

Comparison Between Passive and Active Control of a Non-Linear Dynamical System

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Vibrations and dynamic chaos should be controlled in structures and machines. The wing of the airplane should be free from vibrations or it should be kept minimum. To do so, two main strategies are used. They are passive and active control methods. In this paper we present a mathematical study of passive and active control in some non-linear differential equations describing the vibration of the wing. Firstly, non-linear differential equation representing the wing system subjected to multi-excitation force is considered and solved using the method of multiple scale perturbation. Secondly, a tuned mass absorber (TMA) is applied to the system at simultaneous primary resonance. Thirdly, the same system is considered with 1:2 internal resonance active control absorber. The approximate solution is derived up to the fourth order approximation, the stability of the system is investigated applying both frequency response equations and phase plane methods. Previous work regarding the wing vibration dealt only with a linear system describing its vibration. Some recommendations are given by the end of the work.

Key words: control, passive, active, absorber, absorber effectiveness

0. Introduction

Vibrations and dynamic chaos are undesired phenomenon in structures. They cause disturbance, discomfort, damage and destruction of the system or the structure. One of the most effective tools for passive vibration control is the dynamic absorber or the damper or the neutralizer [1]. Eissa [2] has shown that the non-linear absorber widens its range of applications, and its damping coefficient should be kept minimum for better performance [3]. Cheng-Tang Lee et al. [4] demonstrated that a dynamic vibration absorber system can be used to reduce speed fluctuations in rotating machinery. Eissa and El-Ganaini [5, 6] studied the control of vibration and dynamic chaos of mechanical structures having quadratic and cubic non-linearities, subjected to harmonic excitation using single and multi-absorbers. Active constrained layer damping (ACLD) has been successfully utilized as effective means of damping out the vibration of various flexible structures [7–12]. In weakly non-linear systems, internal resonances may occur if the linear natural frequencies are commensurate or nearly commensurate, and internal resonances provide coupling and energy exchange among the vibration modes [13, 14]. If two natural

frequencies of a system with quadratic non-linearities are in the ratio 1:2 there exists a saturation phenomenon [13]. When the system is excited at a frequency near the higher natural frequency, the structure responds at the excitation frequency and the amplitude of the response increases linearly with the excitation amplitude. However, when the high-frequency modal amplitude reaches a critical value, this mode saturates and all additional energy added to the system by increasing the excitation amplitude overflows into the low-frequency mode. Recently the use of internal resonance and saturation phenomena in non-linear control has been extensively studied [15, 18]. This method is based on an approach originally introduced by Golnaraghi [15] and thoroughly investigated by Nayfeh et al. [17]. To control transient vibrations, reference [15] used a second-order controller coupled to a vibrating system via quadratic or cubic terms. References [16, 17] used the saturation phenomenon to successfully control the motion of a d.c. motor with a rigid beam attached.

Aerospace structures are required to be light in weight and hence composite structures are increasingly used. Moreover, aerospace composite structures (e.g. helicopter rotor blades) are often designed with built in elastic bending torsion couplings to improve aerodynamic efficiency. To actively control such structures without too much added weight, PZT (lead zirconate titanate) sensors and actuators are attractive because of their mechanical simplicity, small volume, light weight, large useful bandwidth, efficient conversion of electrical to mechanical energy, ability to perform shape control, and ability to be easily integrated with the structure [19]. Wen, et al. [20] investigated non-linear saturation control, non-linear internal resonance control and linear position-feedback control of steady-state and transient vibrations of a cantilever beam using PZT patches as sensors and actuators.

The aim of this work is to apply both linear passive, and active non-linear vibration absorbers using higher-order internal resonances and saturation phenomena to suppress the steady-state vibrations of a cantilever skew aluminum plate representing the wing of an aircraft [21], [22]. Higher-order internal resonances are introduced using quadratic terms to couple the controller with the plate. The multiple time scale perturbation technique is applied throughout. An approximate solution is derived up to third order approximation. The stability of the system is investigated applying both frequency response functions (FRFs) and phase-plane methods. The effects of the absorber on system behavior are studied numerically. Optimum working conditions of the system are obtained applying both passive and active control methods. Both control methods are demonstrated and compared numerically.

1. Mathematical Modeling

The considered system is shown schematically in Fig. 1.

The investigated equation is the modified non-linear differential equation de-

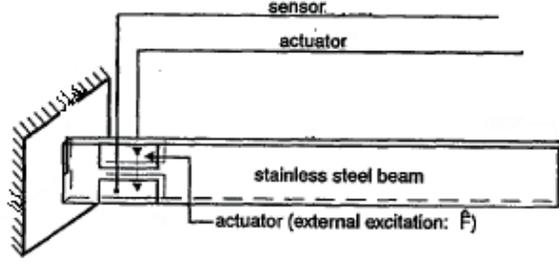


Fig. 1. The non-linear cantilevered skew aluminium plate with a 1:2 non-linear vibration absorber for controlling the first-mode vibration.

describing the vibration of an aircraft wing [22] which is given by:

$$\ddot{u} + 2\bar{\zeta}\omega\dot{u} + \omega^2u + \bar{\alpha}u^2 + \bar{\beta}u^3 = \sum_{j=1}^N \bar{F}_j \cos \Omega_j t \quad (1)$$

where u , \dot{u} and \ddot{u} represent displacement, velocity and acceleration of the vibrating system, ω is the natural frequency, $\bar{\zeta}$ is the damping coefficient, $\bar{\alpha}$ and $\bar{\beta}$ are non-linear stiffness coefficients, \bar{F}_j is the forcing amplitudes and Ω_j are the excitation frequencies.

1.1. Perturbation analysis

We seek a first-order approximate solution of equation (1) by using the method of multiple scales [2] in the form

$$u(T_0, T_n; \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k u_k(T_0, T_n) \quad (2)$$

where ε is a small dimensionless parameter used for book-keeping only; $T_0 = t$ and $T_n = \varepsilon^n t$ where $n = (1, 2, \dots)$ are the fast and slow time scales respectively. To make dampings, non-linearities, and the primary resonance force appear in the same perturbation equations, we order that $\bar{\zeta} = \varepsilon\zeta$, $\bar{\alpha} = \varepsilon\alpha$, $\bar{\beta} = \varepsilon\beta$, $\bar{F}_j = \varepsilon F_j$. Substituting equation (2) into equation (1) and equating coefficients of like powers of ε , we obtain the following set of ordinary differential equations:

$$(D_0^2 + \omega^2)u_0 = 0 \quad (3)$$

$$(D_0^2 + \omega^2)u_1 = -2D_0D_1u_0 - 2\zeta\omega D_0u_0 - \alpha u_0^2 - \beta u_0^3 + \sum_{j=1}^N F_j \cos \Omega_j T_0 \quad (4)$$

$$(D_0^2 + \omega^2)u_2 = -D_1^2u_0 - 2D_0D_1u_1 - 2\zeta\omega(D_1u_0 + D_0u_1) - 2\alpha u_0u_1 - 3\beta u_0^2u_1 \quad (5)$$

$$(D_0^2 + \omega^2)u_3 = -D_1^2u_1 - 2D_0D_1u_2 - 2\zeta\omega(D_1u_1 + D_0u_2) - \alpha(u_1^2 + 2u_0u_2) - 3\beta(u_1^2u_0 + u_0^2u_2) \quad (6)$$

where

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots, \quad D_i = \frac{\partial}{\partial T_i}, \quad i = 0, 1, 2, \dots$$

For our considered case we have the external excitation frequency will be Ω_j , $j = 1, 2, 3, 4$ for simplicity.

The solution of equation (3) can be expressed in the form

$$u_0(T_0, T_n) = A_0(T_n)e^{i\omega T_0} + cc \quad (7)$$

where A_0 is a complex function in T_n and cc represents the complex conjugate. Substituting equation (7) into equation (4), yields

$$\begin{aligned} (D_0^2 + \omega^2)u_1 = & [-2i\omega(D_1 A_0 + \zeta\omega A_0) - 3\beta A_0^2 \bar{A}_0]e^{i\omega T_0} - \alpha A_0^2 e^{2i\omega T_0} \\ & - 2\alpha A_0 \bar{A}_0 - \beta A_0^3 e^{3i\omega T_0} + \frac{1}{2} \sum_{j=1}^4 F_j e^{i\Omega_j T_0} + cc \end{aligned} \quad (8)$$

Eliminating the secular terms, then the first order approximation is given by

$$u_1(T_0, T_n) = E_1 e^{i\omega T_0} + E_2 e^{2i\omega T_0} + E_3 e^{3i\omega T_0} + E_4 + \sum_{j=1}^4 E_{(4+j)} e^{i\Omega_j T_0} + cc \quad (9)$$

where E_m ($m = 1, 2, 3, 4$) are complex functions in T_n . From equation (7) and (9) into equation (5), yields

$$\begin{aligned} (D_0^2 + \omega^2)u_2 = & \left[-D_1^2 A_0 - 2i\omega D_1 A_1 - 2\zeta\omega D_1 A_0 - 2i\zeta\omega^2 A_1 + \frac{10\alpha^2}{3\omega^2} A_0^2 \bar{A}_0 \right. \\ & \left. - \frac{3\beta^2}{8\omega^2} A_0^3 \bar{A}_0^2 6\beta A_0 \bar{A}_0 A_1 - 3\beta \bar{A}_1 A_0^2 \right] e^{i\omega T_0} \\ & + \left[-\frac{8i\alpha}{3\omega} A_0 D_1 A_0 - \frac{4i\alpha}{3} \zeta A_0^2 + \frac{15\alpha\beta}{4\omega^2} A_0^3 \bar{A}_0 - 2\alpha A_0 A_1 \right] e^{2i\omega T_0} \\ & - \left[\frac{9i\beta}{4\omega} A_0^2 D_1 A_0 + \frac{3i\beta}{4} \zeta A_0^3 + \frac{2\alpha^2}{3\omega^2} A_0^3 + \frac{3\beta^2}{4\omega^2} A_0^4 \bar{A}_0 + 3\beta A_0^2 A_1 \right] e^{3i\omega T_0} \\ & - \sum_{j=1}^4 \frac{F_j}{\omega^2 - \Omega_j^2} \left[(i\zeta\omega\Omega_j + 3\beta A_0 \bar{A}_0) e^{i\Omega_j T_0} + \alpha A_0 (e^{i(\omega+\Omega_j)T_0} + e^{i(\omega-\Omega_j)T_0}) \right. \\ & \left. + \frac{3}{2} \beta A_0^2 (e^{i(2\omega+\Omega_j)T_0} + e^{i(2\omega-\Omega_j)T_0}) \right] - \frac{5\alpha\beta}{4\omega^2} A_0^4 e^{4i\omega T_0} - \frac{3\beta^2}{8\omega^2} A_0^5 e^{5i\omega T_0} \\ & + \frac{10\alpha\beta}{\omega^2} A_0^2 \bar{A}_0^2 + A_0 \bar{A}_1 + A_1 \bar{A}_0 + cc \end{aligned} \quad (10)$$

To obtain a bounded solution the secular terms should be eliminated, then the second order approximation is given by

$$u_2(T_0, T_n) = H_1 e^{i\omega T_0} + H_2 e^{2i\omega T_0} + H_3 e^{3i\omega T_0} + H_4 e^{4i\omega T_0} + H_5 e^{5i\omega T_0}$$

$$\begin{aligned}
& + \sum_{j=1}^4 [H_{(5+j)} e^{i\Omega_j T_0} + H_{(9+j)} e^{i(\omega+\Omega_j)T_0} + H_{(13+j)} e^{i(\omega-\Omega_j)T_0} \\
& + H_{(17+j)} e^{i(2\omega+\Omega_j)T_0} + H_{(21+j)} e^{i(2\omega-\Omega_j)T_0}] + H_{26} + cc \quad (11)
\end{aligned}$$

where H_m ($m = 1, 2, \dots, 26$) are complex functions in T_n . From (7), (9) and (11) into (6), we get the third order approximation as:

$$\begin{aligned}
u_3(T_0, T_n) = & P_1 e^{i\omega T_0} + P_2 e^{2i\omega T_0} + P_3 e^{3i\omega T_0} + P_4 e^{4i\omega T_0} + P_5 e^{5i\omega T_0} + P_6 e^{6i\omega T_0} \\
& + P_7 e^{7i\omega T_0} + P_8 e^{i\Omega_1 T_0} + P_9 e^{i\Omega_2 T_0} + P_{10} e^{i\Omega_3 T_0} + P_{11} e^{i\Omega_4 T_0} \\
& + P_{12} e^{2i\Omega_1 T_0} + P_{13} e^{2i\Omega_2 T_0} + P_{14} e^{2i\Omega_3 T_0} + P_{15} e^{2i\Omega_4 T_0} + P_{16} e^{i(2\Omega_1+\omega)T_0} \\
& + P_{17} e^{i(2\Omega_2+\omega)T_0} + P_{18} e^{i(2\Omega_3+\omega)T_0} + P_{19} e^{i(2\Omega_4+\omega)T_0} + P_{20} e^{i(2\Omega_1-\omega)T_0} \\
& + P_{21} e^{i(2\Omega_2-\omega)T_0} + P_{22} e^{i(2\Omega_3-\omega)T_0} + P_{23} e^{i(2\Omega_4-\omega)T_0} + P_{24} e^{i(\Omega_1+\omega)T_0} \\
& + P_{25} e^{i(\Omega_2+\omega)T_0} + P_{26} e^{i(\Omega_3+\omega)T_0} + P_{27} e^{i(\Omega_4+\omega)T_0} + P_{28} e^{i(\Omega_1-\omega)T_0} \\
& + P_{29} e^{i(\Omega_2-\omega)T_0} + P_{30} e^{i(\Omega_3-\omega)T_0} + P_{31} e^{i(\Omega_4-\omega)T_0} + P_{32} e^{i(\Omega_1+2\omega)T_0} \\
& + P_{33} e^{i(\Omega_2+2\omega)T_0} + P_{34} e^{i(\Omega_3+2\omega)T_0} + P_{35} e^{i(\Omega_4+2\omega)T_0} + P_{36} e^{i(\Omega_1-2\omega)T_0} \\
& + P_{37} e^{i(\Omega_2-2\omega)T_0} + P_{38} e^{i(\Omega_3-2\omega)T_0} + P_{39} e^{i(\Omega_4-2\omega)T_0} + P_{40} e^{i(\Omega_1+3\omega)T_0} \\
& + P_{41} e^{i(\Omega_2+3\omega)T_0} + P_{42} e^{i(\Omega_3+3\omega)T_0} + P_{43} e^{i(\Omega_4+3\omega)T_0} + P_{44} e^{i(\Omega_1-3\omega)T_0} \\
& + P_{45} e^{i(\Omega_2-3\omega)T_0} + P_{46} e^{i(\Omega_3-3\omega)T_0} + P_{47} e^{i(\Omega_4-3\omega)T_0} + P_{48} e^{i(\Omega_1+4\omega)T_0} \\
& + P_{49} e^{i(\Omega_2+4\omega)T_0} + P_{50} e^{i(\Omega_3+4\omega)T_0} + P_{51} e^{i(\Omega_4+4\omega)T_0} + P_{52} e^{i(\Omega_1-4\omega)T_0} \\
& + P_{53} e^{i(\Omega_2-4\omega)T_0} + P_{54} e^{i(\Omega_3-4\omega)T_0} + P_{55} e^{i(\Omega_4-4\omega)T_0} \\
& + P_{56} [e^{i(\Omega_2+\Omega_1)T_0} + e^{i(\Omega_2-\Omega_1)T_0}] + P_{57} [e^{i(\Omega_3+\Omega_1)T_0} + e^{i(\Omega_3-\Omega_1)T_0}] \\
& + P_{58} [e^{i(\Omega_3+\Omega_2)T_0} + e^{i(\Omega_3-\Omega_2)T_0}] + P_{59} [e^{i(\Omega_4+\Omega_1)T_0} + e^{i(\Omega_4-\Omega_1)T_0}] \\
& + P_{60} [e^{i(\Omega_4+\Omega_2)T_0} + e^{i(\Omega_4-\Omega_2)T_0}] + P_{61} [e^{i(\Omega_4+\Omega_3)T_0} + e^{i(\Omega_4-\Omega_3)T_0}] \\
& + P_{62} e^{i[(\Omega_2+\Omega_1)+\omega]T_0} + P_{63} e^{i[(\Omega_2+\Omega_1)-\omega]T_0} + P_{64} e^{i[(\Omega_2-\Omega_1)+\omega]T_0} \\
& + P_{65} e^{i[(\Omega_2-\Omega_1)-\omega]T_0} + P_{66} e^{i[(\Omega_3+\Omega_1)+\omega]T_0} + P_{67} e^{i[(\Omega_3+\Omega_1)-\omega]T_0} \\
& + P_{68} e^{i[(\Omega_3-\Omega_1)+\omega]T_0} + P_{69} e^{i[(\Omega_3-\Omega_1)-\omega]T_0} + P_{70} e^{i[(\Omega_3+\Omega_2)+\omega]T_0} \\
& + P_{71} e^{i[(\Omega_3+\Omega_2)-\omega]T_0} + P_{72} e^{i[(\Omega_3-\Omega_2)+\omega]T_0} + P_{73} e^{i[(\Omega_3-\Omega_2)-\omega]T_0} \\
& + P_{74} e^{i[(\Omega_4+\Omega_1)+\omega]T_0} + P_{75} e^{i[(\Omega_4+\Omega_1)-\omega]T_0} + P_{76} e^{i[(\Omega_4-\Omega_1)+\omega]T_0} \\
& + P_{77} e^{i[(\Omega_4-\Omega_1)-\omega]T_0} + P_{78} e^{i[(\Omega_4+\Omega_2)+\omega]T_0} + P_{79} e^{i[(\Omega_4+\Omega_2)-\omega]T_0} \\
& + P_{80} e^{i[(\Omega_4-\Omega_2)+\omega]T_0} + P_{81} e^{i[(\Omega_4-\Omega_2)-\omega]T_0} + P_{82} e^{i[(\Omega_4+\Omega_3)+\omega]T_0} \\
& + P_{83} e^{i[(\Omega_4+\Omega_3)-\omega]T_0} + P_{84} e^{i[(\Omega_4-\Omega_3)+\omega]T_0} + P_{85} e^{i[(\Omega_4-\Omega_3)-\omega]T_0} \\
& + P_{86} + cc \quad (12)
\end{aligned}$$

where P_m ($m = 1, 2, \dots, 55, 62, 63, \dots, 86$) are complex functions in T_n . From the above analysis the solution of u is given by

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + O(\varepsilon^4) \quad (13)$$

From the above-derived solutions, the reported resonance cases are deduced as shown in Table 1.

Table 1.

No.	Type of Resonances	Case
1	trivial resonance	$\Omega_j \cong \omega = 0$
2	primary resonance	$\Omega_j \cong \omega$
3	sub-harmonic resonances	$\Omega_j \cong n\omega, \quad n = 2, 3, 4$
4	super-harmonic resonances	$\Omega_j \cong \omega/2$
5	combined resonances	$\pm\omega \cong \pm\Omega_1 \pm \Omega_2, \pm\omega \cong \pm\Omega_1 \pm \Omega_3,$ $\pm\omega \cong \pm\Omega_1 \pm \Omega_4, \pm\omega \cong \pm\Omega_2 \pm \Omega_3,$ $\pm\omega \cong \pm\Omega_2 \pm \Omega_4, \pm\omega \cong \pm\Omega_3 \pm \Omega_4$ $\pm\omega \cong \frac{1}{2}(\pm\Omega_1 \pm \Omega_2), \pm\omega \cong \frac{1}{2}(\pm\Omega_1 \pm \Omega_3)$ $\pm\omega \cong \frac{1}{2}(\pm\Omega_1 \pm \Omega_4), \pm\omega \cong \frac{1}{2}(\pm\Omega_2 \pm \Omega_3)$ $\pm\omega \cong \frac{1}{2}(\pm\Omega_2 \pm \Omega_4), \pm\omega \cong \frac{1}{2}(\pm\Omega_3 \pm \Omega_4)$
6	simultaneous resonance	any combination of the two resonance cases are classified as simultaneous resonance

1.2. Stability of the system

After studying numerically the different resonance cases, one of the worst cases has been chosen to study the system stability. The selected resonance case is the primary resonance one where $\Omega_j \cong \omega$. In this case we introduce a detuning parameter σ such that

$$\Omega_j = \omega + \varepsilon\sigma, \quad j = 1, 2, 3, 4 \quad (14)$$

Eliminating the secular terms of the first order approximation given by equation (8) leads to the solvability conditions for the first order approximation. Using equation (14) and noting that A_0 is a function in T_1 only, we get

$$-2i\omega D_1 A_0 - 2i\zeta\omega^2 A_0 - 3\beta A_0^2 \bar{A}_0 + \frac{1}{2} F_j e^{i\sigma T_1} = 0 \quad (15)$$

Substituting the polar form $A_0 = \frac{1}{2}a(T_1)e^{i\gamma(T_1)}$ into equation (15), we get

$$-2i\omega \left(\frac{1}{2}a' + \frac{1}{2}ia\gamma' \right) e^{i\gamma} - 2i\zeta\omega^2 \left(\frac{1}{2}ae^{i\gamma} \right) - 3\beta \left(\frac{1}{8}a^3 e^{i\gamma} \right) + \frac{1}{2} F_j e^{i\sigma T_1} = 0 \quad (16)$$

which yields:

$$a' + ia\gamma' + \zeta\omega a - \frac{3i\beta a^3}{8\omega} + \frac{iF_j}{2\omega}(\cos \mu + i \sin \mu) = 0 \quad (17)$$

where $\mu = \sigma T_1 - \gamma$, $\mu' = \sigma - \gamma'$, separating real and imaginary parts in equation (17), we get

$$a' + \zeta\omega a - \frac{F_j}{2\omega} \sin \mu = 0 \quad a(\sigma - \mu') - \frac{3\beta a^3}{8\omega} + \frac{F_j}{2\omega} \cos \mu = 0 \quad (18)$$

Then for steady state solution $a' = \mu' = 0$, and equation (18) becomes

$$\zeta\omega = \frac{F_j}{2a\omega} \sin \mu \quad \sigma - \frac{3\beta a^2}{8\omega} = -\frac{F_j}{2a\omega} \cos \mu \quad (19)$$

squaring equation (19), and adding the results, we get

$$\sigma^2 - \frac{3\beta a^2}{4\omega} \sigma + \left(\zeta^2 \omega^2 + \frac{9\beta^2 a^4}{64\omega^2} - \frac{F_j^2}{4a^2 \omega^2} \right) = 0 \Rightarrow \sigma = \frac{3\beta a^2}{8\omega} \pm \sqrt{F_j^2 - 4a^2 \zeta^2 \omega^4} \quad (20)$$

Stability of the fixed points

To analyze the stability of the fixed points, one lets

$$a = b_0 + b_1(T_1), \quad \mu = \mu_0 + \mu_1(T_1) \quad (21)$$

where b_0 and μ_0 satisfy equation (18) and b_1 and μ_1 are perturbation terms which are assumed to be small compared to b_0 and μ_0 , substituting equation (21) into equation (18), the following are obtained

$$\dot{b}_1 + \Gamma_1 b_1 + \Gamma_2 \mu_1 = 0 \quad (22)$$

$$\dot{\mu}_1 + \Gamma_1 \mu_1 + \Gamma_3 b_1 = 0 \quad (23)$$

where

$$\Gamma_1 = \zeta\omega, \quad \Gamma_2 = b_0 \left(\sigma - \frac{3\beta b_0^2}{8\omega} \right) \quad \text{and} \quad \Gamma_3 = \frac{1}{b_0} \left(\dot{\mu}_0 + \frac{9\beta b_0^2}{8\omega} - \sigma \right)$$

The eigenvalues of the above system of equations (22) and (23) can be obtained as follows

$$\begin{vmatrix} (\lambda + \Gamma_1) & \Gamma_2 \\ \Gamma_3 & (\lambda + \Gamma_1) \end{vmatrix} = 0$$

i.e.

$$\lambda^2 + 2\lambda\Gamma_1 + (\Gamma_1^2 - \Gamma_2\Gamma_3) = 0 \quad \text{or} \quad \lambda = -\Gamma_1 \pm \sqrt{\Gamma_2\Gamma_3}$$

Hence, the non-linear solution is stable if and only if the real part of $\lambda < 0$, i.e. $\Gamma_1^2 > \Gamma_2\Gamma_3$ otherwise it is unstable.

1.3. Results and discussion

Results are presented in graphical forms as steady state amplitude against detuning parameter (σ) and as time history. A good criterion of both stability and dynamic chaos presence is the phase-plane trajectories, which are shown for some considered cases. The solutions of the frequency response equations regarding the stability of the system is shown in Fig. 2. It can be seen from the figure that maximum steady state amplitude occurs at primary resonance when $\Omega_j \cong \omega$. Figs. 2 (a)–(d) show the effects of the damping coefficient ζ , the non-linear parameter β , the natural frequency ω and the excitation amplitudes F_j on the steady

state amplitude “a”. Fig. 2 (a) shows that the curve is bent to the right, leading to the occurrence of the jump phenomena and the effect of increasing or decreasing ζ is insignificant denoting the occurrence of the saturation phenomena. Fig. 2 (b) shows that the positive and negative values of β , produce either hard or soft spring respectively as the curve is either bent to the right or the left, leading to the appearance of the jump phenomenon. Fig. 2 (c) indicates that the steady state amplitude is monotonic decreasing function in ω also for the decreasing natural frequency ω , the curve is bent to the right leading to the occurrence of the jump phenomena. Fig. 2 (d) shows that the steady state amplitude is a monotonic increasing function in the excitation amplitude F_j .

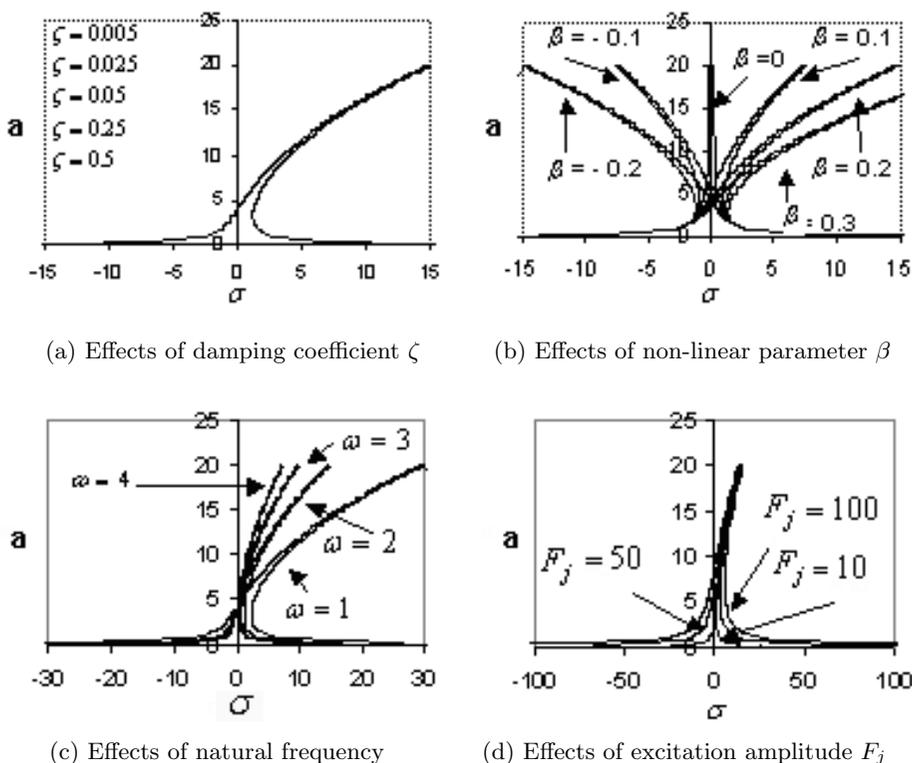


Fig. 2.

Fig. 3 illustrates the response and the phase plane for the non-resonant system at some practical values of the equation parameters. It can be seen from the figure that the maximum steady state amplitude is about 30% of the maximum excitation amplitude F_1 . The phase-plane shows approximately fine limit cycle denoting that the system is free of chaos.

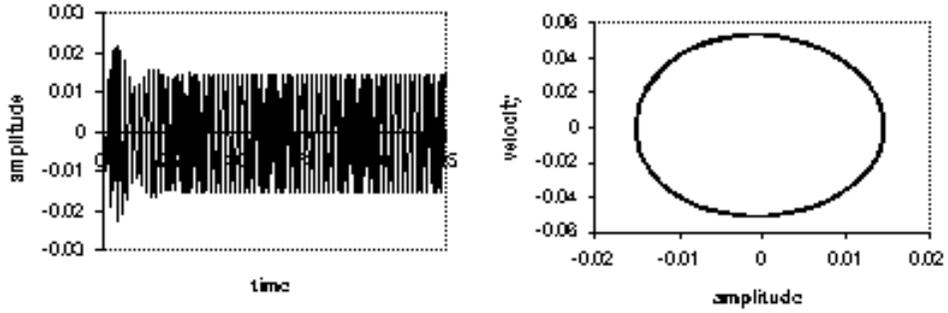
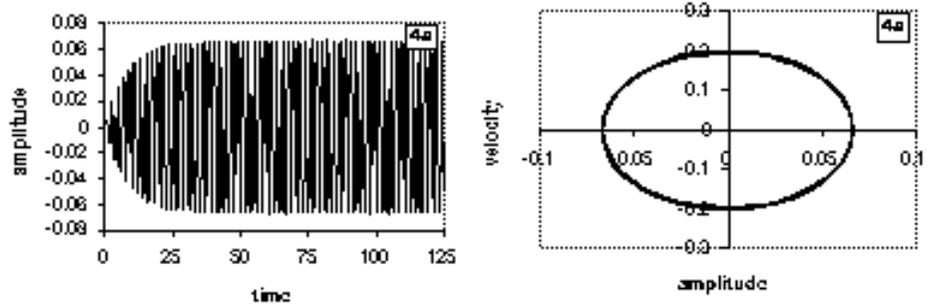


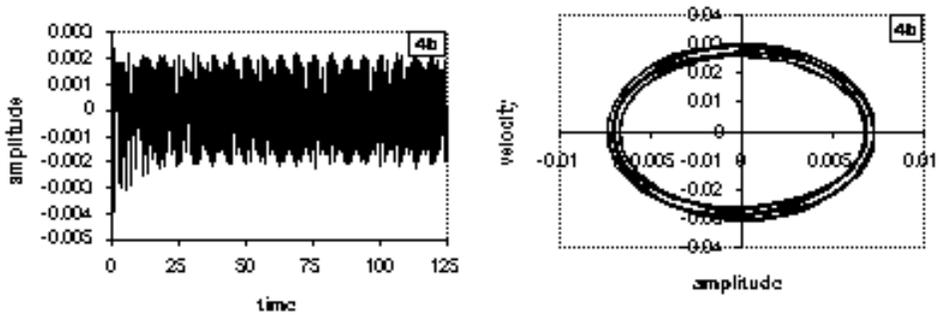
Fig. 3. Non-resonance system behavior (basic case). $\zeta = 0.08$, $\omega = 3$, $\alpha = 0.05$, $\beta = 0.2$, $F_1 = 0.05$, $F_2 = 0.0125$, $F_3 = 0.0025$, $F_4 = 0.0005$, $\Omega_1 = 3.5$, $\Omega_2 = 7$, $\Omega_3 = 10.5$, $\Omega_4 = 14$.

1.4. Resonance cases

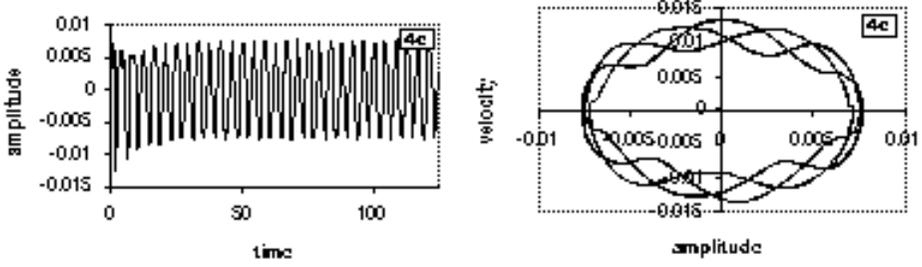
Some of the deduced resonance cases are confirmed numerically (Fig. 4). Table 2 summarizes the different considered cases. It can be seen that the primary resonance case or cases represent the system worst case.



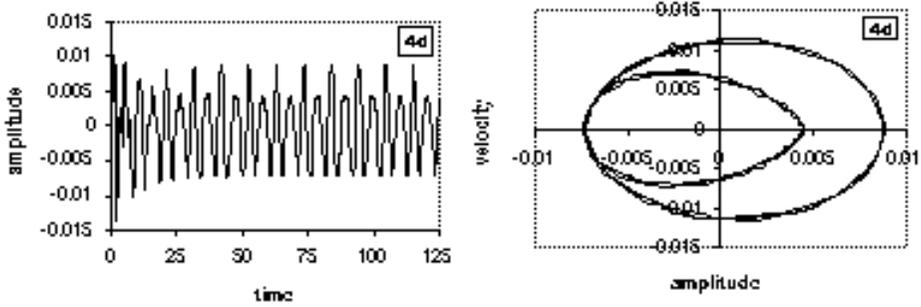
(a) Primary resonance $\Omega_1 \cong \omega$



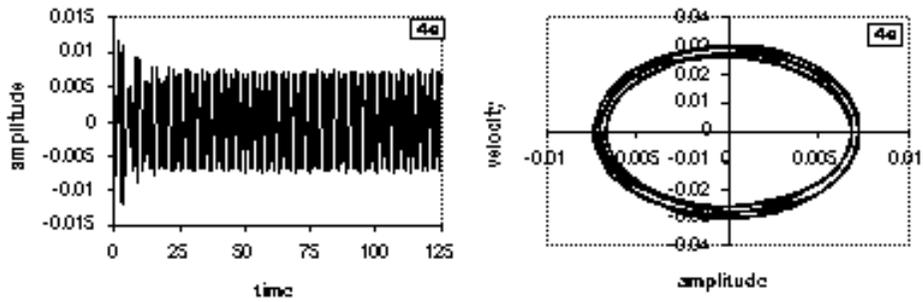
(b) Sub-harmonic resonance $\Omega_1 \cong 2\omega$



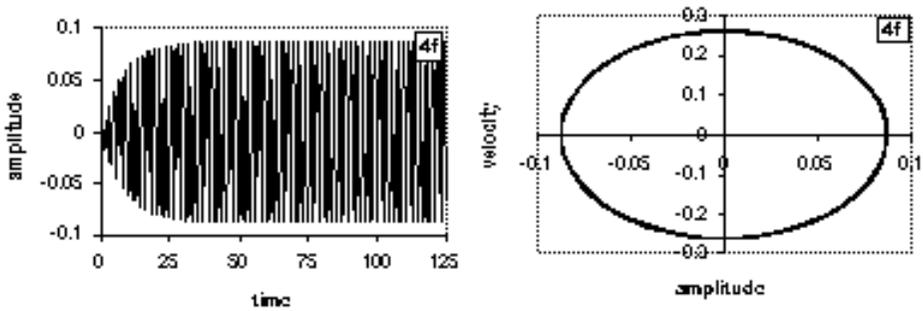
(c) Super-harmonic resonance $\Omega_1 \cong \omega/2$



(d) Combined resonance $\omega \cong \Omega_1 + \Omega_2$



(e) Combined resonance $\omega \cong \Omega_2 - \Omega_1$



(f) Simultaneous resonance $\Omega_1 \cong \Omega_2 \cong \Omega_3 \cong \Omega_4 \cong \omega$

Fig. 4.

Table 2. Summary of some investigated resonance cases.

No	Resonance type	Case	Amplitude ratio	Chaos
1		Non-resonant	100%	Limit cycle
2	Primary resonance	$\Omega_1 \cong \omega$	460%	Limit cycle
3		$\Omega_2 \cong \omega$	215%	Multi-Limit cycle
4	Sub-harmonic resonance	$\Omega_1 \cong 2\omega$	15%	Multi-Limit cycle
5		$\Omega_2 \cong 2\omega$	105%	Double-Limit
6		$\Omega_1 \cong 3\omega$	10%	Multi-Limit cycle
7		$\Omega_2 \cong 3\omega$	105%	Limit cycle
8		$\Omega_1 \cong 4\omega$	5%	Multi-Limit cycle
9		$\Omega_2 \cong 4\omega$	100%	Limit cycle
10	Super-harmonic resonance	$\Omega_1 \cong \omega/2$	50%	Multi-Limit cycle
11		$\Omega_2 \cong \omega/2$	115%	Multi-Limit cycle
12	Combined resonance	$\omega \cong \Omega_1 + \Omega_2$	60%	Double-Limit cycle
13		$\omega \cong \Omega_2 - \Omega_1$	105%	Limit cycle
14		$2\omega \cong \Omega_1 + \Omega_2$	150%	Multi-Limit cycle
15		$2\omega \cong \Omega_2 - \Omega_1$	105%	Limit cycle
16	Simultaneous resonance	$\omega \cong \Omega_1 \cong \Omega_2$	570%	Limit cycle

2. Passive Control

Using a linear tuned mass absorber (TMA) connected to the system, equations of motions can be written in the following form:

$$\begin{aligned} \ddot{u}_2 + 2\varepsilon\zeta\omega\dot{u}_2 + \omega^2u_2 + \varepsilon\alpha u_2^2 + \varepsilon\beta u_2^3 + \varepsilon\zeta_1(\dot{u}_2 - \dot{u}_1) + \varepsilon\gamma(u_2 - u_1) \\ = \varepsilon \sum_{j=1}^N F_j \cos \Omega_j t \end{aligned} \quad (24)$$

$$\ddot{u}_1 + 2\varepsilon\zeta_2\omega_1(\dot{u}_1 - \dot{u}_2) + \omega_1^2(u_1 - u_2) = 0 \quad (25)$$

where u_1 denotes the response of the second-order controller, u_2 represents one of the modal co-ordinates of a structure, ω and ω_1 are the natural frequencies, ζ and ζ_2 are the damping coefficients, α and β are non-linear coefficients of the wing, ε is a small perturbation parameter, F_j the forcing amplitudes and Ω_j are the excitation frequencies, $j = 1, 2$ for simplicity. Following the same procedure as in part 1, we get:

$$u_{20}(T_0, T_n) = A_0(T_n)e^{i\omega T_0} + cc \quad (26)$$

$$u_{10}(T_0, T_n) = B_0(T_n)e^{i\omega_1 T_0} + C_0(T_n)e^{j\omega T_0} + cc \quad (27)$$

$$u_{21}(T_0, T_n) = K_1 e^{i\omega T_0} + K_2 e^{2i\omega T_0} + K_3 e^{3i\omega T_0} + K_4 e^{i\omega_1 T_0} + K_5 e^{i\Omega_1 T_0} \\ + K_6 e^{i\Omega_2 T_0} + K_7 + cc \quad (28)$$

$$u_{11}(T_0, T_n) = K_8 e^{i\omega_1 T_0} + K_9 e^{i\omega T_0} + K_{10} e^{2i\omega T_0} + K_{11} e^{3i\omega T_0} + K_{12} e^{i\Omega_1 T_0} \\ + K_{13} e^{i\Omega_2 T_0} + K_{14} + cc \quad (29)$$

$$u_{22}(T_0, T_n) = R_1 e^{i\omega T_0} + R_2 e^{2i\omega T_0} + R_3 e^{3i\omega T_0} + R_4 e^{4i\omega T_0} + R_5 e^{5i\omega T_0} + R_6 e^{i\omega_1 T_0} \\ + R_7 e^{i(\omega_1+\omega)T_0} + R_8 e^{i(\omega_1-\omega)T_0} + R_9 e^{i(\omega_1+2\omega)T_0} + R_{10} e^{i(\omega_1-2\omega)T_0} \\ + R_{11} e^{i\Omega_1 T_0} + R_{12} e^{i\Omega_2 T_0} + R_{13} e^{i(\omega+\Omega_1)T_0} + R_{14} e^{i(\omega+\Omega_2)T_0} \\ + R_{15} e^{i(\omega-\Omega_1)T_0} + R_{16} e^{i(\omega-\Omega_2)T_0} + R_{17} e^{i(2\omega+\Omega_1)T_0} + R_{18} e^{i(2\omega+\Omega_2)T_0} \\ + R_{19} e^{i(2\omega-\Omega_1)T_0} + R_{20} e^{i(2\omega-\Omega_2)T_0} + R_{21} + cc \quad (30)$$

$$u_{12}(T_0, T_n) = P_1 e^{i\omega T_0} + P_2 e^{2i\omega T_0} + P_3 e^{3i\omega T_0} + P_4 e^{4i\omega T_0} + P_5 e^{5i\omega T_0} + P_6 e^{i\omega_1 T_0} \\ + P_7 e^{i(\omega_1+\omega)T_0} + P_8 e^{i(\omega_1-\omega)T_0} + P_9 e^{i(\omega_1+2\omega)T_0} + P_{10} e^{i(\omega_1-2\omega)T_0} \\ + P_{11} e^{i\Omega_1 T_0} + P_{12} e^{i\Omega_2 T_0} + P_{13} e^{i(\omega+\Omega_1)T_0} + P_{14} e^{i(\omega+\Omega_2)T_0} \\ + P_{15} e^{i(\omega-\Omega_1)T_0} + P_{16} e^{i(\omega-\Omega_2)T_0} + P_{17} e^{i(2\omega+\Omega_1)T_0} + P_{18} e^{i(2\omega+\Omega_2)T_0} \\ + P_{19} e^{i(2\omega-\Omega_1)T_0} + P_{20} e^{i(2\omega-\Omega_2)T_0} + P_{21} + cc \quad (31)$$

$$u_{23}(T_0, T_n) = q_1 e^{i\omega T_0} + q_2 e^{2i\omega T_0} + q_3 e^{3i\omega T_0} + q_4 e^{4i\omega T_0} + q_5 e^{5i\omega T_0} + q_6 e^{6i\omega T_0} \\ + q_7 e^{7i\omega T_0} + q_8 e^{i\omega_1 T_0} + q_9 e^{2i\omega_1 T_0} + q_{10} e^{i(\omega_1+\omega)T_0} + q_{11} e^{i(\omega_1-\omega)T_0} \\ + q_{12} e^{i(\omega_1+2\omega)T_0} + q_{13} e^{i(\omega_1-2\omega)T_0} + q_{14} e^{i\Omega_1 T_0} + q_{15} e^{i\Omega_2 T_0} \\ + q_{16} e^{2i\Omega_1 T_0} + q_{17} e^{2i\Omega_2 T_0} + q_{18} e^{i(\omega+\Omega_1)T_0} + q_{19} e^{i(\omega+\Omega_2)T_0} \\ + q_{20} e^{i(\omega-\Omega_1)T_0} + q_{21} e^{i(\omega-\Omega_2)T_0} + q_{22} e^{i(2\omega+\Omega_1)T_0} + q_{23} e^{i(2\omega+\Omega_2)T_0} \\ + q_{24} e^{i(2\omega-\Omega_1)T_0} + q_{25} e^{i(2\omega-\Omega_2)T_0} + q_{26} e^{i(\Omega_1+\Omega_2)T_0} + q_{27} e^{i(\Omega_2-\Omega_1)T_0} \\ + q_{28} e^{i(\omega+(\Omega_1+\Omega_2))T_0} + q_{29} e^{i(\omega+(\Omega_2-\Omega_1))T_0} + q_{30} e^{i(\omega-(\Omega_1+\Omega_2))T_0} \\ + q_{31} e^{i(\omega-(\Omega_2-\Omega_1))T_0} + q_{32} + cc \quad (32)$$

$$u_{13}(T_0, T_n) = g_1 e^{i\omega T_0} + g_2 e^{2i\omega T_0} + g_3 e^{3i\omega T_0} + g_4 e^{4i\omega T_0} + g_5 e^{5i\omega T_0} + g_6 e^{6i\omega T_0} \\ + g_7 e^{7i\omega T_0} + g_8 e^{i\omega_1 T_0} + g_9 e^{2i\omega_1 T_0} + g_{10} e^{i(\omega_1+\omega)T_0} + g_{11} e^{i(\omega_1-\omega)T_0} \\ + g_{12} e^{i(\omega_1+2\omega)T_0} + g_{13} e^{i(\omega_1-2\omega)T_0} + g_{14} e^{i\Omega_1 T_0} + g_{15} e^{i\Omega_2 T_0} \\ + g_{16} e^{2i\Omega_1 T_0} + g_{17} e^{2i\Omega_2 T_0} + g_{18} e^{i(\omega+\Omega_1)T_0} + g_{19} e^{i(\omega+\Omega_2)T_0} \\ + g_{20} e^{i(\omega-\Omega_1)T_0} + g_{21} e^{i(\omega-\Omega_2)T_0} + g_{22} e^{i(2\omega+\Omega_1)T_0} + g_{23} e^{i(2\omega+\Omega_2)T_0} \\ + g_{24} e^{i(2\omega-\Omega_1)T_0} + g_{25} e^{i(2\omega-\Omega_2)T_0} + g_{26} e^{i(\Omega_1+\Omega_2)T_0} + g_{27} e^{i(\Omega_2-\Omega_1)T_0} \\ + g_{28} e^{i(\omega+(\Omega_1+\Omega_2))T_0} + g_{29} e^{i(\omega+(\Omega_2-\Omega_1))T_0} + g_{30} e^{i(\omega-(\Omega_1+\Omega_2))T_0} \\ + g_{31} e^{i(\omega-(\Omega_2-\Omega_1))T_0} + g_{32} + cc \quad (33)$$

where $A_0, B_0, C_0, K_i, R_m, P_m, q_n, g_n$ ($i = 1, 2, \dots, 14$), ($m = 1, 2, \dots, 21$) and ($n = 1, 2, \dots, 32$) are complex functions in T_n . From the above analysis the general solutions of u_2 and u_1 is given by

$$u_2 = u_{20} + \varepsilon u_{21} + \varepsilon^2 u_{22} + \varepsilon^3 u_{23} + O(\varepsilon^4) \text{ and } u_1 = u_{10} + \varepsilon u_{11} + \varepsilon^2 u_{12} + \varepsilon^3 u_{13} + O(\varepsilon^3)$$

Many resonance cases are shown in Table 3.

Table 3.

No.	Type of Resonances	Case
1	trivial resonance	$\Omega_1 \cong \Omega_2 \cong \omega \cong \omega_1 = 0$
2	primary resonance	$\Omega_1 \cong \omega, \Omega_2 \cong \omega, \Omega_1 \cong \omega_1, \Omega_2 \cong \omega_1$
3	sub-harmonic resonances	$\Omega_1 \cong n\omega, \Omega_2 \cong n\omega, \omega_1 = m\omega$ $n = 2, 3$ and $m = 2, 3, \dots, 7$
4	super-harmonic resonances	$\Omega_1 \cong \omega/2, \Omega_2 \cong \omega/2, \Omega_1 \cong \omega/3, \Omega_2 \cong \omega/3,$ $\omega_1 = \omega/2, \Omega_1 \cong \omega_1/2, \Omega_2 \cong \omega_1/2$
5	combined resonances	$\pm\omega \cong \pm\Omega_1 \pm \Omega_2, \pm\omega_1 \cong \pm\Omega_1 \pm \Omega_2,$ $\pm\omega \cong \frac{1}{2}(\pm\Omega_1 \pm \Omega_2), \pm\Omega_1 \cong \pm\omega \pm \omega_1,$ $\pm\Omega_2 \cong \pm\omega \pm \omega_1, \pm\Omega_1 \cong \pm 2\omega \pm \omega_1,$ $\pm\Omega_2 \cong \pm 2\omega \pm \omega_1, \pm\Omega_1 \pm \Omega_2 = \pm\omega \pm \omega_1$
6	simultaneous resonance	any combination of the two resonance cases are classified as simultaneous resonance

2.1. Stability of the system

We study the stability of the system at the simultaneous primary resonance $\Omega_1 \cong \omega, \Omega_2 \cong \omega_1$. Using the detuning σ_1 and σ_2 such that

$$\Omega_1 \cong \omega + \varepsilon\sigma_1 \quad \text{and} \quad \Omega_2 \cong \omega_1 + \varepsilon\sigma_2 \tag{34}$$

Eliminating the secular terms of equations (28) and (29), leads to the solvability conditions for the first order approximation and noting that A_0 and B_0 are a function in T_1 only, we get

$$\left[-2i\omega(D_1A_0 + \zeta\omega A_0) - 3\beta A_0^2 \bar{A}_0 - i\omega\zeta_1 A_0 - \gamma A_0 + \frac{\omega_1^2 A_0}{\omega_1^2 - \omega^2} (i\omega\zeta_1 + \gamma) \right] e^{i\omega T_0} + \frac{1}{2} F_1 e^{i\Omega_1 T_0} = 0 \tag{35}$$

$$- \left[2i\omega_1(D_1B_0 + \zeta_2\omega_1 B_0) + \frac{\omega_1^2 B_0}{\omega_1^2 - \omega^2} (i\omega_1\zeta_1 + \gamma) \right] e^{i\omega_1 T_0} + \frac{\omega_1^2 F_2}{2(\omega^2 - \Omega_2^2)} e^{i\Omega_2 T_0} = 0 \tag{36}$$

putting

$$A_0 = \frac{1}{2} a_1(T_1) e^{im(T_1)} \quad \text{and} \quad B_0 = \frac{1}{2} a_2(T_1) e^{in(T_1)}$$

similar to the former system, there are three possibilities in addition to the trivial solution. They are

- (1) $a_1 \neq 0, a_2 = 0$ (2) $a_2 \neq 0, a_1 = 0$ (3) $a_1 \neq 0, a_2 \neq 0$

The corresponding frequency response function (FRF) for each case are:

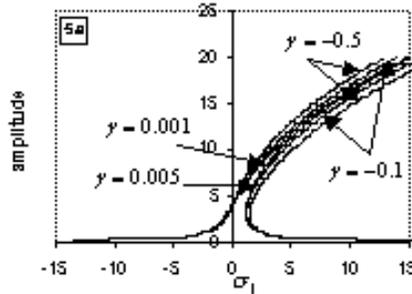
$$\begin{aligned} \sigma_1^2 - \left(\frac{3\beta a_1^2}{4\omega} + \frac{\gamma}{\omega} - \frac{\gamma\omega_1^3}{\omega^2(\omega_1^2 - \omega^2)} \right) \sigma_1 + \left[\frac{9\beta^2 a_1^4}{64\omega^2} + \frac{\gamma^2}{4\omega^2} + \frac{\gamma^2\omega_1^4}{4\omega^2(\omega_1^2 - \omega^2)^2} \right. \\ \left. + \frac{3\beta\gamma a_1^2}{4\omega^2} - \frac{3\beta\gamma\omega_1^2 a_1^2}{8\omega^2(\omega_1^2 - \omega^2)} - \frac{\gamma^2\omega_1^2}{2\omega^2(\omega_1^2 - \omega^2)} + \zeta^2\omega^2 + \frac{\zeta_1^2}{4} + \frac{\omega_1^4\zeta_1^2}{4(\omega_1^2 - \omega^2)^2} \right. \\ \left. + \zeta_1\omega - \frac{\omega\omega_1^2\zeta_1}{\omega_1^2 - \omega^2} - \frac{\omega_1^2\zeta_1^2}{2(\omega_1^2 - \omega^2)} - \frac{F_1^2}{4\omega^2 a_1^2} \right] = 0 \end{aligned} \tag{37}$$

$$\begin{aligned} \sigma_2^2 - \frac{\gamma\omega_1}{\omega_1^2 - \omega^2} \sigma_2 + \left(\frac{\omega_1^2\gamma^2}{4(\omega_1^2 - \omega^2)^2} + \zeta_2^2\omega_1^2 + \frac{\omega_1^4\zeta_1^2}{4(\omega_1^2 - \omega^2)^2} \right. \\ \left. + \frac{\omega_1^3\zeta_1\zeta_2}{\omega_1^2 - \omega^2} - \frac{F_2^2\omega_1^2}{4a_2^2(\omega^2 - \Omega_2^2)^2} \right) = 0 \end{aligned} \tag{38}$$

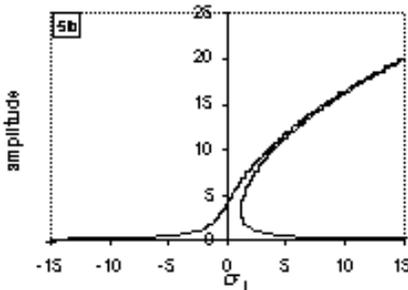
and case (3) is represented by equations (37) and (38).

2.2. Results and discussion

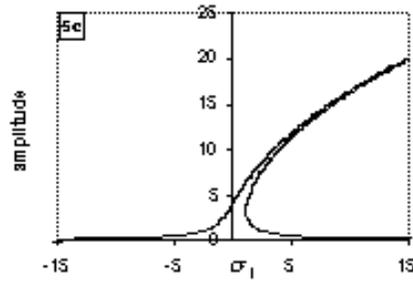
Fig. 5 shows the effects of the detuning parameter σ_1 on the steady state amplitude of the main system a_1 for the stability first case where $a_1 \neq 0$ and $a_2 = 0$. Fig. 5(a) shows that the curves converges or diverges as γ is decreasing or increasing, leading to the appearance of the jump phenomenon. Figs. 5(b) and



(a) Effects of non-linear parameter γ



(b) Effects of natural frequency ω_1

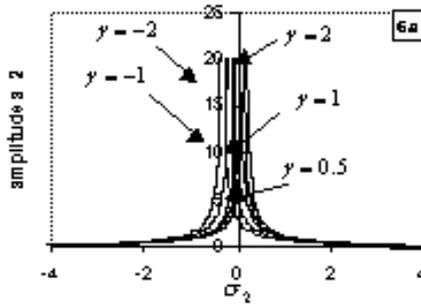


(c) Effects of damping coefficient ζ_1

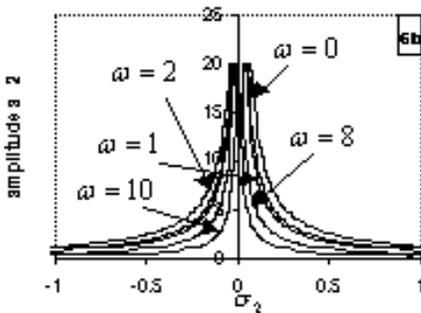
Fig. 5.

(c) show that the curves are bent to the right leading to the appearance of the jump phenomenon and the effect of increasing or decreasing ω_1 and ζ_1 trivial.

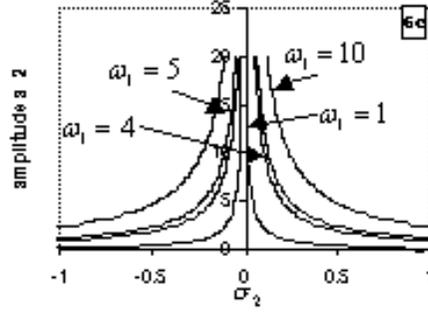
Fig. 6 shows the effects of the detuning parameter σ_2 on the steady state amplitude of the main system a_2 for the stability second case $a_2 \neq 0$ and $a_1 = 0$, not leading to the appearance of the jump phenomenon. Fig. 6 (a) shows that the positive and negative values of γ , the curve is neither bent to the right nor the left. Figs. 6 ((b) and (g)) indicate that the steady state amplitude is monotonic decreasing function in natural frequency ω and in the excitation frequency Ω_2 . Figs. 6 ((c)



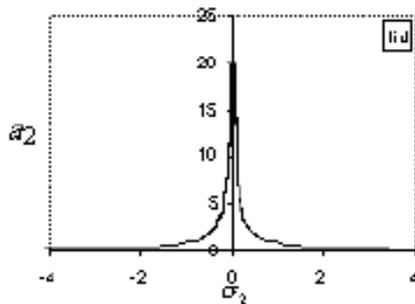
(a) Effects of non-linear parameter γ



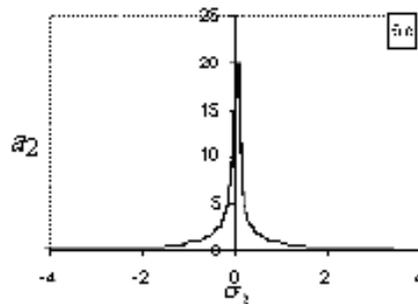
(b) Effects of natural frequency ω



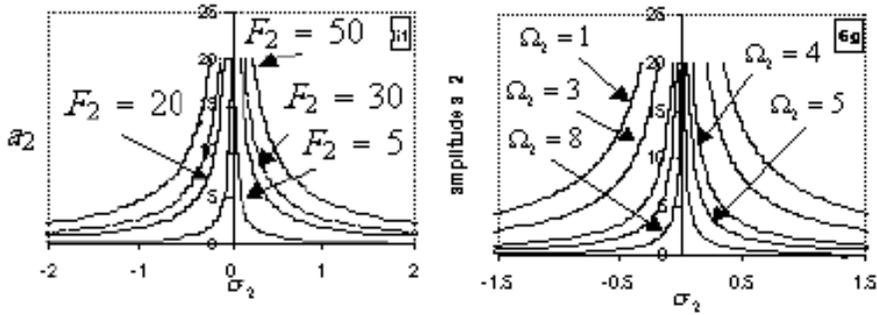
(c) Effects of natural frequency ω_1



(d) Effects of damping coefficient ζ_1



(e) Effects of damping coefficient ζ_2



(f) Effects of excitation amplitude F_2 (g) Effects of excitation frequency Ω_2

Fig. 6.

and (f) show that the steady state amplitude a_2 is a monotonic decreasing function in the natural frequency ω_1 and increasing function in the excitation amplitude F_2 . Figs. 6 ((d) and (e)) show increasing or decreasing ζ_1 and ζ_2 do not effect the steady state amplitude a_2 .

Fig. 7 shows that the steady state amplitude without absorber at primary resonance where, $\Omega_1 = \omega$ is about 100% of the maximum excitation amplitude F_1 , the system is stable and free of dynamic chaos.

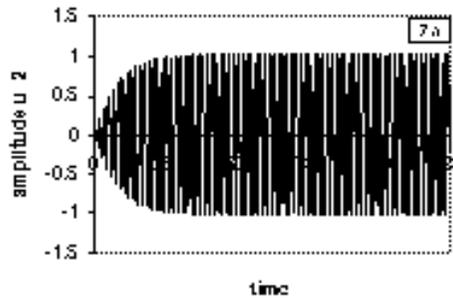


Fig. 7. System behavior at primary resonance.

Effects of the TMA

Fig. 8 illustrates the response for the system with absorber at the simultaneous primary resonance $\Omega_1 \cong \omega \cong \omega_1$. The effectiveness of the absorber E_a (steady state of the wing without absorber /steady state with absorber) is increased about 125. The phase plane is approximately limit cycle.

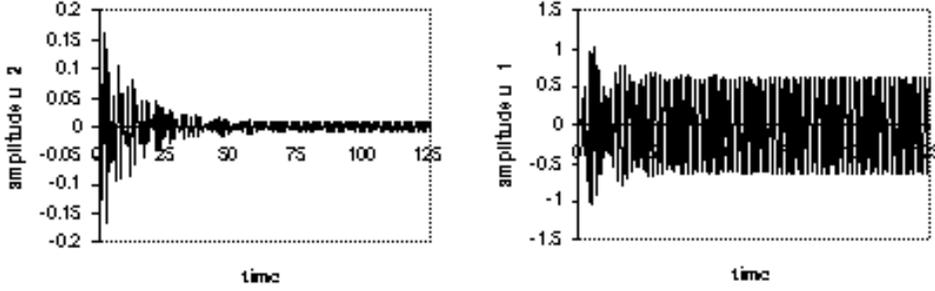


Fig. 8. System behavior with absorber at the simultaneous primary resonance $\Omega_1 \cong \omega \cong \omega_1$. $\omega = \omega_1 = \Omega_1 = 4$, $\alpha = 0.1$, $\beta = 0.1$, $\gamma = 1.6$, $\zeta = 0.03$, $\zeta_1 = 0.005$, $\zeta_2 = 0.00625$, $F_1 = 1$, $F_2 = 0.0025$, $\Omega_2 = 7.5$.

3. Active Control

Here, active control is applied to improve the behavior of the system. A second order non-linear controller is added to the main system which represents a 1:2 internal resonance active control absorber of a vibrating aircraft wing in the following form [20], [21] and [22].

$$\ddot{u}_1 + 2\varepsilon\zeta_1\omega_1\dot{u}_1 + \omega_1^2u_1 = \varepsilon g_{12}u_1u_2 \quad (39)$$

$$\ddot{u}_2 + 2\varepsilon\zeta_2\omega_2\dot{u}_2 + \omega_2^2u_2 + \varepsilon\alpha u_2^2 + \varepsilon\beta u_2^3 = \varepsilon g_{11}u_1^2 + \varepsilon \sum_{j=1}^2 F_j \cos(\Omega_j T_0) \quad (40)$$

where u_1 denotes the response of a second-order controller, ω_1 is the natural frequency of the controller, ζ_1 is the damping ratio of the controller, u_2 represents one of the modal co-ordinates of the wing, ω_2 is this modal frequency, ζ_2 is the damping ratio, g_{11} and g_{12} are positive gain constants, F is the amplitude of the external excitation force, Ω_j are the external excitation frequencies, ω_2 is close to $2\omega_1$ (i.e. 1:2), Ω_j is close to ω_2 , t is the time, α and β are coefficients of non-linear terms, ε is a small perturbation parameter. The analytic solutions of equations (39) and (40) are obtained assuming that the coefficients of $\zeta_1\omega_1$, $\zeta_2\omega_2$, g_{12} , g_{11} , F , α and β are small. Following the same procedure, we get:

$$u_{10}(T_0, T_n) = A_0(T_n)e^{i\omega_1 T_0} + cc \quad (41)$$

$$u_{20}(T_0, T_n) = B_0(T_n)e^{i\omega_2 T_0} + cc \quad (42)$$

$$u_{11}(T_0, T_n) = C_1e^{i\omega_1 T_0} + C_2e^{i(\omega_1+\omega_2)T_0} + C_3e^{i(\omega_1-\omega_2)T_0} + cc \quad (43)$$

$$u_{21}(T_0, T_n) = C_4e^{i\omega_2 T_0} + C_5e^{2i\omega_2 T_0} + C_6e^{3i\omega_2 T_0} + C_7e^{i\Omega_1 T_0} + C_8e^{i\Omega_2 T_0} + C_9 + cc \quad (44)$$

$$u_{12}(T_0, T_n) = F_1e^{i\omega_1 T_0} + F_2e^{i(\omega_1+\omega_2)T_0} + F_3e^{i(\omega_1-\omega_2)T_0} + F_4e^{i(\omega_1+2\omega_2)T_0}$$

$$\begin{aligned}
& + F_5 e^{i(\omega_1 - 2\omega_2)T_0} + F_6 e^{i(\omega_1 + 3\omega_2)T_0} + F_7 e^{i(\omega_1 - 3\omega_2)T_0} + F_8 e^{i(\omega_1 + \Omega_1)T_0} \\
& + F_9 e^{i(\omega_1 - \Omega_1)T_0} + F_{10} e^{i(\omega_1 + \Omega_2)T_0} + F_{11} e^{i(\omega_1 - \Omega_2)T_0} + cc \tag{45}
\end{aligned}$$

$$\begin{aligned}
u_{22}(T_0, T_n) = & g_1 e^{i\omega_2 T_0} + g_2 e^{2i\omega_2 T_0} + g_3 e^{3i\omega_2 T_0} + g_4 e^{4i\omega_2 T_0} + g_5 e^{5i\omega_2 T_0} + g_6 e^{2i\omega_1 T_0} \\
& + g_7 e^{i(\omega_2 + 2\omega_1)T_0} + g_8 e^{i(\omega_2 - 2\omega_1)T_0} + g_9 e^{i\Omega_1 T_0} + g_{10} e^{i\Omega_2 T_0} \\
& + g_{11} e^{i(\omega_2 + \Omega_1)T_0} + g_{12} e^{i(\omega_2 - \Omega_1)T_0} + g_{13} e^{i(\omega_2 + \Omega_2)T_0} + g_{14} e^{i(\omega_2 - \Omega_2)T_0} \\
& + g_{15} e^{i(2\omega_2 + \Omega_1)T_0} + g_{16} e^{i(2\omega_2 - \Omega_1)T_0} + g_{17} e^{i(2\omega_2 + \Omega_2)T_0} \\
& + g_{18} e^{i(2\omega_2 - \Omega_2)T_0} + g_{19} + cc \tag{46}
\end{aligned}$$

$$\begin{aligned}
u_{13}(T_0, T_n) = & h_1 e^{i\omega_1 T_0} + h_2 e^{3i\omega_1 T_0} + h_3 e^{i(\omega_1 + \omega_2)T_0} + h_4 e^{i(\omega_1 - \omega_2)T_0} + h_5 e^{i(\omega_1 + 2\omega_2)T_0} \\
& + h_6 e^{i(\omega_1 - 2\omega_2)T_0} + h_7 e^{i(\omega_1 + 3\omega_2)T_0} + h_8 e^{i(\omega_1 - 3\omega_2)T_0} + h_9 e^{i(\omega_1 + 4\omega_2)T_0} \\
& + h_{10} e^{i(\omega_1 - 4\omega_2)T_0} + h_{11} e^{i(\omega_1 + 5\omega_2)T_0} + h_{12} e^{i(\omega_1 - 5\omega_2)T_0} \\
& + h_{13} e^{i(\omega_1 + \Omega_1)T_0} + h_{14} e^{i(\omega_1 - \Omega_1)T_0} + h_{15} e^{i(\omega_1 + \Omega_2)T_0} + h_{16} e^{i(\omega_1 - \Omega_2)T_0} \\
& + h_{17} e^{i(\Omega_1 + (\omega_1 + \omega_2))T_0} + h_{18} e^{i(\Omega_1 - (\omega_1 + \omega_2))T_0} + h_{19} e^{i(\Omega_2 + (\omega_1 + \omega_2))T_0} \\
& + h_{20} e^{i(\Omega_2 - (\omega_1 + \omega_2))T_0} + h_{21} e^{i(\Omega_1 + (\omega_1 - \omega_2))T_0} + h_{22} e^{i(\Omega_1 - (\omega_1 - \omega_2))T_0} \\
& + h_{23} e^{i(\Omega_2 + (\omega_1 - \omega_2))T_0} + h_{24} e^{i(\Omega_2 - (\omega_1 - \omega_2))T_0} + h_{25} e^{i(\Omega_1 + (\omega_1 + 2\omega_2))T_0} \\
& + h_{26} e^{i(\Omega_1 - (\omega_1 + 2\omega_2))T_0} + h_{27} e^{i(\Omega_2 + (\omega_1 + 2\omega_2))T_0} + h_{28} e^{i(\Omega_2 - (\omega_1 + 2\omega_2))T_0} \\
& + h_{29} e^{i(\Omega_1 + (\omega_1 - 2\omega_2))T_0} + h_{30} e^{i(\Omega_1 - (\omega_1 - 2\omega_2))T_0} + h_{31} e^{i(\Omega_2 + (\omega_1 - 2\omega_2))T_0} \\
& + h_{32} e^{i(\Omega_2 - (\omega_1 - 2\omega_2))T_0} + h_{33} e^{i(3\omega_1 + \omega_2)T_0} + h_{34} e^{i(3\omega_1 - \omega_2)T_0} + cc \tag{47}
\end{aligned}$$

$$\begin{aligned}
u_{23}(T_0, T_n) = & k_1 e^{i\omega_2 T_0} + k_2 e^{2i\omega_2 T_0} + k_3 e^{3i\omega_2 T_0} + k_4 e^{4i\omega_2 T_0} + k_5 e^{5i\omega_2 T_0} + k_6 e^{6i\omega_2 T_0} \\
& + k_7 e^{7i\omega_2 T_0} + k_8 e^{i\omega_1 T_0} + k_9 e^{2i\omega_1 T_0} + k_{10} e^{3i\omega_1 T_0} + k_{11} e^{4i\omega_1 T_0} \\
& + k_{12} e^{i(2\omega_1 + \omega_2)T_0} + k_{13} e^{i(2\omega_1 - \omega_2)T_0} + k_{14} e^{i(2\omega_1 + 2\omega_2)T_0} \\
& + k_{15} e^{i(2\omega_1 - 2\omega_2)T_0} + k_{16} e^{i(2\omega_1 + 3\omega_2)T_0} + k_{17} e^{i(2\omega_1 - 3\omega_2)T_0} \\
& + k_{18} e^{i(2\omega_1 + 4\omega_2)T_0} + k_{19} e^{i(2\omega_1 - 4\omega_2)T_0} + k_{20} e^{i\Omega_1 T_0} + k_{21} e^{i\Omega_2 T_0} \\
& + k_{22} e^{2i\Omega_1 T_0} + k_{23} e^{2i\Omega_2 T_0} + k_{24} e^{i(\Omega_1 + \Omega_2)T_0} + k_{25} e^{i(\Omega_2 - \Omega_1)T_0} \\
& + k_{26} e^{i(\Omega_1 + 2\omega_1)T_0} + k_{27} e^{i(\Omega_1 - 2\omega_1)T_0} + k_{28} e^{i(\Omega_2 + 2\omega_1)T_0} \\
& + k_{29} e^{i(\Omega_2 - 2\omega_1)T_0} + k_{30} e^{i(\Omega_1 + \omega_2)T_0} + k_{31} e^{i(\Omega_1 - \omega_2)T_0} \\
& + k_{32} e^{i(\Omega_2 + \omega_2)T_0} + k_{33} e^{i(\Omega_2 - \omega_2)T_0} + k_{34} e^{i(\Omega_1 + 2\omega_2)T_0} \\
& + k_{35} e^{i(\Omega_1 - 2\omega_2)T_0} + k_{36} e^{i(\Omega_2 + 2\omega_2)T_0} + k_{37} e^{i(\Omega_2 - 2\omega_2)T_0} \\
& + k_{38} e^{i(\Omega_1 + 3\omega_2)T_0} + k_{39} e^{i(\Omega_1 - 3\omega_2)T_0} + k_{40} e^{i(\Omega_2 + 3\omega_2)T_0} \\
& + k_{41} e^{i(\Omega_2 - 3\omega_2)T_0} + k_{42} e^{i(\Omega_1 + 4\omega_2)T_0} + k_{43} e^{i(\Omega_1 - 4\omega_2)T_0} \\
& + k_{44} e^{i(\Omega_2 + 4\omega_2)T_0} + k_{45} e^{i(\Omega_2 - 4\omega_2)T_0} + k_{46} e^{i(2\Omega_1 + \omega_2)T_0} \\
& + k_{47} e^{i(2\Omega_1 - \omega_2)T_0} + k_{48} e^{i(2\Omega_2 + \omega_2)T_0} + k_{49} e^{i(2\Omega_2 - \omega_2)T_0} \\
& + k_{50} e^{i(\omega_2 + (\Omega_1 + \Omega_2))T_0} + k_{51} e^{i(\omega_2 - (\Omega_1 + \Omega_2))T_0} + k_{52} e^{i(\omega_2 + (\Omega_1 - \Omega_2))T_0} \\
& + k_{53} e^{i(\omega_2 - (\Omega_1 - \Omega_2))T_0} + k_{54} e^{i(\Omega_1 + (2\omega_1 + \omega_2))T_0} + k_{55} e^{i(\Omega_1 - (2\omega_1 + \omega_2))T_0} \\
& + k_{56} e^{i(\Omega_2 + (2\omega_1 + \omega_2))T_0} + k_{57} e^{i(\Omega_2 - (2\omega_1 + \omega_2))T_0} + k_{58} e^{i(\Omega_1 + (2\omega_1 - \omega_2))T_0}
\end{aligned}$$

$$\begin{aligned}
 &+ k_{59}e^{i(\Omega_1-(2\omega_1-\omega_2))T_0} + k_{60}e^{i(\Omega_2+(2\omega_1-\omega_2))T_0} + k_{61}e^{i(\Omega_2-(2\omega_1-\omega_2))T_0} \\
 &+ k_{62}e^{i(4\omega_1+\omega_2)T_0} + k_{63}e^{i(4\omega_1-\omega_2)T_0} + k_{64} + cc
 \end{aligned} \tag{48}$$

where $A_0, B_0, C_i, F_m, g_n, h_s, k_l$: ($i = 1, 2, \dots, 9$), ($m = 1, 2, \dots, 11$), ($n = 1, 2, \dots, 19$), ($s = 1, 2, \dots, 34$) and ($l = 1, 2, \dots, 64$) are complex functions in T_n . From the above analysis the general solutions of u_1 and u_2 are:

$$u_1 = u_{10} + \varepsilon u_{11} + \varepsilon^2 u_{12} + \varepsilon^3 u_{13} + O(\varepsilon^4) \tag{49}$$

$$u_2 = u_{20} + \varepsilon u_{21} + \varepsilon^2 u_{22} + \varepsilon^3 u_{23} + O(\varepsilon^4) \tag{50}$$

From the above solutions many resonance cases are shown in Table 4.

Table 4.

No.	Type of Resonances	Case
1	trivial resonance	$\Omega_1 \cong \Omega_2 \cong \omega_1 \cong \omega_2 = 0$
2	primary resonance	$\omega_2 \cong \omega_1, \omega_2 \cong \Omega_1, \omega_2 \cong \Omega_2$
3	sub-harmonic resonances	$\Omega_1 \cong n\omega_2, \Omega_2 \cong n\omega_2, \omega_2 \cong m\omega_1, \Omega_1 \cong 2\omega_1,$ $n = 2, 3, 4, 5$ and $m = 2, 3, 4$
4	super-harmonic resonances	$\omega_2 = \omega_1/2, \omega_2 = 2\omega_1/3, \omega_2 = 2\omega_1/5,$ $\Omega_1 \cong \omega_2/2, \Omega_2 \cong \omega_2/2$
5	combination resonances	$\pm\omega_2 \cong \pm\Omega_1 \pm \Omega_2, \pm\omega_2 \cong (\pm\Omega_1 \pm \Omega_2)/2,$ $\pm\Omega_1 \cong \pm\omega_2 \pm 2\omega_1, \pm\Omega_2 \cong \pm\omega_2 \pm 2\omega_1,$ $\pm\Omega_1 \cong \pm 2(\omega_2 \pm \omega_1), \pm\Omega_2 \cong 2(\pm\omega_2 \pm \omega_1)$
6	simultaneous resonance	any combination of the two resonance cases are classified as simultaneous resonance

Following the same procedure as in part 2, we get

$$\sigma_1 = -\frac{3\beta a_2^2}{8\omega_2} \pm \sqrt{F_1^2 - 4a_2^2 \zeta_2^2 \omega_2^4}$$

Fig. 9 (a)–(d) show the effects of the non-linear parameter β , the natural frequency ω_2 , the damping coefficient ζ_2 , and the excitation amplitudes F_1 on the steady state amplitude “ a_2 ”. Fig. 9 (a) shows that the positive and negative values of β , produce either hard or soft spring respectively as the curve is either bent to the left or the right, leading to the appearance of the jump phenomenon. Fig. 9 (b) indicates that the steady state amplitude is a monotonic increasing function in ω_2 also for the decreasing natural frequency ω_2 , the curve is bent to the left leading to the occurrence of the jump phenomena. Fig. 9 (c) shows that the curve is bent to the left, leading to the occurrence of the jump phenomena and the effect of increasing or decreasing ζ_2 is insignificant indicating the occurrence of saturation phenomena. Fig. 9 (d) shows that the steady state amplitude is a monotonic increasing function in the excitation amplitude F_1 .

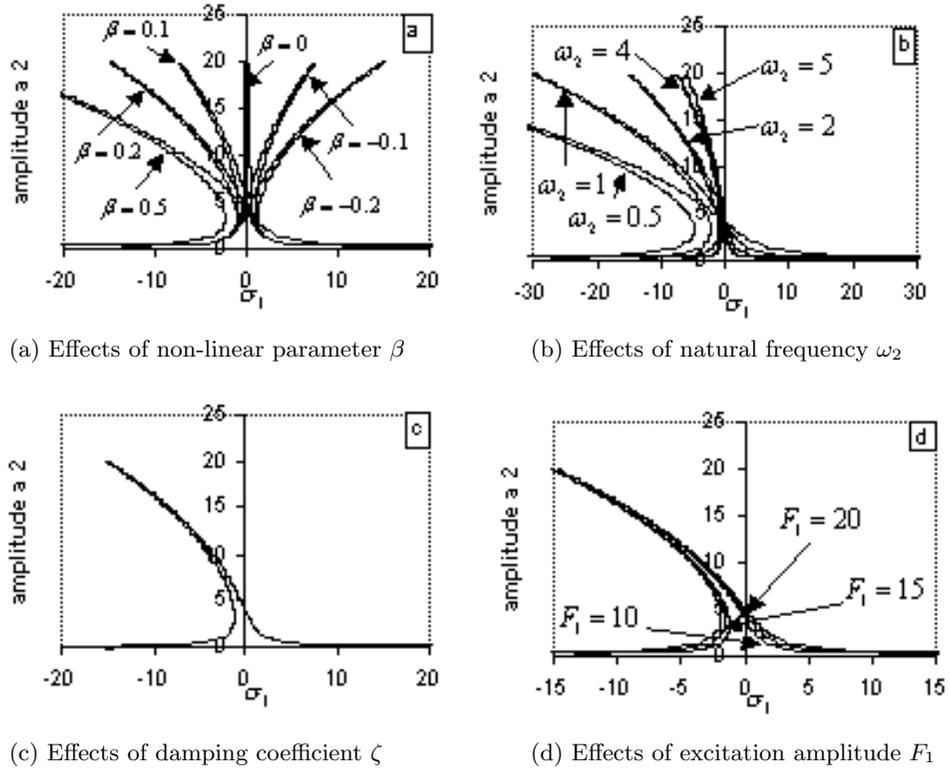


Fig. 9.

Fig. 10 illustrates the response for the system with active controller at primary resonance where $\Omega_1 \cong \omega_2$ and internal resonance where $\omega_2 = 2\omega_1$. The effectiveness of the absorber E_a is about 8.

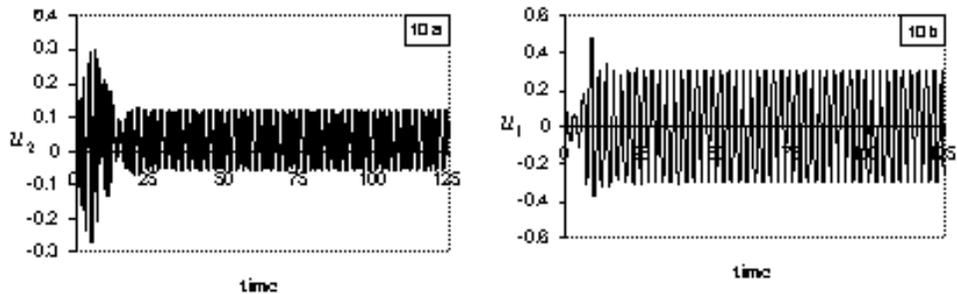


Fig. 10. System behavior with active controller at primary resonance where $\Omega_1 \cong \omega_2$ and internal resonance where $\omega_2 = 2\omega_1$. $\zeta_2 = 0.05$, $\omega_2 = 5$, $\alpha = 0.1$, $\beta = 0.2$, $\zeta_1 = 0.05$, $F_1 = 1$, $F_2 = 0.005$, $\Omega_1 = 5$, $\Omega_2 = 8$, $\omega_1 = 2.5$, $g_{12} = 20$, $g_{11} = 18$.

4. Conclusions

The main system of non-linear differential equation representing the vibration of an aircraft wing subjected to multi-excitation force is considered and solved using the method of multiple scale perturbation. Two main strategies are used. They are passive and active control methods. A tuned mass absorber is applied to the system at simultaneous primary resonance (passive control) and the same system is considered with 1:2 internal resonance controller (active control). The analytical solution is derived up to the fourth order approximation and its stability is obtained and studied applying frequency response functions. From the above study the following conclusion can be deduced.

- (i) For the main system:
 1. The steady state amplitudes is monotonic increasing function to the maximum excitation amplitude F_1 .
 2. The effects on steady state amplitudes is insignificant for increasing or decreasing values of ζ . The curve is bent to the right, leading to the occurrence of the jump phenomena.
 3. The positive and negative values of β , produce either hard or soft spring respectively as the curve is either bent to the right or the left, leading to the appearance of the jump phenomenon.
 4. For the small values of natural frequency the curve is bent to the right leading to the appearance of the jump phenomenon.
- (ii) For passive control
 1. The steady state amplitudes is monotonic increasing functions to the maximum excitation amplitude F_1 .
 2. The steady state amplitude insignificant for increasing or decreasing values of ζ_1 . The curve is bent to the right, leading to the occurrence of the jump phenomena.
 3. The positive and negative values of β , produce either hard or soft spring respectively as the curve is either bent to the right or the left, leading to the appearance of the jump phenomenon.
 4. For the small values of natural frequency the curve is bent to the right leading to the appearance of the jump phenomenon.
 5. The effectiveness of the absorber is about $E_a = 125$ at primary resonance where $\Omega_1 = \omega$ and internal resonance $\omega = \omega_1$.
- (iii) For active control
 1. The steady state amplitudes is monotonic increasing function to the maximum excitation amplitude F_1 .
 2. The steady state amplitudes is insignificant for increasing or decreasing value of ζ_2 . The curve is bent to the left, leading to the occurrence of the jump phenomena.

3. The positive and negative values of β , produce either hard or soft spring respectively as the curve is either bent to the left or the right, leading to the appearance of the jump phenomenon and vice versa at active control.
4. For the small values of natural frequency the curve is bent to the left leading to the appearance of the jump phenomenon.
5. The effectiveness of the absorber is reduce to about $E_a = 8$ at primary resonance $\Omega_1 = \omega_2$ and internal resonance $\omega_2 = 2\omega_1$.

It is worth to mention that passive control if possible has the priority over the active control from both the economical and stability problems point if view.

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