Asymptotic Behavior of the Solutions for One-Dimensional Equations of a Viscous Reactive Gas

Dedicated to Professor Atusi Tani on his sixtieth birthday

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We consider the asymptotic behavior of the complete system of equations governing a heat-conductive, reactive, compressible viscous gas bounded by two infinite parallel plates. The motion is proved to tend towards the corresponding constant state, as time tends to infinity. Moreover, the decay rate is investigated.

Key words: heat-conductive gas, reactive gas, compressible viscous gas, asymptotic stability

1. Introduction

In this paper, we investigate the asymptotic behavior of the complete system of equations governing a heat-conductive, reactive, compressible viscous gas bounded by two parallel infinite plates. We assume that its describing parameters vary spatially only in one direction perpendicular to the plates. Then, in the Lagrangian mass coordinate, such a model is well-formulated by the system of equations (see [5]):

$$\begin{cases} v_t - u_x = 0, \\ u_t + k\left(\dfrac{\theta}{v}\right)_x = \lambda_1\left(\dfrac{u_x}{v}\right)_x, \\ \theta_t = \lambda_2\left(\dfrac{\theta_x}{v}\right)_x + \lambda_1\dfrac{u_x^2}{v} - k\dfrac{\theta u_x}{v} + \delta f(v,\theta,z), \\ z_t = \lambda_3\left(\dfrac{z_x}{v^2}\right)_x - f(v,\theta,z), \end{cases} \tag{1.1}$$

in $Q = \{(x,t) \mid 0 < x < 1,\ t > 0\}$, with the boundary conditions

$$u(0,t) = u(1,t) = \theta_x(0,t) = \theta_x(1,t) = z_x(0,t) = z_x(1,t) = 0, \quad t \geq 0, \tag{1.2}$$

and the initial conditions

$$v(x,0) = v_0(x), \quad u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \quad z(x,0) = z_0(x), \quad x \in [0,1], \tag{1.3}$$

where v, u, θ, and z denote the specific volume, the velocity, the temperature, and the concentration of the unburned fuel, respectively, while k, λ_i $(i = 1, 2, 3)$ and δ are positive constants. The function f represents the intensity of the chemical reaction, whose typical example is (see [5])

$$f(v, \theta, z) = \alpha k v^{-m+1} z^m \exp\left(\frac{\theta - 1}{\alpha\theta}\right), \tag{1.4}$$

where α and m are positive constants. We note that the exponent m means the overall sum of the individual reaction orders for fuel and oxidizer, which will be an important parameter of our calculation.

As basic assumptions, we assume

$$v_0, \theta_0, z_0 \in H^1(0, 1), \quad u_0 \in H_0^1(0, 1), \tag{1.5}$$

$$C_0^{-1} \le v_0(x) \le C_0, \quad C_0^{-1} \le \theta_0(x) \le C_0, \quad 0 \le z_0(x) \le C_0, \quad x \in [0, 1], \tag{1.6}$$

with some constant $C_0 > 1$, and normalize as

$$\int_0^1 v_0(x)\, dx = 1. \tag{1.7}$$

Furthermore, we assume (1.4) with

$$0 < m \le 2. \tag{1.8}$$

For this initial-boundary value problem (1.1)–(1.3), the existence and the uniqueness of a generalized global-in-time solution was studied by many authors including Bressan [1], Bebernes–Bressan [2, 3], Bebernes–Eberly [4], and so on. Among them, Bebernes–Bressan [3] showed that for arbitrary fixed $T > 0$, there exists a unique set of functions (v, u, θ, z), belonging to

$$v \in L^\infty(0, T; H^1(0, 1)), \quad v_t \in L^\infty(0, T; L^2(0, 1)), \tag{1.9}$$

$$(u, \theta, z) \in L^\infty(0, T; H^1(0, 1)) \cap L^2(0, T; H^2(0, 1)), \quad (u_t, \theta_t, z_t) \in L^2(0, T; L^2(0, 1)), \tag{1.10}$$

satisfying equations (1.1) almost everywhere and initial-boundary conditions (1.2), (1.3) in the sense of traces. The essential point of their proof was to obtain a priori bound of the form

$$C^{-1} \le v(x, t) \le C, \quad C^{-1} \le \theta(x, t) \le C, \quad 0 \le z(x, t) \le C, \quad (x, t) \in [0, 1] \times [0, T], \tag{1.11}$$

with some constant $C > 1$. We note that their constant C depends on T, so that they did not obtain uniform boundedness of the solution with respect to time, or the asymptotic behavior of the solution.

From this points of view, Yanagi [6] showed that the solution tends exponentially towards a constant state as time tends to infinity, provided that the function f is given by (1.4) with $m = 1$; namely, $f = \alpha k z \exp\left(\frac{\theta-1}{\alpha\theta}\right)$.

In the current paper, we aim to extend this results for $0 < m \leq 2$. To state our results precisely, let us consider the following stationary problem associated with (1.1)–(1.3):

$$\begin{cases} -\tilde{u}_x = 0, \\ k\left(\dfrac{\tilde{\theta}}{\tilde{v}}\right)_x = \lambda_1\left(\dfrac{\tilde{u}_x}{\tilde{v}}\right)_x, \\ \lambda_2\left(\dfrac{\tilde{\theta}_x}{\tilde{v}}\right)_x + \lambda_1\dfrac{\tilde{u}_x^2}{\tilde{v}} - k\dfrac{\tilde{\theta}\tilde{u}_x}{\tilde{v}} + \alpha\delta k\tilde{v}^{-m+1}\tilde{z}^m\exp\left(\dfrac{\tilde{\theta}-1}{\alpha\tilde{\theta}}\right) = 0, \\ \lambda_3\left(\dfrac{\tilde{z}_x}{\tilde{v}^2}\right)_x - \alpha k\tilde{v}^{-m+1}\tilde{z}^m\exp\left(\dfrac{\tilde{\theta}-1}{\alpha\tilde{\theta}}\right) = 0, \end{cases} \tag{1.12}$$

$$\tilde{u}(0) = \tilde{u}(1) = \tilde{\theta}'(0) = \tilde{\theta}'(1) = \tilde{z}'(0) = \tilde{z}'(1) = 0, \tag{1.13}$$

with conditions

$$\int_0^1 \tilde{v}\, dx = 1, \tag{1.14}$$

and

$$\int_0^1 \left(\frac{1}{2}\tilde{u}^2 + \tilde{\theta} + \delta\tilde{z}\right) dx = A, \tag{1.15}$$

where

$$A = \int_0^1 \left(\frac{1}{2}u_0^2 + \theta_0 + \delta z_0\right) dx. \tag{1.16}$$

We note that these conditions come from the following (see [6]):

Proposition 1.1. *We have*

$$\int_0^1 v\, dx = 1, \tag{1.17}$$

$$\int_0^1 \left(\frac{1}{2}u^2 + \theta + \delta z\right) dx = A, \tag{1.18}$$

for any $t \geq 0$.

This stationary problem (1.12)–(1.16) can be solved by the following procedure: it is easily seen from $(1.12)_1$ and (1.13) that $\tilde{u} = 0$. Integrating $(1.12)_4$ (or $(1.12)_3$) over $[0,1]$ together with (1.13) yields $\int_0^1 \tilde{v}^{-m+1}\tilde{z}^m \exp\left(\frac{\tilde{\theta}-1}{\alpha\tilde{\theta}}\right) dx = 0$. Noting that $C^{-1} \leq \tilde{v} \leq C$, $\tilde{z} \geq 0$, which are consequences of (1.11), we have $\tilde{z} = 0$. Therefore, it follows from $(1.12)_3$ and (1.13) that $\tilde{\theta} = \text{const.}$, and from $(1.12)_2$ that $\tilde{v} = \text{const.}$ Using (1.14) and (1.15), we conclude $\tilde{v} = 1$ and $\tilde{\theta} = A$.

Our main result is the following:

Theorem 1.1. *Assume* (1.5)–(1.8). *Let* (v, u, θ, z) *be a unique generalized solution to* (1.1)–(1.3) *in* Q. *Then there exist positive constants* C *and* ν *such that the following inequality is satisfied:*

$$\|(v-1, u, \theta - A, z)(\,\cdot\,, t)\|_1 \leq \begin{cases} C \exp\left(-\nu t\right) & \text{for } 0 < m \leq 1, \\ C(1+t)^{-\frac{1}{2(m-1)}} & \text{for } 1 < m \leq 2, \end{cases} \tag{1.19}$$

where $\|\cdot\|_1$ *denotes a norm in* $H^1(0,1)$.

In the following section, we shall obtain the uniform boundedness of the solution, namely the a priori bound of the form (1.11) with the constant C being independent of T. The proof of Theorem 1.1 shall be done in Sec. 3.

2. Uniform boundedness of the solution

In this section, we shall prove the following theorem:

Theorem 2.1. *Under the same assumptions as in* Theorem 1.1, *we have*

$$\sup_{t \geq 0} \|(v, u, \theta, z)(\,\cdot\,, t)\|_1 \leq C, \tag{2.1}$$

$$\begin{aligned} &C^{-1} \leq v(x,t) \leq C, \quad |u(x,t)| \leq C, \quad C^{-1} \leq \theta(x,t) \leq C, \\ &0 \leq z(x,t) \leq C, \quad \text{for all } (x,t) \in \overline{Q}, \end{aligned} \tag{2.2}$$

where, and in what follows, the letter C denotes a positive universal constant depending only on the given data. We begin with the following four propositions.

Proposition 2.1. *There exists a constant* $C > 1$ *such that*

$$C^{-1} \leq \bar{\theta}, \quad t \geq 0, \tag{2.3}$$

and

$$\int_0^\infty \int_0^1 \left(\frac{\theta_x^2}{v\theta^2} + \frac{u_x^2}{v\theta} + \frac{f}{\theta} \right) dx\, dt \leq C, \tag{2.4}$$

where

$$\bar{\theta} = \int_0^1 \theta \, dx. \tag{2.5}$$

Proposition 2.2. *We have*

$$\theta(x,t) \le C(1 + |v|_\infty I(t)), \tag{2.6}$$

for any $(x,t) \in \overline{Q}$, *where* $|\cdot|_\infty$ *denotes a sup-norm in* $\overline{Q}$, *while*

$$I(t) = \int_0^1 \frac{\theta_x^2}{v\theta^2}\,dx. \tag{2.7}$$

Here, we note that the estimation

$$\int_0^\infty I(t)\,dt \le C, \tag{2.8}$$

which will be used frequently, is already obtained in (2.4).

Proposition 2.3. *There exists a constant* $C > 1$ *such that*

$$C^{-1} \le v(x,t) \le C \tag{2.9}$$

holds for all $(x,t) \in \overline{Q}$.

Corollary 2.1. *There exists a positive constant* C *such that*

$$\int_0^\infty \max_{[0,1]} u^2\,dt \le C. \tag{2.10}$$

Corollary 2.2. *There exists a positive constant* C *such that*

$$\left|\theta^{1/2} - \bar\theta^{1/2}\right| \le C I(t)^{1/2}. \tag{2.11}$$

Proposition 2.4. *We have*

$$\int_0^\infty \int_0^1 (\theta - \bar\theta)^2\,dx\,dt \le C. \tag{2.12}$$

Since the proofs of the propositions 2.1–2.4 are similar to those in reference [6], we omit the details.

Proposition 2.5. *For any* $l \ge 0$, *there exists a constant* $C > 0$ *such that*

$$\int_0^1 z^l\,dx + \int_0^\infty \int_0^1 (z^{m+l} + z_x^2)\,dx\,dt \le C. \tag{2.13}$$

Proof. Multiplying $(1.1)_4$ by z^l and integrating it over $[0,1]$ implies

$$\frac{1}{l+1}\frac{d}{dt}\int_0^1 z^{l+1}\,dx + \lambda_3 l \int_0^1 \frac{z^{l-1}z_x^2}{v^2}\,dx + \int_0^1 f z^l\,dx = 0, \tag{2.14}$$

which together with (2.9) gives

$$\int_0^1 z^{l+1}\,dx + \int_0^\infty \int_0^1 z^{l-1} z_x^2\,dx\,dt \le C, \tag{2.15}$$

for any $l \ge 0$. Especially, we have

$$\int_0^\infty \int_0^1 z_x^2\,dx\,dt \le C, \tag{2.16}$$

and since $z^l \le C(1 + z^{l+1})$, we can easily obtain

$$\int_0^1 z^l\,dx \le C. \tag{2.17}$$

Next, it follows from (2.14) that

$$\frac{1}{l+1}\frac{d}{dt}\int_0^1 z^{l+1}\,dx + \beta\int_0^1 g(\theta)v^{-m+1}z^{m+l}\,dx \le 0, \tag{2.18}$$

where $\beta = \alpha k \exp\left(\frac{1}{\alpha}\right)$, and $g(\theta) = \exp\left(\frac{-1}{\alpha\theta}\right)$. Since $g(\bar\theta) - g(\theta) \le 0$ for $\theta \ge \bar\theta$, and $g'(\eta) \le 4\alpha e^{-2}$ for $\eta > 0$, we have from (2.18)

$$\begin{aligned}
&\frac{1}{l+1}\frac{d}{dt}\int_0^1 z^{l+1}\,dx + \beta\int_0^1 g(\bar\theta)v^{-m+1}z^{m+l}\,dx \\
&\le \beta\int_0^1 (g(\bar\theta) - g(\theta))v^{-m+1}z^{m+l}\,dx \\
&\le \beta\int_{\theta<\bar\theta} (g(\bar\theta) - g(\theta))v^{-m+1}z^{m+l}\,dx \\
&= \beta\int_{\theta<\bar\theta}\left(\int_0^1 g'(\theta + \sigma(\bar\theta - \theta))\,d\sigma\right)(\bar\theta - \theta)v^{-m+1}z^{m+l}\,dx \\
&\le 4\alpha\beta e^{-2}\int_{\theta<\bar\theta}(\bar\theta - \theta)v^{-m+1}z^{m+l}\,dx \\
&= 4\alpha\beta e^{-2}\int_{\theta<\bar\theta}\big(\bar\theta^{1/2} + \theta^{1/2}\big)\big(\bar\theta^{1/2} - \theta^{1/2}\big)v^{-m+1}z^{m+l}\,dx \\
&\le 8\sqrt{A}\alpha\beta e^{-2}\int_{\theta<\bar\theta}\big(\bar\theta^{1/2} - \theta^{1/2}\big)v^{-m+1}z^{m+l}\,dx \\
&\le CI(t)^{1/2}\int_0^1 z^{m+l}\,dx \\
&\le \varepsilon\int_0^1 z^{m+l}\,dx + CI(t)\int_0^1 z^{m+l}\,dx \\
&\le \varepsilon\int_0^1 z^{m+l}\,dx + CI(t)
\end{aligned} \tag{2.19}$$

for any $\varepsilon > 0$, where in the seventh line, we have used (1.18), in the eighth line, we have used (2.9), (2.11), and in the final line, we have used (2.17). Considering (2.3), (2.9), and choosing ε sufficiently small, we get

$$\frac{d}{dt}\int_0^1 z^{l+1}\,dx + C^{-1}\int_0^1 z^{m+l}\,dx \le CI(t), \tag{2.20}$$

which together with (2.8) implies

$$\int_0^\infty\int_0^1 z^{m+l}\,dx\,dt \le C, \tag{2.21}$$

which completes the proof of Prop. 2.5. □

PROPOSITION 2.6. *We have*

$$\int_0^\infty\int_0^1 (A-\theta)^2\,dx\,dt \le C. \tag{2.22}$$

Proof. From (2.12), (1.18), (2.10), (2.13), and (1.8), the inequality (2.22) is derived as follows:

$$\begin{aligned}\int_0^\infty\int_0^1 (A-\theta)^2\,dx\,dt &\le C\int_0^\infty\int_0^1 \{(A-\bar\theta)^2 + (\theta-\bar\theta)^2\}\,dx\,dt\\ &\le C + C\int_0^\infty\int_0^1 (u^4+z^2)\,dx\,dt\\ &\le C + C\int_0^\infty \max_{[0,1]} u^2\,dt + C\int_0^\infty\int_0^1 z^2\,dx\,dt\\ &\le C.\end{aligned} \tag{2.23}$$

□

PROPOSITION 2.7. *There exists a constant $C>0$ such that*

$$\int_0^1 (u^2+v-\log v-1)\,dx + \int_0^\infty\int_0^1 u_x^2\,dx\,dt \le C. \tag{2.24}$$

Proof. We rewrite $(1.1)_2$ as

$$u_t + k\left(\frac{\theta-A}{v}\right)_x + k\left(\frac{A}{v}\right)_x = \lambda_1\left(\frac{u_x}{v}\right)_x. \tag{2.25}$$

Multiplying (2.25) by u and integrating it over $[0,1]$ gives

$$\begin{aligned}
&\frac{d}{dt}\int\left\{\frac{1}{2}u^2+kA(v-\log v-1)\right\}dx+\lambda_1\int_0^1\frac{u_x^2}{v}\,dx\\
&=k\int_0^1\frac{(\theta-A)u_x}{v}\,dx\\
&\le\frac{\lambda_1}{2}\int_0^1\frac{u_x^2}{v}\,dx+C\int_0^1(\theta-A)^2\,dx, \qquad (2.26)
\end{aligned}$$

which together with (2.22) completes the proof of Prop. 2.7. □

Proposition 2.8. *We have*

$$\int_0^1\{(u^2+\theta-\bar\theta+z)^2+u^4\}\,dx+\int_0^\infty\int_0^1(u^2u_x^2+\theta_x^2)\,dx\,dt\le C. \qquad (2.27)$$

Proof. Multiplying $(1.1)_2$ by u, multiplying $(1.1)_3$ by -1 and integrating it over $[0,1]$, multiplying $(1.1)_4$ by δ, adding these three results together with $(1.1)_3$, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}u^2+\theta-\bar\theta+\delta z\right)\\
&=\left(-k\frac{u\theta}{v}+\lambda_1\frac{uu_x}{v}+\lambda_2\frac{\theta_x}{v}+\delta\lambda_3\frac{z_x}{v^2}\right)_x\\
&\quad-\lambda_1\int_0^1\frac{u_x^2}{v}\,dx+k\int_0^1\frac{\theta u_x}{v}\,dx-\delta\int_0^1 f\,dx. \qquad (2.28)
\end{aligned}$$

Multiplying (2.28) by $\frac{1}{2}u^2+\theta-\bar\theta+\delta z$, integrating it over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{1}{2}u^2+\theta-\bar\theta+\delta z\right)^2dx+\int_0^1\left(\lambda_1\frac{u^2u_x^2}{v}+\lambda_2\frac{\theta_x^2}{v}+\delta^2\lambda_3\frac{z_x^2}{v^2}\right)dx\\
&+\lambda_1\int_0^1\frac{u_x^2}{v}\,dx\int_0^1\left(\frac{1}{2}u^2+\delta z\right)dx+\delta\int_0^1 f\,dx\int_0^1\left(\frac{1}{2}u^2+\delta z\right)dx\\
&=\int_0^1\left(k\frac{u^2u_x\theta}{v}+k\frac{u\theta\theta_x}{v}+k\delta\frac{u\theta z_x}{v}-\lambda_1\frac{uu_x\theta_x}{v}-\delta\lambda_1\frac{uu_xz_x}{v}\right.\\
&\qquad\left.-\lambda_2\frac{uu_x\theta_x}{v}-\delta\lambda_2\frac{\theta_xz_x}{v}-\delta\lambda_3\frac{uu_xz_x}{v^2}-\delta\lambda_3\frac{\theta_xz_x}{v^2}\right)dx\\
&\quad+k\int_0^1\frac{\theta u_x}{v}\,dx\int_0^1\left(\frac{1}{2}u^2+\delta z\right)dx\\
&\le\varepsilon\int_0^1\frac{\theta_x^2}{v}\,dx\\
&\quad+C(1+\varepsilon^{-1})\int_0^1\{u^2u_x^2+u^2\theta^2+z_x^2+(\theta-\bar\theta)^2+u_x^2+u^4+z^2\}\,dx, \qquad (2.29)
\end{aligned}$$

for any $\varepsilon > 0$, where we have used (1.18). Choosing ε properly, we get

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \left(\frac{1}{2}u^2 + \theta - \bar{\theta} + z\right)^2 dx + C^{-1}\int_0^1 \theta_x^2\, dx \\ &\le C\int_0^1 \{u^2u_x^2 + u^2\theta^2 + z_x^2 + (\theta - \bar{\theta})^2 + u_x^2 + u^4 + z^2\}\, dx. \end{aligned} \tag{2.30}$$

Multiplying $(1.1)_2$ by u^3 and integrating it over $[0,1]$ yields

$$\begin{aligned} \frac{1}{4}\frac{d}{dt}\int_0^1 u^4\, dx + 3\lambda_1\int_0^1 \frac{u^2u_x^2}{v}\, dx &= 3k\int_0^1 \frac{u^2u_x\theta}{v}\, dx \\ &\le \lambda_1\int_0^1 \frac{u^2u_x^2}{v}\, dx + C\int_0^1 \frac{u^2\theta^2}{v}\, dx, \end{aligned} \tag{2.31}$$

which immediately leads to

$$\frac{d}{dt}\int_0^1 u^4\, dx + C^{-1}\int_0^1 u^2u_x^2\, dx \le C\int_0^1 u^2\theta^2\, dx. \tag{2.32}$$

Multiplying (2.30) by ε_0, adding it to (2.32), choosing ε_0 sufficiently small, we have

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \left\{\varepsilon_0\left(\frac{1}{2}u^2 + \theta - \bar{\theta} + z\right)^2 + u^4\right\} dx + C^{-1}\int_0^1 (u^2u_x^2 + \theta_x^2)\, dx \\ &\le C\int_0^1 \{u^2\theta^2 + z_x^2 + (\theta - \bar{\theta})^2 + u_x^2 + u^4 + z^2\}\, dx \\ &\le C\int_0^1 u^2\{1 + (\theta - \bar{\theta})^2\}\, dx \\ &\quad + C\int_0^1 \{z_x^2 + (\theta - \bar{\theta})^2 + u_x^2 + u^4 + z^2\}\, dx \\ &\le C\max_{[0,1]} u^2 \int_0^1 (\theta - \bar{\theta})^2\, dx \\ &\quad + C\int_0^1 \{u^2 + z_x^2 + (\theta - \bar{\theta})^2 + u_x^2 + u^4 + z^2\}\, dx. \end{aligned} \tag{2.33}$$

Considering (2.10), (2.13), (2.12), and (2.24), we have from integrating (2.33) over $[0,t]$

$$\begin{aligned} &\int_0^1 \{(u^2 + \theta - \bar{\theta} + z)^2 + u^4\}\, dx + \int_0^t\int_0^1 (u^2u_x^2 + \theta_x^2)\, dx\, d\tau \\ &\le C + C\int_0^t \left(\max_{[0,1]} u^2\right)(\tau)\int_0^1 (\theta - \bar{\theta})^2\, dx\, d\tau, \end{aligned} \tag{2.34}$$

which together with (2.13) gives

$$\int_0^1 (\theta-\bar\theta)^2\,dx \le 2\int_0^1 (u^2+\theta-\bar\theta+z)^2 + 2\int_0^1 (u^2+z)^2\,dx$$
$$\le C + C\int_0^t \Big(\max_{[0,1]} u^2\Big)(\tau)\int_0^1 (\theta-\bar\theta)^2\,dx\,d\tau. \qquad (2.35)$$

Using (2.10) and Gronwall's inequality, we have from (2.35)

$$\int_0^1 (\theta-\bar\theta)^2\,dx \le C. \qquad (2.36)$$

This together with (2.34) and (2.10) completes the proof of Prop. 2.8. □

Proposition 2.9. *There exists a constant $C>0$ such that*

$$\int_0^\infty \max_{[0,1]} (\theta-\bar\theta)^2\,dt \le C. \qquad (2.37)$$

Proof. Using the point $x(t)\in[0,1]$ which satisfies $\theta(x(t),t)=\bar\theta$, we obtain

$$|\theta-\bar\theta| = \left|\int_{x(t)}^x \theta_x\,dx'\right| \le \left(\int_0^1 v\theta^2\,dx\right)^{1/2}\left(\int_0^1 \frac{\theta_x^2}{v\theta^2}\,dx\right)^{1/2}, \qquad (2.38)$$

from which, we get

$$(\theta-\bar\theta)^2 \le CI(t)\int_0^1 \theta^2\,dx \le CI(t)\int_0^1 \{1+(\theta-\bar\theta)^2\}\,dx \le CI(t), \qquad (2.39)$$

where we have used (2.36). It is easily seen from (2.39) and (2.8) that (2.37) is satisfied. □

Proposition 2.10. *We have*

$$\int_0^1 v_x^2\,dx + \int_0^\infty\int_0^1 \theta v_x^2\,dx\,dt \le C. \qquad (2.40)$$

Proof. Multiplying $(1.1)_2$ by $\frac{v_x}{v}$ and integrating it over $[0,1]$ implies

$$\frac{d}{dt}\int_0^1 \left(\frac{\lambda_1}{2}\frac{v_x^2}{v^2} - \frac{uv_x}{v}\right)dx + k\int_0^1 \frac{\theta v_x^2}{v^3}\,dx$$
$$= k\int_0^1 \frac{\theta_x v_x}{v^2}\,dx + \int_0^1 \frac{u_x^2}{v}\,dx$$
$$\le \frac{k}{2}\int_0^1 \frac{\theta v_x^2}{v^3}\,dx + \frac{1}{2k}\int_0^1 \frac{\theta_x^2}{v\theta}\,dx + \int_0^1 \frac{u_x^2}{v}\,dx, \qquad (2.41)$$

from which, we obtain

$$\frac{d}{dt}\int_0^1 \left(\frac{\lambda_1}{2}\frac{v_x^2}{v^2} - \frac{uv_x}{v}\right) dx + \frac{k}{2}\int_0^1 \frac{\theta v_x^2}{v^3}\, dx \le C\int_0^1 \left(\frac{\theta_x^2}{v\theta^2} + \theta_x^2 + u_x^2\right) dx. \quad (2.42)$$

Noting (2.4), (2.27), (2.24), and the inequality $\left|\frac{uv_x}{v}\right| \le \frac{\lambda_1}{4}\frac{v_x^2}{v^2} + \frac{1}{\lambda_1}u^2$, we conclude (2.40) by integrating (2.42) over $[0,\infty)$. □

Proposition 2.11. *There exists a constant $C > 0$ such that*

$$\int_0^1 u_x^2\, dx + \int_0^\infty\int_0^1 u_{xx}^2\, dx\, dt \le C. \quad (2.43)$$

Proof. Multiplying $(1.1)_2$ by $-u_{xx}$, integrating it over $[0,1]$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 u_x^2\, dx + \lambda_1\int_0^1 \frac{u_{xx}^2}{v}\, dx \\ &= \int_0^1 \left(k\frac{\theta_x}{v} - k\frac{\theta v_x}{v^2} + \lambda_1\frac{u_x v_x}{v^2}\right) u_{xx}\, dx \\ &\le \varepsilon\int_0^1 \frac{u_{xx}^2}{v}\, dx + C\varepsilon^{-1}\int_0^1 (\theta_x^2 + \theta^2 v_x^2 + u_x^2 v_x^2)\, dx, \end{aligned} \quad (2.44)$$

for any $\varepsilon > 0$. Here, the third and the last terms in the right hand side of (2.44) can be estimated as

$$\begin{aligned} \int_0^1 \theta^2 v_x^2\, dx &\le \int_0^1 \{(\theta-\bar\theta)^2 + 2\theta\bar\theta\} v_x^2\, dx \\ &\le C\max_{[0,1]}(\theta-\bar\theta)^2 + C\int_0^1 \theta v_x^2\, dx, \end{aligned} \quad (2.45)$$

$$\begin{aligned} \int_0^1 u_x^2 v_x^2\, dx &\le \int_0^1 v_x^2\, dx \left(\varepsilon^2\int_0^1 u_{xx}^2\, dx + C\varepsilon^{-2}\int_0^1 u_x^2\, dx\right) \\ &\le C\varepsilon^2\int_0^1 \frac{u_{xx}^2}{v}\, dx + C\varepsilon^{-2}\int_0^1 u_x^2\, dx, \end{aligned} \quad (2.46)$$

where we have used (2.40). Substituting (2.45) and (2.46) into (2.44), choosing ε sufficiently small, we have (2.43) from (2.27), (2.37), (2.40), and (2.24). □

By using the equality: $u = \int_0^x u_x\, dx'$, the following is easily obtained from Prop. 2.11:

Corollary 2.3.

$$|u|_\infty \le C. \quad (2.47)$$

Proposition 2.12. *We have*

$$\int_0^1 \theta_x^2\, dx + \int_0^\infty\int_0^1 \theta_{xx}^2\, dx\, dt \le C. \quad (2.48)$$

Proof. Multiplying $(1.1)_3$ by $-\theta_{xx}$ and integrating it over $[0,1]$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \theta_x^2\,dx + \lambda_2\int_0^1 \frac{\theta_{xx}^2}{v}\,dx \\
&= \int_0^1 \left(\lambda_2\frac{v_x\theta_x}{v^2} - \lambda_1\frac{u_x^2}{v} + k\frac{\theta u_x}{v} - \delta f\right)\theta_{xx}\,dx \\
&\le \varepsilon\int_0^1 \frac{\theta_{xx}^2}{v}\,dx + C\varepsilon^{-1}\int_0^1 (v_x^2\theta_x^2 + u_x^4 + \theta^2 u_x^2 + z^{2m})\,dx, \qquad (2.49)
\end{aligned}$$

for any $\varepsilon > 0$. Here, the second, the third, and the fourth terms in the right hand side of (2.49) can be estimated as

$$\begin{aligned}
\int_0^1 v_x^2\theta_x^2\,dx &\le C\int_0^1 v_x^2\,dx\left(\varepsilon^2\int_0^1 \theta_{xx}^2\,dx + \varepsilon^{-2}\int_0^1 \theta_x^2\,dx\right) \\
&\le C\varepsilon^2\int_0^1 \frac{\theta_{xx}^2}{v}\,dx + C\varepsilon^{-2}\int_0^1 \theta_x^2\,dx, \qquad (2.50) \\
\int_0^1 u_x^4\,dx &\le C\int_0^1 u_x^2\,dx\left(\int_0^1 u_x^2\,dx + \int_0^1 u_{xx}^2\,dx\right) \\
&\le C\int_0^1 u_x^2\,dx + C\int_0^1 u_{xx}^2\,dx, \qquad (2.51) \\
\int_0^1 \theta^2 u_x^2\,dx &\le C\int_0^1 \{1 + (\theta - \bar{\theta})^2\}u_x^2\,dx \\
&\le C\int_0^1 u_x^2\,dx + C\max_{[0,1]}(\theta - \bar{\theta})^2, \qquad (2.52)
\end{aligned}$$

where we have used (2.40) and (2.43). Substituting (2.50)–(2.52) into (2.49), choosing ε sufficiently small, we get (2.48) from (2.27), (2.24), (2.43), (2.37), and (2.13). □

From (2.38) and this proposition, we get

COROLLARY 2.4.

$$\theta(x,t) \le C, \quad (x,t) \in \overline{Q}. \qquad (2.53)$$

PROPOSITION 2.13. *We have*

$$\int_0^1 z_x^2\,dx + \int_0^\infty\int_0^1 z_{xx}^2\,dx\,dt \le C. \qquad (2.54)$$

Proof. Multiplying $(1.1)_4$ by $-z_{xx}$ and integrating it over $[0,1]$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 z_x^2\,dx + \lambda_3\int_0^1 \frac{z_{xx}^2}{v^2}\,dx \\
&= 2\lambda_3\int_0^1 \frac{v_x z_x z_{xx}}{v^3} + \int_0^1 f z_{xx}\,dx \\
&\le \varepsilon\int_0^1 \frac{z_{xx}^2}{v^2}\,dx + C\varepsilon^{-1}\int_0^1 (v_x^2 z_x^2 + z^{2m})\,dx, \qquad (2.55)
\end{aligned}$$

for any $\varepsilon > 0$. Here, the second term in the right hand side of (2.55) can be estimated as

$$\begin{aligned}
\int_0^1 v_x^2 z_x^2\,dx &\le C\int_0^1 v_x^2\,dx\left(\varepsilon^2\int_0^1 z_{xx}^2\,dx + \varepsilon^{-2}\int_0^1 z_x^2\,dx\right) \\
&\le C\varepsilon^2\int_0^1 \frac{z_{xx}^2}{v^2}\,dx + C\varepsilon^{-2}\int_0^1 z_x^2\,dx, \qquad (2.56)
\end{aligned}$$

where we have used (2.40). Substituting (2.56) into (2.55), choosing ε sufficiently small, we get (2.54) from (2.13). □

From (1.11), (1.18), and this proposition, we get

Corollary 2.5.

$$0 \le z \le C, \quad (x,t) \in \overline{Q}. \qquad (2.57)$$

Proposition 2.14. *We have*

$$\int_0^1 \frac{1}{\theta^2}\,dx \le C. \qquad (2.58)$$

Proof. Multiplying $(1.1)_3$ by $-\frac{1}{\theta^3}$ and integrating it over $[0,1]$ implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \frac{1}{\theta^2}\,dx + \int_0^1\left(3\lambda_2\frac{\theta_x^2}{v\theta^4} + \lambda_1\frac{u_x^2}{v\theta^3} + \delta\frac{f}{\theta^3}\right)dx \\
&= k\int_0^1 \frac{u_x}{v\theta^2}\,dx = k\int_0^1 \frac{u_x}{v}\left(\frac{1}{A^2} + \frac{A^2-\theta^2}{A^2\theta^2}\right)dx \\
&\le \frac{k}{A^2}\frac{d}{dt}\int_0^1 (\log v - v + 1)\,dx + C\int_0^1 |u_x|\,|A-\theta|\frac{1}{\theta^2}\,dx, \qquad (2.59)
\end{aligned}$$

where we have used (1.17) and (2.53). Therefore, we get

$$\int_0^1 \frac{1}{\theta^2}\,dx \le C + C\int_0^t\int_0^1 \{u_x^2 + (A-\theta)^2\}\frac{1}{\theta^2}\,dx\,d\tau. \qquad (2.60)$$

Applying Gronwall's inequality to (2.60), considering

$$\int_0^\infty \max_{[0,1]} u_x^2\,dt \le \int_0^\infty\int_0^1 u_{xx}^2\,dx\,dt \le C, \qquad (2.61)$$

and

$$\begin{aligned}\int_0^\infty \max_{[0,1]}(A-\theta)^2\,dt &\le C\int_0^\infty \max_{[0,1]}\{(A-\bar\theta)^2+(\bar\theta-\theta)^2\}\,dt\\ &\le C\int_0^\infty \max_{[0,1]}(\bar\theta-\theta)^2\,dt + C\int_0^\infty\int_0^1 (u^4+z^2)\,dx\,dt\\ &\le C, \end{aligned} \tag{2.62}$$

we complete the proof of (2.58). □

From this proposition, it is not difficult to obtain the boundedness of θ from below.

Proposition 2.15. *There exists a constant $C>1$ such that*

$$\theta(x,t) \ge C^{-1} \tag{2.63}$$

is satisfied for any $(x,t)\in\overline{Q}$.

Proof. By using a point $x(t)$ which satisfies $\theta(x(t),t)=\bar\theta$, (2.63) is easily obtained by the following inequality:

$$|\log\theta| = \left|\log\bar\theta + \int_{x(t)}^x \frac{\theta_x}{\theta}\,dx\right| \le C + C\int_0^1\left(\theta_x^2+\frac{1}{\theta^2}\right)dx \le C, \tag{2.64}$$

where we have used (2.3), (2.48), and (2.58). □

3. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. In what follows, we shall obtain eight differential inequalities.

Proposition 3.1.

$$\frac{d}{dt}\int_0^1 (z+z^2)\,dx + C^{-1}\int_0^1 (z^m+z^{m+1}+z_x^2)\,dx \le 0. \tag{3.1}$$

Proof. Integrating $(1.1)_4$ over $[0,1]$ implies

$$\frac{d}{dt}\int_0^1 z\,dx + \int_0^1 f\,dx = 0. \tag{3.2}$$

Multiplying $(1.1)_4$ by z, integrating it over $[0,1]$, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 z^2\,dx + \lambda_3\int_0^1 \frac{z_x^2}{v^2}\,dx + \int_0^1 fz\,dx = 0. \tag{3.3}$$

It follows from (1.4), (2.9), and (2.63) that $f \ge C^{-1}z^m$, which together with (3.2)–(3.3) leads to (3.1). □

Put $\hat{\theta} = A - \theta$. Then we have

PROPOSITION 3.2.

$$\frac{d}{dt}\int_0^1 \{\hat{\theta}^2 + Au^2 + 2kA^2(v - 1 - \log v)\}\,dx + C^{-1}\int_0^1 \left(\hat{\theta}^2 + \hat{\theta}_x^2 + u_x^2\right)dx$$
$$\le C\int_0^1 (u_x^4 + z^m)\,dx. \tag{3.4}$$

Proof. Repeating the same arguments as in Prop. 2.6, we have

$$\int_0^1 \hat{\theta}^2 dx \le C\int_0^1 \left(\hat{\theta}_x^2 + u^4 + z^2\right)dx \le C\int_0^1 \left(\hat{\theta}_x^2 + u_x^2 + z^2\right)dx. \tag{3.5}$$

Multiplying the equation:

$$\hat{\theta}_t = \lambda_2\left(\frac{\hat{\theta}_x}{v}\right)_x - \lambda_1\frac{u_x^2}{v} + kA\frac{u_x}{v} - k\hat{\theta}\frac{u_x}{v} - \delta f, \tag{3.6}$$

which comes from $(1.1)_3$, by $\hat{\theta}$, integrating it over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \hat{\theta}^2\,dx + \lambda_2\int_0^1 \frac{\hat{\theta}_x^2}{v}\,dx \\
&= \int_0^1 \left(-\lambda_1\frac{u_x^2\hat{\theta}}{v} + kA\frac{u_x\hat{\theta}}{v} - k\frac{u_x\hat{\theta}^2}{v} - \delta f\hat{\theta}\right)dx \\
&\le \varepsilon\int_0^1 \hat{\theta}^2\,dx + C\varepsilon^{-1}\int_0^1 (u_x^4 + z^{2m})\,dx + kA\int_0^1 \frac{u_x\hat{\theta}}{v}\,dx,
\end{aligned} \tag{3.7}$$

where we have estimated the third term in the second line in (3.7) as $\left|k\frac{u_x\hat{\theta}^2}{v}\right| \le \varepsilon\hat{\theta}^{8/3} + C\varepsilon^{-1}u_x^4$, and have used the boundedness of $\hat{\theta}$. Multiplying $(1.1)_2$ by u and integrating it over $[0,1]$ implies

$$\frac{d}{dt}\int_0^1 \left\{\frac{1}{2}u^2 + kA(v - 1 - \log v)\right\}dx + \lambda_1\int_0^1 \frac{u_x^2}{v}\,dx = -k\int_0^1 \frac{u_x\hat{\theta}}{v}\,dx. \tag{3.8}$$

Considering $z^2, z^{2m} \le Cz^m$, we conclude (3.4) by multiplying (3.8) by A, multiplying (3.5) by ε', adding the results together with (3.7), and finally choosing ε and ε' properly. □

PROPOSITION 3.3.

$$\frac{d}{dt}\int_0^1 \left\{u\int_0^x (v-1)\,dx' + \lambda_1(v - 1 - \log v)\right\}dx + C^{-1}\int_0^1 (v-1)^2\,dx$$
$$\le C\int_0^1 \left(\hat{\theta}^2 + u_x^2\right)dx. \tag{3.9}$$

Proof. Multiplying $(1.1)_2$ by $\int_0^x (v-1)\,dx'$ and integrating it over $[0,1]$ yields

$$\frac{d}{dt}\int_0^1 u\,dx \int_0^x (v-1)\,dx' - \int_0^1 u^2\,dx - k\int_0^1 \left(\frac{\theta}{v} - \theta + \theta - A\right)(v-1)\,dx$$
$$= -\lambda_1 \int_0^1 \frac{u_x}{v}(v-1)\,dx = -\lambda_1 \int_0^1 (v-1-\log v)_t\,dx, \tag{3.10}$$

from which, we get

$$\frac{d}{dt}\int_0^1 \left\{ u\int_0^x (v-1)\,dx' + \lambda_1(v-1-\log v)\right\} dx + k\int_0^1 \frac{\theta}{v}(v-1)^2\,dx$$
$$= \int_0^1 \{k(\theta - A)(v-1) + u^2\}\,dx$$
$$\le \frac{k}{2}\int_0^1 \frac{\theta}{v}(v-1)^2\,dx + C\int_0^1 \hat{\theta}^2\,dx + \int_0^1 u_x^2\,dx. \tag{3.11}$$

This completes the proof of Prop. 3.3. □

PROPOSITION 3.4.

$$\frac{d}{dt}\int_0^1 \left(\theta - A - A\log\frac{\theta}{A}\right) dx + C^{-1}\int_0^1 \left(\frac{\hat{\theta}_x^2}{v\theta^2} + \frac{u_x^2}{v\theta}\right) dx$$
$$\le C\int_0^1 \left(u_x^2 + \hat{\theta}^2 + z^m\right) dx. \tag{3.12}$$

Proof. Multiplying $(1.1)_3$ by $1 - \frac{A}{\theta}$ and integrating it over $[0,1]$ immediately leads to (3.12). □

Finally, repeating the same computations as in Prop. 2.10–2.13, we have

PROPOSITION 3.5.

$$\frac{d}{dt}\int_0^1 \left(\frac{v_x^2}{v^2} - \frac{uv_x}{v}\right) dx + C^{-1}\int_0^1 v_x^2\,dx \le C\int_0^1 \left(\hat{\theta}_x^2 + u_x^2\right) dx, \tag{3.13}$$
$$\frac{d}{dt}\int_0^1 u_x^2\,dx + C^{-1}\int_0^1 u_{xx}^2\,dx \le C\int_0^1 \left(\hat{\theta}_x^2 + v_x^2\right) dx, \tag{3.14}$$
$$\frac{d}{dt}\int_0^1 \hat{\theta}_x^2\,dx + C^{-1}\int_0^1 \hat{\theta}_{xx}^2\,dx \le C\int_0^1 \left(\hat{\theta}_x^2 + u_x^2 + u_x^4 + z^m\right) dx, \tag{3.15}$$
$$\frac{d}{dt}\int_0^1 z_x^2\,dx + C^{-1}\int_0^1 z_{xx}^2\,dx \le C\int_0^1 (z^m + z_x^2)\,dx. \tag{3.16}$$

Now, multiplying (3.4) by ε_1, (3.9) by ε_2, (3.12) by ε_3, (3.13) by ε_4, (3.14) by ε_5, (3.15) by ε_6, (3.16) by ε_7, adding all these results together with (3.1), and choosing ε_i $(i = 1, 2, \dots, 7)$ properly, we obtain

$$\frac{d}{dt}E_1(t) + \nu E_2(t) \le C\int_0^1 u_x^4\,dx, \tag{3.17}$$

for some constant $\nu > 0$, where

$$E_1(t) = \int_0^1 \{(v-1)^2 + v_x^2 + u^2 + u_x^2 + \hat{\theta}^2 + \hat{\theta}_x^2 + z + z^2 + z_x^2\}\, dx, \tag{3.18}$$

and

$$E_2(t) = \int_0^1 \{(v-1)^2 + v_x^2 + u^2 + u_x^2 + \hat{\theta}^2 + \hat{\theta}_x^2 + z^m + z^{m+1} + z_x^2\}\, dx. \tag{3.19}$$

Here, for simplicity, we have replaced all coefficients in E_1 and E_2 by 1. Noting that $\int_0^1 u_x^4\, dx \le |u_x|_\infty^2 \int_0^1 u_x^2\, dx \le CE_1(t) \int_0^1 u_{xx}^2\, dx$, we get from (3.17)

$$\frac{d}{dt}E_1(t) + \nu E_2(t) \le CE_1(t) \int_0^1 u_{xx}^2\, dx. \tag{3.20}$$

First, we consider the case $0 < m \le 1$. Since $E_1(t) \le CE_2(t)$ with some constant $C > 0$, we have

$$\frac{d}{dt}E_1(t) + \nu E_1(t) \le CE_1(t) \int_0^1 u_{xx}^2\, dx, \tag{3.21}$$

which is equivalent to

$$\frac{d}{dt}\{\exp(\nu t)E_1(t)\} \le C\exp(\nu t)E_1(t) \int_0^1 u_{xx}^2\, dx, \tag{3.22}$$

where we have also denoted $C^{-1}\nu$ by ν. From which, we have

$$\frac{d}{dt}\log(\exp(\nu t)E_1(t)) \le C \int_0^1 u_{xx}^2\, dx. \tag{3.23}$$

It follows from (2.43) and (3.21) that

$$E_1(t) \le C\exp(-\nu t), \tag{3.24}$$

which completes the proof of Theorem 1.1 for $0 \le m \le 1$.

Next, we consider the case $1 < m \le 2$. By using Schwarz inequality and the boundedness of $E_2(t)$, it follows that $E_1(t) \le CE_2(t)^{1/m}$. Then we get from (3.20)

$$\frac{d}{dt}E_1(t) + \nu E_1^m(t) \le CE_1(t) \int_0^1 u_{xx}^2\, dx, \tag{3.25}$$

where, we have also denoted $C^{-m}\nu$ by ν. We put $F(t) = E_1^{1-m}(t)$, then this inequality is reduced to

$$\frac{d}{dt}F(t) + C(m-1)F(t) \int_0^1 u_{xx}^2\, dx \ge (m-1)\nu. \tag{3.26}$$

Therefore, we obtain

$$
\begin{aligned}
F(t) \geq\ & F(0)\exp\left(-C(m-1)\int_0^t\int_0^1 u_{xx}^2\,dx\,ds\right)\\
&\quad + (m-1)\nu\int_0^t \exp\left(-C(m-1)\int_\tau^t\int_0^1 u_{xx}^2\,dx\,ds\right)d\tau\\
\geq\ & C^{-1}(1+t),
\end{aligned}
\tag{3.27}
$$

where we have used (2.43). Thus, we have

$$
E_1(t) \leq C(1+t)^{-\frac{1}{m-1}}, \tag{3.28}
$$

which completes the proof of Theorem 1.1.

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