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BOOK REVIEW

Lectures on Kähler Manifolds, by Werner Ballmann, European Mathematical Society, 2006, viii + 172pp., ISBN 978-3-03719-025-8.

This is an advanced book on Kähler geometry based on the lectures of a course delivered at University of Bonn and Erwin Schrödinger Institute in Wien. The author is a well-known name in Riemannian geometry.

To recall what a Kähler manifold is we start with a given complex manifold M with complex structure J. A Riemannian metric $\langle , \rangle$ on M is called Hermitian, if $\langle JX, JY\rangle = \langle X, Y\rangle$. Then $\omega(X, Y) = \langle JX, Y\rangle$ is a two-form, called the Kähler form of M. The metric is called *Kähler* if $\mathrm{d}\omega = 0$. The geometry of Kähler metrics is a central topic in the differential geometry since its appearance in 1930's. It is related to fundamental branches of mathematics and mathematical physics like algebraic geometry, nonlinear partial differential equations, topology, string theory and others. In recent years the topic is flourishing, thanks to the developments in such areas as extremal Kähler metrics and mirror symmetry. So it is not surprising that several books based on Kähler geometry appeared on these topics.

The book under review is an addition to the existing literature emphasizing on three topics: Calabi conjecture, Kähler hyperbolic spaces and Kodaira embedding theorem. These topics are classical, but after reading the book one should be prepared to enter the recent developments mentioned above. There is certain amount of prerequisites to the reader, which is natural for an advanced topic: basic facts about elliptic partial differential operators, including Sobolev and Hölder spaces, Hodge theory and L^2-index theorem. Some familiarity with differential geometric constructions like vector bundles and characteristic classes is helpful since their discussion is rather brief.

The book contains nine chapters and three appendices. The first six chapters contain general material, necessary to understand the content of the last three. The first chapter sets some notations and conventions for smooth manifolds, vector bundles, Lie derivatives, Riemannian metrics, covariant derivatives, Laplace operator. Here are introduced the Hodge decomposition and the Weitzenböck formulas. It is interesting to note that the curvature tensor is not explicitly defined and the reader may consult some other sources to clear the conventions. The second chapter concerns the basic facts about complex manifolds. It contains a good amount of examples including the blow-up (at a point) construction. The notions of holomorphic vector fields, Hermitian metrics and type decomposition of differential forms together with the $\overline{\partial}$-operator are discussed and relation between the automorphisms of a complex surface and a blow-up surface is mentioned without proof. Chapter three is about holomorphic vector bundles and contains the basic material about Hermitian metrics, Hermitian connections and Dolbeault's cohomology of a holomorphic vector bundle. There are also some facts about holomorphic line bundles and their relations with complex hypersurfaces, although the full definition of divisor is not given for the sake of clarity. The canonical bundle is defined and its change after a blow-up is explained. Chapter four is about the definition of the basic topic of the book - Kähler manifolds. The main characterization in terms of Levi-Civita connection is given. The basic symmetries of the curvature tensor are given and the notions of holomorphic sectional, bisectional and Ricci curvatures are defined as well as the first Chern form. Here is also the relation between holomorphic and Killing vector fields. The non-vanishing of the even-degree cohomology groups and the Wirtinger's inequality are proven. Chapter five is devoted to the cohomology properties of the Kähler manifolds based on the Lefschetz decomposition and Hard Lefschetz theorem. Hodge-Riemann bilinear relations are mentioned and the Hodge index theorem is proven. The dd^c-lemma is proven and its consequences on the formality of the deRham differential algebra is explained as in the Deligne-Griffiths-Morgan-Sullian's paper. Finally the Kodaira vanishing theorem is proven for the higher cohomology groups of positive holomorphic line bundle. Chapter six is the last of the general chapters and contains the material about the topological consequences of the positivity of the Ricci tensor. In particular two results are proven: the one of Bochner about the vanishing of the Dolbeault's cohomology groups of type $(p, 0)$ for $p > 0$ and the other of Kobayashi that such manifold is simply connected and its arithmetic genus (or the holomorphic Euler characteristic) is equal to one. An estimate of the first eigenvalue of the Laplace operator of Kähler manifold with strictly positive Ricci tensor is given, as well as its relation to the result of Lichnerowicz-Matsushima about the identification between the lowest eigenspace of the Laplace operator and the space of holomorphic vector fields.

Chapter seven is about the first one of the three applications of the theory of Kähler manifolds. It contains the proof by S. T. Yau of the Calabi conjecture. The conjecture states that a Kähler manifold of negative or vanishing first Chern class admits a Kähler - Einstein metric. It is explained how the problem relates to Monge-Ampere equation and the uniqueness and existence of solution is proved. This is the place where Hölder spaces and Sobolev Embedding theorem are used. The chapter ends with discussion of the possible obstructions in the positive definite case, including the Calabi-Futaki invariant. It should be noted that this is one of the topics of the most active research recently. It is based on the relation between stability and generalizations of the Futaki functional, as developed by G. Tian, S. Donaldson, S. S. Chern and others. Here it is extended to the so-called extremal Kähler metrics. The reader may consult the recent book by G. Tian on the subject.

Chapter eight is based on Gromov's definition of Kähler-hyperbolic spaces and the approach is based on L^2-deRham theory. It includes a discussion of Atiyah's L^2-index theorem and focuses on the fact that the L^2 cohomology of the Kähler-hyperbolic spaces are concentrated in the middle dimension.

Chapter nine is about the proof of the Kodaira embedding theorem. The theorem states that every compact complex manifold admitting an ample line bundle can be embedded into the complex projective space. Here one needs the blow-up construction from chapter three. At its final step the proof uses some sheaf theory. Overall it follows the standard steps, as in e.g. Griffiths-Harris or Wells' books.

The book ends with three Appendices. The first one is about the Chern-Weil theory, where the chracteristic classes of complex vector bundles are defined via the curvature. The second is about the Riemannian and Hermitian symmetric spaces and contains some good examples, although does not finish the classification of the simply-connected symmetric spaces. The third one contains some remarks about the differential operators defined on vector bundles.

In the reviewer's opinion the book is aimed to an advanced graduate students or mathematicians who are experts in different but related fields, and who want to learn the basics and some applications of the Kähler geometry. Its main advantage is the good choice of examples and the straightforward exposition, which makes the reading fast and fruitful.

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