



## BOOK REVIEW

*Lie Groups, Physics, and Geometry*, by Robert Gilmore, Cambridge University Press, Cambridge, 2008, ix + 319pp., ISBN 978-0-521-88400-6.

This is the second book on the Lie groups and Lie algebras by R. Gilmore who is a well-known expert in this field. It is aimed as an introduction to the subject for physicists, engineers and chemists and was intended as a shrunk and appropriately restructured revision of his first book [1] with some accents on the most useful for physicists (and students) aspects, avoiding too technical developments.

In the first chapter the author presents the elements of the Galois theory by describing how the structure of the discrete symmetry group of a polynomial equation determines whether the equation is solvable in radicals and notes that Galois theory motivated Marius Sophus Lie to develop analogous theory for the ordinary differential equations.

In the second chapter the basic algebraic and topological properties of Lie groups are listed and illustrated via the unimodular group  $SL(2, \mathbb{R})$ . The third chapter is devoted to the matrix groups  $GL(n, \mathbb{F})$ , where  $\mathbb{F}$  could be any of the field of real/complex numbers or quaternions and their various subgroups. The author underlines that most of the Lie groups used in applied mathematics and physics are matrix groups, matrix groups over finite fields being however out of the scope of the book.

Chapters 4 and 5 are devoted to the study of Lie algebras. The discussion begins with the explanation that the Lie algebras are useful as they linearize the Lie group in the neighborhood of any of its point. The study of the Lie algebra is easier since it is a linear vector space and one can use all available standard tools for such spaces. It is underlined also that the Lie algebra of a given group retains most but not all of its properties. The inverse operation, called the EXPonentiation, maps (locally) the Lie algebra to the group manifold and parameterizes the Lie group. Basic structure properties and notions of Lie algebras are introduced, such as structure constants, Hilbert-Schmidt and Cartan-Killing inner products and the related invariant metric and measure on the group manifold. The group  $SL(2, \mathbb{R})$  is

again considered as an illustration. In the fifth chapter the Lie algebras of several matrix groups, described previously in Chapter 3, are constructed by linearizing the corresponding constraints defining the group.

Chapter 6 concerns the ladder representations of matrix Lie algebras on the examples of (finite number of) bosonic and fermionic annihilation and creation operators and the first order differential operators  $x_i \partial_j$ ,  $i, j = 1, 2, \dots, n$ . Ladder representations, first introduced by Schwinger in the case of angular momentum operators, are of great use in quantum mechanics and quantum optics. Several explicit examples of boson operator algebras are included at the end of the chapter as problems. These include angular momentum operators and related orthonormalized states, boson oscillator Hamiltonian and related Fock states. In particular here it is pointed out that the linear boson canonical transformations belong to concern Lie group  $\text{Sp}(2n, \mathbb{R})$ . Here I however miss the harmonic oscillator coherent states and symmetric problems with the fermion operator algebras and the group  $\text{SO}(2n, \mathbb{R})$  that preserves the fermionic (anti)commutation relations.

In Chapter 7 the problem of EXPonential, introduced briefly in Chapter 4, is considered in greater detail. At the very beginning the answers to the three fundamental questions about the EXPonential mapping, raised in Chapter 4 are presented. First on verbal level shortly, and then in more technical way. These questions and their answers in fact summarize in a nice pedagogical manner the material developed in this and preceding chapters.

Important physical applications of the EXPonential mapping are considered in the Section 7.5. These include the dynamic evolution of quantum systems and their thermal expectation values. The time evolutions governed by the Hamiltonians which are linear combinations of spin, boson ladder and number operators are computed as examples.

Chapter 8 begins with a systematical study of the Lie algebra structure. According to Levy-Maltsev theorem any Lie algebra  $\mathfrak{g}$  admits decomposition as a semidirect sum of a semisimple Lie algebra and a radical in  $\mathfrak{g}$ . Here the semisimple part and the maximal solvable invariant subalgebra are determined by inspection of the regular representation, which is brought to a 'suitable canonical form' by a similarity transformation. On the way it is shown how to use Cartan-Killing inner product to determine which subalgebras are nilpotent, solvable or semisimple, compact or noncompact. The decomposition of the one mode two-photon algebra (important in the field of squeezed states in quantum optics) is performed as an example in Section 8.5, where the semisimple part is given by the  $\mathfrak{su}(1, 1)$  algebra (spanned by the number operator and the two squared ladder operators) and the maximal solvable invariant subalgebra is the nilpotent Heisenberg-Weyl algebra spanned

the two photon ladder operators and the identity. Several physically interesting problems related to boson and fermion ladder operator algebras are formulated in the last section.

Chapters 9 and 10 deal with the structure theory of the complex semisimple Lie algebras. In these chapters the author uses power tool of the eigenoperator decomposition (of any Lie algebra), which is effected through the secular equation determined from the matrix in the regular representation of the Lie algebra. The ‘eigenoperator equation’ for two Lie algebra elements  $X$  and  $Z$  reads  $[Z, X] = \lambda X$ . With  $X = a^i X_i$ ,  $Z = z^i X_i$  and  $[Z, X_i] = R(Z)_i^j X_j$ ,  $i = 1, 2, \dots, N$ , one gets a system of equations for  $a^i$  by expanding the secular equation  $\det(R(Z) - \lambda I_N) = 0$ . The coefficients  $\phi_j(z^i)$  in front of  $\lambda^j$  provide under replacements  $z^i \rightarrow X_i$  the invariant operators (Casimir operators  $\phi_j(X_i)$ ), and the number of the roots of the secular equation equals the rank of the semisimple algebra. The properties of the roots are listed and explained. Graphics of the root system are provided, which help to understand the general situation and the classification of all possible Dynkin diagrams. The author notes that more complete treatment can be found elsewhere [1, 2].

The real forms of complex simple Lie algebras are studied in Chapter 11. In Section 11.3 the Cartan procedure for the construction of the real forms is described and illustrated by taking as an example the  $\mathfrak{su}(2)$  Lie algebra. It is shown that real forms of any simple classical matrix Lie algebra can be obtained by systematically relying on a mapping  $T$  of matrices into themselves which obeys to  $T^2 = 1$  and three different procedures - block submatrix decomposition, subfield restriction and field embedding. The results are summarized respectively in Tables 11.1 and 11.2. For completeness the real forms of the five exceptional Lie algebras are also presented. More than 20 problems revealing further details of the subject are formulated at the end of the chapter.

In Chapter 12 Riemannian symmetric spaces are reviewed exploiting the geometrical role of the EXP mapping, i.e., by considering the homogeneous spaces  $G/H = \text{EXP}(\mathfrak{p})$  and  $G'/H = \text{EXP}(\mathfrak{ip})$ , generated by the parabolic part  $\mathfrak{p}$  in the Cartan decomposition of the Lie algebra. The Cartan-Killing metric in these cases is negative, respectively positive definite, and thus  $\text{EXP}(\mathfrak{p})$  respectively  $G'/H = \text{EXP}(\mathfrak{ip})$  is globally symmetric Riemannian space. The Christoffel symbols and the curvature tensors of such compact spaces are computed in Section 12.6. Similarly, the procedure for construction of pseudo-Riemannian spaces, associated to the noncompact Lie algebras is presented in Section 12.7. Further details of the geometric properties of the symmetric Riemannian spaces are provided in the form of problems (Section 12.9). (Two misprints were noted: 1)  $G/K$  in

the first line of 12.8 evidently should be replaced by  $\text{EXP}(\mathfrak{p})$ ; 2) The string “metric tensor  $R_{\mu\sigma,\alpha\beta}$ ” between equations (12.15) and (12.16) should read “curvature tensor  $R_{\mu\sigma,\alpha\beta}$ ”). In Chapter 13 the techniques of the group and Lie algebra contractions are developed. Using the approach of Inönü and Wigner the contractions of the group of rotations  $\text{SO}(3)$  to the group of motion on the plane  $\text{ISO}(2)$ , de Sitter group  $\text{SO}(4,1)$  to the Poincaré group  $\text{ISO}(3,1)$ , and  $\text{SO}(4) \rightarrow \text{ISO}(3)$  are performed. The contraction of compact unitary group  $\text{U}(2)$  to the solvable group  $\text{H}_4$  is given in a slightly different way, along with the contraction of the Casimir operators, basis states, matrix elements etc. What I miss here is the contraction of the relativistic Poincaré group to the non-relativistic Galilei group by using the speed of light as a contraction parameter. (The interested reader can find it in the first chapter of [3]). Chapter 14 is devoted to the important problem of the symmetry of physical systems. In the beginning the author provides in a comprehensible manner the formulation of two basic principles - the principle of relativity of observers and the principle of equivalence of states. Then the quantization of the energy of a spinless particle in the Coulomb field (the Hydrogen atom) according to Schrödinger and Klein-Gordon equations are described and discussed in parallel. The group  $\text{SO}(4)$  generated by the operators associated with the components of angular momentum and the Runge-Lenz vector are shown to generate the dynamical symmetry group (called also the invariance group) of the non-relativistic atom. This group transforms bound states into another bound states with the same energy. For the scattering states the corresponding group is shown to be  $\text{SO}(3,1)$ . The group that transforms all bound states among themselves is explained to be the de Sitter group  $\text{SO}(4,1)$ . This is the so-called *spectrum generating group* (sometimes called also *characteristic non-invariance group* for the system. The largest group that leave the states of hydrogen atom invariant is the conformal group  $\text{SO}(4,2)$ . This property of  $\text{SO}(4,2)$  transformations is first established by Malkin and Man’ko [4]. For further details and references see [3, 5].

In Chapter 14 the free field Maxwell equations are derived by means of group theoretical methods. Maxwell equations and the gravitational radiation equations emerge as constraints equations on the  $2(2j + 1)$  components of a manifestly covariant fields transforming under the  $\text{D}^{j^0+0j}$  representation of the inhomogeneous Lorentz group. The constraints are inferred by comparison of the transformation properties of the  $2(2j + 1)$  states that carry the manifestly covariant (but nonunitary) representations of the inhomogeneous Lorentz group with the transformation properties of the two helicity  $j$  states that carry the irreducible unitary representations of this group. For  $j = 1$  these are the electric and magnetic fields, and for  $j = 2$  these are the second order tracefree symmetric gravito-electric and

gravito-magnetic field tensors. These new derivations are due to J. Ramos and R. Gilmore [6]. It is well-known that the symmetry group of the Maxwell equations is the conformal group  $SO(4, 2)$  and this is the largest group that leaves the equations in free space invariant.

In the last Chapter 16 the Lie's method for solving the ordinary differential equations, regarded as an extension of the Galois approach to the algebraic equations, discussed in the first chapter, is outlined. It is described in detail via the Bernoulli equation which is of the form  $xydy/dx + y - xy^2 = 0$ . The extensions of Lie's method to higher degree and higher order ordinary differential equations are briefly sketched. The application of the method to partial differential equations is illustrated on the examples of the Laplace and the heat equations with sources. In conclusion, the new Gilmore's book provides a very comprehensive introduction to the Lie group theory and its application in physics. The exposition is very clear and easy to acquire and assimilate by students and young researchers. The numerous problems with which each chapter ends not only illustrate and help for the understanding the presented material but also reveal new features and applications. Some new original ideas and results are also contained in the book, e.g. the group theoretical derivation of Maxwell equations using the concept of photons. The book could be also useful for specialists, who want to refresh their knowledge on Lie group applications to physics. Finally it is recommended to everyone who wishes to enter into this interesting and fascinating field.

## References

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