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BOOK REVIEW

Differential Geometry of Curves and Surfaces, by Victor Andreevich Toponogov, Birkhäuser, Basel, 2005, xi + 206 pp., ISBN 0-8176-4384-2

The author of this book is the remarkable Russian mathematician V. A. Toponogov (1930–2004), specialist in Riemannian geometry in the large and one of the founders of CAT(k)– spaces theory.

The book is devoted to differential geometry of curves and surfaces in three-dimensional Euclidean space. It consists of three chapters. The first chapter – *Theory of curves in three-dimentional Euclidean space and in plane* is structured into twelve sections. The second chapter – *Extrinsic geometry of surfaces in three-dimentional Euclidean space* is splited into ten sections and the last chapter *Intrinsic geometry of surfaces* consists also of ten sections. The material is presented in two parallel streams.

The first stream treats the standard theoretical material on differential geometry of curves and surfaces complemented by certain number of exercises and problems of a local nature. It includes eight sections of first chapter and six sections of second chapter.

The *first Chapter* contains the following standard themes of differential geometry of curves: definition and methods of presentation of curves, tangent line, osculating plane, length of a curve, curvature of a curve, torsion of a curve, the Frenet formulas and the natural equations of a curve. All definitions are geometrical. For example: a plane α is called an osculating plane to a curve γ at a point P if $\lim\limits_{d\to 0} \dfrac{h}{d^2} = 0$ where d is the length of the chord of γ joining the points P and $\tilde{P}$, and h is the length of the perpendicular drown from $\tilde{P}$ onto the plane α. Equations for tangent line and osculating plane are deduced. Formulas for calculation of the length, of the curvature and of the torsion of a curve are derived. The fundamental theorem of curves theory is proved.

The *second Chapter* continues with the others standard themes of differential geometry of surfaces: definition of (embedded and immersed) surface, tangent plane, first fundamental form, second and third fundamental forms of a surface. Defined and studied are the principal notions related to the first fundamental form: a metric on a surface, the length of a curve on a surface, an angle between the curves on a surface, and the area of a region on a surface. A definition of isometric surfaces and necessary and sufficient condition for such surfaces are given. Definitions and formulas for calculations of the normal curvature, the principal curvatures, the Gaussian curvature and the mean curvature of a surface are given. The Meusnier's theorem, the Rodrigues' theorem and the Gauss theorem for a spherical map are proved. The definition and examples of parallel surfaces, an elliptic point (or point, of convexity), a hyperbolic (saddle) point, a parabolic (or cylindrical) point, an umbilical point, a planar point and Euler's formula are given. The lines of curvature, the asymptotic curves and the geodesics on a surface are studied. The main equations of surface theory are deduced and the fundamental theorem of surfaces is formulated.

The second stream contains more difficult material and formulation of some complicated but important theorems of a global nature. It includes four sections of the first chapter (§1.5, §1.7, §1.10 and §1.11), four sections (§2.6, §2.7.4, §2.8.3, §2.9) and Theorems 2.8.2 – 2.8.4 of the second chapter, and the whole of Chapter three. In Section §1.5 the solutions of five problems for convex plan curves and in Section §1.10 the solutions of six problems for space curves are given. In Section §1.7 28 very interesting problems for curvature or plane curves, for parallel curves, for evolutes and evolvents and for curves of constant width are considered. Section §1.11 is devoted to phase length of a curve and the Fenchel–Reshetnyak inequality. In Section §2.6 the next classes of surfaces are considered: surfaces of revolution with constant Gaussian curvature, ruled and developable surfaces, convex surfaces and saddle surfaces. The theorem of S. Bernstein and the theorem of N. Efimov for saddle surfaces are formulated. In Sections §2.7.4 and §2.8.3 very interesting problems of curves on a surface, of the principal curvatures and of the Gaussian curvature are given. The theorems or H. Liebmann for complete surface with constant Gaussian curvature $K > 0$ and for a surface with positive Gaussian curvature and constant mean curvature, and the theorem or D. Hilbert for nonexistence of complete surface with constant Gaussian curvature $K < 0$ are proved. Section §2.9 is devoted to indicatrix of a surface of revolution.

In *Chapter three* the foundations of intrinsic geometry of a two-dimensional surface are introduced. The geometric properties and objects that can be determined only in terms of the first fundamental form of a surface are called intrinsic geometric properties and objects. The collection of these geometric properties and objects forms the subject of intrinsic geometry of a surface. The chapter contains the following themes of intrinsic geometry of a surface: a covariant derivative and parallel translation of a vector along a curve on a surface, geodesics, shortest paths and geodesics, special coordinate systems (Riemannian, normal, geodesic polar and semigeodesic) on a surface. The theorem of Hopf-Rinow for geodesically complete surface, the Gauss-Bonnet theorem and the comparison theorem for the angles of a triangle, local comparison theorems for triangles and Aleksandrof's comparison theorem for the angles of a triangle are proved. Various and interesting problems are considered. The material in the *Chapter three* is presented in such a form which allows that this chapter be considered as an introduction to the theory of n-dimensional Riemannian manifolds.

The bibliography of the book presents 66 entries. There are also index pages.

A distinctive feature of the book is a large collection (80 to 90) of nonstandard and original problems. Most of these problems are new and are not to be found in other textbooks or books of problems. The key of these problems is the notion of curvature – the curvature of a curve, the principal curvatures and the Gaussian curvature of a surface. Almost all of the problems are given with their solutions. In some cases only short instructions are provided. The large number of original problems makes this text book interesting and useful.

As a whole the book is well written, very well ordered and illustrated and I enjoyed reading it. This concise guide to the geometry of curves and surfaces can be recommended to first-year graduate students, strong senior students, students specializing in geometry, and all interested readers.

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