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BOOK REVIEW

Analytic Hyperbolic Geometry – Mathematical Foundations and Applications, by Abraham A. Ungar, World Scientific, 2005, xvii + 463pp., ISBN 981-256-457-8.

The book under review provides an efficient algebraic formalism for studying the hyperbolic geometry of Bolyai and Lobachevsky, which underlies Einstein special relativity. More precisely, it extracts the properties of Einstein addition of relativistically admissible velocities, varying over the ball $B_c^3 = \{v \in \mathbb{R};\ ||v|| < c\}$ of radius c, equal to the vacuum speed of light, and introduces the notion of a gyrocommutative gyrogroup $(B_c^3, \oplus_E)$. The Einstein addition $\oplus_E$ is not associative and its deviation from associativity is measured by the so called Thomas gyrations. These are rotations from $\mathrm{SO}(3)$, representing the relativistic Thomas precession of the motion of the moon. Moreover, the relativistically admissible velocities $u \in B_c^3$ are associated with Lorentz boosts $L(u)$, i.e., with linear transformations $L(u)$ of the space-time, without a rotation. The deviation of the product of two Lorentz boosts from a Lorentz boost is a space-time rotation, called Thomas precession. As another application of the gyrooperations, the velocity parameter of the Lorentz link between two relativistic space-time events with equal space-time norms is expressed as a cogyrodifference of the velocities of the arguments. Extending the gyroaddition of natural number of copies of one and a same gyrovector to a scalar multiplication by real numbers, the author turns any abstract gyrocommutative gyrogroup $(G, \oplus)$ into a gyrovector space $(G, \oplus, \otimes)$.

Classically, the vector algebra provides the necessary algebraic formalism for analytic description of the Euclidean geometry. In a similar vein, Ungar's gyrovector spaces constitute the algebraic background for solving qualitative and quantitative problems from the hyperbolic geometry. In physics, the Newtonian mechanics is modelled on the Euclidean geometry, so that the usual vector addition and its associated analytic expression account for the addition of Newtonian velocities. Respectively, Einstein's relativistic space is assumed to be hyperbolic and the gyroaddition of the gyrovectors extract the properties of Einstein's addition of relativistically admissible velocities.

Concerning the applications to the special relativity, the center of momentum of several particles $p_1, \ldots, p_k$ with equal rest mass m_o and relativistically admissible velocities $v_1, \ldots, v_k \in B_c^3$ turns to coincide with the gyrocentroid of the gyrosimplex, generated by $p_1, \ldots, p_k$. The points $v \in (B_c^3, \oplus_E, \otimes_E)$ of Einstein gyrovector space represent all the inertial frames Σ_v with relativistically admissible velocities v with respect to a rest frame Σ_0. For two particles with arbitrary rest masses t_1, t_2 and admissible velocities $a_1, a_2 \in B_c^3$, the center of momentum velocity $a_{12} \in B_c^3$ coincides with the so called $(t_1 \, ; \, t_2)$-gyromidpoint. More precisely, a_{12} is such a relativistic velocity that the ratio between the "proper speed" of a frame Σ_{a_2} relative to $\Sigma_{a_{12}}$ and the “proper speed” of a frame Σ_{a_1} relative to $\Sigma_{a_{12}}$ is $t_1 \, ; \, t_2$. The pair $(t_1 \, ; \, t_2)$ is referred to as homogeneous gyrobarycentric coordinates of a_{12}, relative to $\{a_1, a_2\}$. The above considerations are generalized to homogeneous gyrobarycentric coordinates of the relativistic center of momentum velocity. Einstein relativistic mass turns to be connected with the gyrobarycentric coordinates. That is one more advantage of the invented gyroformalism, which makes it an indispensable and extremely powerful tool for description of Einstein’s special relativity.

The gyrovector spaces have an application also to quantum computation theory. All postulates of quantum mechanics of a two state quantum system, called a qubit, can be formulated in terms of density matrices $\rho(v) = \frac{1}{2}(I_2 + v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3)$, parametrized by their Bloch vectors $v \in B_1^3$ and expressed by the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$. Ungar has discovered that $v \in B_1^3$ behave rather as gyrovectors, instead of vectors. This conclusion is based on the fact that if $\oplus$ is the Möbius addition on B_1^3 then the matrix $R(u, v) = \rho^{-1}(u \oplus v)\rho(u)\rho(v)$ belongs to $PSU(2)$ and plays the role of a gyration of $\rho(u)$ and $\rho(v)$, reflecting the evolution of a closed quantum system.

Let us give a brief account of the specific mathematical topics, covered by the book. *Chapter 2* studies the properties of the gyrogroups, gyroautomorphism groups and their gyrosemidirect products, which turn to be groups. In analogy with the associativity of the group operation, the gyrogroups are shown to satisfy the right gyroassociative law. The next chapter provides necessary and sufficient conditions for the gyrocommutativity of a gyrogroup and describes some properties of the gyrocommutative gyrogroups. It introduces the gyromidpoint and generalizes the Möbius addition on the unit disc in $\mathbb{C}$ to Einstein addition on a ball $B_r^n = \{v \in \mathbb{R}^n \, ; \, ||v|| < r\}$ and to Ungar’s Proper Velocity gyrogroup on an inner product real vector space. The fourth chapter deals with the gyrogroup factors and their associated gyrogroup extensions. The gyrocommutativity and the gyroautomorphisms, preserving the gyrofactor, are shown to be inherited from the gyrogroup extensions. The basic example here is the gyrocommutative gyrogroup of the Lorentz boosts, which is a gyrogroup extension of Einstein's gyrogroup of relativistic velocities. At the end of the chapter are introduced the homogeneous gyrobarycentric coordinates. *Chapter 5* starts with rooted gyrovectors in a gyrocommutative gyrogroup and passing to their equivalence classes defines the gyrovectors. *Chapter 6* is devoted to the properties of the real inner product gyrovector spaces, their automorphism groups, the left gyrotranslations. In analogy with the Euclidean lines are constructed the gyrolines and the cogyrolines. The gyrometric is shown to coincide with the usual hyperbolic metric on the corresponding model. The seventh chapter describes the gyrovector spaces from a differential geometric point of view. For instance, it studies the gyrodistance, the gyrometric, the gyroline element, the gyrocurvature and other features of the various models of the hyperbolic geometry. The next chapter is devoted to the gyrotrigonometry, i.e., the relationship among the gyrolengths of the sides and the gyroangles of a gyrotriangle. It develops various sufficient conditions for congruence of gyrotriangles. In a distinction with Euclidean geometry, if two gyrotriangles have congruent pairs of gyroangles then they are proved to be congruent. Furthermore, the sides of a gyrotriangle are expressed explicitly by the gyroangles. The exposition presents also the Law of the gyrocosines, the Law of the gyrosines, describes the properties of the parallel transport. It provides the Möbius hyperbolic Pythagorean Theorem, which results in the usual Euclidean Pythagorean Theorem when the radius of the ball model of the hyperbolic space is pushed to infinity. If α, β, γ are the gyroangles of a gyrotriangle then $\delta := \pi - (\alpha + \beta + \gamma) > 0$ is called a gyrotriangular defect. The gyroarea of a gyrotriangle in the Möbius gyrovector space is shown to be completely determined by $\tan\left(\frac{\delta}{2}\right)$ and the Gaussian curvature. *Chapter 9* focuses on the application of the gyrovector spaces to the quantum computation theory, while the last chapter reveals the relativistic origin and applications of the developed algebraic gyroformalism.

The book is of interest both to mathematicians, working in the field of geometry, and the physicists, specialized in relativity or quantum computation theory. The exposition is self-contained and based on the usual undergraduate courses in algebra, analytic and differential geometry, as well as in trigonometry. The gyrogeometric language is a useful abstract tool for studying the hyperbolic geometry and its applications. The book is recommended to graduate students and researchers, interested in the interrelations among the non-associative algebra, the hyperbolic and differential geometry, Einstein relativity theory and the quantum computation theory.

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