



BOOK REVIEW

Topology and Geometry for Physics, by Helmut Eschrig, Springer, Heidelberg 2011, xii +390 pp, ISBN 978-3-642-14699-2.

There are so many books in mathematical physics and ones devoted particularly to widely understood mathematical methods of theoretical physics that every new edition is a challenge and should be somehow explained, not to say justified. Fortunately, the book by Helmut Eschrig passes the test and its author expresses in detail and correctly its novelty and characteristic features. This is done both verbally in Preface, and proved in every chapter just by the way of presentation. He claims that majority of texts written by mathematicians is rather non-accessible to physicists because of their strictly mathematical, deductive style of presentation and also because of their relatively far distance from physical applications, even if some physical aspects are mentioned. Being theoretical physicist, the author presents some intermediate approach, avoiding the deductive style based on a sequence of theorems, as it is usually the case in books written by professional mathematicians. He claims, correctly, that his approach is nevertheless rigorous from the mathematical point of view, although with rather sketched proofs and with the main stress on physical applications. This style of presentation is suited to mathematical abilities of theoretical physicists which, in spite of their more or less familiarity with topics like Lie groups theory, tensor analysis and variational calculus etc., still have some fundamental problems with more abstract, non-computational things. In author's opinion this fact proves why it was almost sixty years before they understood that not every physical observable may be expressed as an operator in Hilbert space (the Berry phase problem).

What concerns the book structure, it consists of nine main chapters, one very short preliminary section with basic notations (and just so entitled) and the final Compendium which summarizes the volume in thirty two pages the basic geometric concepts and facts used in the book.

Chapter 1 is very short and devoted mainly to some sketch-like introduction of basic topological concepts like the global and local structure of the Cartesian product, metric spaces, topological spaces, and the basis of mathematical analysis. This is thought on as a preliminary study to further sections. The author stresses that unlike the standard ideas concerning differentiation and integration as in a sense

mutually reciprocal operations, there is a subtle difference, because differentiation in linear spaces needs only affine concepts, while integration is based on the definition of measure. Nevertheless, as author claims, the integration of differential forms in manifold is a coordinate-independent operation free of any a priori measure concepts, therefore, in a sense the standard rule of mutual reciprocity still exists.

The Chapter 2, “Topology”, is relatively long and introduces all basic topological concepts based on the standard approach using open subsets as a primary family of subsets subject to the standard rules. It is shown how the usual concepts of continuity of mappings, base of the topology, metric, norm, differentiation in topological linear spaces etc are introduced. The special and very important case of compactness is deeply discussed. Besides of the usual concepts of the point-set topology also some elementary concepts usually discussed within the area of algebraic topology are introduced. As usual in his book, the author makes some comments concerning physical applications, e.g., ones in quantum mechanics and topological charges in physics. Also the general idea of the fixed point method in solving nonlinear equations is presented.

The Chapter 3, “Manifolds”, introduces in a standard way all basic concepts of the geometry of manifolds like vector fields, differential forms, Frobenius theorem and related geometric problems. Being faithful to his main geometric-physical credo, the author introduces here interesting physical examples of applications, mainly in Section 3.7 “Examples from Physics”. This contains some material from the classical and quantum mechanics. For example, discussed is the interesting problem of degenerate Lagrange mechanics with the vanishing velocity-Hessian of the Lagrange function. It is well-known that the usual Hamiltonian formalism does not work then and one has to modify it according to certain ideas invented many years ago by Dirac. The author presents the geometric formulation of the Dirac procedure. The chapter is finished with a brief discussion of non-holonomic constraints in mechanics. The procedure is based on the Lagrange formalism, so one might perhaps to regret that no comments were given to so-called “vakonomy”. It is still not very clear to everybody that there is some conceptual discontinuity between holonomic and non-holonomic constraints. In the holonomic case the usual d’Alembert procedure and variational formulation based on the conditional extremum of action give identical results. When dealing with non-holonomic constraints, it is no longer the case. The author seems to be aware of the problem, namely when he tells that the theory of non-holonomic constraints “comprises a huge in large parts unexplored field”.

Let me mention some interesting point I was faced with in the subsection 3.7.1. I had a similar problem in my own investigations of Hamiltonian mechanics. Namely,

the author at some moment gives up from the co-vector notation of canonical momentum and introduces the unique, so-to-speak “contravariant” notation for velocities and momenta. Indeed, there are some difficulties when working in the phase-space manifold, where coordinates and canonical momenta are basic coordinates and their time derivatives form contravariant vectors in the symplectic manifold. There is no place to discuss this, in a sense, apparent failure of the kernel-index convention of tensor analysis. I preferred to preserve the covariant notation for covectors, but I am aware of certain difficulties resulting from this.

Chapter 4 “Tensor Fields” introduces in a concise geometric way all basic concepts of the tensor algebra, if I well understood, for finite-dimensional linear spaces. This is applied then to tensor fields. Besides, the exterior algebra is introduced with all its internal operations, including differential ones, independent of the concept of affine connection. The symplectic formulation of Hamiltonian mechanics is quoted as a physical example.

Chapter 5 “Integration, Homology and Cohomology” comprises all the basic concepts of integration of differential forms. This includes the basic facts which belong to algebraic topology. As an application the theory of critical points is discussed. Those are problems which are important for the catastrophe theory. As a physical example the very important for statistical mechanics Liouville measure on the phase space is discussed. Besides, the structure of Maxwell equations is discussed in detail. Relatively unexpected application concerns Fermi surfaces in solid state physics.

Chapter 6 “Lie Groups” contains a modern exposition of the theory of Lie groups and Lie algebras. It contains as an important physical application the review of all basic subgroups of the complete linear group, like the special linear, orthogonal, unitary and symplectic groups. As an important physical example the Lorentz group and its covering $SL(2, \mathbb{C})$ are discussed in more detail.

Chapter 7 “Bundles and Connections” presents a standard approach to principal and associate fiber bundles and to connection forms. The concept of parallel transport, holonomy and exterior covariant differentiation are concisely introduced and expressed in analytical terms. In particular, the traditional analytical concepts concerning affine connection, covariant differentiation and Bianchi identities are presented in detail. It is shown that they really are rather far, secondary expressions of geometric rules correctly expressed in terms of principal fiber bundles.

Chapter 8 “Parallelism, Holonomy, Homotopy and (Co)homology” plays in a sense a central role in authors plans of expressing fundamental physical facts interesting him in terms of geometry. In a sense, it is a direct application of the formerly introduced concepts. The discussed applications comprise the gauge fields on the four-dimensional space-time, like electromagnetism, more general connection fields, Belavin-Polyakov instanton. The detailed analysis of characteristic classes

is given. But the main point is the discussion of the section 8.6 “Geometric Phases in Quantum physics”. This discussion contains the Berry phase, Berry-Simon connection, electrical polarization and orbital magnetism. It is shown that for all those concepts and phenomena the basic tools are just ones elaborated in the theory of connections in fiber bundles over the space-time manifold.

The last Chapter 9 “Riemannian Geometry” is slightly different in character from the previous ones. It presents Riemannian and pseudo-Riemannian geometry in a more traditional way, although the section 9.7 “Complex, Hermitian and Kahlerian Manifolds” resembles more the style of former chapters. The main idea of the author in this chapter is to present the basic ideas of gravitation theory, more recisely, of the Einstein-Hilbert theory of relativity, on the basis of pseudo-Riemannian geometry.

Summarizing, I would say that the author really has achieved his purpose, namely he wrote a book rigorous in a mathematical sense, nevertheless not over-loaded by the physically non-necessary proofs and mathematical details, and at the same time giving to the reader a rigorous mathematical explanation of physical phenomena, often quite recently discovered and not commonly known.

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