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A Note on Discrete Convexity and Local Optimality*

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One of the most important properties of a convex function is that a local optimum is also a global optimum. This paper explores the discrete analogue of this property. We consider arbitrary locality in a discrete space and the corresponding local optimum of a function over the discrete space. We introduce the corresponding notion of discrete convexity and show that the local optimum of a function satisfying the discrete convexity is also a global optimum. The special cases include discretely-convex, integrally-convex, M-convex, M$^\natural$-convex, L-convex, and L$^\natural$-convex functions.

Key words: discrete optimization, convex function, quasiconvex function, Nash equilibrium, potential game

1. Introduction

The concept of convexity for sets and functions plays a central role in continuous optimization. The importance of convexity relies on the fact that a local optimum of a convex function is a global optimum. In the area of discrete optimization, on the other hand, discrete analogues of convexity, or "discrete convexity" for short, have been considered. There exist several different types of discrete convexity. Examples include "discretely-convex functions" by Miller [5], "integrally-convex functions" by Favati and Tardella [3], "M-convex functions" by Murota [7], "L-convex functions" by Murota [8], "M$^\natural$-convex functions" by Murota and Shioura [12], and "L$^\natural$-convex functions" by Fujishige and Murota [4]. While these functions also have the property that a local optimum is a global optimum, the type of local optimum (i.e. the definition of locality) depends upon the type of discrete convexity.

The purpose of this paper is to elucidate the relationship between discrete convexity and local optimality by asking what type of discrete convexity is required by a given type of local optimality. We consider arbitrary locality in a discrete space and the corresponding local optimum of a function over the discrete space. We then introduce the corresponding notion of discrete convexity and show that a function satisfying the discrete convexity has the property that the local optimum is a global optimum. Finally, we argue that the special classes of functions satisfying discrete convexity include discretely-convex, integrally-convex, M-convex, M$^\natural$-convex, L-

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convex, and L$^\natural$-convex functions. Thus, we can understand the local optimality conditions for these functions in a unified framework. We also argue that a sufficient condition for the uniqueness of Nash equilibrium in the class of strategic potential games [6] obtained by [14] can be seen as a special case of our results.

2. Results

We denote by $\mathbb{R}$ the set of reals, and by $\mathbb{Z}$ the set of integers. Let n be a positive integer and denote $N = \{1, \ldots, n\}$. The characteristic vector of a subset $S \subseteq N$ is denoted by $\chi_S \in \{0,1\}^N$:

$$\chi_S(i) = \begin{cases} 1 & \text{if } \ i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

We use the notation $\mathbf{0} = \chi_\emptyset$, $\mathbf{1} = \chi_N$, and $\chi_i = \chi_{\{i\}}$ for $i \in N$. For a vector $x \in \mathbb{R}^N$, let $\|x\|_1 = \sum_{i \in N} |x(i)|$ be the ℓ_1-norm.

A function $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is convex if $\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$ for all $x, y \in \mathbb{R}^N$ and $\lambda \in (0,1)$. If f is convex, then $\max\{f(x), f(y)\} > f(\lambda x + (1-\lambda)y)$ for all $x, y \in \mathbb{R}^N$ with $f(x) \neq f(y)$ and $\lambda \in (0,1)$. A function f satisfying this condition is said to be semistrictly quasiconvex. Note that f is semistrictly quasiconvex if and only if

$$\max\{f(x), f(y)\} > \min\{f(x + \Delta), f(y - \Delta)\}$$

for all $x, y \in \mathbb{R}^N$ with $f(x) \neq f(y)$ where $\Delta = \lambda(y - x)$ and $\lambda \in (0,1)$. It is known that a local minimum of a semistrictly quasiconvex function is also a global minimum.[1] We consider discrete analogues of convexity and semistrict quasiconvexity having a similar property.

Fix $\mathcal{D} \subseteq \{-1,0,1\}^N \backslash \{\mathbf{0}\}$ such that $\chi_i \in \mathcal{D}$ for each $i \in N$ and $-d \in \mathcal{D}$ for all $d \in \mathcal{D}$. For $x \in \mathbb{Z}^N$, we write $\mathcal{D}(x) = \{z \in \mathbb{Z}^N : z = x + d, \ d \in \mathcal{D}\}$, which is interpreted as a neighborhood of x. Note that $y \in \mathcal{D}(x)$ if and only if $x \in \mathcal{D}(y)$. For a function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$, we say that $x \in \mathbb{Z}^N$ is a $\mathcal{D}$-*local minimum* of f if $f(x) \leq f(y)$ for all $y \in \mathcal{D}(x)$.

For $x, y \in \mathbb{Z}^N$, we write

$$\mathcal{R}(x,y) = \{z \in \mathbb{Z}^N : x \wedge y \leq z \leq x \vee y\}$$

where $(x \wedge y)(i) = \min\{x(i), y(i)\}$ and $(x \vee y)(i) = \max\{x(i), y(i)\}$ for each $i \in N$. Note that $\|x - z\|_1 + \|y - z\|_1 = \|x - y\|_1$ if and only if $z \in \mathcal{R}(x,y)$.

For a function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$, let $\operatorname{dom} f = \{x \in \mathbb{Z}^N : f(x) < +\infty\}$ be the effective domain. We say that $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is $\mathcal{D}$-*convex* if, for any $x, y \in \mathbb{Z}^N$ with $x \neq y$,

$$f(x) + f(y) \geq \min_{x' \in \mathcal{D}(x) \cap \mathcal{R}(x,y)} f(x') + \min_{y' \in \mathcal{D}(y) \cap \mathcal{R}(x,y)} f(y'). \tag{1}$$

[1] See Avriel et al. [2] for more accounts on quasiconvexity.

Note that the above inequality is trivially true when $y \in \mathcal{D}(x)$ and $x \in \mathcal{D}(y)$. We say that $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is *semistrictly quasi $\mathcal{D}$-convex* if, for any $x, y \in \mathbb{Z}^N$ with $f(x) \neq f(y)$,

$$\max\{f(x), f(y)\} > \min\left\{\min_{x' \in \mathcal{D}(x) \cap \mathcal{R}(x,y)} f(x'), \min_{y' \in \mathcal{D}(y) \cap \mathcal{R}(x,y)} f(y')\right\}. \tag{2}$$

Note that the above inequality is trivially true when $y \in \mathcal{D}(x)$ and $x \in \mathcal{D}(y)$ with $f(x) \neq f(y)$. A $\mathcal{D}$-convex function is semistrictly quasi $\mathcal{D}$-convex.

The following proposition is the main result of this paper.

PROPOSITION 1. *Suppose that $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ is semistrictly quasi $\mathcal{D}$-convex. Then, $x \in \mathbb{Z}^N$ is a $\mathcal{D}$-local minimum of f if and only if it is a global minimum of f, i.e.,*

$$f(x) \leq f(y) \ \ \textit{for all} \ \ y \in \mathcal{D}(x) \Leftrightarrow f(x) \leq f(y) \ \ \textit{for all} \ \ y \in \mathbb{Z}^N.$$

Proof. The "if" part is obvious and we show the "only if" part by induction. Let $x \in \mathbb{Z}^N$ be a $\mathcal{D}$-local minimum of f. Then, $f(x) \leq f(y)$ for all $y \in \mathbb{Z}^N$ with $\|x - y\|_1 = 1$ because $x \pm \chi_i \in \mathcal{D}(x)$ for each $i \in N$. Suppose that $f(x) \leq f(y)$ for all $y \in \mathbb{Z}^N$ with $\|x - y\|_1 \leq k$ where $k \geq 1$. Let $y \in \mathbb{Z}^N$ be such that $\|x - y\|_1 = k + 1$. We show that $f(x) \leq f(y)$. Seeking a contradiction, suppose that $f(y) < f(x)$. Since x is a $\mathcal{D}$-local minimum, $f(x) \leq \min_{x' \in \mathcal{D}(x) \cap \mathcal{R}(x,y)} f(x')$. Since f is semistrictly quasi $\mathcal{D}$-convex,

$$\begin{aligned} f(x) &= \max\{f(x), f(y)\} \\ &> \min\left\{\min_{x' \in \mathcal{D}(x) \cap \mathcal{R}(x,y)} f(x'), \min_{y' \in \mathcal{D}(y) \cap \mathcal{R}(x,y)} f(y')\right\} = \min_{y' \in \mathcal{D}(y) \cap \mathcal{R}(x,y)} f(y'). \end{aligned}$$

Note that $\|x - y'\|_1 < \|x - y\|_1 = k + 1$ for all $y' \in \mathcal{D}(y) \cap \mathcal{R}(x,y)$. Thus, by the induction hypothesis, $f(x) \leq \min_{y' \in \mathcal{D}(y) \cap \mathcal{R}(x,y)} f(y')$, a contradiction. ■

The following proposition, which we will use later, provides a sufficient condition for semistrict quasi $\mathcal{D}$-convexity in terms of a local condition, where one point is in the local area of another if neighborhoods of the two points have a non-empty intersection.

PROPOSITION 2. *Suppose that, for any $x, y \in \mathbb{Z}^N$ with $y \notin \mathcal{D}(x)$, $x \notin \mathcal{D}(y)$, and $\mathcal{D}(x) \cap \mathcal{D}(y) \cap \mathcal{R}(x,y) \neq \emptyset$,*

$$\min_{z \in \mathcal{D}(x) \cap \mathcal{D}(y) \cap \mathcal{R}(x,y)} f(z) \begin{cases} < \max\{f(x), f(y)\} & \textit{if} \ \ f(x) \neq f(y), \\ \leq f(x) = f(y) & \textit{otherwise.} \end{cases} \tag{3}$$

Then, f is semistrictly quasi $\mathcal{D}$-convex.

Proof. For $x, y \in \mathbb{Z}^N$ with $y \in \mathcal{D}(x)$, $x \in \mathcal{D}(y)$, and $f(x) \neq f(y)$, (2) is trivially true. For $x, y \in \mathbb{Z}^N$ with $y \notin \mathcal{D}(x)$, $x \notin \mathcal{D}(y)$, and $f(x) \neq f(y)$, construct a sequence $\{x_k \in \mathcal{R}(x,y)\}_{k=0}^m$ such that $x_0 = x$ and $x_m = y$ by the following steps: set $x_{k+1} \in \mathcal{D}(x_k) \cap \mathcal{R}(x_k, y)$ for $k = 0, \ldots, m-1$ such that

- $x_{k+1} \in \arg \min_{z \in \mathcal{D}(x_k) \cap \mathcal{R}(x_k,y)} f(z)$,
- $\|x_{k+1} - x_k\|_1 \geq \|x' - x_k\|_1$ for all $x' \in \arg \min_{z \in \mathcal{D}(x_k) \cap \mathcal{R}(x_k,y)} f(z)$.

Since $x_k \pm \chi_i \in \mathcal{D}(x_k)$ for all $i \in N$ and $x_k \notin \mathcal{D}(x_k)$, we have $\|x_0 - y\|_1 > \|x_1 - y\|_1 > \cdots > \|x_{m-1} - y\|_1 > \|x_m - y\|_1 = 0$. Thus, this sequence is well defined. By construction, $x_0(i) \leq x_1(i) \leq \cdots \leq x_m(i)$ if $x(i) \leq y(i)$ and $x_0(i) \geq x_1(i) \geq \cdots \geq x_m(i)$ if $x(i) \geq y(i)$. This implies that $x_{k+1} \in \mathcal{R}(x_k, x_{k+2}) \subseteq \mathcal{R}(x_k, y)$ for all $k \leq m-2$. We also have $x_{k+1} \in \mathcal{D}(x_{k+2})$ because $\pm(x_{k+1} - x_{k+2}) \in \mathcal{D}$. Therefore,

$$f(x_{k+1}) = \min_{z \in \mathcal{D}(x_k) \cap \mathcal{R}(x_k,y)} f(z) = \min_{z \in \mathcal{D}(x_k) \cap \mathcal{D}(x_{k+2}) \cap \mathcal{R}(x_k,x_{k+2})} f(z).$$

By (3), if $x_{k+2} \notin \mathcal{D}(x_k)$ then

$$f(x_{k+1}) \begin{cases} < \max\{f(x_k), f(x_{k+2})\} & \text{if } f(x_k) \neq f(x_{k+2}), \\ \leq f(x_k) = f(x_{k+2}) & \text{otherwise.} \end{cases} \tag{4}$$

If $x_{k+2} \in \mathcal{D}(x_k)$ (and thus $x_{k+2} \in \mathcal{D}(x_k) \cap \mathcal{R}(x_k, y)$), we must have $f(x_{k+1}) < f(x_{k+2})$. To see this, recall that $\|x_{k+1} - x_k\|_1 \geq \|x' - x_k\|_1$ for all $x' \in \arg\min_{z \in \mathcal{D}(x_k) \cap \mathcal{R}(x_k,y)} f(z)$. Since $\|x_{k+1} - x_k\|_1 < \|x_{k+1} - x_k\|_1 + \|x_{k+2} - x_{k+1}\|_1 = \|x_{k+2} - x_k\|_1$, it must be true that $x_{k+2} \notin \arg\min_{z \in \mathcal{D}(x_k) \cap \mathcal{R}(x_k,y)} f(z)$ and thus $f(x_{k+1}) < f(x_{k+2})$. Therefore, to summarize, (4) is true for all k.

The condition (4) implies that if $f(x_k) < f(x_{k+1})$ then $f(x_{k+1}) < f(x_{k+2})$, which further implies $f(x_{k+2}) < f(x_{k+3})$. Therefore, if $f(x_k) < f(x_{k+1})$ then $f(x_l) < f(x_{l+1})$ for all $l \geq k$. Symmetrically, if $f(x_k) < f(x_{k-1})$ then $f(x_l) < f(x_{l-1})$ for all $l \leq k$. Using this property, we show that (2) is true.

If $f(x_0) < f(x_m)$, there exists $k \leq m-1$ such that $f(x_k) < f(x_{k+1})$. By the above argument, we must have $f(x_{m-1}) < f(x_m)$. Therefore,

$$\begin{aligned}
&\max\{f(x), f(y)\} \\
&\;= \max\{f(x_0), f(x_m)\} \\
&\;= f(x_m) > f(x_{m-1}) \\
&\;\geq \min\{f(x_1), f(x_{m-1})\} \\
&\;= \min\left\{\min_{x' \in \mathcal{D}(x_0) \cap \mathcal{D}(x_2) \cap \mathcal{R}(x_0,x_2)} f(x'), \min_{y' \in \mathcal{D}(x_{m-2}) \cap \mathcal{D}(x_m) \cap \mathcal{R}(x_{m-2},x_m)} f(y')\right\} \\
&\;\geq \min\left\{\min_{x' \in \mathcal{D}(x) \cap \mathcal{R}(x,y)} f(x'), \min_{y' \in \mathcal{D}(y) \cap \mathcal{R}(x,y)} f(y')\right\},
\end{aligned}$$

which implies (2). Similarly, we can also show that if $f(x_m) < f(x_0)$ then (2) is true. Therefore, f is semistrictly quasi $\mathcal{D}$-convex. ■

Note that the condition in this proposition is not necessary for semistrict quasi $\mathcal{D}$-convexity. For example, let $f : \mathbb{Z}^3 \to \mathbb{R} \cup \{+\infty\}$ be such that $\operatorname{dom} f = \{0,1\}^3$ and, for each $x \in \operatorname{dom} f$,

$$f(x) = \begin{cases} 1 & \text{if} \;\; x = (0,0,0), (1,1,0), (0,0,1), \\ 0 & \text{otherwise.} \end{cases}$$

A function f is semistrictly quasi $\mathcal{D}$-convex with $\mathcal{D} = \{\pm\chi_1, \pm\chi_2, \pm\chi_3\} \cup \{\pm(\chi_1 + \chi_2)\}$ but does not satisfy (3) for $x = (0,0,0)$ and $y = (1,1,1)$.

3. Examples

3.1. Coordinatewise locality and Nash equilibrium

Let $\mathcal{D}_{\mathrm{C}} = \{\pm\chi_i : i \in N\}$. If f is semistrictly quasi $\mathcal{D}_{\mathrm{C}}$-convex, then Proposition 1 implies that[2]

$$f(x) \le f(x \pm \chi_i) \;\; \text{for all} \;\; i \in N \Leftrightarrow f(x) \le f(y) \;\; \text{for all} \;\; y \in \mathbb{Z}^N. \tag{5}$$

For example, suppose that, for any $x, y \in \mathbb{Z}^N$ with $\|x - y\|_1 = 2$,

$$\min_{z : \|x-z\|_1 = \|y-z\|_1 = 1} f(z) \begin{cases} < \max\{f(x), f(y)\} & \text{if} \;\; f(x) \neq f(y), \\ \le f(x) = f(y) & \text{otherwise.} \end{cases} \tag{6}$$

Then, by Proposition 2, f is semistrictly quasi $\mathcal{D}_{\mathrm{C}}$-convex and thus (5) is true. It is easy to check that a separable convex function satisfies the above condition and thus it is semistrictly quasi $\mathcal{D}_{\mathrm{C}}$-convex. Note that a semistrictly quasi $\mathcal{D}_{\mathrm{C}}$-convex function is not necessarily separable convex.

The above argument has an application to game theory. A game consists of a set of players $N = \{1, \ldots, n\}$, a set of strategies $X_i = \mathbb{Z}$ for $i \in N$, and a payoff function $g_i : X \to \mathbb{R} \cup \{-\infty\}$ for $i \in N$ where $X = \prod_{i \in N} X_i = \mathbb{Z}^N$. Simply denote a game by $\mathbf{g} = (g_i)_{i \in N}$. We write $X_{-i} = \prod_{j \neq i} X_j$ and $x_{-i} = (x_j)_{j \neq i} \in X_{-i}$, and denote $(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) \in X$ by (x_i', x_{-i}). A strategy profile $x \in X$ is a Nash equilibrium of $\mathbf{g}$ if $g_i(x_i, x_{-i}) \ge g_i(x_i', x_{-i})$ for all $x_i' \in X_i$ and $i \in N$.

A game $\mathbf{g}$ is a potential game [6] if there exists a potential function $p : X \to \mathbb{R} \cup \{-\infty\}$ satisfying $g_i(x_i, x_{-i}) - g_i(x_i', x_{-i}) = p(x_i, x_{-i}) - p(x_i', x_{-i})$ for all $x_i, x_i' \in X_i$, $x_{-i} \in X_{-i}$, and $i \in N$. If $x \in X$ maximizes a potential function p, then $p(x_i, x_{-i}) \ge p(x_i', x_{-i})$ for all $x_i' \in X_i$ and $i \in N$, which is equivalent to $g_i(x_i, x_{-i}) \ge g_i(x_i', x_{-i})$ for all $x_i' \in X_i$ and $i \in N$. This implies that if $x \in X$ maximizes p, then it is a Nash equilibrium. Note that every Nash equilibrium does not necessarily maximize p. However, if it holds that

$$p(x) \ge p(x \pm \chi_i) \;\; \text{for all} \;\; i \in N \Leftrightarrow p(x) \ge p(y) \;\; \text{for all} \;\; y \in \mathbb{Z}^N,$$

[2] Altman et al. [1, Corollary 2.2] states that if f is multimodular then (5) is true. Murota [11], however, finds a counterexample against it and provides a correct local optimality condition.

then every Nash equilibrium maximizes p. To see this, let $x \in X$ be a Nash equilibrium. Then, $g_i(x) - g_i(x_i \pm 1, x_{-i}) = g_i(x) - g_i(x \pm \chi_i) = p(x) - p(x \pm \chi_i) \geq 0$ for all $i \in N$. This implies that $p(x) \geq p(y)$ for all $y \in X$. The following result reported in [14] is an immediate consequence of the above discussion.

Proposition 3. *Let* $\mathbf{g}$ *be a potential game with a potential function* p. *Suppose that* $f \equiv -p$ *satisfies* (6) *for any* $x, y \in \mathbb{Z}^N$ *with* $\|x - y\|_1 = 2$. *Then,* $x \in X$ *maximizes* p *if and only if it is a Nash equilibrium. Thus, if a potential maximizer is unique, so is a Nash equilibrium.*

3.2. M-convex, M$^\natural$-convex, L-convex, and L$^\natural$-convex functions

Recently, Murota [8, 10] advocates "discrete convex analysis," where M-convex and L-convex functions, introduced respectively by Murota [7] and Murota [8], play central roles. M$^\natural$-convex and L$^\natural$-convex functions, introduced respectively by Murota and Shioura [13] and Fujishige and Murota [4], are variants of M-convex and L-convex functions. By choosing appropriate $\mathcal{D}$, we can show that these functions are $\mathcal{D}$-convex.

Let $\mathrm{supp}^+(x) = \{i : x(i) > 0\}$ be the positive support and $\mathrm{supp}^-(x) = \{i : x(i) < 0\}$ be the negative support of $x \in \mathbb{Z}^N$. A function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is said to be an M-convex function [7] if, for any $x, y \in \mathrm{dom}\, f$ and $i \in \mathrm{supp}^+(x - y)$, there exists $j \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).$$

It is known that this inequality implicitly imposes the condition that the effective domain of an M-convex function lies on a hyperplane $\{x \in \mathbb{Z} : \sum_{i \in N} x(i) = r\}$ for some $r \in \mathbb{Z}$ and, accordingly, we may consider the projection of an M-convex function along a coordinate axis. A function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ is said to be an M$^\natural$-convex function [13] if the function $\tilde{f} : \mathbb{Z}^{\{0\} \cup N} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & \text{if } x_0 = -\sum_{i \in N} x(i), \\ +\infty & \text{otherwise} \end{cases}$$

is an M-convex function. The following proposition characterizes an M$^\natural$-convex function [10, Theorem 6.2].

Proposition 4. *A function* $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ *is an* M$^\natural$*-convex function if and only if, for any* $x, y \in \mathrm{dom}\, f$ *and* $i \in \mathrm{supp}^+(x - y)$,

$$f(x) + f(y) \geq \min\Big\{ f(x - \chi_i) + f(y + \chi_i), \min_{j \in \mathrm{supp}^-(x-y)} f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j) \Big\}.$$

This proposition and the definition of $\mathcal{D}$-convexity imply that an M$^\natural$-convex function is $\mathcal{D}_\mathrm{M}$-convex with $\mathcal{D}_\mathrm{M} = \{\pm\chi_i : i \in N\} \cup \{\chi_i - \chi_j : i \neq j\}$. Thus, by

Proposition 1, if $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ is an M$^\natural$-convex function, then

$$f(x) \leq \begin{cases} f(x \pm \chi_i) & \text{for all} \;\; i \in N \\ f(x + \chi_i - \chi_j) & \text{for all} \;\; i, j \in N \end{cases} \Leftrightarrow f(x) \leq f(y) \;\; \text{for all} \;\; y \in \mathbb{Z}^N.$$

This result is reported in Murota [7]. Proposition 4 says that an M-convex function is an M$^\natural$-convex function. Thus, an M-convex function is also $\mathcal{D}_{\mathrm{M}}$-convex.[3]

A function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is said to be an L-convex function [8] if

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$$

for all $x, y \in \mathbb{Z}^N$ and there exists $r \in \mathbb{R}$ such that $f(x + \mathbf{1}) = f(x) + r$ for all $x \in \mathbb{Z}^N$. Since an L-convex function is linear in the direction of $\mathbf{1}$, we may dispense with this direction as far as we are interested in its nonlinear behavior. A function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ is said to be an L$^\natural$-convex function [4] if the function $\tilde{f} : \mathbb{Z}^{\{0\} \cup N} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\tilde{f}(x_0, x) = f(x - x_0 \mathbf{1})$$

for $x_0 \in \mathbb{Z}$ and $x \in \mathbb{Z}^N$ is an L-convex function. The following proposition characterizes an L$^\natural$-convex function [10, Theorem 7.7].

Proposition 5. *A function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ is an L$^\natural$-convex function if and only if, for any $x, y \in \mathbb{Z}^N$ with $\mathrm{supp}^+(x - y) \neq \emptyset$,*

$$f(x) + f(y) \geq f(x - \chi_S) + f(y + \chi_S) \quad \textit{where} \quad S = \arg\max_{i \in N} (x(i) - y(i)).$$

This proposition and the definition of $\mathcal{D}$-convexity imply that an L$^\natural$-convex function is $\mathcal{D}_{\mathrm{L}}$-convex with $\mathcal{D}_{\mathrm{L}} = \{\pm \chi_S : S \subseteq N, \; S \neq \emptyset\}$. Thus, by Proposition 1, if $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ is an L$^\natural$-convex function, then

$$f(x) \leq f(x \pm \chi_S) \;\; \text{for all} \;\; S \subseteq N \Leftrightarrow f(x) \leq f(y) \;\; \text{for all} \;\; y \in \mathbb{Z}^N.$$

This result is reported in Murota [9]. It is known that an L-convex function is an L$^\natural$-convex function [10, Theorem 7.3]. Thus, an L-convex function is also $\mathcal{D}_{\mathrm{L}}$-convex.[4]

Murota and Shioura [13] introduced semistrictly quasi M-convex and L-convex functions. It can be readily shown that a semistrictly quasi M-convex function is semistrictly quasi $\mathcal{D}_{\mathrm{M}}$-convex and that a semistrictly quasi L-convex function is semistrictly quasi $\mathcal{D}_{\mathrm{L}}$-convex. Murota and Shioura [13] obtained the local optimality conditions for semistrictly quasi M-convex and L-convex functions, which are weaker than those for $\mathcal{D}_{\mathrm{M}}$-convex and $\mathcal{D}_{\mathrm{L}}$-convex functions, respectively.

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[3] One can obtain the local optimality condition for M-convex functions by weakening that for M$^\natural$-convex functions. See Murota [10, Theorem 6.26] for more accounts on this issue.

[4] One can obtain the local optimality condition for L-convex functions by weakening that for L$^\natural$-convex functions. See Murota [10, Theorem 7.14] for more accounts on this issue.

3.3. Discretely-convex and integrally-convex functions

For $x \in \mathbb{R}^N$, let $N(x) = \{z \in \mathbb{Z}^N : \lfloor x \rfloor \leq z \leq \lceil x \rceil\}$ where $\lfloor x \rfloor$ denotes the vector obtained by rounding down and $\lceil x \rceil$ by rounding up the components of x to the nearest integers. A function $f : \mathbb{Z}^N \to \mathbb{R} \cup \{+\infty\}$ is a discretely-convex function [5] if, for any $x, y \in \operatorname{dom} f$, it holds that

$$\lambda f(x) + (1-\lambda) f(y) \geq \min_{z \in N(\lambda x + (1-\lambda) y)} f(z) \ (\forall \lambda \in [0,1]). \tag{7}$$

Let $\mathcal{D}_{\mathrm{A}} = \{-1, 0, 1\}^N \setminus \{\mathbf{0}\}$. The following lemma connects a discretely-convex function to a semistrictly quasi $\mathcal{D}_{\mathrm{A}}$-convex function.

Lemma 6. *Let $x, y \in \mathbb{Z}^N$ be such that $y \notin \mathcal{D}_{\mathrm{A}}(x)$, $x \notin \mathcal{D}_{\mathrm{A}}(y)$, and $\mathcal{D}_{\mathrm{A}}(x) \cap \mathcal{D}_{\mathrm{A}}(y) \cap \mathcal{R}(x, y) \neq \emptyset$. Then, $N((x+y)/2) \subseteq \mathcal{D}_{\mathrm{A}}(x) \cap \mathcal{D}_{\mathrm{A}}(y) \cap \mathcal{R}(x, y)$.*

Proof. By the assumption, there exist $d, d' \in \mathcal{D}_{\mathrm{A}}$ such that $y = x + d + d'$, $d + d' \neq \mathbf{0}$, and $d + d' \notin \mathcal{D}_{\mathrm{A}}$. This implies that $|d(i) + d'(i)| \leq 2$ for all $i \in N$ and $|d(i) + d'(i)| = 2$ for some $i \in N$. Thus, if $\lfloor (d+d')/2 \rfloor \leq \delta \leq \lceil (d+d')/2 \rceil$ then $\delta \in \{-1, 0, 1\}^N \setminus \{\mathbf{0}\} = \mathcal{D}_{\mathrm{A}}$.

Let $z \in N((x+y)/2)$. Then, $\lfloor (x+y)/2 \rfloor \leq z \leq \lceil (x+y)/2 \rceil$. Thus, $z \in \mathcal{D}_{\mathrm{A}}(x)$ because $(x+y)/2 = x + (d+d')/2$. Similarly, $z \in \mathcal{D}_{\mathrm{A}}(y)$. Since $z \in \mathcal{R}(x, y)$, we have $N((x+y)/2) \subseteq \mathcal{D}_{\mathrm{A}}(x) \cap \mathcal{D}_{\mathrm{A}}(y) \cap \mathcal{R}(x, y)$. ∎

Let $x, y \in \operatorname{dom} f$ satisfy the condition in the above lemma. Assume that f is discretely-convex. Then, we have

$$f(x) + f(y) \geq 2 \min_{z \in N((x+y)/2)} f(z) \geq 2 \min_{z \in \mathcal{D}_{\mathrm{A}}(x) \cap \mathcal{D}_{\mathrm{A}}(y) \cap \mathcal{R}(x,y)} f(z)$$

where the first inequality is due to (7) and the second inequality is due to Lemma 6. This implies that (3) is true for all $x, y \in \mathbb{Z}^N$ with $y \notin \mathcal{D}_{\mathrm{A}}(x)$, $x \notin \mathcal{D}_{\mathrm{A}}(y)$, and $\mathcal{D}_{\mathrm{A}}(x) \cap \mathcal{D}_{\mathrm{A}}(y) \cap \mathcal{R}(x, y) \neq \emptyset$. Thus, we have the following proposition by Proposition 2.

Proposition 7. *A discretely-convex function is semistrictly quasi $\mathcal{D}_{\mathrm{A}}$-convex.*

Therefore, by Proposition 1, if $f : \mathbb{Z}^N \to \mathbb{R}$ is a discretely-convex function, then

$$f(x) \leq f(x + \chi_S - \chi_T) \text{ for all } S, T \subseteq N \Leftrightarrow f(x) \leq f(y) \text{ for all } y \in \mathbb{Z}^N.$$

This result is reported in [5].

Favati and Tardella [3] introduced integrally-convex functions and showed that these functions form a special class of discretely-convex functions. Thus, an integrally-convex function is also semistrictly quasi $\mathcal{D}_{\mathrm{A}}$-convex.

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