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ERRATUM TO 'DICOVERING SPACES'

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Abstract

In [**1**] (L. Fajstrup, Dicovering spaces, *Homology Homotopy Appl.* **5** (2003), no. 2, 1–17), we study coverings in the setting of directed topology. Unfortunately, there is a condition missing in the definition of a directed covering. Some of the results in [**1**] require this extra condition and in fact it was claimed to follow from the original definition. It is the purpose of this note to give the right definition and point out how this affects the statements in that paper. Moreover, we give an example of a dicovering in the sense of [**1**], which does not satisfy the extra condition. Fortunately, with the extra condition, the subsequent results are now correct.

1. Introduction

In [**1**], we give a construction $\pi : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ which is denoted a universal dicovering. Moreover, we define dicoverings as maps $p : (Y, y_0) \to (X, x_0)$ with certain lifting properties. We claim the existence of a map $\phi : (\tilde{X}, \tilde{x}_0) \to (Y, y_0)$ s.t. $\pi = p \circ \phi$; and it is in this sense, that the universal dicovering is universal. However, for the map ϕ to be well defined, we need an extra lifting condition in the definition 2.4 of a dicovering. The purpose of this note is to make this clear.

2. The problem and the solution

The directed spaces in [**1**] are locally partially ordered spaces. We will work in **d-Top**, the category of d-spaces, here. The approach through locally partially ordered spaces as in [**1**] requires extra conditions on the spaces, i.e., longer definitions, and hence would make this note longer; moreover, **d-Top** seems by now to be the right category for directed topology.

Definition 2.1. A *d*-space is a topological space X with a set of paths $\vec{P}(X) \subset X^I$ such that

- $\vec{P}(X)$ contains all constant paths.
- $\gamma, \mu \in \vec{P}(X)$ implies $\gamma \star \mu \in \vec{P}(X)$, where $\star$ is concatenation.

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- If $\phi : I \to I$ is monotone, that is, $t \leqslant s \Rightarrow \phi(t) \leqslant \phi(s)$, and $\gamma \in \vec{P}(X)$, then $\gamma \circ \phi \in \vec{P}(X)$

A d-map or dimap $f : X \to Y$ is a continuous map, such that if $\alpha \in \vec{P}(X)$ then $f \circ \alpha \in \vec{P}(Y)$.

The set of distinguished paths, $\vec{P}(X)$ are called the *dipaths*. They are d-maps from the ordered interval $\vec{I}$ to X. The category of d-spaces is denoted **d-Top**

Definition 2.2. For a d-space $(X, \vec{P}(X))$ and A, B subsets of X, let $\vec{P}(X, A, B)$ be the set of dipaths $\gamma : \vec{I} \to X$, $\gamma(0) \in A$, $\gamma(1) \in B$.

Let $I \times \vec{I}$ be the unit square with discrete order (equality) in one coordinate and the standard order along the other. Dipaths are increasing in the second and constant in the first coordinate. Let x, y be points in X and $\gamma, \mu \in \vec{P}(X, x, y)$,. Then γ is dihomotopic to μ if there is a d-map $H : I \times \vec{I} \to X$ with $H(0, t) = \gamma(t)$, $H(1, t) = \mu(t)$ and $H(s_0, t) \in \vec{P}(X, x, y)$ for all s_0.

H is a dihomotopy with fixed endpoints and the equivalence classes are called dihomotopy classes.

$\vec{\pi}_1(X, x, y)$ is the set of equivalence classes.

We define a universal dicovering as in [**1**] disregarding the extra conditions on the topology given in [**1**] on X, since these are not relevant for this note. See Rem. 2.8

Definition 2.3. *[Following [1] Def. 3.1]* Let $((X, x_0), \vec{P}(X)))$ be a pointed d-space and let $\mathcal{U}$ be a basis for the topology on X. The universal dicovering space $(\widetilde{X}, \tilde{x}_0)$ of X with respect to x_0 is the set

$$\{[\gamma] | \gamma : (\vec{I}, 0) \to (X, x_0)\},$$

where $[\gamma]$ is the dihomotopy class of γ with fixed endpoints and $\tilde{x}_0$ is the dihomotopy class of the constant dipath $[x_0]$. The topology on $(\widetilde{X}, \tilde{x}_0)$ is given by the following subbasis:

For $\gamma : (\vec{I}, 0) \to (X, x_0)$ such that $\gamma(1) \in U$, where $U \in \mathcal{U}$, let

$$U_{[\gamma]} = \{[\eta] \in \cup_{y \in U} \vec{\pi}_1(X, x_0, y) | [\eta] \sim_U [\gamma]\},$$

where $[\gamma] \sim_U [\eta]$ if there is a dimap $H : I \times \vec{I} \to X$ such that $H(0, t) = \gamma(t)$, $H(1, t) = \eta(t)$, $H(s, 0) = x_0$ and $H(s, 1) \in U$ for all $s \in I$.

The set of dipaths $\vec{P}(\widetilde{X})$ is the closure of $\{\Gamma(t) = [\gamma|_{[0,t]}]$, where $\gamma\colon (\vec{I}, 0) \to (X, x_0)\}$ under composition, reparametrization and subpath. Let $\pi : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be the endpoint map, $\pi([\gamma]) = \gamma(1)$.

A directed covering is defined in terms of lifting properties in [**1**]. We did not require condition 3, i.e., that dihomotopies with fixed endpoints and initial point x_0 lift to dihomotopies with fixed endpoints. Condition 2 is a lifting property for dihomotopies with fixed initial point.

Definition 2.4. *[Following [1] Def. 4.1]* Let $\Pi : \hat{X} \to X$ be a d-map. Then Π is a dicovering with respect to $x_0 \in X$ if for every $y_0 \in \Pi^{-1}(x_0)$:

1. For every dipath $\gamma : \vec{I} \to X$ such that $\gamma(0) = x_0$, there is a unique lift $\hat{\gamma} : \vec{I} \to \hat{X}$, such that $\Pi \circ \hat{\gamma} = \gamma$ and $\hat{\gamma}(0) = y_0$.

2. For every d-map $H : I \times \vec{I} \to X$ with $H(s,0) = x_0$ there is a unique lift $\hat{H} : I \times \vec{I} \to \hat{X}$ s.t. $\Pi \circ \hat{H} = H$ and $\hat{H}(s,0) = y_0$.
3. For every d-map $H : I \times \vec{I} \to X$ with $H(s,0) = x_0$ and $H(s,1) = x_1$ there is a unique lift $\hat{H} : I \times \vec{I} \to \hat{X}$ s.t. $\Pi \circ \hat{H} = H$, $\hat{H}(s,0) = y_0$ and $\hat{H}(s,1)$ is constant.

When $X = \uparrow_X x_0 := \{x \in X \mid \exists \gamma : (\vec{I}, 0) \to (X, x_0) : \gamma(1) = x\}$, $\Pi^{-1}(x_0) = \hat{x}_0$ and $\hat{X} = \uparrow_{\hat{X}} \hat{x}_0$, the dicovering is a *simple* dicovering, and all dipaths lift uniquely, not only the ones initiating in x_0.

Our universal dicovering is universal in the sense of

Proposition 2.5. [[**1**] 4.6] *Let* $\Pi : (\hat{X}, \hat{x}_0) \to (X, x_0)$ *be a dicovering w.r.t.* $x_0 \in X$, *such that* $\Pi^{-1}(x_0) = \hat{x}_0$. *Then there is a map* $\phi : (\widetilde{X}, \tilde{x}_0) \to (\hat{X}, \hat{x}_0)$ *covering the identity, i.e.,* $\Pi \circ \phi = \pi$.

Proof. Let $\phi([\gamma]) = \hat{\gamma}(1)$, where $\hat{\gamma}$ is the unique lift of γ with initial point $\hat{x}_0$. This is well defined, since if $[\lambda] = [\gamma] \in \widetilde{X}_{x_0}$, λ is dihomotopic to γ. Since dihomotopies with fixed endpoints lift to dihomotopies with fixed endpoints, it follows that $\hat{\lambda}(1) = \hat{\gamma}(1)$. □

The map ϕ is not well defined if we remove condition 3 in Def. 2.4, and hence $\widetilde{X}$ is not "universal" in that setting. In [**1**] we claimed that condition 3 follows from condition 2 and continuity of dihomotopies. This claim is made, and condition 3 is needed, in the proof of [**1**] Prop. 4.5 as well:

Proposition 2.6. *[[**1**] 4.5] Let* $\Pi : (\hat{X}, \hat{x}_0) \to (X, x_0)$ *be a simple dicovering. Then for* $x \in X$, $|\Pi^{-1}(x)| \leqslant |\vec{\pi}_1(X, x_0, x)|$.

However, as the following example shows, condition 3 does not follow from the original definition. In particular, Prop. 2.6 does not hold in this example.

Example 2.7. (See Fig.1) Let $\hat{X}$ be the quotient of $I \times \vec{I}$ under the relation $(s,0) \sim (0,0)$ and let X be the quotient of $\hat{X}$ under the relation $(s,1) \sim (0,1)$. Let $p : (\hat{X}, (0,0)) \to (X, (0,0))$ be the quotient map. Then Def. 2.4 1 and 2 certainly hold with $x_0 = (0,0)$, i.e., dipaths and dihomotopies initiating in $(0,0)$ lift uniquely. But the dihomotopy (with fixed endpoints) $H : I \times \vec{I} \to X$ given by projection to the quotient X lifts to projection to the quotient $\hat{X}$, which does not have fixed endpoint $H(s,1)$. Hence this example is a dicover with regard to the original definition but not to the one given here.

$$
\begin{tikzcd}
(0,0) \arrow[r, hook] \arrow[d, hook] & I \times \vec{I}/_{(s,0)\sim(0,0)} \arrow[d, "p"] \\
I \times \vec{I} \arrow[ur] \arrow[r, "H"] & I \times \vec{I}/^{(s,1)\sim(0,1)}_{(s,0)\sim(0,0)}
\end{tikzcd}
$$

Figure 1: All maps are projections or inclusions

All examples of dicoverings in [**1**] in fact do satisfy the extra lifting condition, hence it is still true that the map ϕ in 2.5 is not necessarily continuous; the "Hawaiian star" example ([**1**] 4.7) proves that.

Remark 2.8. In [**1**], we study universal dicoverings with base space X a locally partially ordered space satisfying some technical requirements on the interplay between the topology and the directed paths. These cases can be studied in the above framework of d-spaces. Since the universal dicovering construction satisfies the present definition of a dicovering, the results, examples and counterexamples are still relevant: For instance, our construction 2.3 will not always provide a Hausdorff universal dicovering even if the base space is Hausdorff, and the fibers are not discrete (by [**1**] Ex. 3.7) but extra requirements on the topology of X will ensure that.

Remark 2.9. In [**3**] we prove that in certain subcategories of **d-Top**, there exists a universal dicovering w.r.t. the original definition. The methods in [**3**] provide existence of a universal dicovering $(\bar{X}, \bar{x}_0) \to (X, x_0)$ w.r.t. the definition here as well, and the resulting d-map $f : \bar{X} \to \widetilde{X}$ is a bijection of sets. We study this in [**2**].

References

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