(x) = E_j(x;\varepsilon) \in X^2_{\rho_1,1}(\boldsymbol{R}_+)$ uniformly in a small positive ε for some positive constant ρ_1 $(j = 1, 2, 3, 4)$. We can easily extend them into the whole interval $\boldsymbol{R}$.

4.4. Normalization of eigenfunctions

We shall normalize eigenfunctions according to Proposition 3.1. Let $\Psi_0 = {}^t(\psi_{0,1}, \psi_{0,2})$, $\Phi_0^* = {}^t(\phi_{0,1}^*, \phi_{0,2}^*)$ and $\Psi_0^* = {}^t(\psi_{0,1}^*, \psi_{0,2}^*)$ be the constructed eigenfunctions in the previous subsections. Then eigenfunctions are given by

$$\Psi = \Psi_0 + k_1 S_x, \quad \Phi^* = k_2 \Phi_0^*, \quad \Psi^* = k_2 \Psi_0^* + k_3 \Phi_0^*$$

with constants k_1, k_2 and k_3. Substituting them into the normalization stated in Proposition 3.1, we have

$$k_1 = -\frac{\langle S_x, \Psi_0 \rangle_2}{\|S_x\|_2^2}, \quad k_2 = \frac{1}{\langle S_x, \Psi_0^* \rangle_2}, \quad k_3 = \frac{-k_1 - k_2 \langle \Psi_0, \Psi_0^* \rangle_2}{\langle \Psi_0, \Phi_0^* \rangle_2}.$$

Here, we summarize the approximate eigenfunctions for $x \in [0,\infty)$ obtained in the previous subsections for convenience:

$$\tau(\varepsilon) = \tau_0 + O(\varepsilon), \quad \tau_0 = \frac{1}{4\sqrt{2D}},$$

$$S(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{2} + \varepsilon^2 E_1(x) + W_0(x/\varepsilon) + \varepsilon W_1(x/\varepsilon) + \varepsilon^2 E_2(x/\varepsilon) \\ V_0(x) + \varepsilon^2 E_3(x) + \varepsilon^2 Y_0(x/\varepsilon) + \varepsilon^3 Y_1(x/\varepsilon) + \varepsilon^4 E_4(x/\varepsilon) \end{pmatrix},$$

$$S_x(x) = \begin{pmatrix} u_x(x) \\ v_x(x) \end{pmatrix} = \begin{pmatrix} \varepsilon^2 E_5(x) + \dfrac{1}{\varepsilon}\dot{W}_0(x/\varepsilon) + \dot{W}_1(x/\varepsilon) + \varepsilon E_6(x/\varepsilon) \\ V_0'(x) + \varepsilon^2 E_7(x) + \varepsilon \dot{Y}_0(x/\varepsilon) + \varepsilon^2 E_8(x/\varepsilon) \end{pmatrix},$$

$$W_0(\xi) = \frac{1}{1 + e^{\xi/\sqrt{2}}}, \quad V_0(x) = -\frac{1}{2}\left(1 - e^{-x/\sqrt{D}}\right), \quad Y_0(\xi) = -\int_\xi^\infty \int_z^\infty W_0(s)\, did,$$

$$\Psi_0(x)=\begin{pmatrix}\psi_{0,1}(x)\\ \psi_{0,2}(x)\end{pmatrix}=\begin{pmatrix}\varepsilon^2E_9(x)+\dfrac{1}{\varepsilon}p_0(x/\varepsilon)+p_1(x/\varepsilon)+\varepsilon E_{10}(x/\varepsilon)\\ \psi_2^0(x)+\varepsilon^2E_{11}(x)+\varepsilon q_0(x/\varepsilon)+\varepsilon^2 q_1(x/\varepsilon)+\varepsilon^3E_{12}(x/\varepsilon)\end{pmatrix},$$

$$\Phi_0^*(x)=\begin{pmatrix}\phi_{0,1}^*(x)\\ \phi_{0,2}^*(x)\end{pmatrix}=\begin{pmatrix}\varepsilon\phi_1^{*,1}(x)+\varepsilon^2E_{13}(x)+p_1^*(x/\varepsilon)+\varepsilon E_{14}(x/\varepsilon)\\ \phi_2^{*,0}(x)+\varepsilon\phi_2^{*,1}(x)+\varepsilon^2E_{15}(x)+\varepsilon q_1^*(x/\varepsilon)+\varepsilon^2E_{16}(x/\varepsilon)\end{pmatrix},$$

$$\Psi_0^*(x)=\begin{pmatrix}\psi_{0,1}^*(x)\\ \psi_{0,2}^*(x)\end{pmatrix}=\begin{pmatrix}\varepsilon\psi_1^{*,1}(x)+\varepsilon^2E_{17}(x)+\hat{p}_1^*(x/\varepsilon)+\varepsilon E_{18}(x/\varepsilon)\\ \psi_2^{*,0}(x)+\varepsilon\psi_2^{*,1}(x)+\varepsilon^2E_{19}(x)+\varepsilon\hat{q}_1^*(x/\varepsilon)+\varepsilon^2E_{20}(x/\varepsilon)\end{pmatrix},$$

where $E_j(x)=E_j(x;\varepsilon)\in X^2_{\rho_1,1}(\boldsymbol{R}_+)$ uniformly in a small positive ε for some positive constant ρ_1 ($j=1,2,\dots,20$). Then we can calculate

$$\begin{aligned}
\|S_x\|_2^2 &= 2\int_0^\infty\left\{\frac{1}{\varepsilon}\dot{W}_0\Big(\frac{x}{\varepsilon}\Big)\right\}^2dx+2\int_0^\infty(V_0'(x))^2\,dx+O(\varepsilon)\\
&=\frac{\sqrt{2}}{12\varepsilon}+\frac{1}{4\sqrt{D}}+O(\varepsilon),\\
\langle S_x,\,\Psi_0\rangle_2 &= \frac{2}{\varepsilon^2}\int_0^\infty p_0\Big(\frac{x}{\varepsilon}\Big)\dot{W}_0\Big(\frac{x}{\varepsilon}\Big)\,dx+\frac{2}{\varepsilon}\int_0^\infty p_1\Big(\frac{x}{\varepsilon}\Big)\dot{W}_0\Big(\frac{x}{\varepsilon}\Big)\,dx\\
&\quad+\frac{2}{\varepsilon}\int_0^\infty p_0\Big(\frac{x}{\varepsilon}\Big)\dot{W}_1\Big(\frac{x}{\varepsilon}\Big)\,dx+2\int_0^\infty V_0'(x)\psi_2^0(x)\,dx+O(\varepsilon)\\
&=\frac{\sqrt{2}}{12\varepsilon}+4\int_0^\infty\dot{W}_0(\xi)\dot{W}_1(\xi)\,d\xi+\frac{7}{16D}+O(\varepsilon),
\end{aligned}$$

which leads to $k_1=-1+O(\varepsilon)$.

Similarly, we have

$$\begin{aligned}
\langle S_x,\,\Psi_0^*\rangle_2 &= \frac{2}{\varepsilon}\int_0^\infty \hat{p}_1^*\Big(\frac{x}{\varepsilon}\Big)\dot{W}_0\Big(\frac{x}{\varepsilon}\Big)\,dx+2\int_0^\infty V_0'(x)\psi_2^{*,0}(x)\,dx+O(\varepsilon)\\
&=\frac{\sqrt{2}}{12}-\frac{7}{192\tau_0\sqrt{D}}+O(\varepsilon),\\
k_2 &= \frac{1}{\langle S_x,\,\Psi_0^*\rangle_2}=\frac{192\tau_0\sqrt{D}}{16\sqrt{2}\tau_0\sqrt{D}-7}+O(\varepsilon)=-8\sqrt{2}+O(\varepsilon)
\end{aligned}$$

by using $\tau_0=\frac{1}{4\sqrt{2D}}$.

Let us obtain the value of k_3. Calculating

$$\begin{aligned}
\langle\Psi_0,\,\Psi_0^*\rangle_2 &= \frac{2}{\varepsilon}\int_0^\infty p_0\Big(\frac{x}{\varepsilon}\Big)\hat{p}_1^*\Big(\frac{x}{\varepsilon}\Big)\,dx+2\int_0^\infty\psi_2^0(x)\psi_2^{*,0}(x)\,dx+O(\varepsilon)\\
&=\frac{\sqrt{2}}{12}-\frac{25}{32\times12\tau_0\sqrt{D}}+O(\varepsilon),
\end{aligned}$$

$$\begin{aligned}\langle \Psi_0, \Phi_0^*\rangle_2 &= \frac{2}{\varepsilon}\int_0^\infty p_0\Big(\frac{x}{\varepsilon}\Big)p_1^*\Big(\frac{x}{\varepsilon}\Big)\,dx + 2\int_0^\infty \psi_2^0(x)\phi_2^{*;0}(x)\,dx + O(\varepsilon)\\ &= \frac{\sqrt{2}}{12} - \frac{7}{192\tau_0\sqrt{D}} + O(\varepsilon),\end{aligned}$$

we have

$$k_1 + k_2\langle \Psi_0, \Psi_0^*\rangle_2 = \frac{-11}{32\sqrt{2}\tau_0\sqrt{D} - 14} + O(\varepsilon)$$

and

$$k_3 = \frac{11\times 192\tau_0\sqrt{D}}{2(16\sqrt{2}\tau_0\sqrt{D} - 7)^2} + O(\varepsilon) = \frac{44\sqrt{2}}{3} + O(\varepsilon).$$

Thus, eigenfunctions satisfying the normalization stated in Proposition 3.1 are explicitly constructed.

5. Constants M_1 and M_2 defined in Section 3

In order to obtain the values of M_1 and M_2, the functions $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ are necessary. Here we remind the definitions of the functions:

$$\begin{aligned}&\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 \in X_1 = \{\boldsymbol{u} \in L^2(\boldsymbol{R});\ \langle \boldsymbol{u}, \Phi^*\rangle_2 = \langle \boldsymbol{u}, \Psi^*\rangle_2 = 0\},\\ &L\boldsymbol{\zeta}_1 + \frac{1}{2}F''(S(x))\Psi\cdot\Psi + \Psi_x = 0, &(5.1)\\ &L\boldsymbol{\zeta}_2 + G_0(S(x)) = 0. &(5.2)\end{aligned}$$

Let $\boldsymbol{\zeta}_1(x) = {}^t\big(\tilde{\zeta}_1(x), \tilde{\zeta}_2(x)\big)$ and $\Psi(x) = {}^t(\psi_1(x), \psi_2(x))$. Then (5.1) is

$$\begin{cases}\varepsilon^2\dfrac{d\tilde{\zeta}_1}{dx^2} + f_u\tilde{\zeta}_1 + f_v\tilde{\zeta}_2 + \dfrac{\varepsilon\tau(\varepsilon)}{2}\{f_{uu}\psi_1^2 + 2f_{uv}\psi_1\psi_2 + f_{vv}\psi_2^2\} + \varepsilon\tau(\varepsilon)\dfrac{d\psi_1}{dx} = 0,\\ D\dfrac{d\tilde{\zeta}_2}{dx^2} + g_u\tilde{\zeta}_1 + g_v\tilde{\zeta}_2 + \dfrac{1}{2}\{g_{uu}\psi_1^2 + 2g_{uv}\psi_1\psi_2 + g_{vv}\psi_2^2\} + \dfrac{d\psi_2}{dx} = 0.\end{cases} \tag{5.3}$$

Here, we note that

$$f_u = \frac{1}{4} - 3u^2(x) + u(x)v(x),\quad f_v = \frac{1}{2}\Big(u(x) + \frac{1}{2}\Big)\Big(u(x) - \frac{1}{2}\Big),\quad g_u = 1,\quad g_v = -1,$$
$$f_{uu} = -6u(x) + v(x),\quad f_{uv} = u(x),\quad f_{vv} = g_{uu} = g_{uv} = g_{vv} = 0.$$

Proposition 5.1. *The function $\boldsymbol{\zeta}_1(x)$ is odd.*

Proof. The inhomogeneous terms of (5.1) are odd functions. Therefore, $\boldsymbol{\zeta}_1$ is written by $\boldsymbol{\zeta}_1 = O(x) + \alpha S_x$, where $O(x)$ is an odd function and α is an arbitrary constant. $\boldsymbol{\zeta}_1 \in X_1 = RX = \{\langle \boldsymbol{u}, \Phi^*\rangle_2 = \langle \boldsymbol{u}, \Psi^*\rangle_2 = 0\}$ implies $\alpha = 0$. □

It suffices to consider (5.3) on $\boldsymbol{R}_+$ with $\boldsymbol{\zeta}_1(0) = 0$. First, we consider outer solutions by expanding

$$\begin{cases} \tilde{\zeta}_1(x;\varepsilon) = \tilde{\zeta}_1^0(x) + \varepsilon\tilde{\zeta}_1^1(x) + \cdots, \\ \tilde{\zeta}_2(x;\varepsilon) = \tilde{\zeta}_2^0(x) + \varepsilon\tilde{\zeta}_2^1(x) + \cdots. \end{cases}$$

From the lowest order terms, we have $\tilde{\zeta}_1^0(x) = 0$ and

$$D\frac{d\tilde{\zeta}_2^0}{dx^2} - \tilde{\zeta}_2^0 = -\frac{x}{4D\sqrt{D}}e^{-\frac{x}{\sqrt{D}}}, \quad x \in \boldsymbol{R}_+$$

with $\tilde{\zeta}_2^0(0) = \tilde{\zeta}_2^0(\infty) = 0$. It leads to

$$\tilde{\zeta}_2^0(x) = \left(\frac{1}{16D\sqrt{D}}x + \frac{x^2}{16D^2}\right)e^{-\frac{x}{\sqrt{D}}}.$$

Similarly, we can calculate $\tilde{\zeta}_1^1$ and $\tilde{\zeta}_2^1$. Especially, $\tilde{\zeta}_1^1$ is solved as $\tilde{\zeta}_1^1(x) = 0$ and $\tilde{\zeta}_2^1$ is so with the boundary conditions $\tilde{\zeta}_2^1(0) = \tilde{A}_1$ and $\tilde{\zeta}_2^1(\infty) = 0$, in which $\tilde{A}_1$ is determined later.

Next, we consider in a neighborhood of $x = 0$ and expand

$$\begin{cases} \tilde{\zeta}_1(x;\varepsilon) = \tilde{\zeta}_1^0(x) + \varepsilon\tilde{\zeta}_1^1(x) + \cdots + \dfrac{1}{\varepsilon}\tilde{p}_0\left(\dfrac{x}{\varepsilon}\right) + \tilde{p}_1\left(\dfrac{x}{\varepsilon}\right) + \cdots, \\ \tilde{\zeta}_2(x;\varepsilon) = \tilde{\zeta}_2^0(x) + \varepsilon\tilde{\zeta}_2^1(x) + \cdots + \varepsilon\tilde{q}_0\left(\dfrac{x}{\varepsilon}\right) + \varepsilon^2\tilde{q}_1\left(\dfrac{x}{\varepsilon}\right) + \cdots. \end{cases}$$

Since $\tilde{\zeta}_1^0 = \tilde{\zeta}_1^1 = 0$, we have from terms of ε^{-1}

$$\begin{cases} \ddot{\tilde{p}}_0 + \left(-3W_0^2 + 3W_0 - \dfrac{1}{2}\right)\tilde{p}_0 = 0, & \xi \in \boldsymbol{R}_+, \\ D\ddot{\tilde{q}}_0 + \tilde{p}_0 = 0, \\ \tilde{p}_0(0) = \tilde{p}_0(\infty) = 0, \\ \tilde{q}_0(0) = \tilde{q}_0(\infty) = 0, \end{cases}$$

which leads to $\tilde{p}_0(\xi) = \tilde{q}(\xi) = 0$.

Terms of order ε^0 gives

$$\begin{cases} \ddot{\tilde{p}}_1 + \left(-3W_0^2 + 2W_0 - \dfrac{1}{2}\right)\tilde{p}_1 = 0, & \xi \in \boldsymbol{R}_+, \\ D\ddot{\tilde{q}}_0 + \tilde{p}_0 = 0, \\ \tilde{p}_1(0) = \tilde{p}_1(\infty) = 0, \\ \tilde{q}_1(\infty) = \dot{\tilde{q}}_1(\infty) = 0. \end{cases} \tag{5.4}$$

Here we used the relation $p_1(\xi) - \dot{W}_1(\xi) = 0$. Clearly we have $\tilde{p}_1(\xi) = \tilde{q}_1(\xi) = 0$.

The boundary condition of $\tilde{\zeta}_2^1(0) = \tilde{A}_1$ is determined by $\tilde{\zeta}_2^1(0) + \tilde{q}_1(0) = 0$, which implies $\tilde{A}_1 = 0$.

Thus, $\boldsymbol{\zeta}_1$ is given by

$$\begin{cases} \tilde{\zeta}_1(x;\varepsilon) = \varepsilon^2 E_1(x) + \varepsilon E_2\Big(\dfrac{x}{\varepsilon}\Big), \\ \tilde{\zeta}_2(x;\varepsilon) = \tilde{\zeta}_2^0(x) + \varepsilon\tilde{\zeta}_2^1(x) + \varepsilon^2 E_3(x) + \varepsilon^3 E_4\Big(\dfrac{x}{\varepsilon}\Big), \end{cases} \quad x \in [0,\infty), \tag{5.5}$$

where $E_j(x) = E_j(x;\varepsilon) \in X^2_{\rho_1,1}(\boldsymbol{R}_+)$ uniformly in a small positive ε for some positive constant ρ_1 $(j = 1,2,3,4)$. We can easily extend them into the whole interval $\boldsymbol{R}$.

On the other hand, we have $\boldsymbol{\zeta}_2(x) = 0$ because of $G_0(S(x)) = 0$ in (5.2).

Finally, we shall obtain the values of M_1 and M_2 according to the definitions in Section 3.

M_1 is given by

$$M_1 = \langle \partial_x \boldsymbol{\zeta}_1, \Phi^* \rangle_2 + \langle F''(S)\Psi \cdot \boldsymbol{\zeta}_1, \Phi^* \rangle_2 + \frac{1}{6}\langle F'''(S)\Psi^3, \Phi^* \rangle_2.$$

The first term in the definition of M_1 is calculated as

$$\begin{aligned}
&\langle \partial_x \boldsymbol{\zeta}_1, \Phi^* \rangle_2 \\
&= \langle \partial_x \boldsymbol{\zeta}_1, k_2 \Phi_0^* \rangle_2 \\
&= 2k_2 \int_0^\infty \big(\partial_x \tilde{\zeta}_1 \phi_{0,1}^* + \partial_x \tilde{\zeta}_2 \phi_{0,2}^*\big)\, dx \\
&= 2k_2 \bigg\{ O(\varepsilon) + \bigg(\int_0^\infty \partial_x \tilde{\zeta}_2^0(x) \phi_2^{*,0}(x)\, dx + O(\varepsilon) \bigg) \bigg\} \\
&= 2k_2 \int_0^\infty \bigg(\frac{1}{16D\sqrt{D}} + \frac{1}{16D^2}x - \frac{x^2}{16D^2\sqrt{D}} \bigg) e^{-\frac{x}{\sqrt{D}}} \cdot \frac{1}{24\tau_0\sqrt{D}} e^{-\frac{x}{\sqrt{D}}}\, dx + O(\varepsilon) \\
&= 2k_2 \cdot \frac{1}{16D\sqrt{D}} \cdot \frac{\sqrt{D}}{2} \frac{1}{24\tau_0\sqrt{D}} + O(\varepsilon) \\
&= 2(-8\sqrt{2} + O(\varepsilon)) \cdot \frac{1}{16 \times 6\sqrt{2}D} + O(\varepsilon) \\
&= -\frac{1}{6D} + O(\varepsilon)
\end{aligned}$$

by noting $\tau_0 = \frac{1}{4\sqrt{2D}}$.

Similarly, $\langle F''(S)\Psi \cdot \boldsymbol{\zeta}_1, \Phi^* \rangle_2$ is calculated. Now, we have

$$S(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{2} + \varepsilon^2 E_1(x) + W_0(x/\varepsilon) + \varepsilon W_1(x/\varepsilon) + \varepsilon^2 E_2(x/\varepsilon) \\ V_0(x) + \varepsilon^2 E_3(x) + \varepsilon^2 E_4(x/\varepsilon) \end{pmatrix},$$

$$\begin{aligned}
\Psi(x) &= \Psi_0(x) + k_1 S_x(x) = \Psi_0(x) - S_x(x) + O(\varepsilon) \\
&= \begin{pmatrix} \varepsilon E_5(x) + \varepsilon E_6(x/\varepsilon) \\ \psi_2^0(x) - V_0'(x) + \varepsilon E_7(x) + \varepsilon E_8(x/\varepsilon) \end{pmatrix},
\end{aligned}$$

$$\boldsymbol{\zeta}_1(x) = \begin{pmatrix} \varepsilon^2 E_9(x) + \varepsilon E_{10}(x/\varepsilon) \\ \tilde{\zeta}_2^0(x) + \varepsilon E_{11}(x) + \varepsilon^2 E_{12}(x/\varepsilon) \end{pmatrix},$$

$$\Phi^*(x) = k_2 \Phi_0^*(x) = k_2 \begin{pmatrix} \varepsilon \phi_1^{*,1}(x) + \varepsilon^2 E_{13}(x) + p_1^*(x/\varepsilon) + \varepsilon E_{14}(x/\varepsilon) \\ \phi_2^{*,0}(x) + \varepsilon E_{15}(x) + \varepsilon E_{16}(x/\varepsilon) \end{pmatrix}$$

for $x \in [0,\infty)$, where $E_j(x) = E_j(x;\varepsilon) \in X^2_{\rho_1,1}(\boldsymbol{R}_+)$ uniformly in a small positive ε for some positive constant ρ_1 $(j = 1, 2, \ldots, 16)$. Since

$$\begin{aligned} F''(S)\Psi \cdot \boldsymbol{\zeta}_1 &= \frac{1}{\varepsilon\tau(\varepsilon)} \begin{pmatrix} f_{up} \cdot \psi_1\tilde{\zeta}_1 + f_{uv} \cdot \psi_1\tilde{\zeta}_2 + f_{vu} \cdot \psi_2\tilde{\zeta}_1 + f_{vv} \cdot \psi_2\tilde{\zeta}_2 \\ 0 \end{pmatrix} \\ &= \frac{1}{\varepsilon\tau(\varepsilon)} \begin{pmatrix} (-6u+v) \cdot \psi_1\tilde{\zeta}_1 + u \cdot \psi_1\tilde{\zeta}_2 + u \cdot \psi_2\tilde{\zeta}_1 \\ 0 \end{pmatrix} \end{aligned}$$

holds, we have

$$\begin{aligned} &\langle F''(S)\Psi \cdot \boldsymbol{\zeta}_1, \Phi^* \rangle_2 \\ &= \left\langle \frac{1}{\varepsilon\tau(\varepsilon)} \begin{pmatrix} (-6u+v) \cdot \psi_1\tilde{\zeta}_1 + u \cdot \psi_1\tilde{\zeta}_2 + u \cdot \psi_2\tilde{\zeta}_1 \\ 0 \end{pmatrix}, k_2 \begin{pmatrix} \phi^*_{0,1} \\ \phi^*_{0,2} \end{pmatrix} \right\rangle_2 \\ &= \frac{2k_2}{\varepsilon\tau(\varepsilon)} \int_0^\infty \big((-6u+v) \cdot \psi_1\tilde{\zeta}_1 + u \cdot \psi_1\tilde{\zeta}_2 + u \cdot \psi_2\tilde{\zeta}_1\big)\phi^*_{0,1}\, dx \\ &= \frac{2k_2}{\varepsilon\tau(\varepsilon)} \cdot O(\varepsilon^2) = O(\varepsilon), \end{aligned}$$

because

$$\begin{aligned} \int_0^\infty (-6u+v)\psi_1\tilde{\zeta}_1\phi^*_{0,1}\, dx &= \int_0^\infty (-6u+v)\Big\{\varepsilon E_{17}(x) + \varepsilon E_{18}\Big(\frac{x}{\varepsilon}\Big)\Big\} \\ &\qquad \times \Big\{\varepsilon^2 E_{19}(x) + \varepsilon E_{20}\Big(\frac{x}{\varepsilon}\Big)\Big\}\phi^*_{0,1}\, dx \\ &= O(\varepsilon^2), \end{aligned}$$

$$\begin{aligned} \int_0^\infty u\psi_1\tilde{\zeta}_2\phi^*_{0,1}\, dx &= \int_0^\infty \Big(-\frac{1}{2} + W_0\Big(\frac{x}{\varepsilon}\Big) + \varepsilon E_{21}\Big(\frac{x}{\varepsilon}\Big)\Big) \\ &\qquad \times \Big\{\varepsilon E_{22}(x) + \varepsilon E_{23}\Big(\frac{x}{\varepsilon}\Big)\Big\}\Big\{\tilde{\zeta}_2^0(x) + \varepsilon E_{24}(x) + \varepsilon^2 E_{25}\Big(\frac{x}{\varepsilon}\Big)\Big\} \\ &\qquad \times \Big\{\varepsilon\phi_1^{*,0}(x) + \dot{W}_0\Big(\frac{x}{\varepsilon}\Big) + \varepsilon^2 E_{26}(x) + \varepsilon E_{27}\Big(\frac{x}{\varepsilon}\Big)\Big\}\, dx \\ &= \tilde{\zeta}_2^0(0) \int_0^\infty \Big(-\frac{1}{2} + W_0(\xi)\Big)\dot{W}_0(\xi)E_{27}(\xi)\, d\xi \cdot O(\varepsilon) + O(\varepsilon^2) \\ &= O(\varepsilon^2) \quad \big(\tilde{\zeta}_2^0(0) = 0\big), \end{aligned}$$

$$\begin{aligned}\int_0^\infty u\psi_2\tilde{\zeta}_1\phi_{0,1}^*\,dx &= \int_0^\infty\Big(-\frac{1}{2}+W_0\Big(\frac{x}{\varepsilon}\Big)+\varepsilon E_{28}\Big(\frac{x}{\varepsilon}\Big)\Big)\\ &\qquad\times\Big\{\psi_2^0(x)-V_0'(x)+\varepsilon E_{29}(x)+\varepsilon E_{30}\Big(\frac{x}{\varepsilon}\Big)\Big\}\\ &\qquad\times\Big\{\varepsilon^2E_{31}(x)+\varepsilon E_{32}\Big(\frac{x}{\varepsilon}\Big)\Big\}\phi_{0,1}^*\,dx\\ &= O(\varepsilon^2),\end{aligned}$$

where $E_j(x)=E_j(x;\varepsilon)\in X^2_{\rho_1,1}(\boldsymbol{R}_+)$ uniformly in a small positive ε for some positive constant ρ_1 $(j=17,18,\dots,32)$.

We shall show $\frac{1}{6}\langle F'''(S)\,\Psi^3,\,\Phi^*\rangle_2=O(\varepsilon)$. It is calculated as

$$\begin{aligned}\frac{1}{6}\langle F'''(S)\,\Psi^3,\,\Phi^*\rangle_2 &= \frac{1}{6}\left\langle\begin{pmatrix}\dfrac{-6}{\varepsilon\tau(\varepsilon)}\psi_1^3+\dfrac{3}{\varepsilon\tau(\varepsilon)}\psi_1^2\psi_2\\ 0\end{pmatrix},k_2\begin{pmatrix}\phi_{0,1}^*\\ \phi_{0,2}^*\end{pmatrix}\right\rangle_2\\ &= \frac{k_2}{\varepsilon\tau(\varepsilon)}\int_0^\infty(-2\psi_1^3+\psi_1^2\psi_2)\phi_{0,1}^*\,dx\\ &= \frac{k_2}{\varepsilon\tau(\varepsilon)}\Big(2\int_0^\infty\varepsilon^3\Big(E_{33}(x)+E_{34}\Big(\frac{x}{\varepsilon}\Big)\Big)^3\phi_{0,1}^*\,dx\\ &\qquad+\int_0^\infty\varepsilon^2\Big(E_{35}(x)+E_{36}\Big(\frac{x}{\varepsilon}\Big)\Big)^2\psi_2\phi_{0,1}^*\,dx\Big)\\ &= \frac{k_2}{\varepsilon\tau(\varepsilon)}O(\varepsilon^2)=O(\varepsilon),\end{aligned}$$

where $E_j(x)=E_j(x;\varepsilon)\in X^2_{\rho_1,1}(\boldsymbol{R}_+)$ uniformly in a small positive ε for some positive constant ρ_1 $(j=33,34,35,36)$. Thus, we have

$$M_1=-\frac{1}{6D}+O(\varepsilon). \tag{5.6}$$

Next, we consider $M_2=\langle\partial_x\boldsymbol{\zeta}_2,\,\Phi^*\rangle_2+\langle F''(S)\,\Psi\cdot\boldsymbol{\zeta}_2,\,\Phi^*\rangle_2+\langle G_0'(S)\,\Psi,\,\Phi^*\rangle_2$. It suffices to obtain $\langle G_0'(S)\,\Psi,\,\Phi^*\rangle_2$ because of $\boldsymbol{\zeta}_2=0$. Since

$$\begin{aligned}G_0'(S)\,\Psi &= -\frac{1}{\varepsilon\tau^2(\varepsilon)}\begin{pmatrix}\varepsilon^2\partial_x^2\psi_1+f_u\cdot\psi_1+f_v\cdot\psi_2\\ 0\end{pmatrix}\\ &= -\frac{1}{\varepsilon\tau^2(\varepsilon)}\begin{pmatrix}-\varepsilon\tau(\varepsilon)u_x\\ 0\end{pmatrix}\\ &= \frac{1}{\varepsilon\tau(\varepsilon)}\begin{pmatrix}\dot{W}_0\Big(\dfrac{x}{\varepsilon}\Big)+E_{37}\Big(\dfrac{x}{\varepsilon}\Big)\\ 0\end{pmatrix},\end{aligned}$$

we have

$$
\begin{aligned}
\langle G_0'(S)\,\Psi,\,\Phi^*\rangle_2 &= \frac{1}{\varepsilon\tau(\varepsilon)}\left\langle\begin{pmatrix}\dot W_0\left(\frac{x}{\varepsilon}\right)+E_{38}\left(\frac{x}{\varepsilon}\right)\\ 0\end{pmatrix}, k_2\,\Phi_0^*\right\rangle_2\\
&= \frac{2k_2}{\varepsilon\tau(\varepsilon)}\int_0^\infty\left\{\dot W_0\left(\frac{x}{\varepsilon}\right)+E_{39}\left(\frac{x}{\varepsilon}\right)\right\}\\
&\qquad\times\left\{\varepsilon\phi_1^{*,1}(x)+\varepsilon^2E_{40}(x)+\dot W_0\left(\frac{x}{\varepsilon}\right)+\varepsilon E_{41}\left(\frac{x}{\varepsilon}\right)\right\}dx\\
&= \frac{2k_2}{\varepsilon\tau(\varepsilon)}\int_0^\infty \dot W_0\left(\frac{x}{\varepsilon}\right)^2dx+O(\varepsilon)\\
&= \frac{2k_2}{\tau(\varepsilon)}\int_0^\infty \dot W_0(\xi)^2\,d\xi+O(\varepsilon)\\
&= \frac{2k_2}{\tau_0}\frac{\sqrt{2}}{24}+O(\varepsilon)\\
&= -\frac{4}{3\tau_0}+O(\varepsilon),
\end{aligned}
$$

where $E_j(x)=E_j(x;\varepsilon)\in X^2_{\rho_1,1}(\boldsymbol{R}_+)$ uniformly in a small positive ε for some positive constant ρ_1 $(j=37,38,\dots,41)$. Thus, M_2 is given

$$M_2=-\frac{4}{3\tau_0}+O(\varepsilon). \tag{5.7}$$

Thus we have the theorem:

Theorem 5.2. *The constants M_1 and M_2 defined in* (3.3) *are given by*

$$M_1=-\frac{1}{6D}+O(\varepsilon),\quad M_2=-\frac{4}{3\tau_0}+O(\varepsilon)$$

for a small positive ε.

Then we easily find the dynamics of solutions for the lowest order reduced ODE $\dot r=M_1r^3+M_2\eta r$ as in Fig. 5.1. For $\eta>0$, $r=0$ is stable, while when $\eta<0$, $r=0$ becomes unstable and $r=\pm\sqrt{-M_2\eta/M_1}$ are stable. This means that $r=0$ is the separator between two stable stationary solutions for $\eta<0$. Note that r corresponds to the velocity of traveling front solutions. Then we have

Corollary 5.3. *Fix a small positive ε arbitrarily. Since both M_1 and M_2 are negative, the bifurcation structure in the neighborhood of $\tau=\tau(\varepsilon)$ $(\tau(0)=\tau_0)$ is a super-critical pitchfork type as in* Fig. 5.2. *That is, the stationary front solution $S(x;\varepsilon)$ which corresponds to the branch with $c=0$ in* Fig. 5.2 *is stable for $\tau>\tau(\varepsilon)$ while it is unstable for $\tau<\tau(\varepsilon)$ and there appear stable traveling front solutions, say $S_\pm(x-ct;\varepsilon)$ with $c=\pm\sqrt{-\frac{M_2\eta}{M_1}}$ as in* Fig. 5.2, *where $\eta=\tau-\tau(\varepsilon)$. Furthermore for $\tau<\tau(\varepsilon)$, the unstable stationary solution $S(x;\varepsilon)$ plays the role of a separator between stable traveling front solutions $S_\pm(x-ct;\varepsilon)$.*

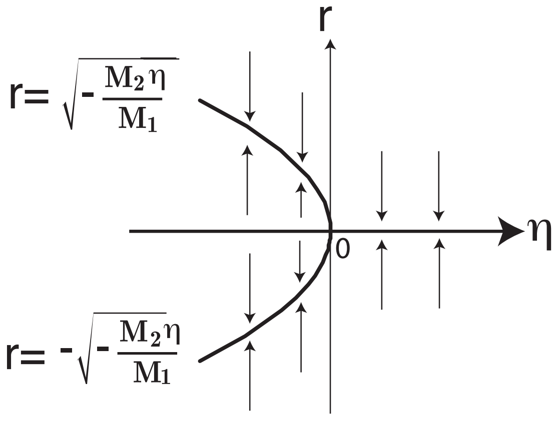

Fig. 5.1. Dynamics of solutions of $\dot{r} = M_1 r^3 + M_2 \eta r$.

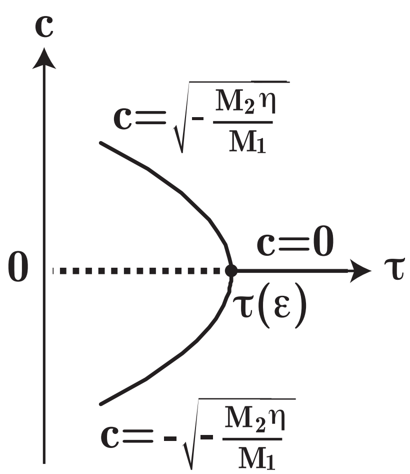

Fig. 5.2. Bifurcation structure of solutions of (1.1) with (1.3) with respect to the velocity, say c of traveling fronts. Solid lines denote stable solutions and broken one does unstable solutions.

6. Application to the dynamics of front solutions with small perturbations

In this section, we will consider

$$\begin{cases} \varepsilon\tau u_t = \varepsilon^2 u_{xx} + f(u,v) + \lambda h_1(x,u,v), \\ v_t = D v_{xx} + g(u,v) + \lambda h_2(x,u,v), \end{cases} \quad t > 0, \quad x \in \boldsymbol{R} \tag{6.1}$$

with (1.3) and sufficiently small $|\lambda|$ as an application to perturbed systems.

First, we will give the general theorem on the dynamics of front solutions. (6.1) is written as

$$\boldsymbol{u}_t = \mathcal{L}(\boldsymbol{u}) + \eta G(\boldsymbol{u};\eta) + \lambda A(\eta) H(x,\boldsymbol{u}),$$

where $A(\eta) := \begin{pmatrix} \frac{1}{\varepsilon(\tau(\varepsilon)+\eta)} & 0 \\ 0 & 1 \end{pmatrix}$ and $H(x,\boldsymbol{u}) := \begin{pmatrix} h_1(x,u,v) \\ h_2(x,u,v) \end{pmatrix}$.

Then quite similarly to Section 3, we have

Theorem 6.1. *Let $l(t)$ and $r(t)$ be the functions stated in* Theorem 3.2. *Then for $|\eta| < \eta^*$, $|r| < r^*$ and $|\lambda| < \lambda^*$,*

$$\begin{aligned} \dot{l} &= r - \lambda\langle A(\eta)H(x+l,S(x)), \boldsymbol{\psi}^*\rangle_2 + O(|r|^3 + |\eta|^{3/2} + |\lambda|^2), \\ \dot{r} &= K(r;\eta) + \lambda\langle A(\eta)H(x+l,S(x)), \boldsymbol{\phi}^*\rangle_2 + O(|r|^4 + |\eta|^2 + |\lambda|^2) \end{aligned}$$

hold as long as $|r(t)| < r^*$, where $K(r;\eta) := M_1 r^3 + M_2 \eta r$.

The proof of this theorem is quite similar to that of Theorem 4.1 in [1] by replacing $\eta g(\boldsymbol{u})$ in [1] with $\eta G(\boldsymbol{u};\eta) + \lambda A(\eta)H(x,\boldsymbol{u})$. We note to take η and λ sufficiently small by fixing small $\varepsilon > 0$.

6.1. Imperfection of bifurcation diagram

First, we consider the following problems with small constant perturbation to analyze the dynamics of solutions more precisely:

$$\begin{cases} \varepsilon\tau u_t = \varepsilon^2 u_{xx} + f(u,v), \\ v_t = D v_{xx} + g(u,v) + \lambda, \end{cases} \quad t > 0, \quad x \in \boldsymbol{R} \tag{6.2}$$

with (1.3) and sufficiently small $|\lambda|$. Then Theorem 6.1 leads (6.2) to

Corollary 6.2. *Let $l(t)$ and $r(t)$ be the functions stated in* Theorem 3.2. *Then for $|\eta| < \eta^*$, $|r| < r^*$ and $|\lambda| < \lambda^*$,*

$$\begin{aligned} \dot{l} &= r + \frac{8\sqrt{D}}{9}\lambda + O(|r|^3 + |\eta|^{3/2} + |\lambda|^2), \\ \dot{r} &= K(r;\eta) - \frac{16\sqrt{D}}{3}\lambda + O(|r|^4 + |\eta|^2 + |\lambda|^2) \end{aligned}$$

hold as long as $|r(t)| < r^*$ as the dynamics of front in (6.2)*, where $K(r;\eta) := M_1 r^3 + M_2 \eta r$.*

This corollary is easily shown by the application of Theorem 6.1 for the case that $H(x,\boldsymbol{u}) = \binom{0}{1}$. Then the equations in Theorem 6.1 become

$$\begin{aligned} \dot{l} &= r - \lambda\langle 1, \psi_2^*\rangle_2 + O(|r|^3 + |\eta|^{3/2} + |\lambda|^2), \\ \dot{r} &= K(r;\eta) + \lambda\langle 1, \phi_2^*\rangle_2 + O(|r|^4 + |\eta|^2 + |\lambda|^2) \end{aligned}$$

and the direct calculations give Corollary 6.2.

Corollary 6.2 shows the imperfection of the pitch-fork bifurcation diagram for front solutions when λ is not zero (Fig. 6.1).

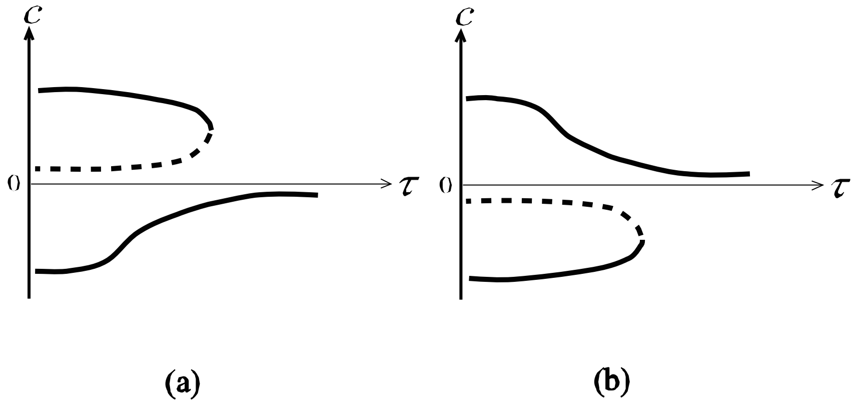

Fig. 6.1. Bifurcation structure of solutions of (6.2) with (1.3) with respect to the velocity, say c of traveling fronts. Solid lines denote stable solutions and broken one does unstable solutions. (a) $\lambda > 0$, (b) $\lambda < 0$.

6.2. Dynamics on heterogeneous media

Next, we consider the problem on heterogeneous media as follows:

$$\begin{cases} \varepsilon\tau u_t = \varepsilon^2 u_{xx} + f(u,v) + \lambda h(x), \\ v_t = Dv_{xx} + g(u,v), \end{cases} \quad t > 0, \quad x \in \boldsymbol{R} \tag{6.3}$$

with (1.3) and sufficiently small $|\lambda|$. Then Theorem 6.1 leads (6.3) to

Corollary 6.3. *Let $l(t)$ and $r(t)$ be the functions stated in* Theorem 3.2. *Then for $|\eta| < \eta^*$, $|r| < r^*$ and $|\lambda| < \lambda^*$,*

$$\begin{aligned} \dot{l} &= r - \frac{\lambda}{\varepsilon\tau(\varepsilon)}\langle h(x+l), \psi_1^* \rangle_2 + O(|r|^3 + |\eta|^{3/2} + |\lambda|^2), \\ \dot{r} &= K(r;\eta) + \frac{\lambda}{\varepsilon\tau(\varepsilon)}\langle h(x+l), \phi_1^* \rangle_2 + O(|r|^4 + |\eta|^2 + |\lambda|^2) \end{aligned}$$

hold as long as $|r(t)| < r^*$, where $K(r;\eta) := M_1 r^3 + M_2 \eta r$.

Now, we assume $\lambda = \Lambda\tau(\varepsilon)|\eta|^{3/2}$ and change the scales by $r = \sqrt{|\eta|}\, R$ and $T = \sqrt{|\eta|}\, t$ for negative η. Then the equations of l and r in Corollary 6.3 become

$$\begin{aligned} l_T &= R - \frac{\Lambda}{\varepsilon}|\eta|\langle h(x+l), \psi_1^* \rangle_2 + O(|\eta|), \\ R_T &= \sqrt{|\eta|}\left\{ (M_1 R^2 - M_2)R + \frac{\Lambda}{\varepsilon}\langle h(x+l), \phi_1^* \rangle_2 \right\} + O(|\eta|) \end{aligned}$$

and the lowest order equations are

$$\begin{cases} l_T = R, \\ R_T = \sqrt{|\eta|}\left\{(M_1R^2 - M_2)R + \dfrac{\Lambda}{\varepsilon}\langle h(x+l), \phi_1^*\rangle_2\right\}. \end{cases} \tag{6.4}$$

Here, we note that terms $K(r;\eta)$ and $\frac{\lambda}{\varepsilon\tau(\varepsilon)}\langle h(x+l), \phi_1^*\rangle_2$ are balanced in the scale of $\lambda = \Lambda\tau(\varepsilon)|\eta|^{3/2}$ since $\langle h(x+l), \phi_1^*\rangle_2$ is $O(\varepsilon)$ as shown below. In other scales of λ, $K(r;\eta)$ or $\frac{\lambda}{\varepsilon\tau(\varepsilon)}\langle h(x+l), \phi_1^*\rangle_2$ is dominant and the dynamics will be almost trivial.

In order to consider behaviors of fronts in detail, we assume $h(x) = H(x)$, the Heaviside function defined by $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$. Since

$$\phi_1^*(x) = -8\sqrt{2}\left\{\dot{W}_0\left(\frac{x}{\varepsilon}\right) + \frac{\varepsilon}{12\sqrt{d}}\frac{e^{-x/\sqrt{D}}}{1+V_0(x)}\right\} + \varepsilon^2E_1(x) + \varepsilon E_2\left(\frac{x}{\varepsilon}\right) \quad (x > 0)$$

and $\phi_1^*(x)$ is an even function, we can easily calculate

$$\begin{aligned} H_2(l) &:= \frac{1}{\varepsilon}\langle H(x+l), \phi_1^*\rangle_2 = \frac{1}{\varepsilon}\int_{-l}^{\infty}\phi_1^*(x)\,dx \\ &= 4\sqrt{2}\left\{\left(1 - \frac{1}{3}\log 2\right) \pm \left(1 - 2W_0\left(\frac{|l|}{\varepsilon}\right) - \frac{1}{3}\log\frac{2}{1+e^{-|l|/\sqrt{D}}}\right)\right\} + O(\varepsilon) \end{aligned} \tag{6.5}$$

for $l > 0$ and $l < 0$, respectively and (6.4) becomes

$$\begin{cases} l_T = R, \\ R_T = \sqrt{|\eta|}\,\{(M_1R^2 - M_2)R + \Lambda H_2(l)\}. \end{cases} \tag{6.6}$$

The graph of $H_2(l)$ is as Fig. 6.2.

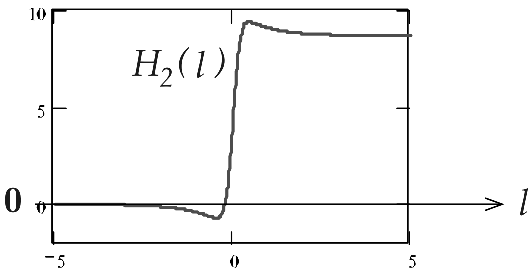

Fig. 6.2. The graph of $H_2(l)$ with $D = 0.5$ and $\varepsilon = 0.07$.

First we consider the case $\Lambda > 0$, which implies the jump-up case at $x = 0$. We should find there appears a negative part in the graph of $H_2(l)$ for $l < 0$. This implies in (6.6) that if $R(0)$ is positive but sufficiently small, $R(T)$ can become negative before $l(T)$ arrives at $l = 0$ as in Fig. 6.4. Since $R(T)$ corresponds to the velocity of front solution of (6.3), this means the front solution can be reflected by the inhomogeneity $\lambda H(x)$. In fact, (6.6) has an unstable equilibrium $P^* = (l^*(\varepsilon), 0)$ satisfying $H_2(l^*(\varepsilon)) = 0$ and $l^*(\varepsilon) < 0$ when $\Lambda > 0$. Fig. 6.3 is a conceptual figure of the flow of (6.6) around P^* in (l, R) plane.

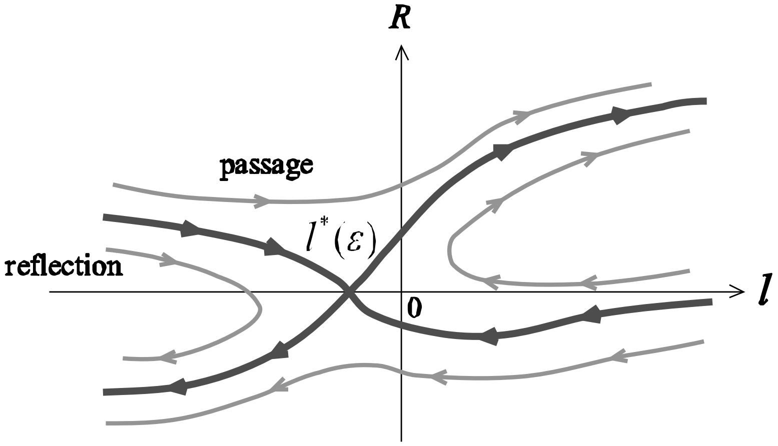

Fig. 6.3. The flow of (6.6) when $\Lambda > 0$ in (l, R) phase plane.

We can also observe in (6.3) that if the initial velocity of a front solution is very small, the front solution goes back even if the initial velocity is positive as in Fig. 6.5. This is a remarkable fact because the inhomogeneity $\lambda H(x)$ is nonnegative and it is intuitively expected that such a nonnegative inhomogeneity merely enhances the velocity of the front solution. Thus, the result of the dynamics is different from intuitive considerations, which shows the importance to know the precise information of eigenfunctions.

For the case $\Lambda < 0$ $(\lambda < 0)$, which implies the jump-down case at $x = 0$, we easily find the front solution is always reflected by the inhomogeneity $\lambda H(x)$ as in Fig. 6.6.

Finally, we briefly show an occurrence of oscillating behaviors by the inhomogeneity when $h(x) := H(x) - a$ for $0 < a < 1$. Then

$$\begin{aligned} H_2(l) &:= \frac{1}{\varepsilon}\langle h(x+l), \phi_1^* \rangle_2 = \frac{1}{\varepsilon}\left\{(1-a)\int_{-l}^{\infty} \phi_1^*(x)\,dx - a\int_{-\infty}^{-l} \phi_1^*(x)\,dx\right\} \\ &= 4\sqrt{2}\left\{(1-2a)\left(1 - \frac{1}{3}\log 2\right) \pm \left(1 - 2W_0\left(\frac{|l|}{\varepsilon}\right) - \frac{1}{3}\log\frac{2}{1+e^{-|l|/\sqrt{D}}}\right)\right\} \\ &\quad + O(\varepsilon) \end{aligned}$$

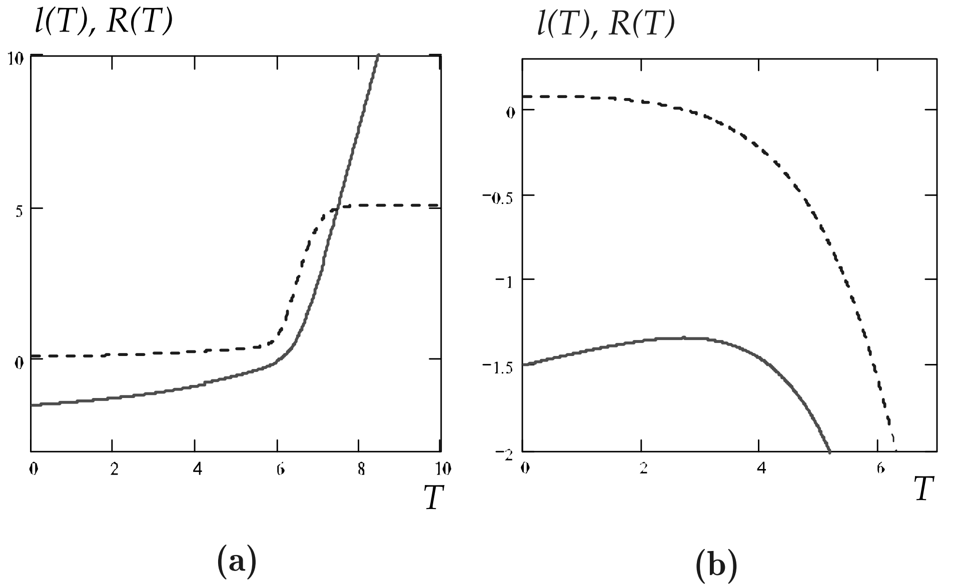

Fig. 6.4. The dynamics of solutions of (6.6) for $\Lambda > 0$ ($\Lambda = 2.0$ and $l(0) = -1.5$). Solid lines denote $l(T)$ and broken lines do $R(T)$. (a) The case when $R(0)$ is not so small ($R(0) = 0.1$). (b) The case when $R(0)$ is positive but sufficiently small ($R(0) = 0.08$).

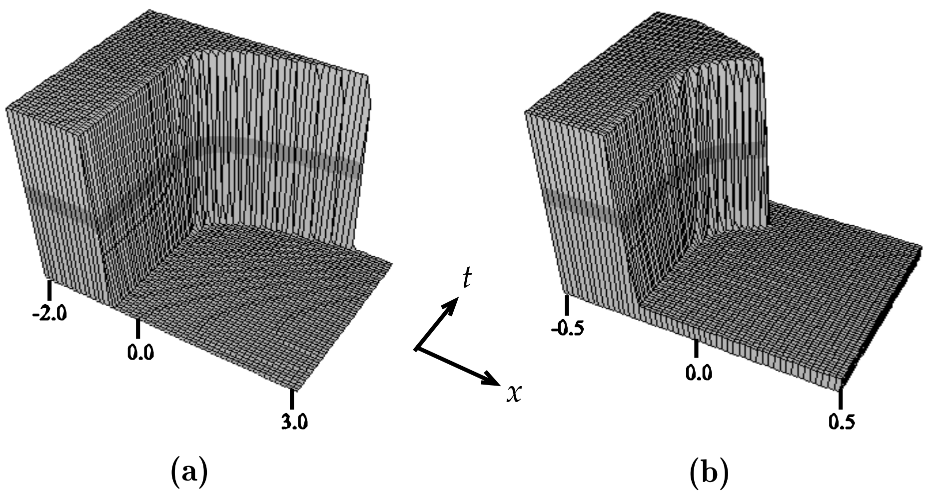

Fig. 6.5. The dynamics of front solutions of (6.3) for $\lambda > 0$ ($\lambda = 3.0 \times 10^{-3}$, which corresponds to the value $\lambda = \Lambda\tau_0|\eta|^{3/2}$). u component is described. (a) The case when the initial velocity is not so small. (b) The case when the initial velocity is positive but sufficiently small.

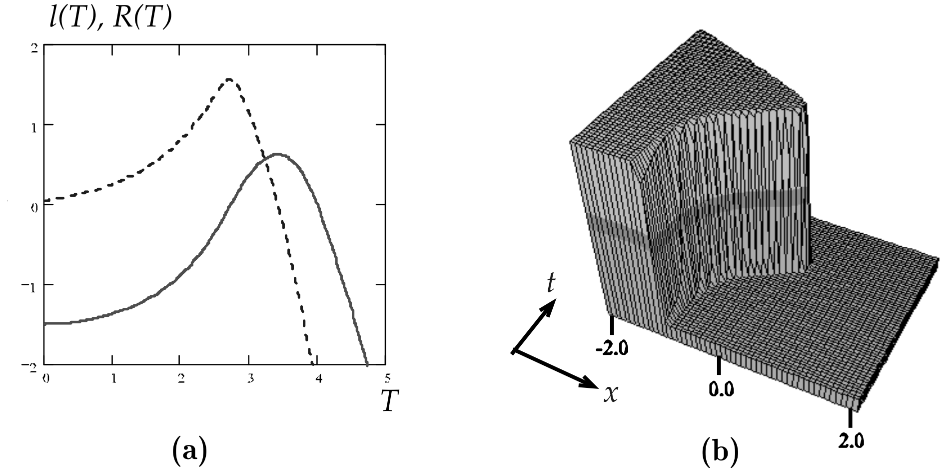

Fig. 6.6. (a) The dynamics of solutions of (6.6) for $\Lambda < 0$ ($\Lambda = -2.0$). Solid lines denote $l(T)$ and broken lines do $R(T)$. (b) The dynamics of front solutions of (6.3) for $\lambda < 0$ ($\lambda = -3.0 \times 10^{-3}$, which corresponds to the value $\lambda = \Lambda\tau_0|\eta|^{3/2}$). u component is described.

for $l > 0$ and $l < 0$, respectively. Since $H_2(+\infty) = 8\sqrt{2}\big(1 - \frac{1}{3}\log 2\big)(1-a) > 0$ and $H_2(-\infty) = -8\sqrt{2}\big(1 - \frac{1}{3}\log 2\big)a < 0$, a negatively large $\Lambda < 0$ implies the occurrence of oscillating motions in (6.6). In fact, an oscillating motion of front solutions for reaction-diffusion system (6.3) is observed as in Fig. 6.7.

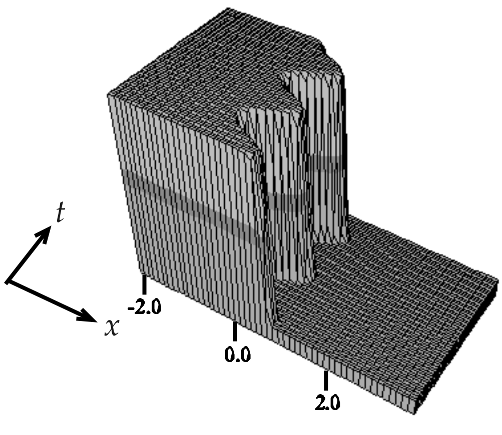

Fig. 6.7. The oscillating motion of front solutions of (6.3) for $\lambda < 0$ and $h(x) = H(x) - 0.5$ with the same value of λ as Fig. 6.6. u component is described.

More detailed analysis of front solutions by using the reduced ODE (6.6) will be mentioned in the forthcoming paper ([2]).

Acknowledgements. Ei thanks for the support of the Grant-in-aid for Scientific Research 16540200. Ikeda also thanks for the support of the Grant-in-aid for Scientific Research 15340038, 18204010 and 19540121.

References

[1] S.-I. Ei, M. Mimura and M. Nagayama, Pulse-pulse interaction in reaction-diffusion systems. Physica D., **165** (2002), 176–198.

[2] S.-I. Ei, H. Ikeda and T. Kusaka, The dynamics of interacting fronts in reaction-diffusion systems with bistable nonlinearity, in preparation.

[3] P. Gridiron, Pattern and waves. Oxford University Press, New York, 1991.

[4] H. Ikeda and T. Ikeda, Bifurcation phenomena from standing pulse solutions in some reaction-diffusion systems. J. Dynam. Diff. Eqs., **12** (2000), 117–167.

[5] H. Ikeda and M. Mimura, Wave-blocking phenomena in bistable reaction-diffusion systems. SIAM J. Appl. Math., **49** (1989), 515–538.

[6] H. Ikeda, M. Mimura and Y. Nishiura, Global bifurcation phenomena of travelling wave solutions for some bistable reaction-diffusion systems. Nonlinear Anal., **13** (1989), 507–526.

[7] A. Hagberg and E. Meron, Pattern formation in non-gradient reaction-diffusion systems: the effects of front bifurcations. Nonlinearity, **7** (1994), 805–835.

[8] A. Hagberg and E. Meron, Complex patterns in reaction-diffusion systems: a tale of two front instabilities. Chaos, **4** (1994), 477–484.

[9] A. Hagberg, E. Meron, I. Rubinstein and B. Maltzman, Order parameter equations for front bifurcations: planar and circular fronts. Phys. Rev. E., **55** (1997), 4450–4457.

[10] S.P. Hastings, On the existence of homoclinic and periodic orbits for the FitzHugh–Nagumo equations. Quart. J. Math. Oxford, **27** (1976), 123–134.

[11] H. Kokubu, Y. Nishiura and H. Oka, Heteroclinic and homoclinic bifurcations in bistable reaction diffusion systems. J. Diff. Eqs., **86** (1990), 260–341.

[12] J.D. Murray, Mathematical biology. Springer-Verlag, Berlin, 1989.

[13] Y. Nishiura, M. Mimura, H. Ikeda and H. Fujii, Singular limit analysis of stability of travelling wave solutions in bistable reaction-diffusion systems. SIAM J. Math. Anal., **21** (1990), 85–122.

[14] Y. Nishiura, Y. Oyama and K.-I. Ueda, Dynamics of traveling pulses in heterogeneous media of jump type. Hokkaido Math. J., **36** (2007), 207–242.

[15] P. Ortoleva and J. Ross, Theory of propagation of discontinuities in kinetic systems with multiple time scales: fronts, front multiplicity, and pulses. J. Chem. Phys., **63** (1975), 3398–3408.

[16] J.P. Pauwelussen, Nerve impulse propagation in a branching nerve system: a simple model. Physica D., **4** (1981), 67–88.

[17] J.P. Pauwelussen, One way traffic pulses in a neuron. J. Math. Biology, **15** (1982), 151–171.

[18] J. Rinzel and D. Terman, Propagation phenomena in a bistable reaction-diffusion system. SIAM J. Appl. Math., **42** (1982), 1111–1137.