Homology, Homotopy and Applications, vol. 16(1), 2014, pp.199–214

EXACT SEQUENCES OF COMMUTATIVE MONOIDS AND SEMIMODULES

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(*communicated by George Janelidze*)

Abstract

Basic homological lemmas well known for modules over rings and, more generally, in the context of abelian categories, have been extended to many other concrete and abstract-categorical contexts by various authors. We propose a new such extension, specifically for commutative monoids and semimodules; these two contexts are equivalent since the forgetful functors from varieties of semimodules to the variety of commutative monoids preserve all limits and colimits.

Introduction

The purpose of this paper is to develop a semimodule version of the collection of basic homological lemmas, including the 5-Lemma, the (3×3)-Lemma, and the Snake Lemma, for commutative monoids and semimodules. In doing so, we use a strong notion of exactness, different from those used by M. Takahashi [**18**], A. Patchkoria [**13**], and K.B. Patil and R.P. Deore [**15**], which are also different from each other. The paper is organized as follows:

- We begin by explaining that both contexts, of commutative monoids and of semimodules, are important in spite of the fact that they are equivalent (Section 1).
- We consider various special classes of morphisms and various notions of an exact sequence of semimodules in Section 2.
- Section 3 is devoted to the above-mentioned homological lemmas.
- Various additional remarks are made in Section 4.

Let us recall here that:

- a semiring $S = (S, 0, +, 1, \cdot)$ is an algebraic structure in which $(S, 0, +)$ and $(S, 1, \cdot)$ are monoids, with commutative $+$ and with $x(y + z) = xy + xz$ (where $xy = x \cdot y$, etc.), $(x + y)z = xz + yz$, and $0x = 0 = x0$ for all $x, y, z \in S$;

The author would like to acknowledge the support provided by the Deanship of Scientific Research (DSR) at King Fahd University of Petroleum & Minerals (KFUPM) for funding this work through project No. FT100004.

Received March 5, 2013, revised December 3, 2013; published on May 23, 2014.

2010 Mathematics Subject Classification: 18A30, 18G60, 16Y60.

Key words and phrases: exact sequence, semimodule, semiring, commutative monoid, homological lemma, snake lemma.

Article available at `http://dx.doi.org/10.4310/HHA.2014.v16.n1.a12`

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- for a semiring S, an S-semimodule is a commutative monoid A equipped with a map $S \times A \longrightarrow A$, written as $(s,a) \mapsto sa$, and satisfying $s(a+b) = sa + sb$, $(s+t)a = sa + ta$, $(s \cdot t)a = s(ta)$, $0a = 0$, and $s0 = 0$ for all $s, t \in S$ and $a, b \in A$.

Our main references for semimodules over semirings and categories are [**6**] and [**12**] (or [**17**]), respectively.

1. Why commutative monoids and semimodules?

1.1. As has been well known for a long time, the following conditions on a variety $\mathbf{A}$ of universal algebras (considered as a category) are equivalent:

(a) $\mathbf{A}$ is enriched in the monoidal closed category of abelian groups; that is, there exist abelian group structures on all hom sets $\mathrm{Hom}_{\mathbf{A}}(A, B)$ $(A, B \in \mathbf{A})$, such that the composition of morphisms distributes over addition on both sides;

(b) $\mathbf{A}$ is an additive category;

(c) $\mathbf{A}$ is an abelian category;

(d) $\mathbf{A}$ is the category of R-modules for some ring R.

1.2. There is a less known similar result on semimodules. It says that the following conditions on a variety $\mathbf{A}$ of universal algebras are equivalent:

(a) $\mathbf{A}$ is enriched in the monoidal closed category of commutative monoids; that is, there exist commutative monoid structures on all hom sets $\mathrm{Hom}_{\mathbf{A}}(A, B)$ $(A, B \in \mathbf{A})$, such that the composition of morphisms distributes over addition on both sides;

(b) $\mathbf{A}$ is the category of S-semimodules for some semiring S.

Both of these results are proved (using different terminology) in [**5**], and, moreover, at least Csákány's proof of 1.2 seems to be the first known such proof.

Observation 1.3. Concerning the so-called basic homological lemmas, we should observe the following. While in the situation 1.1 they have clear unique formulations that belong to classical homological algebra, in the situation 1.2 the formulations might depend on the chosen notion of an exact sequence. However, as soon as the notion of an exact sequence is defined categorically, using limits and colimits (only), each such lemma will hold for semimodules if and only if it holds for commutative monoids. This follows from the fact that the forgetful functor

$$U\colon S\text{-}\mathbf{SMod} \longrightarrow \mathbf{CMon}$$

from the category S-**SMod** of S-semimodules to the category **CMon** of commutative monoids preserves limits and colimits and reflects isomorphisms; in particular, a diagram $A \longrightarrow B \longrightarrow C$ in S-**SMod** is an exact sequence (in the chosen sense) if and only if so is its U-image $U(A) \longrightarrow U(B) \longrightarrow U(C)$. Therefore, although all our results apply to the situation 1.2, it suffices to prove them for commutative monoids, making both these structures fundamentally important for our purposes.

2. Four notions of an exact sequence of semimodules

Considering semimodules over an arbitrary fixed semiring S, we keep in mind Observation 1.3, according to which every argument we use reduces to the case of commutative monoids; that is, to the case where S is the semiring of natural numbers. Proposition 2.2 below is well known; it describes several categorically defined classes of morphisms in S-**SMod** in classically algebraic terms. The readers less familiar with category theory can use these descriptions as definitions. Before formulating Proposition 2.2, let us explain our notation and terminology for kernels and cokernels:

2.1. On the one hand, we shall use classical-algebraic notation for kernels and cokernels: for a morphism $f\colon A \longrightarrow B$ in S-**SMod**, we write

$$\mathrm{Ker}(f) = \{a \in A \mid \ f(a) = 0\},$$
$$\mathrm{Coker}(f) = B/\{(b_1, b_2) \in B \times B \mid (\exists a_1)(\exists a_2)(b_1 + f(a_1) = b_2 + f(a_2))\},$$

and write $\mathrm{ker}(f)\colon \mathrm{Ker}(f) \longrightarrow A$ and $\mathrm{coker}(f)\colon B \longrightarrow \mathrm{Coker}(f)$ for the corresponding canonical maps. On the other hand, we shall use the categorical notation and terminology, according to which $\mathrm{ker}(f)\colon \mathrm{Ker}(f) \longrightarrow A$ and $\mathrm{coker}(f)\colon B \longrightarrow \mathrm{Coker}(f)$ are defined (up to isomorphism) via their universal properties; we will also say that $\mathrm{ker}(f)$ is the kernel of f and $\mathrm{coker}(f)$ is the cokernel of f. The classical-algebraic and the categorical-algebraic notations agree up isomorphism, of course.

Proposition 2.2. *A morphism $f\colon A \longrightarrow B$ in S-**SMod** is:*

(a) *a monomorphism, if and only if f an injective map;*

(b) *a normal monomorphism, that is, a kernel of some morphism, if and only if (f is injective and) whenever $b + f(a_1) = f(a_2)$ for $a_1, a_2 \in A$ and $b \in B$, there exists a unique $a \in A$ with $f(a) = b$;*

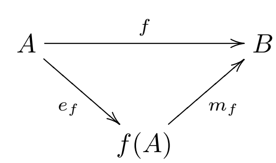
(c) *a regular epimorphism if and only if f is a surjective map;*

(d) *a normal epimorphism, that is, a cokernel of some morphism, if and only if it is surjective and*

$$f(a_1) = f(a_2) \text{ if and only if there exist } k_1, k_2 \in \mathrm{Ker}(f) \text{ with } a_1 + k_1 = a_2 + k_2.$$

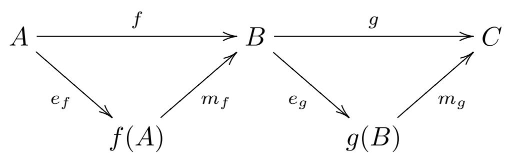
(e) *a pullback stable normal epimorphism, whenever it is a normal epimorphism.*

For an arbitrary morphism $f\colon A \longrightarrow B$ in S-**SMod**, we have the canonical factorization



in which $e_f\colon A \longrightarrow f(A)$ is induced by f and $m_f\colon f(A) \longrightarrow B$ is the inclusion map.

2.3. For morphisms $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ in S-**SMod**, consider the diagram



For its top row $A \longrightarrow B \longrightarrow C$ we have

(a) $A \longrightarrow B \longrightarrow C$ is Takahashi exact, that is, exact in the sense of [**18**], if and only if the kernel of g is the normal closure of m_f, that is, $\mathrm{Ker}(g) = \mathrm{Ker}(\mathrm{coker}(f))$;

(b) $A \longrightarrow B \longrightarrow C$ is Patchkoria exact, that is, exact in the sense of [**13**], if and only if m_f is the kernel of g, that is, $\mathrm{Ker}(g) = f(A)$;

(c) $A \longrightarrow B \longrightarrow C$ is Patil–Deore exact, that is, exact in the sense of [**15**], if and only if e_g is the cokernel of f.

This shows the Patchkoria exactness and the Patil–Deore exactness dual to each other and makes both of them more restrictive than Takahashi exactness. We are going, however, to use an even more restrictive notion, namely:

Definition 2.4. In the notation of 2.3, we will say that $A \xrightarrow{f} B \xrightarrow{g} C$ is *exact* if m_f is the kernel of g and e_g is the cokernel of f, or, equivalently, $A \xrightarrow{f} B \xrightarrow{g} C$ is Patchkoria exact and Patil–Deore exact at the same time.

As follows from previous observations, this notion of exactness has the following classical-algebraic reformulation:

Proposition 2.5. *In the notation of 2.3, $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $\mathrm{Ker}(g) = f(A)$ and whenever $g(b_1) = g(b_2)$ there exist $a_1, a_2 \in A$ with $b_1 + f(a_1) = b_2 + f(a_2)$.*

2.6. As usually:

(a) a diagram $\ldots \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \ldots$ is said to be exact in B if $A \longrightarrow B \longrightarrow C$ is exact;

(b) a diagram $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is said to be a short exact sequence, if it is exact in A, in B, and in C.

(c) a diagram $A_0 \longrightarrow A_1 \longrightarrow \ldots \longrightarrow A_{n-1} \longrightarrow A_n$ is an exact sequence, if it is exact in A_i for every $i = 1, \ldots, n-1$.

Corollary 2.7. *In the notation of 2.3, we have:*

(a) *$0 \longrightarrow A \longrightarrow B$ is exact if and only if the map $A \longrightarrow B$ is injective;*

(b) *$B \longrightarrow C \longrightarrow 0$ is exact if and only if the map $B \longrightarrow C$ is surjective.*

3. Homological lemmas

In this section we prove basic homological lemmas for semimodules over an arbitrary fixed semiring S. As explained in Section 1, and mentioned again at the beginning of Section 2, we could equivalently do that for commutative monoids, that is, in the special case where S is the semiring of natural numbers.

Note that according to Proposition 2.2(b), the normal closure $\overline{A}$ of a subsemimodule A of a semimodule B is

$$\overline{A} = \operatorname{Ker}(\operatorname{coker}(A \longrightarrow B)) = \{b \in B \mid \text{ there exist } a_1, a_2 \in A \text{ with } b + f(a_1) = fa_2)\}.$$

More precisely, we will say that $\overline{A}$ is the normal closure of A in B.

Next, we will say that a morphism $\varphi\colon X \longrightarrow Y$ of semimodules is *cancellative* if so are all elements of its image, that is, if, for x in X and y_1 and y_2 in Y, $x + y_1 = x + y_2$ always implies $y_1 = y_2$. Let us also agree that, in our diagram-chasing arguments below, we shall sometimes use elements of semimodules not mentioning the semimodules they belong to, since it will be clear from the context.

The Five Lemma

The following result can be easily proved using *diagram chasing* (compare (b) with [**14**, Lemma 1.9]).

Lemma 3.1. *Let*

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 \\
\alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow \\
L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

be a commutative diagram of semimodules, in which the first and the third columns are exact. Then

(a) *If α_2 is a regular epimorphism and the first row (formed by f_1 and g_1) is exact, then the second row (formed by f_2 and g_2) is exact.*

(b) *If α_2 is a monomorphism and the second row is exact, then the first row is exact.*

(c) *If a_2 is an isomorphism, then the first row is exact if and only if the second row is exact.*

Lemma 3.2. *Let*

$$\begin{array}{ccccc}
L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 \\
\alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow \\
L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2
\end{array}$$

be a commutative diagram of semimodules with exact rows. Then

(a) *If g_1 and α_1 are regular epimorphisms and α_2 is a monomorphism, then α_3 is a monomorphism.*

(b) *If f_2 is a monomorphism, α_2 is a regular epimorphism, and α_3 has zero kernel, then α_1 is a regular epimorphism.*

(c) *If f_2, α_1, and α_3 have zero kernels, then α_2 also has zero kernel.*

(d) *If f_1 and α_2 are cancellative while f_2, α_1 and α_3 are monomorphisms, then α_2 also is a monomorphism.*

(e) *If g_1, α_1, and α_3 are regular epimorphisms, then the normal closure of $\alpha_2(M_1)$ in M_2 is M_2 itself.*

Proof. (a) Suppose that $\alpha_3(n_1) = \alpha_3(n_1')$ for some $n_1, n_1' \in N_1$. Since g_1 is a regular epimorphism, $n_1 = g_1(m_1)$ and $n_1' = g_1(m_1')$ for some $m_1, m_1' \in M_1$. It follows that $(g_2 \circ \alpha_2)(m_1) = (g_2 \circ \alpha_2)(m_1')$. Since the second row is exact, there exist $l_2, l_2' \in L_2$ such that $\alpha_2(m_1) + f_2(l_2) = \alpha_2(m_1') + f_2(l_2')$. By assumption, α_1 is a regular epimorphism and so there exist $l_1, l_1' \in L_1$ such that $\alpha_1(l_1) = l_2$ and $\alpha_1(l_1') = l_2'$. Now, we successively obtain the following equalities:

$$\begin{aligned} \alpha_2(m_1) + (f_2 \circ \alpha_1)(l_1) &= \alpha_2(m_1') + (f_2 \circ \alpha_1)(l_1'), \\ \alpha_2(m_1) + (\alpha_2 \circ f_1)(l_1) &= \alpha_2(m_1') + (\alpha_2 \circ f_1)(l_1'), \\ m_1 + f_1(l_1) &= m_1' + f_1(l_1'), && (\alpha_2 \text{ is a monomorphism}) \\ g_1(m_1) &= g_1(m_1'), && (g_1 \circ f_1 = 0) \\ n_1 &= n_1'. \end{aligned}$$

(b) Let $l_2 \in L_2$. Since α_2 is a regular epimorphism, there exists $m_1 \in M_1$ such that $f_2(l_2) = \alpha_2(m_1)$. It follows that $0 = (g_2 \circ f_2)(l_2) = (g_2 \circ \alpha_2)(m_1) = (\alpha_3 \circ g_1)(m_1)$, whence $g_1(m_1) = 0$ (since α_3 has zero kernel). Since the first row is exact, $m_1 = f_1(l_1)$ for some $l_1 \in L_1$ and so $f_2(l_2) = \alpha_2(m_1) = (\alpha_2 \circ f_1)(l_1) = (f_2 \circ \alpha_1)(l_1)$. Since f_2 is a monomorphism, we have $l_2 = \alpha_1(l_1)$.

(c) Suppose that $\alpha_2(m_1) = 0$ for some $m_1 \in M_1$. We have $(\alpha_3 \circ g_1)(m_1) = (g_2 \circ \alpha_2)(m_1) = 0$, whence $g_1(m_1) = 0$ (since α_3 has zero kernel). Since the first row is exact, $m_1 = f_1(l_1)$ for some $l_1 \in L_1$. So, $0 = \alpha_2(m_1) = (\alpha_2 \circ f_1)(l_1) = (f_2 \circ \alpha_1)(l_1)$, whence $l_1 = 0$ (since f_2 and α_1 have zero kernels); consequently, $m_1 = f_1(l_1) = 0$.

(d) Suppose that $\alpha_2(m_1) = \alpha_2(m_1')$ for some $m_1, m_1' \in M_1$. We have $(\alpha_3 \circ g_1)(m_1) = (g_2 \circ \alpha_2)(m_1) = (g_2 \circ \alpha_2)(m_1') = (\alpha_3 \circ g_1)(m_1')$, whence $g_1(m_1) = g_1(m_1')$ since α_3 is a monomorphism. Since the first row is exact, there exist $l_1, l_1' \in L_1$ such that $m_1 + f_1(l_1) = m_1' + f_1(l_1')$. Now, we successively obtain the following equalities:

$$\begin{aligned} \alpha_2(m_1) + (\alpha_2 \circ f_1)(l_1) &= \alpha_2(m_1') + (\alpha_2 \circ f_1)(l_1'), \\ \alpha_2(m_1') + (f_2 \circ \alpha_1)(l_1) &= \alpha_2(m_1') + (f_2 \circ \alpha_1)(l_1'), \\ (f_2 \circ \alpha_1)(l_1) &= (f_2 \circ \alpha_1)(l_1'), && (\alpha_2 \text{ is cancellative}) \\ l_1 &= l_1', && (f_2 \text{ and } \alpha_1 \text{ are monomorphisms}) \\ m_1 + f_1(l_1') &= m_1' + f_1(l_1'), \\ m_1 &= m_1'. && (f_1 \text{ is cancellative}). \end{aligned}$$

(e) Let $m_2 \in M_2$. Since g_1 and α_3 are regular epimorphisms, there exists $m_1 \in M_1$ such that $g_2(m_2) = (\alpha_3 \circ g_1)(m_1) = (g_2 \circ \alpha_2)(m_1)$. Since the second row is exact and α_1 is a regular epimorphism, there exist $l_1, l_1' \in L_1$ such that

$$\begin{aligned} m_2 + (f_2 \circ \alpha_1)(l_1) &= \alpha_2(m_1) + (f_2 \circ \alpha_1)(l_1'), \\ m_2 + \alpha_2(f_1(l_1)) &= \alpha_2(m_1 + f_1(l_1')), \end{aligned}$$

that is, $m_2 \in \overline{\alpha_2(M_1)}$. ∎

Proposition 3.3 (The Short Five Lemma). *Let*

$$\begin{array}{ccccccccc} & & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\ 0 & \longrightarrow & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 & & \end{array}$$

be a commutative diagram of semimodules with exact rows, and assume that M_1 and M_2 are cancellative. If $\alpha_2(M_1) = \overline{\alpha_2(M_1)}$ while α_1 and α_3 are isomorphisms, then α_2 also is an isomorphism.

Corollary 3.4. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\ 0 & \longrightarrow & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 & \longrightarrow & 0 \end{array}$$

be a commutative diagram of semimodules with exact rows, and assume that M_1 and M_2 are cancellative. If $\alpha_2(M_1) = \overline{\alpha_2(M_1)}$ and any two of α_1, α_2, and α_3 are isomorphisms, then the third also is an isomorphism.

Lemma 3.5. *Let*

$$\begin{array}{ccccccccc} U_1 & \xrightarrow{d_1} & L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 & \xrightarrow{h_1} & V_1 \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \delta} \\ U_2 & \xrightarrow{d_2} & L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 & \xrightarrow{h_2} & V_2 \end{array}$$

be a commutative diagram of semimodules with exact rows. Then:

(a) *If γ is a regular epimorphism, α_1 is a monomorphism and α_3 has zero kernel, then α_2 also has zero kernel.*

(b) *If γ is a regular epimorphism, f_1 and α_2 are cancellative while α_1 and α_3 are monomorphisms, then α_2 also is a monomorphism.*

(c) *If δ has zero kernel while α_1 and α_3 are regular epimorphisms, then the normal closure of $\alpha_2(M_1)$ in M_2 is M_2 itself.*

(d) *If γ is a regular epimorphism, δ is a monomorphism, f_1 and α_2 are cancellative while α_1 and α_3 are isomorphisms, then α_2 is a monomorphism and the normal closure of $\alpha_2(M_1)$ is M_2 is M_2 itself.*

Proof. (a) Suppose that $\alpha_2(m_1) = 0$ for some $m_1 \in M_1$. We have $(\alpha_3 \circ g_1)(m_1) = (g_2 \circ \alpha_2)(m_1) = 0$. Since α_3 has zero kernel, $g_1(m_1) = 0$ and so $m_1 = f_1(l_1)$ for some $l_1 \in L_1$. It follows that $0 = \alpha_2(m_1) = (\alpha_2 \circ f_1)(l_1) = (f_2 \circ \alpha_1)(l_1)$, whence $\alpha_1(l_1) = (d_2 \circ \gamma)(u_1) = (\alpha_1 \circ d_1)(u_1)$ for some $u_1 \in U_1$ (since γ is a regular epimorphism and $\mathrm{Ker}(f_2) = d_2(U_2)$). Since α_1 is a monomorphism, $l_1 = d_1(u_1)$, whence $m_1 = f_1(l_1) = (f_1 \circ d_1)(u_1) = 0$.

(b) Suppose that $\alpha_2(m_1) = \alpha_2(m_1')$ for some $m_1, m_1' \in M_1$. We have $(\alpha_3 \circ g_1)(m_1) = (g_2 \circ \alpha_2)(m_1) = (g_2 \circ \alpha_2)(m_1') = (\alpha_3 \circ g_1)(m_1')$, whence $g_1(m_1) = g_1(m_1')$ because

α_3 is a monomorphism. Since $L_1 \xrightarrow{f_1} M_1 \xrightarrow{g_1} N_1$ is exact, there exist $l_1, l_1' \in L_1$ such that $m_1 + f_1(l_1) = m_1' + f_1(l_1')$. Now, we successively obtain the following equalities:

$$\begin{aligned}
\alpha_2(m_1) + (\alpha_2 \circ f_1)(l_1) &= \alpha_2(m_1') + (\alpha_2 \circ f_1)(l_1'), \\
\alpha_2(m_1') + (f_2 \circ \alpha_1)(l_1) &= \alpha_2(m_1') + (f_2 \circ \alpha_1)(l_1'), \\
f_2(\alpha_1(l_1)) &= f_2(\alpha_1(l_1')), && (\alpha_2 \text{ is cancellative}) \\
\alpha_1(l_1) + k_2 &= \alpha_1(l_1') + k_2', && (U_2 \xrightarrow{d_2} L_2 \xrightarrow{f_2} M_2 \text{ is exact}) \\
\alpha_1(l_1) + (d_2 \circ \gamma)(u_1) &= \alpha_1(l_1') + (d_2 \circ \gamma)(u_1'), && (\gamma \text{ is a regular epimorphism}) \\
\alpha_1(l_1) + (\alpha_1 \circ d_1)(u_1) &= \alpha_1(l_1') + (\alpha_1 \circ d_1)(u_1'), \\
l_1 + d_1(u_1) &= l_1' + d_1(u_1'), && (\alpha_1 \text{ is a monomorphism}) \\
f_1(l_1) &= f_1(l_1'), \ (f_1 \circ d_1 = 0) \\
m_1 + f_1(l_1) &= m_1 + f_1(l_1'), \\
m_1' + f_1(l_1') &= m_1 + f_1(l_1'), \\
m_1' &= m_1. && (f_1 \text{ is cancellative}).
\end{aligned}$$

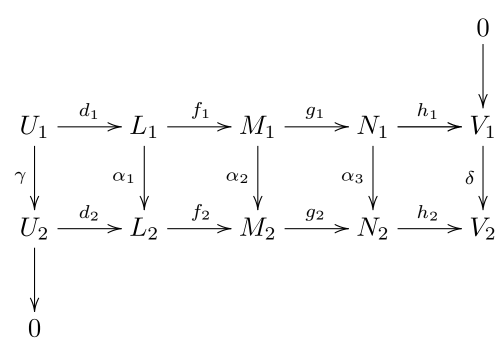
(c) Let $m_2 \in M_2$. Since α_3 is a regular epimorphism, there exists $n_1 \in N_1$ such that $g_2(m_2) = \alpha_3(n_1)$. It follows that $0 = (h_2 \circ g_2)(m_2) = (h_2 \circ \alpha_3)(n_1) = (\delta \circ h_1)(n_1)$, whence $h_1(n_1) = 0$ (since δ has zero kernel). Since $g_1(M_1) = \mathrm{Ker}(h_1)$, we have $n_1 = g_1(m_1)$ for some $m_1 \in M_1$. Notice that $(g_2 \circ \alpha_2)(m_1) = (\alpha_3 \circ g_1)(m_1) = \alpha_3(n_1) = g_2(m_2)$. Since $L_2 \xrightarrow{f_2} M_2 \xrightarrow{g_2} N_2$ is exact and α_1 is a regular epimorphism, there exists $l_1, l_1' \in L_1$ such that

$$\begin{aligned}
\alpha_2(m_1) + (f_2 \circ \alpha_1)(l_1) &= m_2 + (f_2 \circ \alpha_1)(l_1'), \\
\alpha_2(m_1 + f_1(l_1)) &= m_2 + \alpha_2(f_1(l_1')),
\end{aligned}$$

that is, $m_2 \in \overline{\alpha_2(M_1)}$.

(d) This is a combination of (a), (b) and (c). □

Proposition 3.6 (The Five Lemma). *Let*



be a commutative diagram of semimodules with exact rows and exact first and fifth columns, and assume that M_1 and M_2 are cancellative. Then:

(a) *If α_1 and α_3 are monomorphisms, then α_2 also is a monomorphism.*

(b) *If $\alpha_2(M_1) = \overline{\alpha_2(M_1)}$ while α_1 and α_3 are regular epimorphisms, then α_2 also is a regular epimorphism.*

(c) *If $\alpha_2(M_1) = \overline{\alpha_2(M_1)}$ while α_1 and α_3 are isomorphisms, then α_2 also is an isomorphism.*

The Nine Lemma

Lemma 3.7. *Let*

$$
\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
L_1 & \xrightarrow{f_1} & M_1 & \xrightarrow{g_1} & N_1 \\
\alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow \\
L_2 & \xrightarrow{f_2} & M_2 & \xrightarrow{g_2} & N_2 \\
\beta_1 \downarrow & & \beta_2 \downarrow & & \beta_3 \downarrow \\
L_3 & \xrightarrow{f_3} & M_3 & \xrightarrow{g_3} & N_3
\end{array}
$$

be a commutative diagram with exact columns and exact second row (formed by f_2 and g_2). Then:

(a) *If f_2 is cancellative and f_3 is a monomorphism, then the first row is exact.*

(b) *If g_2 and β_1 are regular epimorphisms and the third row is exact, then the normal closure of $g_1(M_1)$ in N_1 is N_1 itself.*

Proof. (a) Notice that $\alpha_3 \circ g_1 \circ f_1 = g_2 \circ \alpha_2 \circ f_1 = g_2 \circ f_2 \circ \alpha_1 = 0$, whence $g_1 \circ f_1 = 0$ since α_3 is a monomorphism. In particular, $f_1(L_1) \subseteq \mathrm{Ker}(g_1)$.

- Suppose that $g_1(m_1) = 0$. We successively obtain the following equalities:

$$
\begin{aligned}
(\alpha_3 \circ g_1)(m_1) &= 0, \\
(g_2 \circ \alpha_2)(m_1) &= 0, \\
\alpha_2(m_1) &= f_2(l_2), \\
0 &= (\beta_2 \circ f_2)(l_2), && (\beta_2 \circ \alpha_2 = 0) \\
0 &= (f_3 \circ \beta_1)(l_2), \\
\beta_1(l_2) &= 0, && (\mathrm{Ker}(f_3) = 0) \\
l_2 &= \alpha_1(l_1), && (\alpha_1(L_1) = \mathrm{Ker}(\beta_1)) \\
f_2(l_2) &= (f_2 \circ \alpha_1)(l_1), \\
\alpha_2(m_1) &= \alpha_2(f_1(l_1)), \\
m_1 &= f_1(l_1) && (\alpha_2 \text{ is a monomorphism})
\end{aligned}
$$

- Suppose that $g_1(m_1) = g_1(m_1')$ for some $m_1, m_1' \in M_1$. We then successively obtain the following equalities:

$$
\begin{aligned}
(\alpha_3 \circ g_1)(m_1) &= (\alpha_3 \circ g_1)(m_1'), \\
(g_2 \circ \alpha_2)(m_1) &= (g_2 \circ \alpha_2)(m_1'), \\
\alpha_2(m_1) + f_2(l_2) &= \alpha_2(m_1') + f_2(l_2') && (\text{2nd row is exact}) \\
(\beta_2 \circ f_2)(l_2) &= (\beta_2 \circ f_2)(l_2') && (\beta_2 \circ \alpha_2 = 0) \\
(f_3 \circ \beta_1)(l_2) &= (f_3 \circ \beta_1)(l_2'), \\
\beta_1(l_2) &= \beta_1(l_2') && (f_3 \text{ is a monomorphism})
\end{aligned}
$$

$$\begin{aligned}
l_2 + \alpha_1(l_1) &= l_2' + \alpha_1(l_1') && \text{(first column is exact)}\\
f_2(l_2) + (f_2 \circ \alpha_1)(l_1) &= f_2(l_2') + (f_2 \circ \alpha_1)(l_1'),\\
f_2(l_2) + (\alpha_2 \circ f_1)(l_1) &= f_2(l_2') + (\alpha_2 \circ f_1)(l_1'),\\
\alpha_2(m_1) + f_2(l_2) + (\alpha_2 \circ f_1)(l_1) &= \alpha_2(m_1) + f_2(l_2') + (\alpha_2 \circ f_1)(l_1')\\
f_2(l_2') + \alpha_2(m_1' + f_1(l_1)) &= f_2(l_2') + \alpha_2(m_1 + f_1(l_1'))\\
\alpha_2(m_1' + f_1(l_1)) &= \alpha_2(m_1 + f_1(l_1')) && (f_2 \text{ is cancellative})\\
m_1' + f_1(l_1) &= m_1 + f_1(l_1') && (\alpha_2 \text{ is a monomorphism})
\end{aligned}$$

The result follows from the fact that $f_1(L_1) \subseteq \mathrm{Ker}(g_1)$.

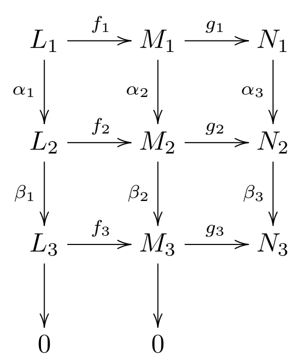
(b) Let $n_1 \in N_1$, and pick $m_2 \in M_2$ such that $g_2(m_2) = \alpha_3(n_1)$ (by assumption g_2 is a regular epimorphism). We successively obtain the following equalities:

$$\begin{aligned}
g_3(\beta_2(m_2)) &= \beta_3(g_2(m_2)),\\
&= (\beta_3 \circ \alpha_3)(m_2),\\
&= 0, && (\beta_3 \circ \alpha_3 = 0)\\
\beta_2(m_2) &= f_3(l_3), && (f_3(L_3) = \mathrm{Ker}(g_3))\\
&= f_3(\beta_1(l_2)), && (\beta_1 \text{ is a regular epimorphism})\\
&= \beta_2(f_2(l_2)),\\
m_2 + \alpha_2(m_1) &= f_2(l_2) + \alpha_2(m_1'), && \text{(2nd column is exact)}\\
g_2(m_2) + (g_2 \circ \alpha_2)(m_1) &= (g_2 \circ \alpha_2)(m_1'), && (g_2 \circ f_2 = 0)\\
\alpha_3(n_1 + g_1(m_1)) &= \alpha_3(g_1(m_1')),\\
n_1 + g_1(m_1) &= g_1(m_1'), && (\alpha_3 \text{ is injective})
\end{aligned}$$

that is, $n_1 \in \overline{g_1(M_1)}$. □

Similarly, one can prove the following result:

Lemma 3.8. *Let*



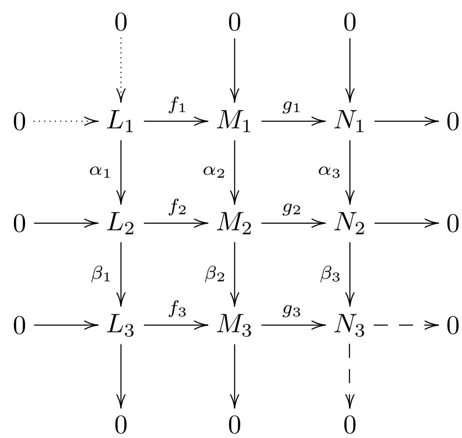
be a commutative diagram with exact columns and exact second row. Then:

(a) *If g_1 is a regular epimorphism and $f_3(L_3) = \overline{f_3(L_3)}$, then the third row is exact.*

(b) *If the first row is exact, α_2 is cancellative while f_2 and α_3 are monomorphisms, then f_3 also is a monomorphism.*

The following result is obtained immediately by combining Lemmas 3.7 and 3.8:

Proposition 3.9 (The Nine Lemma)**.** *Let*

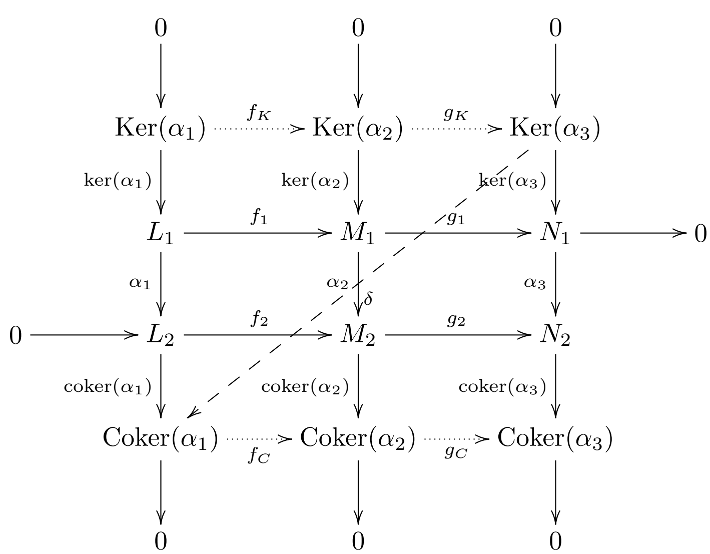


be a commutative diagram with exact columns and exact second row where M_2 *is cancellative. If* $f_3(L_3) = \overline{f_3(L_3)}$ *and* $g_1(M_1) = \overline{g_1(M_1)}$*, then the first row is exact if and only if the third row is exact.*

The Snake Lemma

One of the basic homological lemmas that are proved usually in categories of modules, or more generally in abelian categories, is the so-called *Kernel-Cokernel Lemma* (*Snake Lemma*). Several versions of this lemma were proved also in non-abelian categories that do not include the category of commutative monoids (e.g., *homological categories* [**4**], *relative homological categories* [**9**], and incomplete relative homological categories [**10**]).

Theorem 3.10 (The Snake Lemma)**.** *Let*



be a diagram of semimodules with exact columns and exact two middle rows, and *assume that the two middle squares are commutative. Then*

(a) *There exist unique morphisms* f_K, g_K, f_C, *and* g_C *which extend the diagram commutatively.*

(b) *If* f_1 *is cancellative, then the first row is exact.*

(c) *If* $f_C(\mathrm{Coker}(\alpha_1)) = \overline{f_C(\mathrm{Coker}(\alpha_1))}$, *then the last row is exact.*

(d) $\underline{\textit{There exists}}$ *a* connecting morphism $\delta\colon \mathrm{Ker}(\alpha_3) \longrightarrow \mathrm{Coker}(\alpha_1)$ *with* $\mathrm{Ker}(\delta) = \overline{g_K(\mathrm{Ker}(\alpha_2))}$, $\delta(\mathrm{Ker}(\alpha_3)) = \mathrm{Ker}(f_C)$, *and assume that* $\delta(k_3) = \delta(k_3')$ *for any* k_3 *and* $k_3' \in \mathrm{Ker}(\alpha_3)$ *implies that* $k_3 + \widetilde{k}_3 = k_3' + \widehat{k}_3$ *for some* $\widetilde{k}_3$, $\widehat{k}_3 \in \mathrm{Ker}(\delta)$.

(e) *If* α_2 *is cancellative, and* $g_K(\mathrm{Ker}(\alpha_2)) = \overline{g_K(\mathrm{Ker}(\alpha_2))}$, *then the following sequence is exact:*

$$\mathrm{Ker}(\alpha_2) \overset{g_K}{\dashrightarrow} \mathrm{Ker}(\alpha_3) \overset{\delta}{\dashrightarrow} \mathrm{Coker}(\alpha_1) \overset{f_C}{\dashrightarrow} \mathrm{Coker}(\alpha_2)$$

Proof. (a) The existence and uniqueness of the morphisms f_K, g_K, f_C, and g_C are guaranteed by the universal properties of the (co)kernels and the commutativity of the middle two squares.

(b) This follows from Lemma 3.7 applied to the first three rows.

(c) This follows from Lemma 3.8 applied to the last three rows.

(d) We show first that δ exists and is well defined.

- We define δ as follows: For $k_3 \in \mathrm{Ker}(\alpha_3)$, we choose $m_1 \in M_1$ and $l_2 \in L_2$ such that $g_1(m_1) = k_3$ and $f_2(l_2) = \alpha_2(m_1)$; notice that this is possible since g_1 is a regular epimorphism and $(g_2 \circ \alpha_2)(m_1) = (\alpha_3 \circ g_1)(m_1) = \alpha_3(k_3) = 0$ whence $\alpha_2(m_1) \in \mathrm{Ker}(g_2) = f_2(L_2)$. Define $\delta(k_3) := \mathrm{coker}(\alpha_1)(l_2) = [l_2]$, the equivalence class of $L_2/\alpha_1(L_1)$ that contains l_2.
- δ is well defined; that is, $\delta(k_3)$ is independent of our choice of $m_1 \in M_1$ and $l_2 \in L_2$ satisfying the stated conditions.

  Suppose that $g_1(m_1) = k_3 = g_1(m_1')$ for some m_1 and $m_1' \in M_1$, and that $f_2(l_2) = \alpha_2(m_1)$, $f_2(l_2') = \alpha_2(m_1')$ for some $l_2, l_2' \in L_2$. Since the second row is exact, there exist $l_1, l_1' \in L_1$ such that $m_1 + f_1(l_1) = m_1' + f_1(l_1')$. We successively obtain the following equalities:

$$\begin{aligned} \alpha_2(m_1) + (\alpha_2 \circ f_1)(l_1) &= \alpha_2(m_1') + (\alpha_2 \circ f_1)(l_1'), \\ f_2(l_2) + (f_2 \circ \alpha_1)(l_1) &= f_2(l_2') + (f_2 \circ \alpha_1)(l_1'), \\ f_2(l_2 + \alpha_1(l_1)) &= f_2(l_2' + \alpha_1(l_1')), \\ l_2 + \alpha_1(l_1) &= l_2' + \alpha_1(l_1'), \qquad (f_2 \text{ is a monomorphism}) \\ [l_2] &= [l_2']. \end{aligned}$$

  Thus l_2 and l_2' lie in the same equivalence class of $L_2/\alpha_1(L_1)$; that is, δ is well defined.
- We prove first that $\overline{g_K(\mathrm{Ker}(\alpha_2))} \subseteq \mathrm{Ker}(\delta)$. Notice that it is enough to prove that $g_K(\mathrm{Ker}(\alpha_2)) \subseteq \mathrm{Ker}(\delta)$. Let $k_3 = g_K(k_2)$ for some $k_2 \in \mathrm{Ker}(\alpha_2)$. Since the definition of δ is independent of the choice of $m_1 \in M_1$ and $l_2 \in L_2$ satisfying the given conditions above and since $g_1(k_2) = g_K(k_2) = k_3$, we can choose $m_1 = k_2$ and $l_2 = 0$. Notice that we have $f_2(l_2) = \alpha_2(m_1) = \alpha_2(k_2) = 0$, whence $l_2 = 0$ (recall that $\mathrm{Ker}(f_2) = 0$). It follows that $\delta(k_3) = [l_2] = [0]$. Consequently, $\overline{g_K(\mathrm{Ker}(\alpha_2))} \subseteq \mathrm{Ker}(\delta)$.

We prove now that $\mathrm{Ker}(\delta) \subseteq \overline{g_K(\mathrm{Ker}(\alpha_2))}$. Let $k_3 \in \mathrm{Ker}(\delta)$, and pick some $m_1 \in M_1$ and $l_2 \in L_2$ such that $g_1(m_1) = k_3$ and $f_2(l_2) = \alpha_2(m_1)$. Then, by assumption, $[l_2] = \delta(k_3) = [0]$; that is, $l_2 + \alpha_1(l_1) = \alpha_1(l_1')$ for some $l_1, l_1' \in L_1$. We successively obtain the following equalities

$$\begin{aligned} f_2(l_2) + (f_2 \circ \alpha_1)(l_1) &= (f_2 \circ \alpha_1)(l_1'), \\ \alpha_2(m_1) + \alpha_2(f_1(l_1)) &= \alpha_2(f_1(l_1')), \\ m_1 + f_1(l_1) + k_2 &= f_1(l_1') + k_2', \\ g_1(m_1) + g_1(k_2) &= g_1(k_2') & (g_1 \circ f_1 = 0) \\ k_3 + g_K(k_2) &= g_K(k_2'). \end{aligned}$$

Consequently, $\overline{g_K(\mathrm{Ker}(\alpha_2))} = \mathrm{Ker}(\delta)$.

- Let $k_3 \in \mathrm{Ker}(\alpha_3)$, and pick some $m_1 \in M_1$, $l_2 \in L_2$ such that $g_1(m_1) = k_3$ and $f_2(l_2) = \alpha_2(m_1)$. It follows that

$$(f_C \circ \delta)(k_3) = f_C([l_2]) = [f_2(l_2)] = [\alpha_2(m_1)] = [0].$$

Consequently, $\delta(\mathrm{Ker}(\alpha_3)) \subseteq \mathrm{Ker}(f_C)$. We claim that $\mathrm{Ker}(f_C) \subseteq \delta(\mathrm{Ker}(\alpha_3))$. Let $[l_2] \in \mathrm{Ker}(f_C)$, so that $[f_2(l_2)] = f_C([l_2]) = [0]$. By assumption, there exist $m_1, m_1' \in M_1$ such that $f_2(l_2) + \alpha_2(m_1) = \alpha_2(m_1')$. Since $\alpha_2(M_1) = \overline{\alpha_2(M_1)}$, there exists $\widetilde{m}_1 \in M_1$ such that $\alpha_2(\widetilde{m}_1) = f_2(l_2)$. It follows that $(\alpha_3 \circ g_1)(\widetilde{m}_1) = (g_2 \circ \alpha_2)(\widetilde{m}_1) = (g_2 \circ f_2)(l_2) = 0$. So $g_1(\widetilde{m}_1) \in \mathrm{Ker}(\alpha_3)$ and $[l_2] = \delta(g_1(\widetilde{m}_1))$. Consequently, $\mathrm{Ker}(f_C) = \delta(\mathrm{Ker}(\alpha_3))$.
- Suppose that $\delta(k_3) = \delta(k_3')$ for some $k_3, k_3' \in \mathrm{Ker}(\alpha_3)$, and pick $m_1, m_1' \in M_1$, $l_2, l_2' \in L_2$ such that $g_1(m_1) = k_3$, $g_1(m_1') = k_3'$, $\alpha_2(m_1) = f_2(l_2)$, and $\alpha_2(m_1') = f_2(l_2')$. By assumption, $[l_2] = [l_2']$; that is, $l_2 + \alpha_1(l_1) = l_2' + \alpha_1(l_1')$ for some $l_1, l_1' \in L_1$ and we successively obtain the following equalities:

$$\begin{aligned} f_2(l_2) + (f_2 \circ \alpha_1)(l_1) &= f_2(l_2') + (f_2 \circ \alpha_1)(l_1'), \\ \alpha_2(m_1) + (\alpha_2 \circ f_1)(l_1) &= \alpha_2(m_1') + (\alpha_2 \circ f_1)(l_1'), \\ m_1 + f_1(l_1) + k_2 &= m_1' + f_1(l_1') + k_2', & \text{(second column is exact)} \\ g_1(m_1) + g_1(k_2) &= g_1(m_1') + g_1(k_2'), & (g_1 \circ f_1 = 0) \\ k_3 + g_K(k_2) &= k_3' + g_K(k_2'). \end{aligned}$$

The last statement in (4) follows since $g_K(\mathrm{Ker}(\alpha_2)) \subseteq \mathrm{Ker}(\delta)$.

(e) If $g_K(\mathrm{Ker}(\alpha_2)) = \overline{g_K(\mathrm{Ker}(\alpha_2))}$, then $\mathrm{Ker}(\delta) = \overline{g_K(\mathrm{Ker}(\alpha_2))} = g_K(\mathrm{Ker}(\alpha_2))$. Suppose that $f_C[l_2] = f_C[l_2']$ for some $l_2, l_2' \in L_2$. By definition, $\exists\, m_1, m_1' \in M_1$ such that $f_2(l_2) + \alpha_2(m_1) = f_2(l_2') + \alpha_2(m_1')$. We successively obtain the following equalities:

$$\begin{aligned} (g_2 \circ \alpha_2)(m_1) &= (g_2 \circ \alpha_2)(m_1') & (g_2 \circ f_2 = 0) \\ (\alpha_3 \circ g_1)(m_1) &= (\alpha_3 \circ g_1)(m_1'), \\ g_1(m_1) + k_3 &= g_1(m_1') + k_3', \\ g_1(m_1 + \widetilde{m}_1) &= g_1(m_1' + \widehat{m}_1) & (g_1 \text{ is surjective}) \\ m_1 + \widetilde{m}_1 + f_1(\widetilde{l}_1) &= m_1' + \widehat{m}_1 + f_1(\widehat{l}_1), \\ \alpha_2(m_1) + \alpha_2(\widetilde{m}_1) + (\alpha_2 \circ f_1)(\widetilde{l}_1) &= \alpha_2(m_1') + \alpha_2(\widehat{m}_1) + (\alpha_2 \circ f_1)(\widehat{l}_1), \end{aligned}$$

$$\begin{aligned}
f_2(l_2') + \alpha_2(m_1) + \alpha_2(\widetilde{m}_1) + (f_2 \circ \alpha_1)(\widetilde{l}_1) &= [f_2(l_2') + \alpha_2(m_1')] + \alpha_2(\widehat{m}_1) \\
&\quad + (f_2 \circ \alpha_1)(\widehat{l}_1), \\
f_2(l_2') + \alpha_2(m_1) + \alpha_2(\widetilde{m}_1) + (f_2 \circ \alpha_1)(\widetilde{l}_1) &= f_2(l_2) + \alpha_2(m_1) + \alpha_2(\widehat{m}_1) \\
&\quad + (f_2 \circ \alpha_1)(\widehat{l}_1), \\
f_2(l_2') + \alpha_2(\widetilde{m}_1) + (f_2 \circ \alpha_1)(\widetilde{l}_1) &= f_2(l_2) + \alpha_2(\widehat{m}_1) + (f_2 \circ \alpha_1)(\widehat{l}_1), \\
f_2(l_2' + \widetilde{l}_2 + \alpha_1(\widetilde{l}_1)) &= f_2(l_2 + \widehat{l}_2 + \alpha_1(\widehat{l}_1)), \\
l_2' + \widetilde{l}_2 + \alpha_1(\widetilde{l}_1) &= l_2 + \widehat{l}_2 + \alpha_1(\widehat{l}_1) \quad (f_2 \text{ is injective}) \\
[l_2'] + [\widetilde{l}_2] &= [l_2] + [\widehat{l}_2], \\
[l_2'] + \delta(k_3) &= [l_2] + \delta(k_3').
\end{aligned}$$

The result follows since $\delta(\mathrm{Ker}(\alpha_3)) \subseteq \mathrm{Ker}(f_C)$. □

4. Additional Remarks

4.1. Our homological lemmas are new, taking into consideration the well-known fact that S-**SMod** is, in general, not *exact* in the sense of [**16**], *not semi-abelian* in the sense of [**11**], and not *homological* in the sense of [**4**]. Moreover, they cannot be obtained via results on *relative homological categories* in the sense of [**9**]; in particular, this applies to the results of [**10**].

4.2. Our homological lemmas allow investigating new notions for semimodules over semirings (e.g., normally flatness [**1**]). This was in fact one of the main motivations behind this paper.

4.3. S-**SMod** is *Barr-exact* [**3**] ([**8**]) with canonical *factorization system* (**Surj**, **Inj**), where **Surj** is the class of surjective morphisms (regular epimorphisms) and **Inj** is the class of injective morphisms (monomorphisms). Moreover, S-**SMod** is *homological* in the sense of [**7**].

4.4. Our definition of exact sequences in S-**SMod** is based on analyzing the notion of an exact sequence in an arbitrary pointed category relative to a given factorization system. It is consistent with the notion of an exact sequence in an arbitrary pointed regular Barr-exact category with finite limits [**4**, 4.1.7]. Moreover, our notion of a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ coincides with that of an *extension* in the sense of [**19**] (see also [**13**]).

4.5. Being a Barr-exact category, a natural tool to study exactness in S-**SMod** is that of an *exact fork* [**3**]. However, since S-**SMod** has additional features, one still expects to deal with exact sequences rather than the more complicated exact forks.

Acknowledgments

The author appreciates the efforts of the editor/referee for the feedback which significantly improved the revised version of the paper. Moreover, he thanks all the mathematicians who clarified to him some issues related to the nature of the categories of semimodules and exact sequences, especially G. Janelidze, F. Linton, A. Patchkoria, H. Porst, and R. Wisbauer.

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