



BOOK REVIEW

Integrable Systems in Celestial Mechanics, by Diarmuid Ó Mathúna, Birkhäuser, Boston – Basel – Berlin 2008, x + 234 pp., ISBN 978-0-8176-4096-5

Many modern books on mechanics are perceived as excessively abstract by the reader, which has the disagreeable impression of an awesome machine able, at most, to integrate exactly the harmonic oscillator. To this regard, the book under review is decisively in countertendency, giving the complete analytical solution of three important systems, relevant in Celestial Mechanics. They are the following.

1. The Kepler problem, i.e., the dynamical problem for the gravitational field of a fixed mass.
2. The Euler problem, where the gravitational field is generated by *two* fixed and in general different masses.
3. The Vinti problem, regarding an integrable potential which gives an excellent approximation to the gravitational field of an oblate planet, for example the Earth.

The book is organized as follows.

Chapter 0 is a general introduction and treats mainly the historical development of the three problems. Particular emphasis is given to the second one, starting from Euler itself (which mentions that the problem had already attracted the attention of some of the greatest analysts of his time, but without success) and Lagrange. They were the first mathematicians able to reduce the problem to the quadrature, recognizing the need to resort to the elliptic functions (whose theory was at the early stage) for the explicit integration. For a simple but modern introduction to the elliptic functions (lacking in the book) see for example [1].

Chapter 1 recalls briefly the Lagrange equations, ignorable coordinates and relative first integrals, separable systems and Liouville form. Notice that further

knowledge on analytical mechanics (such as Hamilton equations, canonical transformations or Hamilton–Jacobi theory) is not required, all the subsequent computations being given at elementary level (provided that elliptic functions are considered elementary matter).

Chapter 2 deals with the Kepler problem, whose solution is given in a form suitable for handling with the Euler problem: indeed, the solution of this latter will clearly result in a generalization of the form of the solution in the Kepler case. The position of the moving point is parametrized by the spherical coordinates r, ϑ, φ (i.e., radial distance, colatitude and longitude) which are expressed as a function of a regularizing parameter f (the true anomaly in this case or a suitable generalization for the other cases). To complete the solution one should give f as a function of the time, but it is not possible to do this in a closed form (it would be equivalent to the inversion of the Kepler equation), so that the inverse, the so-called “time–angle” relation is deduced. For a more general study of the Kepler problem see [2].

Chapter 3, which is the central chapter of the book, gives a full analysis of the planar Euler problem, yielding an exact solution form in terms of the Jacobian elliptic functions $\operatorname{sn}, \operatorname{cn}, \operatorname{dn}$, complemented by the “time–angle” relation. We recall that these last functions are a sort of generalized circular functions, to which they reduce when one of the two arguments vanishes. (To appreciate the previous status of the problem see [3].) The starting point is the choice of the (planar prolate) spheroidal coordinate R, σ in the vertical plane XOZ

$$x = r \sin \vartheta = \pm \sqrt{R^2 - b^2} \sin \sigma, \quad z = r \cos \vartheta = R \cos \sigma$$

which convey in the polar coordinates when b , the semidistance between the two attracting masses, vanishes. The potential function of the two attracting masses m_1, m_2 is

$$U_{\text{Euler}} = \mu \frac{R + \beta b \cos \sigma}{R^2 - b^2 \cos^2 \sigma}, \quad \mu = G(m_1 + m_2), \quad \beta = \frac{m_1 - m_2}{m_1 + m_2}$$

and the corresponding equations result separable. After a long and detailed analysis one reaches the conclusion that in the bounded motion case there are four types of orbits: 1) about the two fixed masses but without to intersect the line segment joining them; 2) as before but with intersection, the orbit resembling a eight; 3) about m_1 ; 4) about m_2 . In all four cases the orbit is contained in a portion of the plane bounded by arcs of confocal ellipses and hyperbolas, whose foci are placed at the two attracting masses.

Chapter 4 deals with the Euler problem in the three–dimensional context. It is shown that, for two of the coordinates, the solution can be reduced by algebraic

manipulation to the planar case. The formula for the third coordinate, involving an elliptic integral of the third kind, is shown to be amenable to an approximate procedure.

Chapter 5 deals with the Vinti potential whose definition and series development in terms of the Legendre polynomials P_k are respectively

$$U_{\text{Vinti}} = \mu \frac{R - \beta b \cos \sigma}{R^2 + b^2 \cos^2 \sigma}$$

and

$$U_{\text{Vinti}} = \frac{\mu}{r} \left[\sum_{k=0}^{\infty} \left(-\frac{b^2}{r^2} \right)^k P_{2k}(\cos \vartheta) - \beta \frac{b}{r} \sum_{k=0}^{\infty} \left(-\frac{b^2}{r^2} \right)^k P_{2k+1}(\cos \vartheta) \right]$$

with μ, β, b adjustable parameters. Clearly, the two replacements $b \rightarrow ib, \beta \rightarrow i\beta$ intertwine U_{Vinti} and U_{Euler} , while the representation of the gravitational potential of an oblate axisymmetric planet (J_n are constants depending on the shape of the planet and r_0 the mean radius)

$$U_{\text{planet}} = \frac{\mu}{r} \left[1 - \sum_{n=1}^{\infty} J_n \left(\frac{r_0}{r} \right)^n P_n(\cos \vartheta) \right]$$

shows that a judicious choice of the parameters can match the two potentials U_{Vinti} and U_{planet} up to the third harmonic. Then, the solution of the dynamical problem relative to the Vinti potential is pursued along the same lines of the previous chapters. An introduction to Vinti's theory can be found in [4].

Chapter 6 deals with certain orbits of the Vinti problem that require or merit special attention, i.e., equatorial orbits, planar orbits and orbits near the so-called "critical inclination". More informations on this argument can be found in [5] and [6].

Lastly, in the Appendix are presented graphs of some orbits based on the analytic results previously derived, primarily in Chapter 3. The work is done by exploiting the capability of MAPLE to treat and plot elliptic functions. Notice however that the interested reader can find, in the CD accompanying the book [2], a complete program which is able to plot the orbit relative to a generic initial condition.

Concluding, the book under review is surely an exhaustive and authoritative reference on the argument and, moreover, may give useful hints in dealing with other algebraically integrable systems. It is addressed mainly to researchers in analytical mechanics but, not requiring deep mathematical knowledge, can be used also by graduate students.

References

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