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On Three Theorems of Lees for Numerical Treatment of Semilinear Two-Point Boundary Value Problems

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This paper is concerned with semilinear tow-point boundary value problems of the form $-(p(x)u')' + f(x,u) = 0$, $a \le x \le b$, $\alpha_0 u(a) - \alpha_1 u'(a) = \alpha$, $\beta_0 u(b) + \beta_1 u'(b) = \beta$, $\alpha_i \ge 0$, $\beta_i \ge 0$, $i = 0, 1$, $\alpha_0+\alpha_1 > 0$, $\beta_0+\beta_1 > 0$, $\alpha_0+\beta_0 > 0$. Under the assumption $\inf f_u > -\lambda_1$, where λ_1 is the smallest eigenvalue of $\mathscr{L}u = -(pu')'$ with the boundary conditions, unique existence theorems of solution for the continuous problem and a discretized system with not necessarily uniform nodes are given as well as error estimates. The results generalize three theorems of Lees for $u'' = f(x,u)$, $0 \le x \le 1$, $u(0) = \alpha$, $u(1) = \beta$.

Key words: tow-point boundary value problems, discretization, existence of solution, error estimates, theorems of Lees

1. Introduction

We will be concerned with a mathematical theory for numerical treatment of semilinear boundary value problem

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + f(x,u) = 0, \quad a \le x \le b, \tag{1.1}$$

$$B_1(u) = \alpha_0 u(a) - \alpha_1 u'(a) = \alpha, \tag{1.2}$$

$$B_2(u) = \beta_0 u(b) + \beta_1 u'(b) = \beta, \tag{1.3}$$

where $p(x) \in C^1[a,b]$, $p(x) > 0$, $f(x,u) \in C([a,b] \times \mathbb{R})$ and α_i, β_i, $i = 0, 1$ are constants which satisfy

$$\alpha_0 \ge 0, \quad \alpha_1 \ge 0, \quad \alpha_0 + \alpha_1 > 0, \tag{1.4}$$

$$\beta_0 \ge 0, \quad \beta_1 \ge 0, \quad \beta_0 + \beta_1 > 0, \tag{1.5}$$

and

$$\alpha_0 + \beta_0 > 0. \tag{1.6}$$

Let $\mathscr{L}u = -(d/dx)(p(x)(du/dx))$ and put $\mathscr{D} = \{u \in C^2[a,b] \mid B_1(u) = B_2(u) = 0\}$. Then, as is easily verified, the Green function for $(\mathscr{L}, \mathscr{D})$ exists under the conditions (1.4)–(1.6). It is known that if $f_u = \partial f/\partial u$ exists, is continuous on $[a, b] \times \mathbb{R}$ and $f_u \geq 0$, then the problem has a unique solution $u \in C^2[a, b]$ (cf. [8]; Remark 2.1). To find a numerical solution, we discretize (1.1)–(1.3) at not necessarily uniform nodes

$$\Delta\colon a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b, \tag{1.7}$$

and put

$$h_i = x_i - x_{i-1}, \quad h = \max_i h_i.$$

The discretized system we consider here is

$$HA\boldsymbol{U} + \widetilde{\boldsymbol{f}}(\boldsymbol{U}) = \boldsymbol{0} \tag{1.8}$$

where, if $\alpha_1\beta_1 \neq 0$, then

$$H = \begin{pmatrix} \omega_0^{-1} & & & \\ & \omega_1^{1} & & \\ & & \ddots & \\ & & & \omega_{n+1}^{-1} \end{pmatrix}, \quad \omega_i = \begin{cases} \frac{1}{2}h_1 & (i = 0) \\ \frac{1}{2}(h_i + h_{j+1}) & (1 \leq i \leq n) \\ \frac{1}{2}h_{n+1} & (i = n+1), \end{cases}$$

$$A = \begin{pmatrix} a_0 + a_1 & -a_1 & & \\ -a_1 & a_1 + a_2 & -a_2 & \\ \ddots & \ddots & \ddots & \\ & & -a_{n+1} & a_{n+1} + a_{n+2} \end{pmatrix}$$

$$a_i = \begin{cases} \frac{\alpha_0}{\alpha_1} p(a) & (i = 0) \\ \left(\int_{x_{i-1}}^{x_i} \frac{dt}{p(t)} \right)^{-1} & (1 \leq i \leq n+1) \\ \frac{\beta_0}{\beta_1} p(b) & (i = n+2), \end{cases}$$

$$\boldsymbol{U} = (U_0, U_1, \ldots, U_{n+1})^t$$

$$\boldsymbol{f}(\boldsymbol{U}) = (f(x_0, U_0), \ldots, f(x_{n+1}, U_{n+1}))^t$$

$$\widetilde{\boldsymbol{f}}(\boldsymbol{U}) = \boldsymbol{f}(\boldsymbol{U}) - \left(\frac{2}{h_1} \cdot \frac{\alpha}{\alpha_1} p(a), 0, \ldots, 0, \frac{2}{h_{n+1}} \cdot \frac{\beta}{\beta_1} p(b) \right)^t.$$

If $\alpha_1\beta_1 = 0$, then (1.2) or (1.3) reduces to the Dirichlet condition $u(a) = 0$ or $u(b) = 0$ so that a modification of (1.8) is necessary. Namely, if $\alpha_1 = 0$ and $\beta_1 \neq 0$, then $\alpha_0 \neq 0$ and we replace H, A, $\boldsymbol{U}$, $\boldsymbol{f}$, and $\widetilde{\boldsymbol{f}}$ by

$$\begin{aligned} H &= \operatorname{diag}\big(\omega_1^{-1}, \omega_2^{-1}, \ldots, \omega_{n+1}^{-1}\big), \\ A &= \begin{pmatrix} a_1 + a_2 & -a_2 & & \\ -a_2 & a_2 + a_3 & -a_3 & \\ \ddots & \ddots & \ddots & \\ & & -a_{n+1} & a_{n+1} + a_{n+2} \end{pmatrix} \\ \boldsymbol{U} &= (U_1, U_2, \ldots, U_{n+1})^t, \\ \boldsymbol{f}(\boldsymbol{U}) &= (f(x_1, U_1), f(x_2, U_2), \ldots, f(x_{x+1}, U_{n+1}))^t \end{aligned} \tag{1.9}$$

and

$$\widetilde{\boldsymbol{f}}(\boldsymbol{U}) = \boldsymbol{f}(\boldsymbol{U}) - \left(\frac{2}{h_1 + h_2}\frac{\alpha}{\alpha_0}a_1, 0, \ldots, 0, \frac{2}{h_{n+1}} \cdot \frac{\beta}{\beta_1}p(b)\right)^t.$$

If $a_1 \neq 0$ and $\beta_1 = 0$, then $\beta_0 \neq 0$ and H, A, $\boldsymbol{U}$, $\boldsymbol{f}$ and $\widetilde{\boldsymbol{f}}$ in (1.8) are replaced by

$$\begin{aligned} H &= \operatorname{diag}\big(\omega_0^{-1}, \omega_1^{-1}, \ldots, \omega_n^{-1}\big), \\ A &= \begin{pmatrix} a_0 + a_1 & -a_1 & & \\ -a_1 & a_1 + a_2 & -a_2 & \\ \ddots & \ddots & \ddots & \\ \ddots & & -a_n & a_n + a_{n+1} \end{pmatrix}, \\ \boldsymbol{U} &= (U_0, U_1, \ldots, U_n)^t, \\ \boldsymbol{f}(\boldsymbol{U}) &= (f_0(x_0, U_0), f(x_1, U_1), \ldots, f(x_n, U_n))^t \end{aligned}$$

and

$$\widetilde{\boldsymbol{f}}(\boldsymbol{U}) = \boldsymbol{f}(\boldsymbol{U}) - \left(\frac{2}{h_1}\frac{\alpha}{\alpha_1}p(a), 0. \ldots, 0, \frac{2}{h_n + h_{n+1}}\frac{\beta}{\beta_0}a_{n+1}\right)^t.$$

Furthermore, if $\alpha_1 = \beta_1 = 0$, then $\alpha_0, \beta_0 \neq 0$ and H, A, $\boldsymbol{U}$, $\boldsymbol{f}$ and $\widetilde{\boldsymbol{f}}$ are replaced by

$$\begin{aligned} H &= \operatorname{diag}\big(\omega_1^{-1}, \omega_2^{-1}, \ldots, \omega_n^{-1}\big), \\ A &= \begin{pmatrix} a_1 + a_2 & -a_2 & & \\ -a_2 & a_2 + a_3 & -a_3 & \\ \ddots & \ddots & \ddots & \\ & & -a_n & a_n + a_{n+1} \end{pmatrix}, \\ \boldsymbol{U} &= (U_1, \ldots, U_n)^t, \\ \boldsymbol{f}(\boldsymbol{U}) &= (f(x_1, U_1), \ldots, f(x_n, U_n))^t \end{aligned}$$

and

$$\widetilde{\boldsymbol{f}}(\boldsymbol{U}) = \boldsymbol{f}(\boldsymbol{U}) - \left(\frac{2}{h_1 + h_2}\frac{\alpha}{\alpha_0}, 0, \ldots, 0, \frac{2}{h_n + h_{n+1}}\frac{\beta}{\beta_0}a_{n+1}\right)^t.$$

It should be remarked here that, in any case, we have $A^{-1} = (G(x_i, x_j))$ (cf. [8]), where $G(x, \xi)$ denotes the Green function for $(\mathscr{L}, \mathscr{D})$ with $\mathscr{L} = -(d/dx)(p(d/dx)[\ ])$.

Observe also that, if the nodes are uniform, i.e., $x_i = ih$, $h = (b-a)/(n+1)$, $p(x) = 1$ and the boundary conditions are of Dirichlet's type $u(a) = \alpha$ and $u(b) = \beta$, then (1.8) reduces to a system of n equations

$$\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ \ddots & \ddots & \ddots & \\ & & -1 & 2 \end{pmatrix}\begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix} + \begin{pmatrix} f(x_1, U_1) - \dfrac{\alpha}{h^2} \\ \vdots \\ f(x_n, U_n) - \dfrac{\beta}{h^2} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{1.10}$$

In [4], M. Lees considered the problem

$$u'' = f(x, u), \quad x \in E = [0, 1], \tag{1.11}$$

$$u(0) = \alpha, \quad u(1) = \beta \tag{1.12}$$

and proved the following three theorems:

THEOREM 1.1. *If f_u exists, is continuous on $E \times \mathbb{R}$ and satisfies*

$$\inf_{E\times\mathbb{R}} f_u = -\eta > -\pi^2, \tag{1.13}$$

then the problem (1.11), (1.12) *has a unique solution $u \in C^2[E]$.*

THEOREM 1.2. *Assume that $u \in C^4[E]$. If f satisfies* (1.13) *and h is sufficiently small, i.e., if $h \le h_0$, where h_0 is a constant satisfying*

$$\eta < \pi^2\left[1 - \frac{h_0^2}{12}\pi^2\right],$$

then (1.10) *has a unique solution $\boldsymbol{U} = (U_1, \dots, U_n)^t \in \mathbb{R}^n$.*

THEOREM 1.3. *Let*

$$\|U\|_D = \sqrt{h\sum_{j=1}^{n+1}\left(\frac{U_j - U_{j-1}}{h}\right)^2},$$

where $U_0 = \alpha$ and $U_{n+1} = \beta$. Then, under the assumption of Theorem 1.2,

$$\|\boldsymbol{u} - \boldsymbol{U}\|_\infty \le \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{U}\|_D \le \frac{1}{12}K(h_0)h^2\|u^{(4)}\|_E,$$

where $\boldsymbol{u} = (u(x_1), \dots, u(x_n))^t$, $K(h_0)$ is a constant and

$$\|u^{(4)}\|_E = \max_{x\in E}|u^{(4)}(x)|.$$

The purpose of this paper is to generalize these result to the problem (1.1)–(1.6) and its discretized system (1.8).

2. Existence of Solution for (1.1)–(1.6)

Observing that the number π^2 in (1.13) is the smallest eigenvalue for the operator $-(d^2/dx^2)$: $\{u \in C^2[0,1] \mid u(0) = u(1) = 0\} \to C[0,1]$, we generalize Theorem 1.1 on the basis of the following three lemmas.

LEMMA 2.1. *Let* $\mathscr{L}u = -(d/dx)(p(du/dx))$ *and* $\mathscr{D} = \{u \in C^2[a,b] \mid B_1(u) = B_2(u) = 0\}$. *We denote by* λ_1 *the smallest eigenvalue of* $(\mathscr{L}, \mathscr{D})$. *Then* λ_1 *is positive and*

$$(\mathscr{L}u, u) \geq \lambda_1 \|u\|^2 \quad \forall u \in \mathscr{D},$$

where $\|\cdot\|$ *denote the* L_2 *norm.*

Proof. The proof is straightforward since $(\mathscr{L}, \mathscr{D})$ has a complete system of orthonrmal eigenfunctinos in $L_2[a,b]$. □

LEMMA 2.2. *Let* $r(x), g(x) \in C[a,b]$ *and*

$$\min_{a \leq x \leq b} r(x) = -\eta > -\lambda_1, \tag{2.1}$$

where λ_1 *is as defined by* Lemma 2.1. *Then the boundary value problem*

$$\begin{aligned} &\mathscr{L}u + r(x)u = g(x), \quad a \leq x \leq b \\ &u \in \mathscr{D} \end{aligned}$$

has a unique solution and

$$\|u\| \leq \frac{\|g\|}{\lambda_1 - \eta}.$$

Proof. Let $\mathscr{L}u + r(x)u = 0$, $u \in \mathscr{D}$. Then

$$\begin{aligned} 0 &= (\mathscr{L}u, u) + (ru, u) \\ &\geq \lambda_1(u,u) + (ru,u) = ((\lambda_1 + r)u, u) \geq (\lambda_1 - \eta)(u,u). \end{aligned}$$

Since $\lambda_1 - \eta > 0$, we obtain $\|u\| = 0$. Hence, $(\mathscr{L} + rI, \mathscr{D})$, where I is the identity, is injective and the problem

$$\mathscr{L}u + ru = g, \quad u \in \mathscr{D} \tag{2.2}$$

has a unique solution. We then have

$$(\lambda_1 - \eta)\|u\|^2 \leq (\mathscr{L}u, u) + (ru, u) = (g, u) \leq \|g\| \cdot \|u\|. \tag{2.3}$$

If $\|u\| > 0$, then (2.3) implies

$$(\lambda_1 - \eta)\|u\| \leq \|g\|$$

or

$$\|u\| \le \frac{\|g\|}{\lambda_1 - \eta}.$$

This inequality holds for $\|u\| = 0$, too. □

LEMMA 2.3. *Let $G(x,\xi)$ be the Green function for $(\mathscr{L}, \mathscr{D})$, where $\mathscr{L}$ and $\mathscr{D}$ are defined in* Lemma 2.1. *Then*

$$\begin{aligned} &G(x,\xi) \\ &= \begin{cases} \dfrac{1}{p(a)p(b)\delta}\left(\alpha_1 + \alpha_0 p(a)\displaystyle\int_a^x \frac{dt}{p(t)}\right)\left(\beta_1 + \beta_0 p(b)\displaystyle\int_\xi^b \frac{dt}{p(t)}\right) & (x \le \xi) \\ \dfrac{1}{p(a)p(b)\delta}\left(\alpha_1 + \alpha_0 p(a)\displaystyle\int_a^\xi \frac{dt}{p(t)}\right)\left(\beta_1 + \beta_0 p(b)\displaystyle\int_x^b \frac{dt}{p(t)}\right) & (x \ge \xi) \end{cases} \\ &\le G(x,x) \end{aligned}$$

where

$$\delta = \alpha_0\left(\beta_0 \int_a^b \frac{dt}{p(t)} + \frac{\beta_1}{p(b)}\right) + \frac{\alpha_1\beta_0}{p(a)} > 0. \tag{2.4}$$

Proof. See [8]. □

We are now in a position to prove the following:

THEOREM 2.1. *If $f(x,u)$ satisfies*

$$\inf_{[a,b]\times\mathbb{R}} f_u(x,u) = -\eta > -\lambda_1, \tag{2.5}$$

where λ_1 is defined in Lemma 2.1, *then the problem* (1.1)–(1.6) *has a unique solution $u \in C^2[a,b]$.*

Proof. Without loss of generality, we may assume $\alpha = \beta = 0$ (cf. [8]).

(i) EXISTENCE. Putting

$$r(x;u) = \int_0^1 f_u(x;\theta u)\, d\theta,$$

we have

$$f(x,u) = f_0(x) + r(x;u)u,$$

where $f_0(x) = f(x,0)$. Then, by Lemma 2.2, given $u \in C[a,b]$, the linear boundary value problem

$$\mathscr{L}w + r(x;u)w = -f_0(x), \quad w \in \mathscr{D} \tag{2.6}$$

has a unique solution $w \in C^2[a,b]$, which satisfies

$$\|w\| \le \frac{\|f_0\|}{\lambda_1 - \eta} \equiv \delta_0 \quad \text{(say)} \tag{2.7}$$

by Lemma 2.2. Furthermore, we have from (2.6)

$$\mathscr{L}w = -f_0(x) - r(x;u)w$$

and

$$w(x) = -\int_a^b G(x,\xi)\{f_0(\xi) + r(\xi;u(\xi))w(\xi)\}\,d\xi$$

so that, using Lemma 2.3, we obtain

$$\begin{aligned}\frac{dw(x)}{dx} &= -\int_a^x \frac{\partial G(x,\xi)}{\partial x}\{f_0(\xi) + r(\xi;u(\xi))w(\xi)\}\,d\xi \\ &\quad - \int_x^b \frac{\partial G(x,\xi)}{\partial x}\{f_0(\xi) + r(\xi;u(\xi))w(\xi)\}\,d\xi \\ &= -\int_a^x \frac{1}{\delta}\left(-\frac{\beta_0}{p(x)}\right)\left(\frac{\alpha}{p(a)} + \alpha_0\int_a^\xi \frac{dt}{p(t)}\right)\{f_0(\xi) + r(\xi;u(\xi))w(\xi)\}\,d\xi \\ &\quad - \int_x^b \frac{1}{\delta}\left(\frac{\alpha_0}{p(x)}\right)\left(\frac{\beta_1}{p(b)} + \beta_0\int_\xi^b \frac{dt}{p(t)}\right)\{f_0(\xi) + r(\xi;u(\xi))w(\xi)\}\,d\xi,\end{aligned}$$

where δ is as defined in (2.4).

Observing that

$$\delta = \beta_0\left(\frac{\alpha_1}{p(a)} + \alpha_0\int_a^b \frac{dt}{p(t)}\right) + \frac{\alpha_0\beta_1}{p(b)} \geq \beta_0\left(\frac{\alpha_1}{p(a)} + \alpha_0\int_a^\xi \frac{dt}{p(t)}\right)$$

and similary

$$\delta \geq \alpha_0\left(\frac{\beta_1}{p(b)} + \beta_0\int_\xi^b \frac{dt}{p(t)}\right),$$

we have

$$\begin{aligned}\left|\frac{dw(x)}{dx}\right| &\leq \int_a^x \frac{1}{p(x)}(|f_0(\xi)| + |r(\xi;u(\xi))|\cdot|w(\xi)|)\,d\xi \\ &\quad + \int_x^b \frac{1}{p(x)}(|f_0(\xi)| + |r(\xi;u(\xi))|\cdot|w(\xi)|)\,d\xi \\ &= \int_a^b \frac{1}{p(x)}(|f_0(\xi)| + |r(\xi;u(\xi))|\cdot|w(\xi)|)\,d\xi \\ &\leq \frac{1}{p_*}\int_a^b \left(\|f_0\|_{[a,b]} + |r(\xi;u(\xi))|\cdot\|w\|_{[a,b]}\right)d\xi,\end{aligned}$$

where $p_* = \min_{a\leq x\leq b} p(x) > 0$ and $\|\cdot\|_{[a,b]}$ denotes the maximum norm in $[a,b]$:

$$\|f_0\|_{[a,b]} = \max_{x\in[a,b]}|f_0(x)|, \quad \|w\|_{[a,b]} = \max_{x\in[a,b]}|w(x)|, \quad \text{etc.}$$

Hence, if $u \in C^2[a,b]$ and $\|u\|_{[a,b]} \le \delta_0$, then, putting

$$\widetilde{K} = \sup_{[a,b]\times[-\delta_0,\delta_0]} |f_u(x,u)|$$

we have

$$\begin{aligned}\|w'\|_{[a,b]} &\le \frac{1}{p_*}\int_a^b \left(\|f_0\|_{[a,b]} + \widetilde{K}\delta_0\right) d\xi \\ &= \frac{1}{p_*}\left(\|f_0\|_{[a,b]} + \widetilde{K}\delta_0\right)(b-a) \equiv \delta_1 \quad \text{(say)}.\end{aligned}$$

Consider a Banach space $X = C^1[a,b]$ equipped with the norm $\|u\|_{C^1} = \|u\|_{[a,b]} + \|u'\|_{[a,b]}$ for $u \in X$ and put

$$S = \left\{u \in X \;\middle|\; \|u\|_{[a,b]} \le \delta_0,\ \|u'\|_{[a,b]} \le \delta_1,\ B_1(u) = B_2(u) = 0\right\}.$$

Then S is a bounded and closed convex set in X. The map $T\colon S \to S \cap C^2[a,b] \subset S$ defined by $Tu = w, u \in S$, is then continuous and it can be shown by Ascoli-Arzela's theorem that $T(S)$ is relatively compact in X (cf. [8]). Hence, Schauder's theorem implies that T has a fixed point $u \in S$. It is clear that $u = Tu$ is a solution of (1.1)–(1.6).

(ii) UNIQUENESS. Let u and v be two solutions of the problem and set $\varphi = u - v$. Then

$$f(x,u) - f(x,v) = r(x;u,v)\varphi$$

where

$$r(x;u,v) = \int_0^1 f_u(x, v + \theta(u-v))\, d\theta.$$

Therefore

$$\begin{aligned}&\mathscr{L}\varphi + r(x;u,v)\varphi = 0, \quad a \le x \le b \\ &\varphi \in \mathscr{D},\end{aligned}$$

where

$$r(x;u,v) \ge -\eta > -\lambda_1.$$

Hence Lemma 2.2 applies to conclude that $\varphi \equiv 0$. □

REMARK 2.1. If $\alpha_1\beta_1 \ne 0$ and $\alpha = \beta = 0$, then the existence of solution for the problem (1.1)–(1.6) follows from the following result which is the one-dimensional version of Theorem 2.3.1 in Sattinger [5]. (Also see Amann [1], [2])

THEOREM 2.2. *Let $\varphi(x)$ and $\psi(x)$ be upper and lower solutions for the problem* (1.1)–(1.6) *with $\alpha = \beta = 0$:*

$$\begin{aligned}&\mathscr{L}\varphi + f(x,\varphi) \ge 0 \quad (a \le x \le b), \quad B_1(\varphi) \ge 0, \quad B_2(\varphi) \ge 0 \\ &\mathscr{L}\psi + f(x,\psi) \le 0 \quad (a \le x \le b), \quad B_1(\psi) \le 0, \quad B_2(\psi) \le 0,\end{aligned}$$

where $\mathscr{L}$ *is as defined in* Lemma 2.1. *If* $\psi \le \varphi$ *in* $[a, b]$, *then there exists a solution* u *for* (1.1)–(1.6) *with* $\psi \le u \le \varphi$.

In fact, if we put $F(x, u) = f(x, u) + \lambda_1 u$ where λ_1 is defined in Lemma 2.1, then

$$F_u(x, u) = f_u(x, u) + \lambda_1 \ge \lambda_1 - \eta > 0.$$

Hence, at each x fixed, F is monotonically increasing in u and $F(x, -\infty) = -\infty$, $F(x, +\infty) = +\infty$, $x \in [a, b]$, so that, by the implicit function theorem, there exists a unique $\Phi(x) \in C[a, b]$ satisfying $F(x, \Phi(x)) = 0$. Take a positive constant m with $-m \le \Phi(x) \le m$ $\forall x \in [a, b]$ and an eigenfunction $v(x)$ corresponding to the eigenvalue λ_1 for $(\mathscr{L}, \mathscr{D})$. Then $v(x) > 0$ in $[a, b]$ since $\alpha_1 \beta_1 \ne 0$.

Furthermore, letting

$$\gamma = \min_{a \le x \le b} v(x) > 0,$$

we have

$$-\frac{m}{\gamma} v(x) \le -m \le \Phi(x) \le m \le \frac{m}{\gamma} v(x) \quad \forall x \in [a, b]$$

and it can be shown that $\varphi(x) = (m/\gamma) v(x)$ and $\psi(x) = -(m/\gamma) v(x)$ are upper and lower solutions with $\psi(x) \le \varphi(x)$ in $[a, b]$.

In fact, we have

$$\begin{aligned} \mathscr{L}\varphi + f(x, \varphi) &= \mathscr{L}\varphi - \lambda_1 \varphi + F(x, \varphi) \\ &= F(x, \varphi) \\ &\ge F(x, \Phi(x)) = 0 \end{aligned}$$

and

$$B_1(\varphi) = B_2(\varphi) = 0.$$

Similarly

$$\mathscr{L}\psi + f(x, \psi) = F(x, \psi) \le F(x, \Phi(x)) = 0$$

and

$$B_1(\psi) = B_2(\psi) = 0.$$

Hence, by Theorem 2.2, the problem (1.1)–(1.6) has a solution u with $\psi \le u \le \varphi$.

3. Existence of Solution for the Discretized System

In this section, we shall show that (1.8) has a unique solution $\boldsymbol{U} \in \mathbb{R}^{n+2}$ for sufficiently small h. Before doing this, we prepare several lemmas. In the following, we assume $\alpha_1 \beta_1 \ne 0$ without loss of generality.

LEMMA 3.1. *Let $\widehat{U}(x)$ be the piecewise linear interpolant for the $n+2$ points (x_i, U_i), $i = 0, 1, 2, \ldots, n+1$. Then*

$$\frac{1}{3}\left(\boldsymbol{U}, H^{-1}\boldsymbol{U}\right) \le \|\widehat{U}\|^2 \le \left(\boldsymbol{U}, H^{-1}\boldsymbol{U}\right).$$

Proof. The trapezoidal rule for numerical integration implies that if $\varphi \in C^2[x_{j-1}, x_j]$, $1 \le j \le n+1$, then there exists $\xi_j \in (x_{j-1}, x_j)$ for each j such that

$$\int_{x_{j-1}}^{x_j} \varphi(x)\, dx = \frac{h_j}{2}\left(\varphi(x_j) + \varphi(x_{j-1})\right) - \frac{h_j^3}{12}\varphi''(\xi_j).$$

Hence

$$\begin{aligned}\int_a^b \varphi(x)\, dx &= \sum_{j=1}^{n+1} \frac{h_j}{2}\left(\varphi(x_j) + \varphi(x_{j-1})\right) - \frac{1}{12}\sum_{j=1}^{n+1} h_j^3 \varphi''(\xi_j) \\ &= \sum_{j=0}^{n+1} \omega_j \varphi(x_j) - \frac{1}{12}\sum_{j=1}^{n+1} h_j^3 \varphi''(\xi_j),\end{aligned}$$

or

$$\sum_{j=0}^{n+1} \omega_j \varphi(x_j) = \int_a^b \varphi(x)\, dx + \frac{1}{12}\sum_{j=1}^{n+1} h_j^3 \varphi''(\xi_j).$$

An application of this formula to $\varphi(x) = \left\{\widehat{U}(x)\right\}^2$ yields

$$\begin{aligned}\left(\boldsymbol{U}, H^{-1}\boldsymbol{U}\right) &= \sum_{j=0}^{n+1} \omega_j U_j^2 = \sum_{j=0}^{n+1} \omega_j \left\{\widehat{U}(x_j)\right\}^2 \\ &= \int_a^b \widehat{U}(x)^2\, dx + \frac{1}{6}\sum_{j=1}^{n+1} h_j^3 \left(\frac{U_j - U_{j-1}}{h_j}\right)^2 \\ &= \|\widehat{U}\|^2 + \frac{1}{6}\sum_{j=1}^{n+1} h_j \left(U_j - U_{j-1}\right)^2 \\ &\ge \|\widehat{U}\|^2,\end{aligned}$$

where we have used the fact that

$$\widehat{U}(x) = \frac{U_j - U_{j-1}}{h_j}(x - x_{j-1}) + U_{j-1}, \quad x \in [x_{j-1}, x_j]$$

and

$$\frac{d^2}{dx^2}\left\{\widehat{U}(x)^2\right\} = \frac{2(U_j - U_{j-1})^2}{h_j^2} \quad \text{(constant)}, \quad x \in [x_{j-1}, x_j].$$

On the other hand

$$\begin{aligned}\int_a^b \widehat{U}(x)^2\,dx &= \sum_{j=1}^{n+1}\int_{x_{j-1}}^{x_j}\left\{\frac{U_j-U_{j-1}}{h_j}(x-x_{j-1})+U_{j-1}\right\}^2 dx\\ &= \sum_{j=1}^{n+1}\left\{(U_j-U_{j-1})^2\frac{h_j}{3}+2(U_j-U_{j-1})U_{j-1}\frac{h_j}{2}+U_{j-1}^2h_j\right\}\\ &= \frac{1}{3}\sum_{j=1}^{n+1}h_j\left(U_j^2+U_jU_{j-1}+U_{j-1}^2\right)\\ &\geq \frac{1}{3}\sum_{j=1}^{n+1}\frac{h_j}{2}\left(U_j^2+U_{j-1}^2\right)\\ &= \frac{1}{3}\sum_{j=0}^{n+1}\omega_jU_j^2=\frac{1}{3}(\boldsymbol{U},H^{-1}\boldsymbol{U}).\end{aligned}$$

This proves Lemma 3.1. □

Lemma 3.2. *The following inequalities hold.*

(i) $\sum_{j=1}^{n+1} h_j|U_j-U_{j-1}|^2 \leq 4(\boldsymbol{U},H^{-1}\boldsymbol{U})$

(ii) $\sum_{j=1}^{n+1} h_j|U_j-U_{j-1}| \leq 2\sqrt{b-a}\sqrt{(\boldsymbol{U},H^{-1}\boldsymbol{U})}$

(iii) $\sum_{j=1}^{n+1} h_j|U_{j-1}|^2 \leq 2(\boldsymbol{U},H^{-1}\boldsymbol{U})$

(iv) $\sum_{j=1}^{n+1} h_j|U_{j-1}| \leq \sqrt{2(b-a)}\sqrt{(\boldsymbol{U},H^{-1}\boldsymbol{U})}$

(v) $\sum_{j=1}^{n+1} h_j\max\left(|U_{j-1}|^2,|U_j|^2\right) \leq 2(\boldsymbol{U},H^{-1}\boldsymbol{U})$

(vi) $\sum_{j=1}^{n+1} h_j\max\left(|U_{j-1}|,|U_j|\right) \leq 2\sqrt{b-a}\sqrt{(\boldsymbol{U},H^{-1}\boldsymbol{U})}$

Proof. For examples, we have

$$\sum_{j=1}^{n+1}h_j\left|U_j-U_{j-1}\right|^2 \leq 2\sum_{j=1}^{n+1}h_j(U_j^2+U_{j-1}^2)=4(\boldsymbol{U},H^{-1}\boldsymbol{U}),$$

$$\begin{aligned}\sum_{j=1}^{n+1}h_j\left|U_j-U_{j-1}\right| &\leq \sqrt{\sum_{j=1}^{n+1}\left(\sqrt{h_j}\right)^2\sum_{J=1}^{n+1}\left(\sqrt{h_j}\right)^2\left|U_j-U_{j-1}\right|^2}\\ &\leq \sqrt{(b-a)\cdot 4(\boldsymbol{U},H^{-1}\boldsymbol{U})}\\ &= 2\sqrt{b-a}\sqrt{(\boldsymbol{U},H^{-1}\boldsymbol{U})},\quad \text{etc.}\end{aligned}$$
□

Lemma 3.3. $\int_a^b\left|\widehat{U}(x)\right|dx \leq (1+\sqrt{2})\sqrt{b-a}\sqrt{(\boldsymbol{U},H^{-1}\boldsymbol{U})}.$

Proof. We have from Lemma 3.2

$$\begin{aligned}\int_a^b\left|\widehat{U}(x)\right|dx &= \sum_{i=1}^{n+1}\int_{x_{i-1}}^{x_i}\left|\widehat{U}(x)\right|dx\\ &\leq \sum_{i=1}^{n+1}\int_{x_{i-1}}^{x_i}\left\{\frac{|U_i-U_{i-1}|}{h_i}(x-x_{i-1})+|U_{i-1}|\right\}dx\end{aligned}$$

$$\leq \sum_{i=1}^{n+1}\left(\frac{|U_i - U_{i-1}|}{2} + |U_{i-1}|\right) h_i$$
$$\leq (1+\sqrt{2})\sqrt{b-a}\sqrt{(\boldsymbol{U}, H^{-1}\boldsymbol{U})}. \qquad \square$$

Since the Green function $G(x,\xi)$ belongs to C^2 class in the regions $\Omega_1 = \{(x,\xi) \mid a \leq x \leq \xi \leq b\}$ and $\Omega_2 = \{(x,\xi) \mid a \leq \xi \leq x \leq b\}$, there exists a constant $M > 0$ such that

$$\left|\frac{\partial^k G(x,\xi)}{\partial x^k}\right| \leq M \quad (0 \leq k \leq 2)$$

in Ω_1 or Ω_2.

Lemma 3.4. *Let* $(\boldsymbol{U}, H^{-1}\boldsymbol{U}) = 1$. *Then*

$$\sum_{j=0}^{n+1} G(x_i, x_j)\omega_j U_j = \int_a^b G(x_i,\xi)\widehat{U}(\xi)\, d\xi + \varepsilon_i, \quad 0 \leq i \leq n+1,$$

where

$$|\varepsilon_i| \leq \varepsilon \equiv \frac{Mh}{6}(2+h)\sqrt{b-a} = O(h).$$

Proof. Let $\widehat{U}(x)$ be as defined in Lemma 3.1 and put

$$\varphi_i(\xi) = G(x_i,\xi)\widehat{U}(\xi).$$

Then

$$\int_a^b G(x_i,\xi)\widehat{U}(\xi)\, d\xi = \sum_{j=1}^{n+1}\int_{x_{j-1}}^{x_j} G(x_i,\xi)\widehat{U}(\xi)\, d\xi$$
$$= \sum_{j=1}^{n+1}\left[\frac{h_j}{2}\{\varphi_i(x_{j-1}) + \varphi_i(x_j)\} - \frac{h_j^3}{12}\varphi_i''(\xi_j)\right]$$
$$(x_{j-1} \leq \xi_j \leq x_j)$$
$$= \sum_{j=0}^{n+1}\omega_j\varphi_i(x_j) - \varepsilon_i,$$

where $\varepsilon_i = (1/12)\sum_{j=1}^{n+1} h_j^3\varphi_i''(\xi_j)$.

If $x_{j-1} < \xi < x_j$, then

$$\varphi''(\xi) = \frac{\partial^2 G(x_i,\xi)}{\partial\xi^2}\widehat{U}(\xi) + 2\frac{\partial G(x_i,\xi)}{\partial\xi}\widehat{U}'(\xi)$$

and

$$|\varphi''(\xi)| \leq M\left|\widehat{U}(\xi)\right| + 2M\frac{|U_j - U_{j-1}|}{h_j}$$
$$\leq M\max(|U_{j-1}|, |U_j|) + 2M\frac{|U_j - U_{j-1}|}{h_j}.$$

Hence we have

$$\frac{h_j^3}{12}\,|\varphi''(\xi_j)| \le \frac{M}{12}h_j^3 \max\left(|U_{j-1}|, |U_j|\right) + \frac{M}{6}h_j^2|U_j - U_{j-1}|$$

and, by Lemma 3.2,

$$\begin{aligned}
|\varepsilon_i| &\le \frac{1}{12}\sum_{j=1}^{n+1} h_j^3\,|\varphi''(\xi_j)| \\
&\le \frac{M}{12}\sum_{j=1}^{n+1} h_j^3 \max\left(|U_{j-1}|, |U_j|\right) + \frac{1}{6}\sum_{j=1}^{n+1} h_j^2|U_j - U_{j-1}| \\
&\le \frac{Mh^2}{12}\sum_{j=1}^{n+1} h_j \max\left(|U_{j-1}|, |U_j|\right) + \frac{Mh}{6}\sum_{j=1}^{n+1} h_j|U_j - U_{j-1}| \\
&\le \frac{Mh^2}{6}\sqrt{b-a}\sqrt{(\boldsymbol{U}, H^{-1}\boldsymbol{U})} + \frac{Mh}{6}2\sqrt{b-a}\sqrt{(\boldsymbol{U}, H^{-1}\boldsymbol{U})} \\
&= \frac{Mh}{6}(h+2)\sqrt{b-a}\sqrt{(\boldsymbol{U}, H^{-1}\boldsymbol{U})} \\
&= \frac{Mh}{6}(h+2)\sqrt{b-a} = \varepsilon.
\end{aligned}$$

□

LEMMA 3.5. *Let* $(\boldsymbol{U}, H^{-1}\boldsymbol{U}) = 1$. *Then*

$$\begin{aligned}
&\int_a^b\int_a^b G(x,\xi)\widehat{U}(\xi)\,d\xi\widehat{U}(x)\,dx \\
&= \sum_{i=0}^{n+1}\sum_{j=0}^{n+1} G(x_i, x_j)(\omega_i U_i)(\omega_j U_j) + O(h).
\end{aligned}$$

Proof. Let

$$\psi(x) = \int_a^b G(x,\xi)\widehat{U}(\xi)\,d\xi\widehat{U}(x).$$

Then

$$\begin{aligned}
&\int_a^b\int_a^b G(x,\xi)\widehat{U}(\xi)\,d\xi\widehat{U}(x)\,dx \\
&= \sum_{i=1}^{n+1}\int_{x_{i-1}}^{x_i} \psi(x)\,dx \qquad (3.1)\\
&= \sum_{i=1}^{n+1}\left\{\frac{h_i}{2}\left[\psi(x_{i-1}) + \psi(x_i)\right] - \frac{h_i^3}{12}\psi''(\eta_i)\right\}, \quad x_{i-1} < \eta_i < x_i.
\end{aligned}$$

If $x_{i-1} < x < x_i$, then

$$\psi''(x) = \left(\int_a^b \frac{\partial^2 G(x,\xi)}{\partial x^2} \widehat{U}(\xi)\, d\xi - \frac{\widehat{U}(x)}{p(x)} \right) \widehat{U}(x) + 2 \int_a^b \frac{\partial G(x,\xi)}{\partial x} \widehat{U}(\xi)\, d\xi \cdot \widehat{U}'(x),$$

and

$$|\psi''(x)| \le \left(M \int_a^b \left|\widehat{U}(\xi)\right| d\xi + \frac{1}{p_*} \max(|U_{i-1}|, |U_i|) \right) \max(|U_{i-1}|, |U_i|) + 2M \int_a^b \left|\widehat{U}(\xi)\right| d\xi \cdot \frac{|U_i - U_{i-1}|}{h_i}.$$

Therefore,

$$\begin{aligned} &\frac{1}{12} \sum_{i=1}^{n+1} h_i^3 |\varphi''(\eta_i)| \\ &\le \left(\frac{M}{12} \int_a^b \left|\widehat{U}(\xi)\right| d\xi \right) \sum_{i=1}^{n+1} h_i^3 \max(|U_{i-1}|, |U_i|) \\ &\quad + \frac{1}{12 p_*} \sum_{i=1}^{n+1} h_i^3 \max(|U_{i-1}|^2, |U_i|^2) \\ &\quad + \frac{M}{6} \int_a^b \left|\widehat{U}(\xi)\right| d\xi \sum_{i=1}^{n+1} h_i^2 |U_i - U_{i-1}| \\ &\le \frac{M}{6} \sqrt{b-a}\, h^2 \int_a^b \left|\widehat{U}(\xi)\right| d\xi + \frac{h^2}{6p_*} \\ &\quad + \frac{Mh}{3} \sqrt{b-a} \int_a^b \left|\widehat{U}(\xi)\right| d\xi \\ &\le \frac{M}{6}(1+\sqrt{2})(b-a)h^2 + \frac{h^2}{6p_*} + \frac{M}{3}(1+\sqrt{2})(b-a)h \\ &= \widetilde{\varepsilon} \quad \text{(say)}. \end{aligned} \tag{3.2}$$

By Lemma 3.4, we have

$$\psi(x_i) = \left(\sum_{j=0}^{n+1} G(x_i, x_j)\omega_j U_j - \varepsilon_i \right) U_i.$$

We obtain from (3.1) that

$$\int_a^b \int_a^b G(x,\xi) \widehat{U}(\xi)\, d\xi \widehat{U}(x)\, dx$$

$$
\begin{aligned}
&= \sum_{i=1}^{n+1} \frac{h_i}{2}\,[\psi(x_{i-1}) + \psi(x_i)] - \frac{1}{12}\sum_{i=1}^{n+1} h_i^3 \psi''(\eta_i) \\
&= \sum_{i=0}^{n+1} \psi(x_i)\omega_i - \frac{1}{12}\sum_{i=1}^{n+1} h_i^3 \psi''(\eta_i) \\
&= \sum_{i=0}^{n+1} \left(\sum_{j=0}^{n+1} G(x_i, x_j)\omega_j U_j - \varepsilon_i \right) U_i\omega_i - \frac{1}{12}\sum_{i=1}^{n+1} h_i^3 \psi''(\eta_i) \\
&= \sum_{i=0}^{n+1}\sum_{j=0}^{n+1} G(x_i, x_j)(\omega_j U_j)(\omega_i U_i) + \sigma,
\end{aligned}
$$

where

$$
\sigma = -\sum_{i=0}^{n+1} \varepsilon_i (U_i\omega_i) - \frac{1}{12}\sum_{i=1}^{n+1} h_i^3 \psi''(\eta_i).
$$

We thus obtain from Lemma 3.4 and (3.2)

$$
\begin{aligned}
|\sigma| &\le \sum_{i=0}^{n+1} |\varepsilon_i| \cdot |U_i|\omega_i + \widetilde{\varepsilon} \\
&\le \varepsilon \sum_{i=0}^{n+1} \omega_i |U_i| + \widetilde{\varepsilon} \\
&= \varepsilon \sum_{i=1}^{n+1} \frac{h_i}{2}(|U_i| + |U_{i-1}|) + \widetilde{\varepsilon} \\
&\le \varepsilon\sqrt{b-a}\,\big(\boldsymbol{U}^{-1}, H^{-1}\boldsymbol{U}\big) + \widetilde{\varepsilon} \\
&= \varepsilon\sqrt{b-a} + \widetilde{\varepsilon} \\
&= O(h)\sqrt{b-a} + O(h) = O(h).
\end{aligned}
$$

□

Lemma 3.6. *For sufficiently small h, the matrix $HA - \eta I$ is an M-matrix, hence nonsingular.*

Proof. Since A is an irreducibly diagonally dominant L-matrix and symmetric, A is a positive definite M-matrix. Then, for any $\boldsymbol{U}$, $\boldsymbol{V} \in \mathbb{R}^{n+2}$, we have

$$
\begin{aligned}
(\boldsymbol{U}, \boldsymbol{V})^2 &= \Big(\sqrt{A}\boldsymbol{U}, \sqrt{A}^{-1}\boldsymbol{V}\Big)^2 \\
&\le \big(\sqrt{A}\boldsymbol{U}, \sqrt{A}\boldsymbol{U}\big)\Big(\sqrt{A}^{-1}\boldsymbol{V}, \sqrt{A}^{-1}\boldsymbol{V}\Big) \\
&= (A\boldsymbol{U}, \boldsymbol{U})\big(A^{-1}\boldsymbol{V}, \boldsymbol{V}\big). \qquad (3.3)
\end{aligned}
$$

Let $\boldsymbol{W} \in \mathbb{R}^{n+2}$ and $\boldsymbol{W} \neq \boldsymbol{0}$. Then $c = \big(\boldsymbol{W}, H^{-1}\boldsymbol{W}\big) > 0$. We put $\boldsymbol{U} = (1/\sqrt{c})\boldsymbol{W}$. Then we have $(\boldsymbol{U}, H^{-1}\boldsymbol{U}) = 1$.

We then have from (3.3), Lemma 3.1 and Lemma 3.5

$$\begin{aligned}
\big((A-\eta H^{-1})\boldsymbol{U},\boldsymbol{U}\big) &= (A\boldsymbol{U},\boldsymbol{U}) - \eta\big(\boldsymbol{U},H^{-1}\boldsymbol{U}\big)\\
&\geq \frac{(\boldsymbol{U},H^{-1}\boldsymbol{U})^2}{(A^{-1}H^{-1}\boldsymbol{U},H^{-1}\boldsymbol{U})} - \eta(\boldsymbol{U},H^{-1}\boldsymbol{U})\\
&\qquad\qquad (\text{Put } \boldsymbol{V} = H^{-1}\boldsymbol{U} \text{ in } (3.3))\\
&= \left[\frac{(\boldsymbol{U},H^{-1}\boldsymbol{U})}{\sum_{i=0}^{n+1}\sum_{j=0}^{n+1} G(x_i,x_j)(\omega_i U_i)(\omega_j U_j)} - \eta\right]\big(\boldsymbol{U},H^{-1}\boldsymbol{U}\big)\\
&\geq \frac{\|\widehat{U}\|^2}{\int_a^b\int_a^b G(x,\xi)\widehat{U}(\xi)\,d\xi\widehat{U}(x)\,dx + O(h)} - \eta\\
&\geq \frac{\|\widehat{U}\|^2}{(1/\lambda_1)\|\widehat{U}\|^2 + O(h)} - \eta\\
&= (\lambda_1-\eta) + \frac{O(h)}{\|\widehat{U}\|^2}.
\end{aligned}$$

By Lemma 3.1, we have $\|\widehat{U}\| = O(1)$ and

$$\lambda_1 - \eta + \frac{O(h)}{\|\widehat{U}\|^2} > 0$$

for sufficiently small $h > 0$ and $((A-\eta H^{-1})\boldsymbol{W},\boldsymbol{W}) > 0$ for any $\boldsymbol{W} \neq \boldsymbol{0}$. Consequently the symmetric matrix $B = A - \eta H^{-1}$ is then positive definite and eigenvalues are all positive. Since B is a Z-matrix, this means that B as well as $HB = HA - \eta I$ is an M-matrix (cf. [7]). □

We are now in a position to prove the following:

THEOREM 3.1. *Under the assumption of* Theorem 2.1, *the discretized system* (1.8) *has a unique solution if* h *is sufficiently small.*

Proof. We again assume $\alpha = \beta = 0$, without loss of generality.

(i) UNIQUENESS. Let $\boldsymbol{U},\boldsymbol{V} \in \mathbb{R}^{n+2}$ be two solutions of (1.8) and put $\boldsymbol{W} = \boldsymbol{U} - \boldsymbol{V} = (W_0, W_1, \ldots, W_{n+1})^t$. Then $\boldsymbol{W}$ satisfies the system of $(n+2)$ equations

$$(HA + D)\boldsymbol{W} = \boldsymbol{0},$$

where

$$D = \operatorname{diag}(d_0, d_1, \ldots, d_{n+1}), \tag{3.4}$$

with

$$d_i = \int_0^1 f_u(x_i, V_i + \theta(U_i - V_i))\,d\theta, \quad 0 \leq i \leq n+1.$$

By Lemma 3.6, we have for sufficiently small h

$$\big((A + H^{-1}D)\boldsymbol{U}, \boldsymbol{U}\big) \geq \big((A - \eta H^{-1})\boldsymbol{U}, \boldsymbol{U}\big) > 0 \quad \forall \boldsymbol{U}\ (\neq \boldsymbol{0}) \in \mathbb{R}^{n+2},$$

and $HA + D$ is nonsingular. We thus obtain $\boldsymbol{W} = \boldsymbol{0}$, which means the uniqueness of the solution.

(ii) EXISTENCE. We write (1.8) as

$$HA\boldsymbol{U} + Z\boldsymbol{U} = -\widetilde{\boldsymbol{f}}(\boldsymbol{0}),$$

where

$$Z = Z(\boldsymbol{U}) = \operatorname{diag}(\zeta_0, \zeta_1, \ldots, \zeta_{n+1}) \tag{3.5}$$

with

$$\zeta_i = \int_0^1 f_u(x_i, \theta U_i)\, d\theta, \quad 0 \leq i \leq n+1,$$

and

$$\begin{aligned}\widetilde{\boldsymbol{f}}(\boldsymbol{0}) &= (f(x_0, 0), f(x_1, 0), \ldots, f(x_{n+1}, 0))^t \\ &= (f_0(x_0), f_0(x_1), \ldots, f_0(x_{n+1}))^t\end{aligned}$$

with $f_0(x) = f(x, 0)$.

Since $Z \geq -\eta I$ (I is the $(n+2)\times(n+2)$ identity), given $\boldsymbol{U} \in \mathbb{R}^{n+2}$, the system of linear equations

$$(HA + Z)\boldsymbol{W} = -\widetilde{\boldsymbol{f}}(\boldsymbol{0})$$

has a unique solution

$$\begin{aligned}\boldsymbol{W} &= -(HA + Z)^{-1}\widetilde{\boldsymbol{f}}(\boldsymbol{0}) \\ &= -(A + H^{-1}Z)^{-1}H^{-1}\widetilde{\boldsymbol{f}}(\boldsymbol{0}),\end{aligned} \tag{3.6}$$

where $A + H^{-1}Z$ is again a symmetric M-matrix. Let $\varphi \in C^2[a, b]$ be the unique solution of the problem

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) - \eta u = 2, \quad u \in \mathscr{D},$$

whose existence is guaranteed by Theorem 2.1. Let

$$\boldsymbol{\tau} = (HA - \eta I)\boldsymbol{\varphi} - 2\boldsymbol{e},$$

where

$$\boldsymbol{\varphi} = (\varphi(x_0), \varphi(x_1), \ldots, \varphi(x_{n+1}))^t \quad \text{and} \quad \boldsymbol{e} = (1, 1, \ldots, 1)^t \in \mathbb{R}^{n+2}.$$

Then a simple computation based upon the usual Taylor expansion for $p(x) \in C^1[a,b]$ and $\varphi(x) \in C^2[a,b]$ yields

$$\|\boldsymbol{\tau}\|_\infty = o(1) \to 0$$

as $h \to 0$, where we apply the midpoint rule to estimate the integrals $a_i = \int_{x_{i-1}}^{x_i}(dt/p(t))$, $1 \le i \le n+1$ and employ the fact that p' and φ'' are uniformly continuous in the interval $[a,b]$. Hence, for sufficiently small h,

$$(HA - \eta I)\boldsymbol{\varphi} = 2\boldsymbol{e} + \boldsymbol{\tau} \ge \boldsymbol{e}$$

or

$$(HA - \eta I)^{-1}\boldsymbol{e} \le \boldsymbol{\varphi}. \tag{3.7}$$

We denote by $|\boldsymbol{W}|$ the vector $(|W_0|, |W_1|, \ldots, |W_{n+1}|)^t$.

Then we have from (3.6)

$$\begin{aligned} |\boldsymbol{W}| &\le \left(A + H^{-1}Z\right)^{-1}H^{-1}\left|\widetilde{\boldsymbol{f}}(\boldsymbol{0})\right| \\ &\le \left(A - \eta H^{-1}\right)^{-1}H^{-1}\left\|\widetilde{\boldsymbol{f}}(\boldsymbol{0})\right\|_\infty \boldsymbol{e} \\ &= \left\|\widetilde{\boldsymbol{f}}(\boldsymbol{0})\right\|_\infty (HA - \eta I)^{-1}\boldsymbol{e} \\ &\le \left\|\widetilde{\boldsymbol{f}}(\boldsymbol{0})\right\|_\infty \boldsymbol{\varphi} \end{aligned}$$

and

$$\|\boldsymbol{W}\|_\infty \le \left\|\widetilde{\boldsymbol{f}}(\boldsymbol{0})\right\|_\infty \|\boldsymbol{\varphi}\|_\infty \le \|f_0\|_{[a,b]}\|\varphi\|_{[a,b]} = C \quad \text{(say)}, \tag{3.8}$$

since we have assumed $\alpha = \beta = 0$ and $\widetilde{\boldsymbol{f}}(\boldsymbol{0}) = (f_0(x_0), f_0(x_1), \ldots, f_0(x_{n+1}))^t$. Hence we put

$$S = \{\boldsymbol{U} \in \mathbb{R}^{n+2} \mid \|\boldsymbol{U}\|_\infty \le C\}$$

and define a map $T\colon S \to S$ by $T\boldsymbol{U} = \boldsymbol{W}$, $\boldsymbol{U} \in S$. Then T is continuous. In fact, we have for $\boldsymbol{U}$, $\widehat{\boldsymbol{U}} \in S$

$$\begin{aligned} T\boldsymbol{U} - T\widehat{\boldsymbol{U}} &= -\left\{(HA + Z)^{-1} - \left(HA + \widehat{Z}\right)^{-1}\right\}\widetilde{\boldsymbol{f}}(\boldsymbol{0}) \\ &= \left(HA + \widehat{Z}\right)^{-1}\left(Z - \widehat{Z}\right)(HA + Z)^{-1}\widetilde{\boldsymbol{f}}(\boldsymbol{0}), \end{aligned} \tag{3.9}$$

where $Z = Z(\boldsymbol{U})$ and $\widehat{Z} = Z\left(\widehat{\boldsymbol{U}}\right)$. It now follows from (3.8) and (3.9) that

$$\left\|T\boldsymbol{U} - T\widehat{\boldsymbol{U}}\right\|_\infty \le \left\|Z - \widehat{Z}\right\|_\infty \|\varphi\|_{[a,b]} C \to 0$$

as $\left\|\boldsymbol{U} - \widehat{\boldsymbol{U}}\right\|_\infty \to 0$, since $\left\|Z - \widehat{Z}\right\|_\infty \to 0$ as $\left\|\boldsymbol{U} - \widehat{\boldsymbol{U}}\right\|_\infty \to 0$ because of the uniform continuity of $f_u(x,u)$ in $[a,b] \times [-C, C]$.

Consequently, we conclude, by Brouwer's theorem, that T has a fixed point $\boldsymbol{U}$ in S, which is a solution of (1.8). $\square$

4. Error Estimates

We still keep the assumption $\alpha_1\beta_1 \neq 0$ without loss of generality.

Let $u = u(x)$ be the solution of the continuous problem (1.1)–(1.6) and $\boldsymbol{u} = (u(x_0), u(x_1), \ldots, u(x_{n+1}))^t$. We put

$$\boldsymbol{\tau} = HA\boldsymbol{u} + \widetilde{\boldsymbol{f}}(\boldsymbol{u}) = (\tau_0, \tau_1, \ldots, \tau_{n+1})^t.$$

Then we have from (1.8)

$$HA(\boldsymbol{u} - \boldsymbol{U}) + \widetilde{\boldsymbol{f}}(\boldsymbol{u}) - \widetilde{\boldsymbol{f}}(\boldsymbol{U}) = \boldsymbol{\tau}$$

or

$$\big(HA + \widetilde{D}\big)(\boldsymbol{u} - \boldsymbol{U}) = \boldsymbol{\tau} \tag{4.1}$$

where $\widetilde{D} = \operatorname{diag}\big(\widetilde{d}_0, \widetilde{d}_1, \ldots, \widetilde{d}_{n+1}\big)^t$ with $\widetilde{d}_i = \int_0^1 f_u(x_i, U_i + \theta(u_i - U_i))\, d\theta$. As is shown in the end of the proof of Lemma 3.6, $HA - \eta I$ is an M-matrix and

$$HA + \widetilde{D} \geq HA - \eta I.$$

Hence, $HA + \widetilde{D}$ is an M-matrix and

$$0 < \big(HA + \widetilde{D}\big)^{-1} \leq (HA - \eta I)^{-1} \tag{4.2}$$

since $HA + \widetilde{D}$ is an irreducible Z-matrix and $\big(HA + \widetilde{D}\big)^{-1}$ is a positive matrix ([7]).

It now follows from (4.1) and (4.2) that

$$\boldsymbol{u} - \boldsymbol{U} = \big(HA + \widetilde{D}\big)^{-1}\boldsymbol{\tau}. \tag{4.3}$$

As is easily seen, we have

$$\|\boldsymbol{\tau}\|_\infty = \begin{cases} o(1) & (\text{if } u \in C^2[a,b]) \\ O(h) & (\text{if } u \in C^{2,1}[a,b],\ p \in C^{1,1}[a,b]) \end{cases}$$

and

$$\begin{aligned} |\boldsymbol{u} - \boldsymbol{U}| &\leq \big(HA + \widetilde{D}\big)^{-1}|\boldsymbol{\tau}| \leq \|\boldsymbol{\tau}\|_\infty \big(HA + \widetilde{D}\big)^{-1}\boldsymbol{e} \\ &\leq \|\boldsymbol{\tau}\|_\infty (HA - \eta)^{-1}\boldsymbol{e} \leq \|\boldsymbol{\tau}\|_\infty \boldsymbol{\varphi} \qquad \text{by (3.7).} \end{aligned}$$

Hence

$$\|\boldsymbol{u} - \boldsymbol{U}\|_\infty = \begin{cases} o(1) & (u \in C^2[a,b]) \\ O(h) & (u \in C^{2,1}[a,b]). \end{cases} \tag{4.4}$$

Furthermore, if $p \in C^{2,1}[a,b]$ and $u \in C^{3,1}[a,b]$, then it can be shown (see [8] pp. 52–56) that

$$\tau_i = \begin{cases} O(h_1) & (i=0) \\ \dfrac{2}{h_i+h_{i+1}}\left(s_{i+(1/2)}h_{i+1}^2 - s_{i-(1/2)}h_i^2\right)u_i' - (h_{i+1}-h_i)\kappa_i + O(h^2) & (1 \le i \le n) \\ O(h_{n+1}) & (i=n+1) \end{cases}$$

where

$$s(x) = \frac{1}{24}\left(\frac{1}{p}\right)'' p^2, \tag{4.5}$$

$$\kappa(x) = \frac{1}{12}\{3p''(x)u'(x) + 6p'(x)u''(x) + 4p(x)u'''(x)\}, \tag{4.6}$$

$$s_{i+(1/2)} = s\left(x_i + \frac{1}{2}h_{i+1}\right),$$

$$s_{i-(1/2)} = s\left(x_i - \frac{1}{2}h_i\right)$$

and

$$\kappa_i = \kappa(x_i).$$

We have from (4.3)

$$\begin{aligned} \boldsymbol{u} - \boldsymbol{U} &= \left(HA + \widetilde{D}\right)^{-1}\boldsymbol{\tau} \\ &= \left[(HA)^{-1} - \left(HA+\widetilde{D}\right)^{-1}\widetilde{D}(HA)^{-1}\right]\boldsymbol{\tau} \\ &= A^{-1}H^{-1}\boldsymbol{\tau} - \left(HA+\widetilde{D}\right)^{-1}\widetilde{D}\left(A^{-1}H^{-1}\boldsymbol{\tau}\right) \end{aligned}$$

and

$$\begin{aligned} |\boldsymbol{u} - \boldsymbol{U}| &\le \left|A^{-1}H^{-1}\boldsymbol{\tau}\right| + \left(HA+\widetilde{D}\right)^{-1}\left|\widetilde{D}\right|\left|A^{-1}H^{-1}\boldsymbol{\tau}\right| \\ &\le \left|A^{-1}H^{-1}\boldsymbol{\tau}\right| + (HA-\eta I)^{-1}\left|\widetilde{D}\right|\left|A^{-1}H^{-1}\boldsymbol{\tau}\right|. \end{aligned} \tag{4.7}$$

Since $\|\boldsymbol{U}-\boldsymbol{u}\|_\infty \to 0$ as $h \to 0$ by (4.4), $\|\boldsymbol{u}\|_\infty \le \|u\|_{[a,b]} \le \delta_0$ (cf. the proof of Theorem 2.1) and, for any $\theta \in [0,1]$,

$$\begin{aligned} \|\boldsymbol{U} + \theta(\boldsymbol{u}-\boldsymbol{U})\|_\infty &= \|\boldsymbol{u} + (1-\theta)(\boldsymbol{U}-\boldsymbol{u})\|_\infty \\ &\le \|\boldsymbol{u}\|_\infty + (1-\theta)\|\boldsymbol{U}-\boldsymbol{u}\|_\infty \\ &\le \|\boldsymbol{u}\|_\infty + \|\boldsymbol{U}-\boldsymbol{u}\|_\infty, \end{aligned}$$

we may assume that $\|\boldsymbol{U} + \theta(\boldsymbol{u}-\boldsymbol{U})\|_\infty \le 2\delta_0 \ \forall\theta \in [0,1]$.

Then

$$\left|\widetilde{d}_i\right| \le \widehat{K} = \max_{[a,b]\times[-2\delta_0,2\delta_0]} |f_u(x,u)| < +\infty \quad \forall i.$$

Therefore we obtain from (4.7) and (3.7)

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{U}\|_\infty &\le \left\|A^{-1}H^{-1}\boldsymbol{\tau}\right\|_\infty + \widehat{K}\left\|A^{-1}H^{-1}\boldsymbol{\tau}\right\|_\infty\|\boldsymbol{\varphi}\|_\infty \\ &\le \left(1+\widehat{K}\|\varphi\|_{[a,b]}\right)\left\|A^{-1}H^{-1}\boldsymbol{\tau}\right\|_\infty. \qquad (4.8)\end{aligned}$$

If $p \in C^{2,1}[a,b]$ and $u \in C^{3,1}[a,b]$, then we have

$$\left(A^{-1}H^{-1}\boldsymbol{\tau}\right)_i = \sum_{j=0}^{n+1} G(x_i,x_j)\omega_j\tau_j$$

and we can show

$$\left\|A^{-1}H^{-1}\boldsymbol{\tau}\right\|_\infty = O(h^2)$$

by noting that the functions $s(x)$ and $\kappa(x)$ defined by (4.5) and (4.6) are Lipschitz continuous in $[a,b]$ (cf. [8]). Hence, from (4.8) we have $\|\boldsymbol{u}-\boldsymbol{U}\|_\infty = O(h^2)$.

Summarizing we have the following result.

Theorem 4.1. *Under the assumption* (2.5), *we have*

$$\|\boldsymbol{u}-\boldsymbol{U}\|_\infty = \begin{cases} o(1) & (u \in C^2[a,b],\ p \in C^1[a,b]) \\ O(h) & (u \in C^{2,1}[a,b],\ p \in C^{1,1}[a,b]) \\ O(h^2) & (u \in C^{3,1}[a,b],\ p \in C^{2,1}[a,b]). \end{cases}$$

5. Remark

In (1.8), if we replace the integral $\int_{x_{i-1}}^{x_i}(dt/p(t))$ by the mid point formula $h_i/p(x_{i-(1/2)})$ $(x_{i-(1/2)} = (1/2)(x_i+x_{i-1}))$ for each i, then the usual finite difference formula arises. We can also derive Theorems 3.1 and 4.1 in this case.

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