



BOOK REVIEW

The Geometry of the Octonions, by Tevian Dray and Corinne A. Manogue, World Scientific, Singapore 2015, xviii+210 pp., ISBN 978-9814401814.

The book *The Geometry of the Octonions* is aimed on the one hand to shed light on the general construction and properties of the largest normed division algebra \mathbb{O} and on the other, to reveal some more subtle connections to other fields of mathematics and physics that have been discovered only recently. Therefore, a part of the exposition resorts on earlier results, while a large portion is based exclusively on the authors' original contribution. The book consists of fifteen chapters organized in three parts.

The first part begins at the very basic level of an undergraduate science student with a brief, but well organized revision on complex numbers and quaternions along with interesting historical remarks. Then, octonions are introduced with their basic properties by means of the Cayley-Dickson process and the split algebras \mathbb{C}' , \mathbb{H}' and \mathbb{O}' are explained as well. Not mentioning the notion of Clifford algebras in this introductory part is a bit of a disappointment - such a reference might have been helpful for the unexperienced reader to grasp the big picture. The book, however, makes up for this by offering an abundance of explicit constructions that are valuable for both beginners and experts, who often tend to underestimate such an approach in the pursuit of abstract ideas and connections.

The second part focuses on Lie groups and algebras and in the authors' view forms "the heart of the book". The content in the beginning is quite standard and might easily be skipped by the specialists, but for the unexperienced reader it is an important link between the basics from the first part and the more advanced constructions to follow. The semi-simple Lie groups and algebras over \mathbb{C} are generalized to \mathbb{H} and \mathbb{O} along with their dual split versions and are thus viewed from a whole new perspective. This reveals a series of interesting relations for the corresponding real forms and in particular some symplectic, spin and conformal groups, which are regularly used in physical models. The exceptional groups and algebras apparently also fit in this construction and their consistent and rather explicit introduction at the end of this part might be considered as a great merit of the book (cf. [1]).

For example, G_2 is constructed as the group of automorphisms of \mathbb{O} . Similarly, F_4 is obtained as the group of special unitary 3×3 octonionic matrices $SU(3, \mathbb{O})$ and E_6 - as the special linear group $SL(3, \mathbb{O})$, with the proper definition. For $E_7 \cong Sp(6, \mathbb{O})$ the authors use a construction based on the Albert algebra and the largest of the exceptional groups E_8 is discussed only briefly.

The third part treats more advanced topics concerning the applications of octonions in various branches of nowadays physics and mathematics. Much of the exposition is based on original research by the authors, (see [2, 3]), while the rest is a compilation of earlier results, such as presented in [5]. Starting with standard constructions, like Hopf bundles and Möbius groups, the authors gradually build an alternative perspective on more complicated topics concerning modern physics. The non-associativity issue is dealt with in an elegant way (via nesting) and some basic constructions over the complex numbers are generalized for octonions, e.g. Gaussian and Hurwitz integers, spinors, elements of projective geometry and the eigenvalue problem. A special section is dedicated to applications in particle physics.

After introducing explicitly the quaternionic spin and the gamma matrices, the authors investigate thoroughly the Dirac equation in octonionic terms. The properties of Dirac, Weyl and Cayley spinors are discussed in this context, the 3- Ψ rule is rewritten as a single associator and the Jordan formulation of quantum mechanics is seen in a new light. This part of the book contains some fascinating from a physical point of view results - in particular, some properties of leptons, viewed from the perspective of Lie algebra representations, seem to receive more natural explanation via this new approach, that leave the reader with the impression that namely octonions might be the key to a unified algebraic description of elementary particles, gauge fields and strings. Finally, as if to provide more evidence in favor of this point, the authors choose to discuss magic squares, thus labeling some of the most applicable to physics spin and conformal groups with the four normed division algebras and their dual split versions. This elegant construction appears to be quite suitable for a conclusion, as it reflects perfectly the overall spirit of the exposition.

In summary, the book offers an interesting compilation of an approachable introductory level review on division algebras and their connection to Lie groups on the one hand, and on the other - some more advanced physical and geometrical applications and interpretations of octonions. The exposition seems to lack mathematical rigor in certain definitions and proofs, constructions that ought to be appreciated by both physicists and mathematicians at different levels and areas of research. Therefore, this short monograph seems to be equally appropriate for inquisitive science students and specialists eager to update their knowledge on

the matter and enjoy a different point of view on some of the most fundamental constructions in mathematics. This category of readers will benefit definitely by consulting also the books (not included in the Bibliography) by Yaglom [7], Kantor and Solodovnikov [4] and Ward [6].

References

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