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3 × 3 LEMMA FOR STAR-EXACT SEQUENCES

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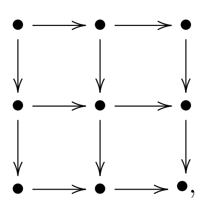
(*communicated by Maria Cristina Pedicchio*)

Abstract

A regular category is said to be *normal* when it is pointed and every regular epimorphism in it is a normal epimorphism. Any abelian category is normal, and in a normal category one can define short exact sequences in a similar way as in an abelian category. Then, the corresponding 3×3 lemma is equivalent to the so-called subtractivity, which in universal algebra is also known as congruence 0-permutability. In the context of non-pointed regular categories, short exact sequences can be replaced with "exact forks" and then, the corresponding 3×3 lemma is equivalent, in the universal algebraic terminology, to congruence 3-permutability; equivalently, regular categories satisfying such 3×3 lemma are precisely the Goursat categories. We show how these two seemingly independent results can be unified in the context of star-regular categories recently introduced in a joint work of A. Ursini and the first two authors.

1. Introduction

In an abelian category, the 3×3 lemma states that given a commutative diagram



where all three columns and the second row are short exact sequences, the top row is a short exact sequence if and only if so is the bottom row. This can be split up into *upper* and *lower* 3×3 lemmas, where the upper 3×3 lemma states only that the short exactness of the top row follows from the short exactness of the bottom one, and the lower 3×3 lemma states the converse implication. There is also a *middle*

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3×3 lemma, which states that if the composite of the two morphisms in the middle row is null and if the top and the bottom rows are short exact, then the middle row is also short exact. It was recently proved by the second author [**18**] that, in any normal category (i.e., a pointed regular category where every regular epimorphism is a normal epimorphism), the upper and the lower 3×3 lemmas are equivalent, and they hold precisely when the normal category is subtractive [**17**]. The middle 3×3 lemma turns out to be stronger than the other two, and it is equivalent to protomodularity [**3**].

The "denormalized 3×3 lemma", studied by D. Bourn in [**5**], replaces short exact sequences with exact forks, i.e., kernel pairs of regular epimorphisms. It was shown by S. Lack in [**19**] that the denormalized 3×3 lemma holds in any Goursat category [**7, 6**] and, more recently, it was proved by the first and third authors [**14**] that in a regular category, the denormalized 3×3 lemma is actually equivalent to the category being a Goursat category. Moreover, just as in the normalized case, the upper and the lower denormalized 3×3 lemmas are equivalent.

These two independent works are now brought together in the present article where we revisit them in the categorical context of a *star-regular category* proposed in [**13**], where it becomes possible to treat the normalized and the denormalized 3×3 lemmas simultaneously. The notion of a star-regular category is in some sense a merger of two notions: that of a *regular category* [**1**] and that of a category equipped with an *ideal* $\mathcal{N}$ of morphisms [**9**]. In [**13**], in a category with an ideal $\mathcal{N}$, we defined a *star* to be an ordered pair of parallel morphisms $(k_1, k_2): K \rightrightarrows X$, where the first morphism in the pair belongs to $\mathcal{N}$. The *star-kernel* (also called *kernel star*) of a morphism $f: X \to Y$ is defined as a universal star such that $fk_1 = fk_2$. Then *star-regularity* refers to the property that every regular epimorphism is a coequalizer of its star-kernel. In the case when $\mathcal{N}$ is the class of zero morphisms in a pointed category, which we call the *pointed context*, star-regular categories become precisely the normal categories, since there the notion of a star-kernel reduces to the usual notion of a kernel. In the case when $\mathcal{N}$ is the class of all morphisms, which we call the *total context*, star-kernels are precisely the kernel pairs, and so star-regular categories are the same as regular categories. Background material regarding stars, star-kernels and star-regularity is presented in Sections 2 and 3 below; at the same time, in Section 3 we clarify the connection between our general context and the quasi-pointed context of [**4**].

Replacing kernel pairs with star-kernels and the Goursat property with the "symmetric saturation property" [**12**], we extend from regular categories to star-regular categories the equivalence of the denormalized 3×3 lemma and the Goursat property (this is carried out in Sections 4 and 5). We achieve this under an additional axiom of the existence of "enough trivial objects" (studied separately in Section 3), which does hold true both in the total and (quasi-)pointed contexts. The main result, Theorem 5.2, states that, for a star-regular category with enough trivial objects, the upper 3×3 lemma, the lower 3×3 lemma and the symmetric saturation property are equivalent. Applying our result to the pointed context, the symmetric saturation property translates into subtractivity and we get precisely the equivalence of subtractivity and the normalized 3×3 lemma in normal categories.

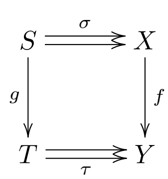
Finally, in Section 6 we extend from normal categories to star-regular categories the equivalence of the short five lemma and the middle 3×3 lemma. In fact, in a star-regular category with enough trivial objects, the short five lemma is equivalent to the short five lemma for regular epimorphisms (Theorem 6.2) and both imply that the middle 3×3 lemma holds (Proposition 6.3). The equivalence between these three lemmas is established in Theorem 6.5 in the case when regular epimorphisms are "saturating" [**12**] — an extra condition which always holds both in the total and pointed contexts. Applying this result to the total context we see that the denormalized middle 3×3 lemma holds true in any regular category. This reveals an interesting "conceptual duality" between the pointed and total contexts, where in the total context the upper/lower 3×3 lemmas are stronger than the middle 3×3 lemma, whereas in the pointed context they are weaker.

2. Stars, constellations and star-regular categories

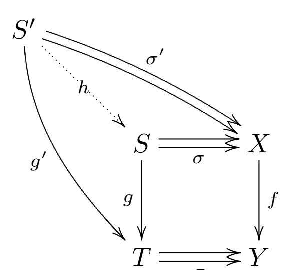
In this section we briefly exhibit some basic notions and properties concerning "stars"; we refer to [**13**] for further details.

Let $\mathbb{C}$ denote a category with finite limits, and $\mathcal{N}$ a distinguished class of morphisms that forms an *ideal*, i.e., for any composable pair of morphisms g, f, if either g or f belongs to $\mathcal{N}$, then the composite gf belongs to $\mathcal{N}$. An $\mathcal{N}$*-kernel* of a morphism $f\colon X \to Y$ is defined as a morphism $k\colon K \to X$ such that $fk \in \mathcal{N}$ and k is universal with this property (note that such k is automatically a monomorphism). A pair of morphisms, denoted by $\sigma = (\sigma_1, \sigma_2)\colon S \rightrightarrows X$ with $\sigma_1 \in \mathcal{N}$ is called a *star*; it is called a *monic* star when the pair (σ_1, σ_2) is jointly monomorphic. A star $\sigma = (\sigma_1, \sigma_2)$ with both $\sigma_1, \sigma_2 \in \mathcal{N}$ is said to be a *bi-star*.

A commutative diagram

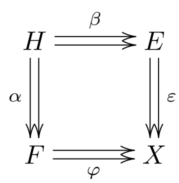


of stars and morphisms (here $f\sigma = \tau g$ means that $f\sigma_1 = \tau_1 g$ and $f\sigma_2 = \tau_2 g$) is called a *star-pullback* when given another such commutative (outer) diagram

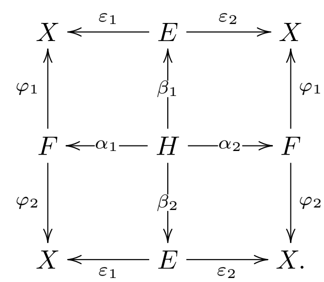


there exists a unique morphism $h\colon S' \to S$ such that $gh = g'$ and $\sigma h = \sigma'$.

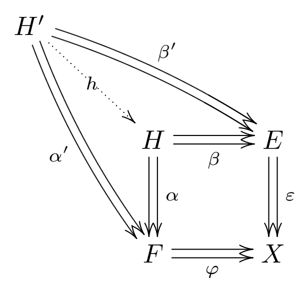
A commutative square of stars



is said to be a *constellation*; the commutativity $\varepsilon\beta = \varphi\alpha$ means that the following diagram commutes:

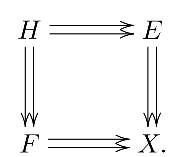


A *universal constellation* (over the stars φ and ε) is a constellation as above such that for any other (outer) constellation



there exists a unique morphism $h\colon H' \to H$ such that $\alpha h = \alpha'$ and $\beta h = \beta'$.

Example 2.1. (The total context) A double equivalence relation is an (internal) equivalence relation in the category of equivalence relations, pictured as a diagram



In particular, it gives a constellation of (monic) stars. Then, the universal constellation is given by the following classical construction (see [**8**], [**16**]):

$$H = F\square E = \{(x, y, t, z) \in X^4 \mid xFy \wedge tFz \wedge xEt \wedge yEz\}.$$

Remark 2.2. (The pointed context) A constellation in the pointed context simply amounts to a commutative square of morphisms

$$\begin{array}{ccc} H & \xrightarrow{\beta_2} & E \\ {\scriptstyle \alpha_2}\downarrow & & \downarrow{\scriptstyle \varepsilon_2} \\ F & \xrightarrow[\varphi_2]{} & X. \end{array}$$

Such a constellation is universal exactly when it is a pullback.

Given a relation $\varrho = (\varrho_1, \varrho_2)\colon R \rightrightarrows X$ on an object X, we denote by ϱ^* the biggest subrelation of ϱ which is a (monic) star; it can be constructed by setting $\varrho^* = (\varrho_1 k, \varrho_2 k)$, where k is the $\mathcal{N}$-kernel of ϱ_1. In particular, if we denote the discrete (equivalence) relation on an object X by $\Delta_X = (1_X, 1_X)\colon X \rightrightarrows X$, then $\Delta_X^* = (k_X, k_X)$, where k_X denotes the $\mathcal{N}$-kernel of 1_X.

The *star-kernel* of a morphism $f\colon X \to Y$ is a universal star $\kappa = (\kappa_1, \kappa_2)\colon K \rightrightarrows X$ with the property $f\kappa_1 = f\kappa_2$ (such a star is then automatically a monic star); it is easy to see that the star-kernel of f coincides with κ_f^*, where κ_f denotes the kernel pair of f.

Following the terminology used in [**13**], a category $\mathbb{C}$ equipped with an ideal $\mathcal{N}$ is called a *multi-pointed* category. If, moreover, every morphism admits an $\mathcal{N}$-kernel, then $\mathbb{C}$ will be called a *multi-pointed category with kernels.*

Definition 2.3. [**13**] A regular multi-pointed category $\mathbb{C}$ with kernels is said to be *star-regular* when every regular epimorphism in $\mathbb{C}$ is a coequalizer of a star.

In the total context, a star-regular category is precisely a regular category. In the pointed context, a star-regular category is the same as a normal category [**18**], i.e., a regular category in which any regular epimorphism is a normal epimorphism.

Throughout the paper, we omit those proofs which closely mimic corresponding results in the total or pointed context that are usually well known. Below is one such result:

Lemma 2.4. [**13**] *In a multi-pointed category, consider a commutative diagram*

$$\begin{array}{ccccc} P & \overset{\lambda}{\rightrightarrows} & X & \xrightarrow{c} & C \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle m} \\ K & \underset{\kappa}{\rightrightarrows} & Y & \xrightarrow[d]{} & Q \end{array}$$

of morphisms and stars.

(a) Suppose κ is a star-kernel of d and m is a monomorphism. Then, the left square is a star-pullback if and only if λ is a star-kernel of c.

(b) Suppose c is a coequalizer of λ and e is an epimorphism. Then, the right square is a pushout if and only if d is a coequalizer of κ.

The following lemma, in the total context, appears as Proposition 1.1 in [**5**]:

Lemma 2.5. *In a star-regular category, consider a commutative diagram*

$$\begin{array}{ccccc} P & \overset{\lambda}{\rightrightarrows} & X & \overset{c}{\longrightarrow} & C \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle m} \\ K & \underset{\kappa}{\rightrightarrows} & Y & \underset{d}{\longrightarrow} & Q \end{array}$$

of stars and morphisms, where κ is a star-kernel of d and c is a regular epimorphism. Any two of the following conditions imply the third one:

(a) m is a monomorphism;

(b) λ is a star-kernel of c;

(c) the left hand side square is a star-pullback.

3. Existence of enough trivial objects

An object X is said to be $\mathcal{N}$*-trivial* when $1_X \in \mathcal{N}$; equivalently, X is $\mathcal{N}$-trivial when any morphism whose domain or codomain is X belongs to $\mathcal{N}$. When $\mathcal{N}$-kernels exist, we have that if a composite fg belongs to $\mathcal{N}$ and g is a strong epimorphism, then also f belongs to $\mathcal{N}$. This implies that $\mathcal{N}$-trivial objects are closed under strong quotients.

Definition 3.1. We say that there are *enough trivial objects* in a multi-pointed category $\mathbb{C}$ when $\mathcal{N}$ is a closed ideal [**15**], i.e., any morphism in $\mathcal{N}$ factors through an $\mathcal{N}$-trivial object, and moreover, the class of $\mathcal{N}$-trivial objects is closed under subobjects and squares (the latter meaning that, for any $\mathcal{N}$-trivial object X, the object $X^2 = X \times X$ is $\mathcal{N}$-trivial).

In a regular multi-pointed category, a sufficient condition for the presence of enough trivial objects is when $\mathcal{N}$ is a closed ideal and $\mathcal{N}$-trivial objects have neither proper subobjects nor proper quotients. More elaborately, we have the following (see also Proposition 3.4 below):

Proposition 3.2. *In a multi-pointed category $\mathbb{C}$ with finite powers, if $\mathcal{N}$ is a closed ideal and $\mathcal{N}$-trivial objects do not have proper subobjects (i.e., every monomorphism with $\mathcal{N}$-trivial codomain is an isomorphism), then the following conditions are equivalent:*

(a) There are enough trivial objects in $\mathbb{C}$.

(b) Every morphism $W \to 1$ from an $\mathcal{N}$-trivial object W to the terminal object is a monomorphism.

(c) Every morphism $W \to X$ whose domain is an $\mathcal{N}$-trivial object, is a monomorphism.

When $\mathbb{C}$ is a regular category, the conditions above are also equivalent to the following one:

(d) $\mathcal{N}$-trivial objects do not have proper regular quotients (i.e., any regular epimorphism with $\mathcal{N}$-trivial domain is an isomorphism).

Proof. (a)⇔(b): The morphism $W \to 1$ is a monomorphism if and only if in the pullback

$$\begin{array}{ccc} W \times W & \xrightarrow{\pi_2} & W \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ W & \longrightarrow & 1 \end{array}$$

π_1 is an isomorphism. Suppose W is $\mathcal{N}$-trivial. If (a) holds, then $W \times W$ is also $\mathcal{N}$-trivial and hence the diagonal $\langle 1_W, 1_W\rangle \colon W \to W \times W$ is an isomorphism. Since π_1 is its right inverse, it follows that π_1 is also an isomorphism. Conversely, if (b) holds, then π_1 is an isomorphism and hence $W \times W$ is $\mathcal{N}$-trivial. This implies that (a) holds, since $\mathcal{N}$-trivial objects do not have proper subobjects, and hence their subobjects are *trivially* $\mathcal{N}$-trivial.

(b)⇔(c): This follows from the fact that if in the commutative triangle

$$\begin{array}{ccc} W & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & 1 & \end{array}$$

the bottom left arrow is a monomorphism, then so is the top one.

(c)⇔(d) is straightforward (for (d)⇒(c) use the fact any morphism decomposes as a regular epimorphism followed by a monomorphism). □

According to the following proposition, under the presence of $\mathcal{N}$-kernels, the case when $\mathcal{N}$ is a closed ideal such that $\mathcal{N}$-trivial objects do not have proper subobjects gives precisely the proto-pointed context in the sense of [**13**] (see also [**22**]).

Proposition 3.3. *For any category $\mathbb{C}$, the following conditions are equivalent:*

- *(a) Every object in $\mathbb{C}$ has smallest subobject and $\mathbb{C}$ has pullbacks of smallest subobjects along arbitrary morphisms.*
- *(b) $\mathbb{C}$ has a closed ideal $\mathcal{N}$ for which $\mathcal{N}$-kernels exist and such that $\mathcal{N}$-trivial objects do not have proper subobjects.*

Moreover, when the above conditions are satisfied, the class $\mathcal{N}$ above is unique and consists of those morphisms f which factor through the smallest subobject of the codomain of f. Further, $\mathcal{N}$-trivial objects are precisely those who do not have proper subobjects.

Proof. (a)⇒(b): Let $\mathcal{N}$ be defined as at the end of the proposition. It is easy to see that $\mathcal{N}$ is an ideal and that $\mathcal{N}$-trivial objects are precisely those who do not have proper subobjects. The $\mathcal{N}$-kernel of a morphism $f \colon X \to Y$ can be constructed as the pullback of the smallest subobject of Y along f. In particular, the $\mathcal{N}$-kernel of an identity morphism $1_X \colon X \to X$ is the smallest subobject $k \colon K \to X$ of X. Since k is the smallest subobject of X, its domain K cannot have a proper subobject and thus K is $\mathcal{N}$-trivial. Since any morphism $n \colon W \to X$ in the class $\mathcal{N}$ factors through k (by definition of $\mathcal{N}$), we can conclude that $\mathcal{N}$ is a closed ideal.

(b)⇒(a): Let X be an object in $\mathbb{C}$ and consider the $\mathcal{N}$-kernel $k_X \colon K_X \to X$ of an identity morphism $1_X \colon X \to X$. Then $k_X \in \mathcal{N}$ and since $\mathcal{N}$ is a closed ideal, it factors through an $\mathcal{N}$-trivial object. The fact that k_X is a monomorphism implies that K_X is a subobject of the same trivial object, which in turn implies that K_X is itself trivial. Now, we prove that k_X is the smallest subobject of X. Let $m\colon M \to X$ be any other subobject. Consider the $\mathcal{N}$-kernel n of m. Since the composite mn belongs to $\mathcal{N}$, it must factor through k_X, and so $k_X l = mn$ for some monomorphism l. Then, l is a subobject of K_X and hence must be an isomorphism, since by assumption $\mathcal{N}$-trivial objects do not have proper subobjects. Thus k_X factors through m as $k_X = mnl^{-1}$. Thus we have shown that $\mathcal{N}$-kernels of identity morphisms are precisely the smallest subobjects, which trivially implies that $\mathcal{N}$ is the class of those morphisms f which factor through the smallest subobject of the codomain of f. Finally, the pullback of a smallest subobject of an object X, along arbitrary morphism $f\colon W \to X$, can be constructed as the $\mathcal{N}$-kernel of f. □

When the existence of $\mathcal{N}$-kernels is required together with the existence of enough trivial objects, we get precisely the quasi-pointed context studied in [**4**]:

Proposition 3.4. *For a finitely complete category $\mathbb{C}$, the following conditions are equivalent:*

- *(a) $\mathbb{C}$ has an initial object 0 such that the morphism $0 \to 1$ from 0 to the terminal object 1 is a monomorphism, i.e., $\mathbb{C}$ is a quasi-pointed category.*
- *(b) $\mathbb{C}$ has a closed ideal $\mathcal{N}$ for which $\mathcal{N}$-kernels exist, $\mathcal{N}$-trivial objects do not have proper subobjects, and there are enough $\mathcal{N}$-trivial objects.*

Moreover, when the above conditions are satisfied, the class $\mathcal{N}$ in (b) above is unique and is equal to the class of those morphisms which factor through an initial object. Further, $\mathcal{N}$-trivial objects are precisely the initial objects.

Proof. (a)⇒(b): Then, for any object X, the unique morphism $0 \to X$ is the smallest subobject of X. Applying Proposition 3.3, we get that the class $\mathcal{N}$ consisting of those morphisms which factor through 0 is a closed ideal for which $\mathcal{N}$-kernels exist and such that $\mathcal{N}$-trivial objects do not have proper subobjects. Further, then objects who do not have proper subobjects are precisely the initial objects, and hence, by Proposition 3.3, initial objects are precisely the $\mathcal{N}$-trivial objects. Now, apply Proposition 3.2 to conclude that there are enough $\mathcal{N}$-trivial objects. This proves (b), while uniqueness of $\mathcal{N}$ follows from Proposition 3.3.

(b)⇒(a): By assumption $\mathcal{N}$-trivial objects are those that do not have proper subobjects. Then, since $\mathcal{N}$ is a closed ideal and $\mathbb{C}$ is nonempty (it contains the terminal object), there must exist at least one such object 0. We show that 0 is an initial object. First, observe that for any object X there can exist at most one morphism $0 \to X$, for, if there are two different morphisms, their equalizer will be a proper subobject of 0. Let $k_X\colon K \to X$ be the smallest subobject of X (which exists by Proposition 3.3 — it is the $\mathcal{N}$-kernel of 1_X as shown in the proof of the same proposition). Then K does not have proper subobjects and so in the pullback

$$\begin{array}{ccc} 0 \times K & \xrightarrow{\pi_2} & K \\ {\scriptstyle \pi_1}\downarrow & & \downarrow \\ 0 & \longrightarrow & 1 \end{array}$$

the morphism π_1, which is a monomorphism since $K \to 1$ is a monomorphism (by Proposition 3.2), must be an isomorphism. This shows that there exists a morphism from 0 to K, and hence there exists a morphism from 0 to X. We have thus shown that 0 is an initial object. By Proposition 3.2, the morphism $0 \to 1$ is a monomorphism. □

Obviously, both in the total and pointed contexts, there are enough trivial objects. As it follows from Proposition 3.4, the proto-pointed context admitting enough trivial objects is precisely the quasi-pointed context. In the quasi-pointed context, for any two objects X and Y there exists at most one morphism $X \to Y$ which belongs to $\mathcal{N}$. Then, the notion of a star-kernel reduces to the notion of an $\mathcal{N}$-kernel, just as in the pointed context. Further, star-exact sequences introduced in Section 4 below become precisely the short exact sequences defined in [**4**] in sequentiable categories, which are protomodular regular quasi-pointed categories (and hence they are also protomodular star-regular categories, as it follows from Proposition 3 in [**4**] which, in our terminology, states that in a protomodular quasi-pointed category a morphism is a regular epimorphism if and only if it is a coequalizer of its star-kernel).

Quasi-pointed varieties of universal algebras which are not pointed are precisely those where the empty set is an algebra. Among these, star-regular ones are those where the identity $x = y$ holds in the corresponding algebraic theory, which forces the category to be equivalent to the ordered set $(\{0, 1\}, \leqslant)$ regarded as a category. This shows that the only "interesting" example of star-regularity in quasi-pointed (one-sorted) varieties of algebras is the star-regularity in the pointed context, which is exactly the context studied in [**11**] (which, in modern terminology, is the context of pointed 0-regular varieties). Some interesting examples of star-regular quasi-pointed categories that are *not* pointed do occur, however, in other algebraic contexts. For instance, the category Grpd_X of groupoids over a fixed set X of objects, which can be presented as a variety of many-sorted algebras (where sorts would be pairs of elements of X), is a quasi-pointed star-regular category which is rarely pointed (see Example 6.6). Similar remarks apply when we consider $\mathrm{Grpd}_X(\mathbb{C})$ internal to any variety $\mathbb{C}$ (see e.g., [**10**], where the special case of Mal'tsev varieties is investigated). In particular, when $\mathbb{C}$ is the variety of groups, $\mathrm{Grpd}_X(\mathbb{C})$ is equivalent to the category of crossed modules over a fixed group X.

We conclude this section with the following result, which will be useful in the subsequent section:

Proposition 3.5. *Let $\mathbb{C}$ be a regular multi-pointed category with kernels. The following conditions are equivalent:*

(a) If a relation $\varrho\colon R \rightrightarrows X$ is a bi-star, then R is an $\mathcal{N}$-trivial object.

(b) If $(s_1, s_2)\colon S \rightrightarrows X$ is a relation such that $s_1 n, s_2 n \in \mathcal{N}$, then $n \in \mathcal{N}$.

(c) In a diagram

$$\begin{array}{ccc} H & \overset{\beta}{\rightrightarrows} & E \\ {\scriptstyle\alpha}\downdownarrows & & \downdownarrows{\scriptstyle\varepsilon} \\ F & \underset{\varphi}{\rightrightarrows} & X \end{array}$$

with the usual commutativity conditions, if ε is a monic star and φ is a star, *then β is a star.*

(d) $\mathbb{C}$ has enough trivial objects.

Proof. The less trivial part of the proof is the implication (d)⇒(a). We begin by considering $k_X\colon K_X \to X$, the $\mathcal{N}$-kernel of 1_X. Since k_X belongs to $\mathcal{N}$, it factors through an $\mathcal{N}$-trivial object T. So K_X is a subobject of T, hence it is also an $\mathcal{N}$-trivial object. Now, if ρ is a bi-star, then both ρ_1 and ρ_2 factor through k_X; say $\rho_1 = k_X\lambda_1$ and $\rho_2 = k_X\lambda_2$. We get a monomorphism $\langle \lambda_1, \lambda_2 \rangle\colon R \to K_X \times K_X$ and, consequently, R is an $\mathcal{N}$-trivial object. □

4. Star-exact sequences and the 3×3 lemma

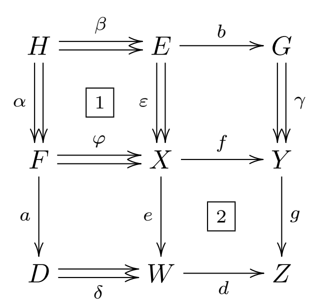
In a star-regular category, a *(short) star-exact sequence* is a diagram

$$K \overset{\kappa}{\rightrightarrows} X \xrightarrow{f} Y,$$

where $\kappa = (\kappa_1, \kappa_2)$ is a star-kernel of f and f is a coequalizer of κ_1 and κ_2 (which, by star-regularity, is the same as to say that f is a regular epimorphism). In the total context, the notion of a star-kernel of a morphism becomes the notion of a kernel pair of a morphism and a star-exact sequence is just an exact fork, while in the pointed context they represent a kernel of a morphism and a short exact sequence, respectively.

In this section we formulate a 3×3 lemma for star-exact sequences. Its diagrammatic shape, therefore, resembles the denormalized 3×3 lemma [**5**], although it captures both the denormalized 3×3 lemma for the total context as well as the 3×3 lemma in the pointed context [**4**] (and more generally, the 3×3 lemma in the "quasi-pointed context", see [**4**]), which in the case of abelian categories gives the classical 3×3 lemma.

In a star-regular category, our 3×3 lemma concerns a commutative diagram



(1)

of stars and morphisms. In particular, the commutativity of $\boxed{1}$ means that $\boxed{1}$ is a constellation. A commutative diagram (1) will be called a 3×3 *diagram* when all columns are star-exact sequences. The *upper* 3×3 *lemma* states that in a 3×3 diagram (1), if the second and third rows are star-exact sequences, then so is the first row; the *lower* 3×3 *lemma* states that, if the first and second rows are star-exact sequences, then so is the third row. The *middle* 3×3 *lemma* states that, in a 3×3 diagram (1), where $f\varphi_1 = f\varphi_2$ and the first and third rows are star-exact sequences, the second row is a star-exact sequence. In the total context, the upper and the lower

3×3 lemmas are equivalent, and they hold in a regular category precisely when it is a Goursat category [**14**]. In the pointed context, they are also equivalent, and they hold precisely when the normal category is subtractive [**18**]. In the total context the middle 3×3 lemma always holds true and, consequently, the denormalized 3×3 lemma for regular categories is usually stated only in the upper and lower formulations. However, in the pointed context, the middle 3×3 lemma is very meaningful since it characterizes protomodular categories [**4, 18**].

In the rest of this section we study conditions which lead to a technical simplification of our upper and lower 3×3 lemmas (see Section 5). This study is essentially a direct generalization of the corresponding study in the total context carried out in [**5**]. For the generalization to work, we need existence of enough trivial objects (see Section 3), which we have both in the total and pointed contexts.

While our approach to the 3×3 lemma for star-exact sequences is very similar to the one adopted in [**5**] for the denormalized 3×3 lemma, it is quite different from the approach to the normalized 3×3 lemma used in [**18**] which is based on the classical diagram chasing method (a direct "denormalization" of the approach used in [**18**] seems to fail).

In a star-regular category, when all columns and rows in a given diagram (1) are star-exact sequences, then certain properties concerning $\boxed{1}$ and $\boxed{2}$ must hold (for the first part, we require the existence of enough trivial objects). In the total context, $\boxed{1}$ necessarily represents the double equivalence relation $F \square E$ and $\boxed{2}$ is a pushout (Proposition 2.1 in [**5**]) and in the pointed context it is easy to see that $\boxed{1}$ must be a pullback and $\boxed{2}$ a pushout. For the general context, these conditions translate into: $\boxed{1}$ is a universal constellation and $\boxed{2}$ is a pushout. We can get the condition on the pushout from the following proposition, which is an immediate consequence of Lemma 2.4 (b):

Proposition 4.1. *In any star-regular category, let* (1) *be a* 3×3 *diagram with a star-exact middle row. The square* $\boxed{2}$ *is a pushout if and only if* d *is a coequalizer of* δ*.*

We get the condition on the universal constellation from the following theorem, which will be proved throughout the rest of this section:

Theorem 4.2. *In a star-regular category with enough trivial objects, consider a commutative diagram of stars and morphisms* (1)*, where the first column is a star-exact sequence,* ε *is a star-kernel of* e*,* φ *is a star-kernel of* f*, and* γ *is monic. Then the following conditions are equivalent:*

(a) δ *is a monic star;*

(b) β *is a star-kernel of* b*;*

(c) $\boxed{1}$ *is a universal constellation.*

The above theorem extends Theorem 2.2 of [**5**] to the star-regular context. Moreover, our proof of the above theorem follows, step-by-step, the proof given in [**5**]. The technical observations contained in this proof, as well as the theorem itself, are used in Section 5 to establish the equivalence of the upper and lower 3×3 lemmas in star-regular categories with enough trivial objects.

We begin by observing that under the presence of enough trivial objects, we have a stability property for star-kernels with respect to products:

Lemma 4.3. *Suppose that $\mathbb{C}$ is a multi-pointed category with enough trivial objects. Then, a pair $\varepsilon = (\varepsilon_1, \varepsilon_2) \colon E \rightrightarrows X$ is a star if and only if so is the pair $\varepsilon \times \varepsilon = (\varepsilon_1 \times \varepsilon_1, \varepsilon_2 \times \varepsilon_2) \colon E \times E \rightrightarrows X \times X$. Moreover, ε is a star-kernel of $e \colon X \to W$ if and only if $\varepsilon \times \varepsilon$ is a star-kernel of $e \times e \colon X \times X \to W \times W$.*

Proof. The non-trivial part of the proof is to show that $\varepsilon \times \varepsilon$ is a star whenever ε is. This follows by applying Proposition 3.5(c) to the diagram

$$\begin{array}{ccc} E \times E & \overset{\varepsilon \times \varepsilon}{\rightrightarrows} & X \times X \\ {\scriptstyle \pi_1} \downdownarrows {\scriptstyle \pi_2} & & {\scriptstyle \pi_1} \downdownarrows {\scriptstyle \pi_2} \\ E & \underset{\varepsilon}{\rightrightarrows} & X. \end{array}$$

□

The following proposition characterizes universal constellations involving a star φ and a monic star ε. The requirement that ε below is a monic star can be dropped in the total context, in which case the result below becomes precisely Remark 2.2 of [**5**].

Proposition 4.4. *In a multi-pointed category $\mathbb{C}$ with enough trivial objects, consider a constellation*

$$\begin{array}{ccc} H & \overset{\beta}{\rightrightarrows} & E \\ {\scriptstyle \alpha} \downdownarrows & & \downdownarrows {\scriptstyle \varepsilon} \\ F & \underset{\varphi}{\rightrightarrows} & X, \end{array} \tag{2}$$

where ε is a monic star. If $\mathbb{C}$ has enough trivial objects, then the following conditions are equivalent:

- *(a) The constellation* (2) *is universal.*
- *(b) The commutative diagram*

$$\begin{array}{ccc} H & \xrightarrow{\langle \beta_1, \beta_2 \rangle} & E \times E \\ {\scriptstyle \alpha} \downdownarrows & & \downdownarrows {\scriptstyle \varepsilon \times \varepsilon} \\ F & \xrightarrow[\langle \varphi_1, \varphi_2 \rangle]{} & X \times X \end{array} \tag{3}$$

  is a star-pullback.

Proof. First, note that diagram (2) commutes if and only if diagram (3) commutes. Lemma 4.3 guarantees that $\varepsilon \times \varepsilon$ is also a star. It is easy to see that the universal property of the constellation is the same as the universal property of the star-pullback.

□

The following propositions characterize universal constellations which are part of diagrams involving stars and morphisms.

Proposition 4.5. *In a star-regular category with enough trivial objects, consider a commutative diagram of stars and morphisms*

$$\begin{array}{ccc} H & \overset{\beta}{\rightrightarrows} & E \\ {\scriptstyle\alpha}\downdownarrows & \boxed{1} & \downdownarrows{\scriptstyle\varepsilon} \\ F & \underset{\varphi}{\rightrightarrows} & X \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e} \\ D & \underset{\delta}{\rightrightarrows} & W, \end{array}$$

where the left column is a star-exact sequence and ε *is the star-kernel of* e*. Then* $\boxed{1}$ *is a universal constellation if and only if* δ *is monic.*

Proof. Use Proposition 4.4, Lemma 4.3 and apply Lemma 2.5 to the following diagram:

$$\begin{array}{ccc} H & \xrightarrow{\langle\beta_1,\beta_2\rangle} & E\times E \\ {\scriptstyle\alpha}\downdownarrows & & \downdownarrows{\scriptstyle\varepsilon\times\varepsilon} \\ F & \xrightarrow[\langle\varphi_1,\varphi_2\rangle]{} & X\times X \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e\times e} \\ D & \xrightarrow[\langle\delta_1,\delta_2\rangle]{} & W\times W. \end{array}$$

□

Using a similar argument as in the proof of the above proposition, we have:

Proposition 4.6. *In a multi-pointed category with enough trivial objects, consider a commutative diagram of stars and morphisms*

$$\begin{array}{ccccc} H & \overset{\beta}{\rightrightarrows} & E & \xrightarrow{b} & G \\ {\scriptstyle\alpha}\downdownarrows & \boxed{1} & \downdownarrows{\scriptstyle\varepsilon} & & \downdownarrows{\scriptstyle\gamma} \\ F & \underset{\varphi}{\rightrightarrows} & X & \xrightarrow[f]{} & Y, \end{array}$$

where φ *is a star-kernel of* f *and* γ *is monic. Then* $\boxed{1}$ *is a universal constellation if and only if* β *is a star-kernel of* b*.*

Proof. Use Proposition 4.4, Lemma 4.3 and apply Lemma 2.4(a) to the following diagram:

$$\begin{array}{ccccc} H & \overset{\beta}{\rightrightarrows} & E & \xrightarrow{b} & G \\ {\scriptstyle\langle\alpha_1,\alpha_2\rangle}\downarrow & & \downarrow{\scriptstyle\langle\varepsilon_1,\varepsilon_2\rangle} & & \downarrow{\scriptstyle\langle\gamma_1,\gamma_2\rangle} \\ F\times F & \underset{\varphi\times\varphi}{\rightrightarrows} & X\times X & \xrightarrow[f\times f]{} & Y\times Y. \end{array}$$

□

Altogether, this proves Theorem 4.2.

5. The equivalence of the upper and lower 3×3 lemmas

Thanks to Theorem 4.2, the upper 3×3 lemma can be equivalently reformulated as follows: in a 3×3 diagram (1), if the second and third rows are star-exact sequences, then b is a regular epimorphism. Similarly, the lower 3×3 lemma becomes: in a 3×3 diagram (1), if the first and second rows are star-exact sequences, then δ (which becomes a relation) is a star-kernel of d. In this section we shall investigate each of these lemmas separately.

We begin by recalling some terminology from [**12**]. By a *diamond* we mean a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle e}\swarrow & & \searrow{\scriptstyle f} & \\ W & & & & Y \\ & {\scriptstyle d}\searrow & & \swarrow{\scriptstyle g} & \\ & & Z. & & \end{array} \qquad (4)$$

(Note that this use of the term "diamond" is different from the one in [**16**].) We say that the diamond (4) is:

- *left saturated* if the direct image $e\langle\kappa_f^*\rangle$ along e of the star-kernel κ_f^* of f is the star-kernel of d:
$$e\langle\kappa_f^*\rangle = \kappa_d^*;$$
- *right saturated* if, symmetrically, $f\langle\kappa_e^*\rangle = \kappa_g^*$;
- *saturated* if it is both left and right saturated;
- *a regular diamond* if all morphisms in the diamond are regular epimorphisms.

Definition 5.1. [**12**] $\mathbb{C}$ is said to have the *symmetric saturation property* if the following equivalent conditions hold:

(a) Any left saturated regular diamond is right saturated.
(b) Any right saturated regular diamond is left saturated.
(c) Left/right saturated regular diamonds are the same as the saturated ones.

Theorem 5.2. *In a star-regular category $\mathbb{C}$ with enough trivial objects, the following conditions are equivalent:*

(a) The upper 3×3 lemma holds in $\mathbb{C}$.

(b) The lower 3×3 *lemma holds in* $\mathbb{C}$.

(c) $\mathbb{C}$ *has the symmetric saturation property.*

Proof. (a)⇔(c): Suppose first the upper 3×3 lemma holds true. From a left saturated regular diamond (4) build a diagram (1) by attaching to the diamond star-kernels of its edges, and the induced factorizations, and completing the top left square in (1) with a universal constellation. By Proposition 4.6 and by the left saturation of the diamond, the first column in (1) is a star-exact sequence. Then (1) is a 3×3 diagram. Applying the upper 3×3 lemma we get that the diamond is right saturated.

The converse is trivial: to get the upper 3×3 lemma, apply the saturation assumption to the diamond that appears as the bottom right square in a 3×3 diagram.

(b)⇔(c): Suppose first the lower 3×3 lemma holds true. From a right saturated regular diamond (4) build a diagram (1) by attaching to the diamond star-kernels of its edges e, f, g, and the induced factorization from the star-kernel of e to the star-kernel of g, which is a regular epimorphism since the diamond is right saturated. Then, take the top left square of (1) to be the universal constellation. Complete the bottom left square of (1) via the regular image of the star-kernel of f along e. By Proposition 4.6, the first column is a star-exact sequence. Now, by Proposition 4.6 again the first row is a star-exact sequence. Applying the lower 3×3 lemma we get precisely the left saturation of the diamond.

Conversely, consider a 3×3 diagram (1) where the first two rows are exact. By Theorem 4.2, δ is monic, and since a is a regular epimorphism, we get that δ is the image of φ under e. Now, the bottom right square is right saturated, and therefore left saturated, which gives the exactness of the bottom row. □

As shown in [**12**], in the total context the symmetric saturation property is equivalent to the Goursat property, while in the pointed context, it is equivalent to subtractivity. So the above result unifies Proposition 1 of [**14**] (which is then exactly Theorem 5.2 in the total context) with Theorem 5.4 of [**18**] (which is the same as Theorem 5.2 in the pointed context).

The following corollary of the above theorem partially refines the theorem:

Corollary 5.3. *In each of the pointed, proto-pointed and total contexts, for a star-regular category* $\mathbb{C}$ *the following conditions are equivalent:*

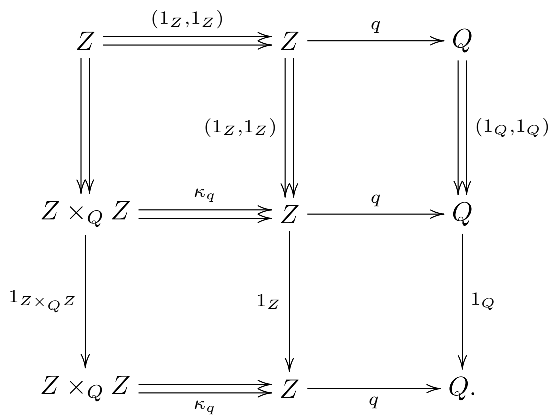
(a) The upper 3×3 *lemma holds in* $\mathbb{C}$.

(b) The lower 3×3 *lemma holds in* $\mathbb{C}$.

(c) $\mathbb{C}$ *has enough trivial objects and has the symmetric saturation property.*

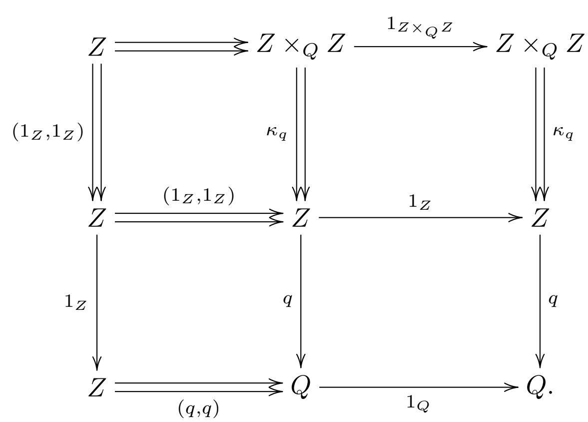
Proof. In the total and in the pointed contexts this becomes precisely Theorem 5.2, since in these contexts there always are enough trivial objects. Also, after Theorem 5.2, to prove the equivalence of the above conditions in the proto-pointed context, it suffices to prove that the upper and the lower 3×3 lemmas each imply the presence of enough trivial objects. According to Proposition 3.2, showing the presence of enough trivial objects is equivalent to showing that trivial objects do not have proper quotients. For this, the following fact, which follows directly from Proposition 3.3, will be needed: for any object X, a morphism $W \to X \times X$ from the class $\mathcal{N}$ always factors through the diagonal $X \to X \times X$. Let Z be a trivial object and let

$q\colon Z \to Q$ be a regular epimorphism. Then the object Q, being a regular quotient of a trivial object is itself trivial. Now, using the fact mentioned above, the following 3×3 diagram can be constructed, with the middle and the bottom rows being star-exact:



(5)

The upper 3×3 lemma implies that the top row is star-exact, and hence, by star-regularity, q is a coequalizer of the pair $(1_Z, 1_Z)\colon Z \rightrightarrows Z$, which shows that q is an isomorphism. To deduce that q is an isomorphism from the lower 3×3 lemma, we should construct the following 3×3 diagram:



(6)

Since the top and the middle rows are star-exact, the lower 3×3 lemma implies that the lower row is star-exact. In particular, this gives that the star $(q, q)\colon Z \rightrightarrows Q$ is monic, and hence q is a monomorphism. Since q is at the same time a regular epimorphism, it follows that q is an isomorphism. □

Remark 5.4. The above corollary can be used to deduce that our upper and lower 3×3 lemmas fail in the proto-pointed context of the category **Rng** of unitary rings. This can be also seen directly, by choosing Z in diagrams (5) and (6) to be the ring $\mathbb{Z}$ of integers, and Q any of its proper quotients. Since **Rng** has a good theory of ideals (see [**22, 13**]), this shows that in general, in a category with a good theory of ideals, both the upper and the lower 3×3 lemmas may fail.

6. The middle 3×3 lemma

In the context of star-regular categories, the *short five lemma* states that, given a commutative diagram of horizontal star-exact sequences

$$\begin{array}{ccccc} F & \overset{\varphi}{\rightrightarrows} & X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle g} \\ D & \underset{\delta}{\rightrightarrows} & W & \xrightarrow[d]{} & Z, \end{array} \tag{7}$$

if a and g are isomorphisms then e is an isomorphism.

In the pointed context, this becomes the classical short five lemma, while in the total context this lemma (which can be called the "denormalized short five lemma") always holds true. Indeed, the fact that φ and δ are both reflexive relations, together with a being an isomorphism, imply that e is both a monomorphism and a split epimorphism, thus an isomorphism.

Lemma 6.1. *In a star-regular category where the short five lemma holds, given a commutative diagram* (7) *of horizontal star-exact sequences, e is a regular epimorphism whenever a and g are regular epimorphisms (we say in this case that the short five lemma for regular epimorphisms holds).*

Proof. Suppose the short five lemma holds. We would like to show that in the diagram

$$\begin{array}{ccccc} K_f & \overset{\kappa_f^*}{\rightrightarrows} & X & \overset{f}{\twoheadrightarrow} & Y \\ {\scriptstyle a}\big\downarrow & & \downarrow{\scriptstyle e} & & \big\downarrow{\scriptstyle g} \\ K_d & \underset{\kappa_d^*}{\rightrightarrows} & W & \underset{d}{\twoheadrightarrow} & Z \end{array}$$

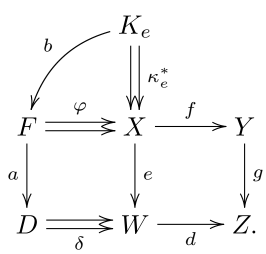
with a and g regular epimorphisms, e is also a regular epimorphism. This diagram can be decomposed as follows, where $ip = e$ is the factorization of e as a regular epimorphism p followed by a monomorphism i:

$$\begin{array}{ccccc} K_f & \overset{\kappa_f^*}{\rightrightarrows} & X & \overset{f}{\twoheadrightarrow} & Y \\ {\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle g} \\ K_d & \overset{\sigma}{\rightrightarrows} & I & \overset{di}{\twoheadrightarrow} & Z \\ {\scriptstyle 1_{K_d}}\big\downarrow & & \downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle 1_Z} \\ K_d & \underset{\kappa_d^*}{\rightrightarrows} & W & \underset{d}{\twoheadrightarrow} & Z. \end{array}$$

In the above diagram we get the induced star $\sigma \colon K_d \rightrightarrows I$ due to the fact that a is a regular epimorphism. Since the left hand side of the bottom part of the above diagram is a star-pullback, applying Lemma 2.4 we get that σ is the star-kernel of di. Now, the short five lemma implies that i is an isomorphism, and hence e is a regular epimorphism, as desired. □

Theorem 6.2. *In a star-regular category with enough trivial objects, the short five lemma holds if and only if the short five lemma for regular epimorphisms holds.*

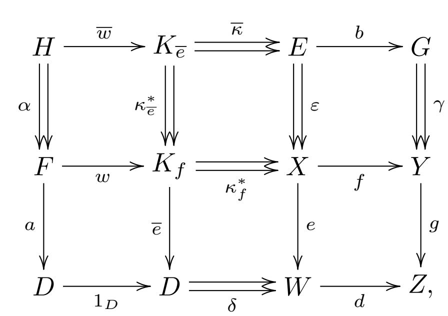
Proof. The "only if" part is given by Lemma 6.1. To prove the "if" part, consider a commutative diagram of horizontal star-exact sequences (7), with a and g being isomorphisms. Then, e is a regular epimorphism. Since the category is star-regular, to prove that e is an isomorphism it suffices to show that the star-kernel κ_e^* of e factors through the star-kernel $\Delta_X^* = (k_X, k_X)$ of 1_X, which is the same as to show that κ_e^* has a diagonal form $\kappa_e^* = (e', e')$. First, observe that κ_e^* factors through φ:



Since the composite $e\kappa_e^* = \delta ab$ is a bi-star, from Proposition 3.5(b) we get $b \in \mathcal{N}$. This implies that κ_e^* is a bi-star. Then, κ_e^* is of the from $\kappa_e^* = (k_X c_1, k_X c_2)$, where k_X denotes the $\mathcal{N}$-kernel of $1_X : X \to X$. We want to show that $c_1 = c_2$. Notice that Δ_X^* factors through φ via some monomorphism m. Using the fact that a is an isomorphism and δ is monic we easily get $mc_1 = b = mc_2$. This implies that $c_1 = c_2$. □

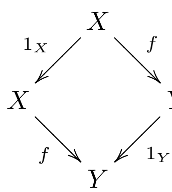
Proposition 6.3. *In a star-regular category with enough trivial objects, if the short five lemma holds then the middle* 3×3 *lemma holds.*

Proof. Let (1) be a 3×3 diagram such that first and third rows are star-exact sequences and $f\varphi_1 = f\varphi_2$. Then f is a regular epimorphism by Lemma 6.1. We can form the following commutative diagram: let $\kappa_f^* : K_f \rightrightarrows X$ be the star-kernel of f and $\overline{e} : K_f \to D$ and $w : F \to K_f$ the induced morphisms such that $\delta\overline{e} = e\kappa_f^*$ and $\kappa_f^* w = \varphi$. Then, by taking the star-kernel $\kappa_{\overline{e}}^* : K_{\overline{e}} \rightrightarrows K_f$ of the regular epimorphism $\overline{e}$ we can generate a commutative diagram



where $\overline{\kappa}\,\overline{w} = \beta$. By Proposition 3.5(c), $\overline{\kappa}$ is a star. Since δ is monic, applying Theorem 4.2 to the right hand side 3×3 diagram above, we can conclude that $\overline{\kappa}$ is the star-kernel of b. Consequently, $\overline{w}$ is an isomorphism. To finish, we just apply the short five lemma to the left part of the above diagram to conclude that w is an isomorphism. This proves that the middle row in diagram (1) is a star-exact sequence. □

Definition 6.4. [**12**] A morphism $f\colon X \to Y$ is said to be *saturating* if the diamond



is right saturated.

Any morphism is saturating in both total and (proto-)pointed contexts.

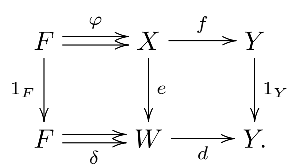
Theorem 6.5. *Let $\mathbb{C}$ be a star-regular category with enough trivial objects and saturating regular epimorphisms. Then the following conditions are equivalent:*

(a) The middle 3×3 lemma holds in $\mathbb{C}$.

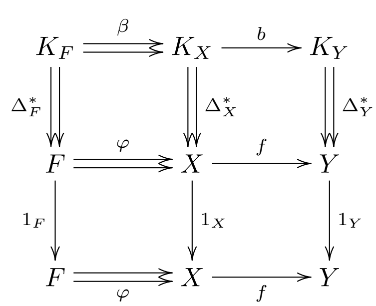
(b) The short five lemma holds in $\mathbb{C}$.

(c) The short five lemma for regular epimorphisms holds in $\mathbb{C}$.

Proof. (a)⇒(b): Consider a commutative diagram of horizontal star-exact sequences



From the top row of the above diagram, construct a commutative diagram



where, by Proposition 3.5(c), β is a star. Then, by Theorem 4.2, β is a star-kernel of b and since f is saturating, b is a regular epimorphism. Therefore, the top row is star-exact. Now, in the above diagram, replace the bottom part with the diagram we had in the beginning:

$$\begin{array}{ccccc} K_F & \overset{\beta}{\rightrightarrows} & K_X & \xrightarrow{b} & K_Y \\ {\scriptstyle \Delta_F^*}\downdownarrows & & \downdownarrows{\scriptstyle \Delta_X^*} & & \downdownarrows{\scriptstyle \Delta_Y^*} \\ F & \overset{\varphi}{\rightrightarrows} & X & \xrightarrow{f} & Y \\ {\scriptstyle 1_F}\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle 1_Y} \\ F & \underset{\delta}{\rightrightarrows} & W & \xrightarrow[d]{} & Y. \end{array}$$

By the middle 3×3 lemma (with the role of rows and columns switched), the middle column is a star-exact sequence. Consequently, e is the coequalizer of $\Delta_X^* = (k_X, k_X)$ and thus it is an isomorphism.

(b)⇒(a) by Proposition 6.3, and (b)⇔(c) by Theorem 6.2. □

Example 6.6. In the quasi-pointed context, Theorem 6.5 gives a characterization of sequentiable categories introduced in [**4**]. Among examples of sequentiable categories are the categories Grpd_X of groupoids over a fixed set X of objects. More generally, the category $\mathrm{Grpd}_X(\mathbb{C})$ of internal groupoids in a regular category $\mathbb{C}$ over a fixed object X of objects is sequentiable, as noted in [**4**]. The category $\mathrm{Grpd}_X(\mathbb{C})$ is pointed if and only if the diagonal $X \to X \times X$ is an isomorphism.

Combining the above result with Theorem 5.2, we obtain:

Corollary 6.7. *Let $\mathbb{C}$ be a star-regular category with enough trivial objects and saturating regular epimorphisms. Then the following conditions are equivalent:*

(a) The complete 3×3 lemma holds in $\mathbb{C}$ (i.e., the lower, upper and middle 3×3 lemmas hold in $\mathbb{C}$).

(b) Any left saturated diamond (4) with regular epimorphic edges f, g, d, is both regular and saturated.

Example 6.8. Among the classes of multi-pointed categories $\mathbb{C}$ satisfying all conditions stated in Corollary 6.7 are the following ones:

- In the total context, they are precisely the Goursat categories, i.e., regular categories where composition of kernel pairs is 3-permutable. Corresponding varieties of universal algebras are known as 3-*permutable varieties* in universal algebra.
- In the proto-pointed context they are precisely the sequentiable categories (this follows from our results as well as from those obtained in [**4**]). In particular, in the pointed context, we get precisely the homological categories [**2**], i.e., pointed regular protomodular categories. Corresponding pointed varieties of universal algebras are precisely the pointed classically ideal determined varieties [**20, 21**].

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References

[1] M. Barr, P. A. Grillet, and D. H. van Osdol, *Exact categories and categories of sheaves, Springer Lecture Notes in Mathematics*, vol. 236, Springer, Berlin, 1971.

[2] F. Borceux and D. Bourn, Mal'cev, protomodular, homological and semi-abelian categories, *Mathematics and its Applications*, vol. 566, Kluwer Academic Publishers, Dordrecht, 2004.

[3] Dominique Bourn, Normalization equivalence, kernel equivalence and affine categories, *Category theory (Como, 1990), Lecture Notes in Math.*, vol. 1488, Springer, Berlin, 1991, 43–62.

[4] D. Bourn, 3×3 lemma and protomodularity, *J. Algebra* **236** (2001), 778–795.

[5] D. Bourn, The denormalized 3×3 lemma, *J. Pure Appl. Algebra* **177** (2003), 113–129.

[6] A. Carboni, G. M. Kelly, and M. C. Pedicchio, Some remarks on Maltsev and Goursat categories, *Appl. Cat. Struct.* **1** (1993), 385–421.

[7] A. Carboni, J. Lambek, and M. C. Pedicchio, Diagram chasing in Malcev categories, *J. Pure Appl. Alg.* **69** (1991), 271–284.

[8] A. Carboni, M. C. Pedicchio, and N. Pirovano, Internal graphs and internal groupoids in Mal'cev categories, *Category theory 1991 (Montreal, PQ, 1991), CMS Conf. Proc.*, vol. 13, Amer. Math. Soc., Providence, RI, 1992, 97–109.

[9] C. Ehresmann, Sur une notion générale de cohomologie, *C. R. Acad. Sci. Paris* **259** (1964), 2050–2053.

[10] T. Everaert and M. Gran, Precrossed modules and Galois theory, *J. Algebra* **297** (2006), 292-309.

[11] K. Fichtner, *Varieties of universal algebras with ideals*, Mat. Sb. (N.S.) **75 (117)** (1968), 445–453. (English translation: *Math. USSR Sbornik* **4** (1968), 411–418).

[12] M. Gran, Z. Janelidze, D. Rodelo, and A. Ursini, Symmetry of regular diamonds, the Goursat property, and subtractivity, *Th. Appl. Categ.* **27** (2012), 80–96.

[13] M. Gran, Z. Janelidze, and A. Ursini, A good theory of ideals in regular multi-pointed categories, *J. Pure Appl. Algebra* **216** (2012), 1905-1919.

[14] M. Gran and D. Rodelo, A new characterisation of Goursat categories, *Appl. Categ. Struct.* **20** (2012), 229–238.

[15] M. Grandis, On the categorical foundations of homological and homotopical algebra, *Cah. Top. Géom. Diff. Catég.* **33** (1992), 135–175.

[16] G. Janelidze and M. C. Pedicchio, Pseudogroupoids and commutators, *Th. Appl. Categ.* **8** (1997), 408–456.

[17] Z. Janelidze, Subtractive categories, *Appl. Categ. Struct.* **13** (2005), 343–350.

[18] Z. Janelidze, The pointed subobject functor, 3×3 lemmas and subtractivity of spans, *Th. Appl. Categ.* **23** (2010), 221–242.

[19] S. Lack, The 3-by-3 lemma for regular Goursat categories, *Homology, Homotopy and Applications* **6** (2004), 1–3.

[20] A. Ursini, Osservazioni sulla varietá BIT, *Bolletino della Unione Matematica Italiana (4)* **7** (1973), 205–211.

[21] A. Ursini, On subtractive varieties, I, *Algebra Universalis* **31** (1994), 204–222.

[22] A. Ursini, Normal subalgebras I, *Appl. Categ. Struct.*, to appear.

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