



BOOK REVIEW

Lie Groups: A Problem-Oriented Introduction via Matrix Groups, by Harriet Pollatsek, The Mathematical Association of America, USA, 2009, ix + 177 pp., ISBN 978-0-88385-759-5.

The author of the book Harriet Pollatsek is a mathematics professor at Mount Holyoke college who has played an important role for young women in the last three decades. She enjoys helping her students to love mathematics and is known to “particularly relish the challenge of changing the opinion of someone who is sure she doesn’t like mathematics.” In 2007 she was awarded the Faculty prize as “that rarest of teachers who excels in virtually every pedagogy invented: lecturing, small groups, laboratory guidance, mentoring.”

The book is distinctive by the low prerequisites (multi-variable calculus and linear algebra) and the development of about 200 problems. It can be used as a supplementary text in linear algebra or as an active reading that helps students make the transition to upper level courses based on differential geometry and topology.

The theory of the Norwegian mathematician Sophus Lie explores the beauty of ideas of symmetry which leads to various applications in mathematics and physics. Conventionally, the study and understanding of Lie theory (Lie groups and Lie algebras) requires deep mathematical knowledge in differential geometry.

This book starts with matrix groups [1]. It aims at making the basics of Lie theory accessible [2] to students who have taken multivariable calculus and linear algebra. It is concentrated on algebraic ideas with enough analysis to define the tangent space and the differential in order to make sense of the exponential map. Topology is excluded except for reference to connected and simply connected sets and closed sets. Quotient structures are also excluded. Groups of Lie type over finite fields are also briefly discussed.

This book aims at teaching matrix groups through very little lecturing but with a lot of working on problems by presenting and discussing their solutions. Throughout the book the point of view of symmetry principles is emphasized and the way mathematics provides a language to understand the world of physics. Since

much of the material is scattered among problems every chapter has a section “Putting the pieces together” where all definitions and results are collected. More advanced theorems are given in this section as well.

A strategy from the book *Introduction to Affine Groups Schemes* by William Waterhouse [3] is adopted which is the way to put the text material in a larger context. At the end of every chapter there is a section called “A broader view” with context which is sometimes mathematical and sometimes historical or both.

The book consists of Introduction, six Chapters, three Appendices, Bibliography.

Chapter 1. Symmetries of Vector Spaces

It begins with understanding symmetries of vector spaces. There is a remark on p.5 explaining that when Sophus Lee “began to develop the ideas that bear his name” he “was actually interested in symmetries of differential equations”. The author accepts the point of view from the classic text *Spacetime Physics* by Taylor and Wheeler [4] which is an introduction to special relativity and puts an emphasis on the fact that the distance is fundamental. The definition of a group of symmetries uses the language of matrices and the language of linear transformations. It connects the ideas of preserving the distance and preserving the dot product.

Chapter 2. Complex Numbers, Quaternions and Geometry

The chapter introduces two number systems that are important for the study of Lie groups, the complex numbers and the quaternions. The complex numbers give another view of the Euclidean geometry and symmetries of \mathbb{R}^2 and the quaternions give another view of the geometry and symmetries of \mathbb{R}^3 .

Chapter 3. Linearization

In this chapter a tangent space for a matrix group is defined. A matrix group is a complicated structure, but its tangent space is a vector space, which is a much simpler mathematical structure. The definition of the tangent space and the differential is motivated with examples from calculus and helps the students to make sense of the exponential map.

Chapter 4. One-parameter Subgroups and the Exponential Map

The definition of the tangent space of a Lie group in the previous chapter has been based on smooth curves passing through the identity. In this section the general procedure for creating such curves is developed. The one-parameter subgroups are introduced. The exponential map is the source of all the curves that are needed to define the tangent space. The key tool for the study of one parameter subgroups is the exponential function. This section also includes instructions how to use Maple for calculating matrix exponential. The relation to linear homogeneous systems of

differential equations is considered as their solutions are given by the exponential function for matrices.

Chapter 5. Lie Algebras

The chapter begins with the observation that the groups $SO(n, \mathbb{R})$ and $O(n, \mathbb{R})$ have the same tangent space, followed by a problem on the $SL(2, \mathbb{R})$ and an isomorphic 3×3 matrix group. The fact that different groups have tangent space that are isomorphic as vector spaces is a motivation to enrich the structure of the tangent space. This is done by adding an operation, the vector product which turns the vector space into a Lie algebra. A lot of problems are considered, and the adjoint map is introduced and illustrated through many example as well.

Chapter 6. Matrix Groups and Other Fields

The chapter begins with the definition of a field. The unitary group is studied in detail which illustrates all the points of the Lie theory approach developed in the previous sections. Matrix groups over finite field are discussed in some detail. In the broader view a section about the Chevalley basis and the associated Dynkin diagrams are introduced. The classification scheme for simple groups is briefly described.

Appendix I. Linear Algebra Facts

Here the basic facts from linear algebra are listed which the students should know in order to be able to follow the text.

Appendix II. Paper Assignments

Possible topics for paper assignment are listed.

Appendix III. Opportunities for Further Study

The appendix extends the suggestions [5–7] for further reading at the end of each chapter. Such are: metric affine geometry, Chevalley groups, octonions and relation to physics, Lie groups as manifolds.

Throughout the book the presentation is via problems which is a good way for students to gain knowledge about Lie theory. Most of the problems need to be treated with imagination and precise thinking. At the end of the book solutions to selected problems are worked out.

This book is an excellent reading for students and young researchers but also for physicists who need refresh their knowledge of Lie theory.

References

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