



BOOK REVIEW

Topics in Physical Mathematics, by Kishore Marathe, Springer, London • Dordrecht • Heidelberg • New York 2010, xxii+442 pp., ISBN-13 978-1-84882-938-1.

Modern natural science gradually understands that it would be wrong to consider Mathematics as something abstract and away from reality, moreover, step by step we understand that mathematics is a part of our knowledge for physical reality, its various parts and directions of development correspond to the various appearances of the physical reality and to the various humans' levels of knowledge about this sometimes awful, sometimes wonderful, but always challenging world. Two basic concepts of mathematics are “element(s)” and “mapping(s)”, and these two concepts express how elements of some kind exist and survive among the elements of other kinds, express, we could say, the harmony of existence and surviving through appropriate changes. It is true that mathematics does not allow an element of a set to be destroyed as physics allows, but it allows families of elements to be built such that we can follow the evolution of the element we have started from to some new element we have come to at the end of the evolution parameter. Interpreting some real process as time parameter we can describe physical processes from the elementary mechanical motion of particles to the complicated internal dynamics of a composite time-stable system and its propagation in space as a whole. In this sense the logical harmony among mathematical concepts represents the harmony of existence among the natural objects, it explains our understanding of the real coexistence, consistency and compatibility of properties.

One of the greatest achievements in this sense is the development of intrinsic consistency and compatibility between algebraic and topological mathematical structures. Clearly, this development is of great importance for physics since algebraic structures allow to identify the subsystems of the time-stable system considered and the topological and differentiable structures allow to follow their time development through physical interaction, i.e. through energy-momentum exchange. Physicists and other natural scientists make discoveries through direct observations, mathematicians help in obtaining appropriate mathematical proficiency, and all this results finally in what we call mathematical physics or physical mathematics, i.e., that part of mathematics, the concepts and theorems of which can serve as logical image of the direct knowledge obtained through observation.

In my view the book under review is totally devoted to represent as far as possible and in the most receptive way the deep truth of this idea, representing most appropriately that part of mathematics which can serve as a logical mirror of our knowledge about the real world. The book is built not following the traditional mathematical way: definitions, theorems, corollaries, examples, it is rather a journey, a trip, a tour through the mathematical world. There is always a signal physical light somewhere ahead in the distance, and the traveller wriggles through this jungle of logical possibilities, trying not to lose the sometimes shimmering, the sometimes steady, light at the always existing end of travel. And from this point of view we have to recognize and highly appreciate the real success of the author.

Let's now follow briefly the author's road through physical mathematics.

The book is parted into eleven chapters, named as follows: Algebra, Topology, Manifolds, Bundles and Connections, Characteristic Classes, Theory of Fields I: Classical, Theory of Fields II: Quantum and Topological, Yang-Mills-Higgs Fields, 4-Manifold Invariants, 3-Manifold Invariants, Knot and Link Invariants. These are followed by an "Epilogue", Four "Appendices", References (419), and an Index.

The Chapter *Algebra* introduces and briefly discusses the following basic concepts: derivation, homomorphisms of algebras, Lie algebras with the Cartan criterion for semi-simpleness, roots, weights and $\mathfrak{sl}(2, \mathbb{C})$ -example. Then Graded, Kac-Moody, Virasoro, Heisenberg algebras are introduced. More attention is paid to Clifford algebras on Euclidean and pseudo-euclidean vector spaces. Hopf Algebras and Quantum Groups are mentioned, finally, Modular groups and Moonshine conjectures are briefly discussed. The author has not included discussion of graded derivations.

The Chapter *Topology* begins with point set topology and metric spaces. Then homotopy groups are introduced (through paths), covering spaces, covering projections, connectedness are discussed. The examples are mainly connected with spheres (Bott periodicity). Then homology and cohomology of chains is considered, also de Rham cohomology and the corresponding Hodge-de Rham operator and Hodge decomposition theorem are defined. The chapter ends with quite brief considerations of the topologies of 2,3,4-topological manifolds together with examples: the Hopf and the Kervaire invariants. There are historical remarks and corresponding references of course.

The Chapter *Manifolds* contains the standard material for smooth atlases, tangent and cotangent tensors, differential forms, Lie derivative and exterior derivative, Stokes theorem for manifolds with boundaries. Pseudo-Riemannian manifolds are introduced, the Hodge-* operator, the coderivative are presented. Symplectic manifolds with corresponding Hamilton equations and Poisson structures are mentioned. The chapter ends with Lie group manifolds and Lie group action on itself and on other manifolds.

The next Chapter *Bundles and Connections* is more or less standard: principal bundles, bundles of frames on a manifold, associated vector bundles, principal connection in terms of horizontal subbundle, or in terms of connection form, exterior covariant derivative and corresponding curvature. Linear connections are defined through the bundle of frames. The interesting from physical point of view Example 4.3 explains the concept of spin structure and spin manifold.

Chapter 5 *Characteristic Classes* explains the concept of a characteristic class of a fibre bundle: Steifel-Whitney, Pontryagin and Chern classes, showing their topological nature and criterion significance for nontriviality of the bundle considered. The author introduces the symmetric algebra on the corresponding Lie algebra, points out the isomorphisms with symmetric functions and homogeneous polynomials and comes to the Weil-Chern homomorphism. Example 5.3 specifies these concepts for $GL(n, \mathbb{R})$ making use of the curvature two-form for a given connection. The $SU(2)$ bundle is considered and the two Chern classes are explicitly written (p.151). The topological charge for compact base manifold is defined also explicitly on the next page. The chapter ends with notes on secondary characteristic classes, K -theory, Atiyah-Singer theorems and Hirzebruch signature theorem.

The next two Chapters: 6 and 7, are devoted to mathematical considerations connected with classical fields, and their quantum and topological aspects. The main concept here is *gauge field*, as the mathematical concept of connection is known and widely used by physicists, rather the projection of a connection form on a principal bundle onto the base manifold by means of a local section. Gauge transformations, the affine structure of the space of gauge potentials and the Dirac monopole are explained. Matter fields are defined as sections of associated vector bundles, curvature lagrangians and instanton numbers, conformal invariance and Yang-Mills action are introduced. Also, curvature product of two symmetric tensors is defined and Einstein gravitation equations are given in these terms. Some quantum and topological aspects of field theory are discussed in Chapter 7. These aspects are specified in the concepts of gauge invariance, instanton number (on base spaces with positive metrics), Wilson loop functional, Donaldson invariants.

Chapter 8 discusses in a more detail the Yang-Mills-Higgs fields. The corresponding gauge formulation of electrodynamics is given and Aharonov-Bohm effect is geometrically considered. The general pure Yang-Mills field is considered and the widely used notions for self-duality and antiself-duality are clarified, but the null standard energy-momentum tensor for these gauge fields is not commented. The concept of spontaneous symmetry breaking is well explained. The chapter gives also some mathematical aspects of the electroweak theory, the so-called standard model and invariant connections.

The last three chapters are devoted to some more sophisticated from mathematical point of view aspects of topological field theory that has been developed during the

last two-three decades. These include the geometry and topology of various Moduli spaces (for instantons, complex structures, etc.). The explanations make use of the introduced and partially studied Donaldson polynomials and their invariants, the Zeiberg-Witten invariants and their possible relation to the Donaldson invariants, Chern-Simons theory, Floer homology. The last Chapter 11 deals mainly with comments on possible usage of the older concepts for Knot and Link invariants in trying to find correspondence with some basic characteristics and properties of physical objects having complicated dynamical structure.

If the author plans to prepare a revised edition of the book, I would allow myself to recommend the concept of curvature to be introduced, illustrated and commented as a measure of nonintegrability of distributions on an *arbitrary* manifold, since physical time-stability and spatial propagation are naturally represented formally by integrability and local symmetries of distributions, and internal interaction among recognizable subsystems is naturally represented by the curvature forms of corresponding nonintegrable subdistributions.

For those readers, who would prefer rigorous mathematical text I would recommend the three-volume monograph [1].

In conclusion, I truly believe that reading this book will bring a real pleasure to all mathematically inclined Young physicists having the necessary mathematical luggage in their minds. Moreover, passing successfully through it will bring the reader to feel the real correspondence between the real harmony in the physical world and the remarkable mathematical harmony among the various algebraic, geometric and topological structures.

References

- [1] Greub W., Halperin S. and Vanstone R., *Connections, Curvature, and Cohomology*, vol. I(1972), vol. II(1973), vol. III(1976), Academic Press, New York.

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