BOOK REVIEW

Index Theory – With Applications to Mathematics and Physics, by David D. Bleecker and Bernhelm Booß-Bavnbeck, International Press, Somerville 2013. xxii+769 pp. ISBN: 978-1-57146-264-0 58-02.

The index theory appears to be a broad and deep theory connecting several mathematical disciplines, which has a significant impact on many areas of physics. Its development initiated a development of a part of K-theory within differential geometry and topology. The monograph "Index Theory with Applications to Mathematics and Physics" contains a thorough description of the index theory and covers a plenty of applications of this theory in the mathematics of PDE's, manifolds, vector bundles, and in theoretical and mathematical physics.

Since the authors track the index theory as it was developed quite faithfully, we can acquaint the content of the book to readers as follows.

The content of the individual chapters will be described in a more detail after that. The index of a linear operator measures the gap between the null space of the operator and the space of elements in the target of the operator which cannot be achieved by the operator (the cokernel). In symbols, if $L : V \to W$ is a map between two k-linear vector spaces, its index is the number $\dim_k \ker L - \dim_k \operatorname{coker} L$ where the cokernel is defined by $\operatorname{coker} L = W/\operatorname{im} L$. In a more abstract way, it can be considered as the element $\ker L - \operatorname{coker} L$ of the K-group of k (k is considered as a ring). If the vector spaces V and W are finite dimensional, the index does not depend on the map L and equals the dimension of V minus the dimension of the target W. Thus the index has a meaning in infinite dimension only.

One of the first question which arises is the following one. For which (infinite dimensional) vector spaces and which (bounded) operators the index may be considered? As the authors do and as it seems to be common, one usually considers the source and target space to be Hilbert spaces. Let us recall that a Fredholm operator is a bounded operator between Hilbert spaces (or Banach spaces eventually) for which the dimensions of its kernel and cokernel are finite – the number index is well defined, and such that its image is closed – the formal subtraction makes

sense in the category the objects of which are Hilbert spaces. The class of Fredholm operators is precisely the one for which the index makes sense as the element of the K-group of k.

Atkinson's theorem gives a more algebraic description of these operators as those bounded operators which have a parametrix. Thus, D is Fredholm if and only if there exists an operator G (the "Green" operator) and compact operators K_1, K_2 such that $GD = 1 + K_1$ and $DG = 1 + K_2$. The index map assigns the number $\dim_k \ker L - \dim_k \operatorname{coker} L$ to each Fredholm operator L. To investigate analytic properties of the index map, one has to choose a suitable topology on the space of the Fredholm operators. In the operator norm topology, the index map appears to be continuous. Being an integer number, it is constant on connected components of the set of the Fredholm operators. When considering the Fredholm operators modulo continuous homotopies, the index appears to be an isomorphism and it follows that it is also stable under compact perturbations.

Another task is to search for examples of Fredholm operators. On a differential manifold M, it is possible to define the set of the so called pseudodifferential operators.

Roughly speaking, these operators are built using a finite number of integrations and partial derivatives (in coordinates given by local charts), and their symbols behave in a specified way. From this class it is further possible to extract the so called elliptic operators using the notion of the symbol. In the theory of PDE's, one is often interested in questions connected to the boundary of a domain $\Omega \subseteq \mathbb{R}^n$ on which we search for a solution. The degree of the smoothness of the boundary, the constraints on the values of the solution on the boundary shall often be settled in order the problem is 'well posed'.

To avoid these complications, one usually restricts himself to closed manifolds only, i.e., compact manifolds without boundary. The pseudodifferential operators act on the space of smooth functions which typically do not form Hilbert spaces. Because of the smoothness of the manifolds, one can define Sobolev spaces similarly as in the case of domains in $\mathbb{R}^n$. The resulting spaces are Banach or Hilbert spaces as in the 'Cartesian' case. A fact is that the pseudodifferential operators can be continuously extended to the Sobolev spaces and also that in the case of the elliptic ones, the kernel on smooth functions coincides with the kernel of the extension to the Sobolev space. One of the important properties of the elliptic operators is that they are Fredholm. Here, the compactness of the manifold, the finite order of the operator and the ellipticity are used for constructing the Green operator and the compact ones. All what we said transfers to operators which act on sections of finite rank vector bundles over manifolds quite straightforwardly.

Now, one is focused to the calculation of the index of a Fredholm operator which arose in this geometric way described above. The famous theorem of Atiyah and Singer can be thought as a generalization of the Riemann-Roch theorem, Lefschetz fixed point formula or also of the classical Gauss-Bonnet theorem. It relates the index of an elliptic operator to topological characterizations of the manifold or vector bundles over the manifold. In this way the index theory connects the manifold topology and the analysis of elliptic PDE's on it. The topological 'characters' are the Chern classes and the Todd class.

The first part (Part I) is devoted to the necessary homological algebra, to properties and definitions of Fredholm, compact and Hilbert-Schmidt operators. The Calkin algebra, the homotopy equivalence and topologies on the space of bounded operators are introduced. The last chapter of Part I is devoted to the so-called Wiener-Hopf operators, a rich source of the Fredholm operators.

In the second part, the authors focus on global analysis. They start with giving information on partial differential operators in the Euclidean space. They treat elliptic differential operators and especially, several operators known from physics. Then these objects are studied on differentiable manifolds. The concept of charts, differential forms, integration on manifolds, connections and covariant derivatives are recalled. In this part, the definition of Sobolev spaces is presented together with the main theorems - on the continuous and the compact embedding.

Part III is devoted to the Atiyah-Singer index formula and its formulation within the K-theory. The index formula is proved in this part as well. The topological K-theory is defined, and a proof of the Bott theorem is presented here.

Part IV is called "Index Theory in Physics and the Local Index Theorem". It is the most extensive part of the book (pages 363-722). If the index formula is the highlight of the logical (deductive) structure of the treatment of the index theory in this book, this chapter is its heart.

The authors recall a formulation of the classical field theory and principles of the quantum theory in the beginning of this part briefly. Then the principle notions of G-bundles, G-connections, associated bundles, gauge transformations, characteristic classes, Bochner-Weitzenböck formulas and holonomy are recalled. These notions are used to explain the mathematics of Yang-Mills theory, instantons in four dimensions (4D), moduli space of self-adjoint connections and its linearized structure. After that, some important preliminaries regarding Dirac operators, as Clifford algebras, spinors, and the twisted or spinorial heat kernels are presented. The last part of Part IV forms a tour through the Seiberg-Witten theory. Its background is explained shortly, the Seiberg-Witten equation is recalled using complex spin structures, and the definition of the Seiberg-Witten invariants are given. The

proof of the compactness of the appropriate moduli space of the Seiberg-Witten equation is presented as well.

The index of notation and the subject index make the book well usable. The readers who are not familiar with Fourier integrals, vector bundles and homotopy equivalence may get familiar with them by the two appendices: Appendix A (Fourier series and Integrals - Fundamental Principles) and Appendix B (Vector Bundles).

The monograph consists of 766 pages. It is useful for mathematicians as well as for physicist. The treatment of the involved topics is self-contained, and most of its parts are understandable for pregradual (undergraduate) students. The history and connections of and in the index theory are covered by the book in an excellent way. The authors give many examples by which they help the reader to clarify his or her understanding of the subject and reveal the application as well. The examples are well elaborated and may serve as a way for understanding the theory. Theorems are formulated with the mathematical rigor and most of them are proved. The proofs are written in a clear way. The theorems which are not proved are explained, and in most cases the authors give a sketch of the proof. Motivations for theorems and definitions are given. In most cases, the authors, choose the historical perspective for treating the subject.

Let us remind that the authors give definitions of notions which are basic in an appropriate field of mathematics or physics (range from homological groups to compact operators, manifolds and vector bundles), but the majority of these terms is not common for most experts working in the individual parts. Putting all these deep themes together without disrupting the integrity of the text, makes this book exceptional among existing text-books on this topic.

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