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STEENROD'S OPERATIONS IN SIMPLICIAL BREDON-ILLMAN COHOMOLOGY WITH LOCAL COEFFICIENTS

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Abstract

In this paper we use Peter May's algebraic approach to Steenrod operations to construct Steenrod's reduced power operations in simplicial Bredon-Illman cohomology with local coefficients of a one vertex G-Kan complex, G being a discrete group.

1. Introduction

The study of cohomology operations has been one of the important areas of research in algebraic topology for a long time. For instance, they have been extensively used to compute obstructions [**20**], to study homotopy type of complexes [**23**] and to show essentiality of maps of spheres [**3**]. A class of basic operations are Steenrod's squares and reduced power operations [**1, 21, 22**]. Steenrod's squares are defined for cohomology with $\mathbb{Z}_2$-coefficients whereas Steenrod's reduced powers are defined in cohomology with coefficients in $\mathbb{Z}_p$, $p \neq 2$ a prime. A very general and useful method of constructing these operations is given in [**14**]. A categorical approach to Steenrod operations can be found in [**6**]. In [**9**], S. Gitler constructed reduced power operations in cohomology with local coefficients. A well-known result of Eilenberg describes cohomology of a space with local coefficients by the cohomology of an invariant subcomplex of its universal cover, equipped with the action of the fundamental group of the space [**5**]. The main idea of Gitler's construction is to lift power operations in this invariant cochain subcomplex and reproduce the operations in cohomology with local coefficients via Eilenberg's description. The relevant local coefficients in this context is obtained by a fixed action of the fundamental group of the space on a fixed cyclic group of prime order $p \neq 2$.

Recently, in [**17**], we introduced simplicial equivariant cohomology with local coefficients, which is the simplicial version of Bredon-Illman cohomology with local coefficients [**15**]. The aim of this paper is to construct Steenrod's reduced power operations in simplicial Bredon-Illman cohomology with local coefficients, where the equivariant local coefficients take values in $\mathbb{Z}_p$-algebras, for a prime $p > 2$. Throughout our method is simplicial. It may be mentioned that, for a space with a topological group action there exists a brace (or homotopy Gerstenhaber) algebra structure [**16**] on

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the Bredon-Illman cochain complex. This brace algebra structure was used in [**8**] to deduce Steenrod's squares in Bredon-Illman cohomology with local coefficients that take values in $\mathbb{Z}_2$-algebras.

We have the notion of a 'universal O_G-covering complex' of a one vertex G-Kan complex X [**17**]. This is defined as a contravariant functor from the category of canonical orbits to the category of one vertex Kan complexes and is the analogue, in the equivariant context, of the universal cover of a one vertex Kan complex [**11**]. This universal O_G-covering complex comes equipped with an action of an O_G-group $\underline{\pi}X$ (see Section 4 for details), and an equivariant analogue of the Eilenberg theorem holds [**18**]. Following Gitler [**9**], we first construct the power operations in the $\underline{\pi}X$-equivariant cohomology of the 'universal O_G-covering complex'. This is done by applying the algebraic description of Steenrod power operations of P. May [**14**]. We then use the equivariant version of the Eilenberg theorem to reproduce Steenrod's reduced power operations in the present context. It may be remarked that our method also applies when $p = 2$ and hence also yields Steenrod squares (cf. Remark 5.15).

The paper is organized as follows: In Section 2, we recall some standard results and fix notations. The notion of equivariant local coefficients of a simplicial set equipped with a simplicial group action is based on fundamental groupoid. In Section 3, we recall these concepts and quickly review the definition of simplicial Bredon-Illman cohomology with local coefficients. In Section 4, we state the equivariant version of the Eilenberg theorem. In Section 5, we briefly recall the algebraic method of P. May and then apply it to construct Steenrod's reduced power operations in simplicial Bredon-Illman cohomology with local coefficients.

2. Preliminaries

In this section we set up our notations and recall some standard facts [**10, 13**].

Throughout, $\mathcal{S}$ will denote the category of simplicial sets and simplicial maps. Let $\Delta[n]$ denote the standard simplicial n-simplex and Δ_n be the unique non-degenerate n-simplex of $\Delta[n]$. We have simplicial maps $\delta_i\colon \Delta[n-1] \to \Delta[n]$ and $\sigma_i\colon \Delta[n+1] \to \Delta[n]$ for $0 \leqslant i \leqslant n$ defined by $\delta_i(\Delta_{n-1}) = \partial_i(\Delta_n)$ and $\sigma_i(\Delta_{n+1}) = s_i(\Delta_n)$. The boundary subcomplex $\partial\Delta[n]$ of $\Delta[n]$ is defined as the smallest subcomplex of $\Delta[n]$ containing the faces $\partial_i\Delta_n$, $i = 0, 1, \ldots, n$.

Definition 2.1. Let G be a discrete group. A G-simplicial set is a simplicial object in the category of G-sets. More precisely, a G-simplicial set is a simplicial set $\{X_n; \partial_i, s_i, 0 \leqslant i \leqslant n\}_{n\geqslant 0}$ such that each X_n is a G-set, and the face maps $\partial_i\colon X_n \to X_{n-1}$ and the degeneracy maps $s_i\colon X_n \to X_{n+1}$ commute with the G-action. A G-simplicial map between G-simplicial sets is a simplicial map which commutes with the G-action.

For a G simplicial set X, we consider $X \times \Delta[1]$ as a G-simplicial set with trivial G-action on $\Delta[1]$.

Definition 2.2. Two G-simplicial maps $f, g\colon X \to Y$ between G-simplicial sets X and Y are G-homotopic if there exists a G-simplicial map $\mathcal{H}\colon X \times \Delta[1] \to Y$ such that

$$\mathcal{H} \circ (\mathrm{id} \times \delta_1) = f, \mathcal{H} \circ (\mathrm{id} \times \delta_0) = g,$$

where $X \times \Delta[0]$ is identified with X. The map $\mathcal{H}$ is called a G-homotopy from f to

g, and we write $\mathcal{H}\colon f \simeq_G g$. If $i\colon X' \subseteq X$ is an inclusion of a subcomplex and f, g agree on X', then we say that f is G-homotopic to g relative to X' if there exists a G homotopy $\mathcal{H}\colon f \simeq_G g$ such that $\mathcal{H} \circ (i \times \mathrm{id}) = \alpha \circ \mathrm{pr}_1$, where $\alpha = f|_{X'} = g|_{X'}$ and $\mathrm{pr}_1\colon X' \times \Delta[1] \to X'$ is the projection onto the first factor. In this case we write $\mathcal{H}\colon f \simeq_G g(\mathrm{rel}\, X')$.

Definition 2.3. A G-simplicial set is a G-Kan complex if for every subgroup $H \subseteq G$ the fixed point simplicial set X^H is a Kan complex.

Remark 2.4. Recall [**2, 7**] that the category $G\mathcal{S}$ of G-simplicial sets and G-simplicial maps between G-simplicial sets has a closed model structure [**19**], where the fibrant objects are the G-Kan complexes and the cofibrant objects are all the G-simplicial sets. From this it follows that G-homotopy on the set of G-simplicial maps $X \to Y$ is an equivalence relation, for every G-simplicial set X and G-Kan complex Y. More generally, relative G-homotopy is an equivalence relation if the target is a G-Kan complex.

We consider $G/H \times \Delta[n]$ as a simplicial set, where $(G/H \times \Delta[n])_q = G/H \times \Delta[n]_q$ with face and degeneracy maps as $\mathrm{id} \times \partial_i$ and $\mathrm{id} \times s_i$. Note that the group G acts on G/H by left translation. With this G-action on the first factor and trivial action on the second factor, $G/H \times \Delta[n]$ is a G-simplicial set.

Let X be any G-simplicial set. A G-simplicial map $\sigma\colon G/H \times \Delta[n] \to X$ is called an equivariant n-simplex of type H in X.

Remark 2.5. We remark that for a G-simplicial set X, the set of equivariant n-simplices of type H in X is in bijective correspondence with n-simplices of X^H. For an equivariant n-simplex σ, the corresponding n-simplex is $\sigma' = \sigma(eH, \Delta_n)$. The simplicial map $\Delta[n] \to X^H$, $\Delta_n \mapsto \sigma'$ will be denoted by $\overline{\sigma}$.

We shall call σ degenerate or non-degenerate according to whether the n-simplex $\sigma' \in X^H_n$ is degenerate or non-degenerate.

Recall that the category of canonical orbits, denoted by O_G, is a category whose objects are cosets G/H, as H runs over the all subgroups of G. A morphism from G/H to G/K is a G-map. Such a morphism determines and is determined by a subconjugacy relation $a^{-1}Ha \subseteq K$ and is given by $\hat{a}(eH) = aK$. We denote this morphism by $\hat{a}$ [**4**].

Definition 2.6. A contravariant functor from O_G to the category of simplicial sets $\mathcal{S}$ is called an O_G-simplicial set. A map between O_G-simplicial sets is a natural transformation of functors.

We shall denote the category of O_G-simplicial sets by $O_G\mathcal{S}$.

For a commutative ring Λ, let Λ-alg denote the category of commutative Λ-algebras with unity and algebra homomorphisms preserving unity. The category of Λ-modules and module maps is denoted by Λ-mod. The category of chain complexes of Λ-modules is denoted by ch_Λ. The notion of O_G-groups, O_G-Λ-algebras or O_G-chain complexes has the obvious meaning replacing $\mathcal{S}$ by $\mathcal{G}rp$ (the category of groups), Λ-alg or ch_Λ, respectively.

For any two O_G-simplicial sets (respectively, O_G-groups) T and T', we define their product $(T \times T') \in O_G\mathcal{S}$ (respectively, O_G-$\mathcal{G}rp$) as

$$(T \times T')(G/H) = T(G/H) \times T'(G/H)$$

for objects G/H of O_G and $(T \times T')(\hat{a}) = T(\hat{a}) \times T'(\hat{a})$ for a morphism $\hat{a}$ of O_G.

For a G-simplicial set X, with a G-fixed 0-simplex v, we have an O_G-group $\underline{\pi}X$ defined as follows: For any subgroup H of G,

$$\underline{\pi}X(G/H) := \pi_1(X^H, v),$$

and for a morphism $\hat{a}\colon G/H \to G/K$, $a^{-1}Ha \subseteq K$, $\underline{\pi}X(\hat{a})$ is the homomorphism of the fundamental groups induced by the simplicial map $a\colon X^K \to X^H$.

Definition 2.7. An O_G-group ρ is said to act on an O_G-simplicial set (O_G-Λ-algebra or O_G-chain complex) T if for every subgroup $H \subseteq G$, $\rho(G/H)$ acts on $T(G/H)$ and this action is natural with respect to maps of O_G. Thus if

$$\phi(G/H)\colon \rho(G/H) \times T(G/H) \to T(G/H)$$

denotes the action of $\rho(G/H)$ on $T(G/H)$, then for each subconjugacy relation $a^{-1}Ha \subseteq K$,

$$\phi(G/H) \circ (\rho(\hat{a}) \times T(\hat{a})) = T(\hat{a}) \circ \phi(G/K).$$

Definition 2.8. Let an O_G-group ρ act on the O_G-simplicial sets T and T'. A map $f\colon T \to T'$ is called ρ-equivariant if

$$f(G/H)(ax) = af(G/H)(x), a \in \rho(G/H), x \in T(G/H),$$

for each subgroup H of G.

Definition 2.9. Let L, L' be O_G-chain complexes. Two natural transformations $\mathsf{v} = \{\mathsf{v}_n\}, \mathsf{w} = \{\mathsf{w}_n\}\colon L \to L'$ are said to be homotopic if there exist natural transformations

$$\mathcal{H}_n\colon \mathsf{v}_n \to \mathsf{w}_{n+1}, n \geqslant 0,$$

such that $\{\mathcal{H}_n(G/H)\}_{n\geqslant 0}$ is a chain homotopy of the chain maps $\mathsf{v}(G/H), \mathsf{w}(G/H)$ for each subgroup H of G. Symbolically we write $\mathcal{H}\colon \mathsf{v} \simeq \mathsf{w}$.

If an O_G-group ρ acts on L, L' and v, w are ρ-equivariant, then v, w are said to be ρ-equivariantly homotopic if there exists a homotopy $\mathcal{H}\colon \mathsf{v} \simeq \mathsf{w}$ which satisfies

$$\mathcal{H}_n(G/H)(ax) = a\mathcal{H}_n(G/H)(x) \quad \text{for} \quad a \in \rho(G/H), x \in \mathsf{v}_n(G/H), H \subseteq G.$$

Definition 2.10. The tensor product $L \otimes L'\colon O_G \to \mathrm{ch}_\Lambda$ of two O_G-chain complexes L and L' is defined as

$$(L \otimes L')(G/H) = L(G/H) \otimes L'(G/H),$$

for each object G/H of O_G and $(L \otimes L')(\hat{a}) = L(\hat{a}) \otimes L'(\hat{a})$ for a morphism $\hat{a}$ of O_G.

Note that a chain complex W can be considered as an O_G-chain complex in the trivial way, that is, $W(G/H) = W$, $W(\hat{a}) = \mathrm{id}$. So the tensor product of W with an O_G-chain complex is defined.

Throughout the paper, unless otherwise mentioned explicitly, all the tensor products are over the ring Λ.

3. Simplicial Bredon-Illman cohomology with local coefficients

In this section we recall [**17**] the relevant notion of a fundamental groupoid of a G-simplicial set X, the notion of equivariant local coefficients on X and the definition of simplicial Bredon-Illman cohomology with local coefficients.

We begin with the notion of a fundamental groupoid. Recall [**10**] that the fundamental groupoid πX of a Kan complex X is a category having as objects all 0-simplexes of X and a morphism $x \to y$ in πX is a homotopy class of 1-simplices $\omega\colon \Delta[1] \to X \operatorname{rel} \partial\Delta[1]$ such that $\omega \circ \delta_0 = \overline{y}$, $\omega \circ \delta_1 = \overline{x}$. If ω_2 represents an arrow from x to y and ω_0 represents an arrow from y to z, then their composite $[\omega_0] \circ [\omega_2]$ is represented by $\Omega \circ \delta_1$, where the simplicial map $\Omega\colon \Delta[2] \to X$ corresponds to a 2-simplex, which is determined by the compatible pair $(\omega_0', \quad, \omega_2')$. For a simplicial set X, the notion of a fundamental groupoid is defined via the geometric realization and the total singular functor.

The fundamental groupoid of a G-simplicial set is defined as follows:

Definition 3.1. Let X be a G-Kan complex. The fundamental groupoid ΠX is a category with objects equivariant 0-simplices

$$x_H\colon G/H \times \Delta[0] \to X$$

of type H, as H varies over all subgroups of G. Given two objects x_H and y_K in ΠX, a morphism from $x_H \longrightarrow y_K$ is defined as follows: Consider the set of all pairs $(\hat{a}, \phi)$ where $\hat{a}\colon G/H \to G/K$ is a morphism in O_G, given by a subconjugacy relation $a^{-1}Ha \subseteq K$, $a \in G$, so that $\hat{a}(eH) = aK$ and $\phi\colon G/H \times \Delta[1] \to X$ is an equivariant 1-simplex such that

$$\phi \circ (\operatorname{id} \times \delta_1) = x_H, \phi \circ (\operatorname{id} \times \delta_0) = y_K \circ (\hat{a} \times \operatorname{id}).$$

The set of morphisms in ΠX from x_H to y_K is a quotient of the set of pairs mentioned above by an equivalence relation '$\sim$', where $(\hat{a}_1, \phi_1) \sim (\hat{a}_2, \phi_2)$ if and only if $a_1 = a_2 = a$ (say), and there exists a G-homotopy $\mathcal{H}\colon G/H \times \Delta[1] \times \Delta[1] \to X$ of G-homotopies such that $\mathcal{H}\colon \phi_1 \simeq_G \phi_2 (\operatorname{rel} G/H \times \partial\Delta[1])$. Since X is a G-Kan complex, by Remark 2.4, $\sim$ is an equivalence relation. We denote the equivalence class of $(\hat{a}, \phi)$ by $[\hat{a}, \phi]$. The set of equivalence classes is the set of morphisms in ΠX from x_H to y_K.

The composition of morphisms in ΠX is defined as follows: Given two morphisms

$$x_H \xrightarrow{[\hat{a}_1, \phi_1]} y_K \xrightarrow{[\hat{a}_2, \phi_2]} z_L,$$

their composition $[\hat{a}_2, \phi_2] \circ [\hat{a}_1, \phi_1]$ is $[\widehat{a_1 a_2}, \psi]\colon x_H \to z_L$, where the first factor is the composition

$$G/H \xrightarrow{\hat{a}_1} G/K \xrightarrow{\hat{a}_2} G/L,$$

and $\psi\colon G/H \times \Delta[1] \to X$ is an equivariant 1-simplex of type H as described below. Let x be a 2-simplex in the Kan complex X^H determined by the compatible pair of 1-simplices $(a_1\phi_2', \quad, \phi_1')$ so that $\partial_0 x = a_1\phi_2'$ and $\partial_2 x = \phi_1'$. Then ψ is given by $\psi(eH, \Delta_1) = \partial_1 x$.

Observe that ϕ' is a 1-simplex in X^H such that $\partial_1\phi' = x'_H$ and $\partial_0\phi' = ay'_K$. Moreover, the 0-simplex ay'_K in X^H corresponds to the composition

$$G/H \times \Delta[0] \xrightarrow{\hat{a}\times \mathrm{id}} G/K \times \Delta[0] \xrightarrow{y_K} X$$

and ϕ is a G-homotopy $x_H \simeq_G y_K \circ (\hat{a} \times \mathrm{id})$ (cf. Remark 2.5 for notations).

It is proved in [**17**] that the composition is well defined. For a version for the fundamental groupoid of a G-space we refer to [**12, 15**].

Observe that if X is a G-simplicial set then $S|X|$ is a G-Kan complex, where for any space Y, SY denotes the total singular complex and for any simplicial set X, $|X|$ denotes the geometric realization of X.

Definition 3.2. For a G-simplicial set X, we define the fundamental groupoid ΠX of X by $\Pi X := \Pi S|X|$.

Note that if $F\colon X \to Y$ is a G-simplicial map then there exists an obvious induced functor $\Pi(F)\colon \Pi X \to \Pi Y$ which assigns to each object x_H of ΠX, the object $F \circ x_H$ of ΠY and a morphism $[\hat{a}, \phi]$ in ΠX to the morphism $[\hat{a}, F\circ\phi]$ of ΠY.

Remark 3.3. Suppose ξ is a morphism from x to y in πX^H, given by a homotopy class $[\overline{\omega}]$, where $\overline{\omega}\colon \Delta[1] \to X^H$ represents the 1-simplex in X^H from x to y. Let x_H and y_H be the objects in πX^H defined respectively by

$$x_H(eH, \Delta_0) = x, y_H(eH, \Delta_0) = y.$$

Then we have a morphism $[\mathrm{id}, \omega]\colon x_H \to y_H$ in ΠX, where $\omega(eH, \Delta_1) = \overline{\omega}(\Delta_1)$. We shall denote this morphism corresponding to ξ by $b\xi$.

Definition 3.4. Equivariant local coefficients on a G-simplicial set X are a contravariant functor from ΠX to the category Λ-alg.

Next, we briefly describe the simplicial version of Bredon-Illman cohomology with local coefficients as introduced in [**17**].

Let X be a G-simplicial set and M equivariant local coefficients on X. For each equivariant n-simplex $\sigma\colon G/H \times \Delta[n] \to X$, we associate an equivariant 0-simplex $\sigma_H\colon G/H \times \Delta[0] \to X$ given by

$$\sigma_H = \sigma \circ (\mathrm{id} \times \delta_{(1,2,\dots,n)}),$$

where $\delta_{(1,2,\dots,n)}$ is the composition

$$\delta_{(1,2,\dots,n)}\colon \Delta[0] \xrightarrow{\delta_1} \Delta[1] \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_n} \Delta[n].$$

The j-th face of σ is an equivariant $(n-1)$-simplex of type H, denoted by $\sigma^{(j)}$, and is defined by

$$\sigma^{(j)} = \sigma \circ (\mathrm{id} \times \delta_j), 0 \leqslant j \leqslant n.$$

Remark 3.5. Note that $\sigma_H^{(j)} = \sigma_H$ for $j > 0$, and $\sigma_H^{(0)} = \sigma \circ (\mathrm{id} \times \delta_{(0,2,\dots,n)})$.

Let $C_G^n(X; M)$ be the Λ-module of all functions f defined on equivariant n-simplexes $\sigma\colon G/H \times \Delta[n] \to X$ such that $f(\sigma) \in M(\sigma_H)$ with $f(\sigma) = 0$, if σ is degenerate. We have a morphism $\sigma_* = [\mathrm{id}, \alpha]$ in ΠX from σ_H to $\sigma_H^{(0)}$ induced by σ, where

$\alpha\colon G/H\times\Delta[1]\to X$ is given by $\alpha=\sigma\circ(\mathrm{id}\times\delta_{(2,\dots,n)})$. Define a homomorphism

$$\delta\colon C^n_G(X;M)\to C^{n+1}_G(X;M);\, f\mapsto\delta f$$

where for any equivariant $(n+1)$-simplex σ of type H,

$$(-1)^{n+1}\delta f(\sigma)=M(\sigma_*)f(\sigma^{(0)})+\sum_{j=1}^{n+1}(-1)^j f(\sigma^{(j)}).$$

A routine verification shows that $\delta\circ\delta=0$. Thus $\{C^*_G(X;M),\delta\}$ is a cochain complex. We are interested in a subcomplex of this cochain complex as described below.

Let $\eta\colon G/H\times\Delta[n]\to X$ and $\tau\colon G/K\times\Delta[n]\to X$ be two equivariant n-simplexes. Suppose there exists a G-map $\hat{a}\colon G/H\to G/K$, $a^{-1}Ha\subseteq K$, such that $\tau\circ(\hat{a}\times\mathrm{id})=\eta$. Then η and τ are said to be compatible under $\hat{a}$. Observe that if η and τ are compatible as described above then η is degenerate if and only if τ is degenerate. Moreover, notice that in this case, we have a morphism $[\hat{a},k]\colon\eta_H\to\tau_K$ in ΠX, where $k=\eta_H\circ(\mathrm{id}\times\sigma_0)$, where $\sigma_0\colon\Delta[1]\to\Delta[0]$ is the simplicial map as described in Section 2. Let us denote this induced morphism by a_*.

Definition 3.6. We define $S^n_G(X;M)$ to be the submodule of $C^n_G(X;M)$ consisting of all functions f such that if η and τ are equivariant n-simplexes in X which are compatible under $\hat{a}$, then $f(\eta)=M(a_*)(f(\tau))$.

If $f\in S^n_G(X;M)$ then one can verify that $\delta f\in S^{n+1}_G(X;M)$. Thus we have a cochain complex of Λ-modules $S_G(X;M)=\{S^n_G(X;M),\delta\}$.

Definition 3.7. Let X be a G-simplicial set with equivariant local coefficients M on it. Then the n-th Bredon-Illman cohomology of X with local coefficients M is defined by

$$H^n_G(X;M):=H^n(S_G(X;M)).$$

Suppose that X,Y are G-simplicial sets and M,N are equivariant local coefficients on X and Y respectively. A map from (X,M) to (Y,N) is a pair (F,γ), where $F\colon X\to Y$ is a G-simplicial map, and $\gamma\colon N\circ\Pi(F)\to M$ is a natural transformation of functors, $\Pi(F)\colon\Pi X\to\Pi Y$ being the map induced by F. A map $(F,\gamma)\colon(X,M)\to(Y,N)$ naturally induces a cochain map $(F,\gamma)^{\#}\colon S^*_G(Y;N)\to S^*_G(X;M)$ as follows: For $f\in S^*_G(Y;N)$ and an equivariant n-simplex σ in X of type H, $(F,\gamma)^{\#}(f)(\sigma)=\gamma(\sigma_H)f(F\circ\sigma)$. Therefore we have an induced map $(F,\gamma)^*\colon H^*_G(Y;N)\to H^*_G(X;M)$ in cohomology.

We now define the cup product in simplicial Bredon-Illman cohomology with local coefficients. Let $\sigma\colon G/H\times\Delta[n+m]\to X$ be an equivariant $(n+m)$-simplex of type H. Then define $\sigma\rfloor_n=\sigma\circ(\mathrm{id}_{G/H}\times\delta_{(n+1,\dots,n+m)})$, $\lfloor_m\sigma=\sigma\circ(\mathrm{id}_{G/H}\times\delta_{(0,\dots,n)})$ where $\delta_{(n+1,\dots,n+m)}\colon\Delta[n]\to\Delta[n+m]$ and $\delta_{(0,\dots,n)}\colon\Delta[m]\to\Delta[n+m]$ are defined as before. For cochains $f\in S^n_G(X;M)$ and $g\in S^m_G(X;M)$, the cup product $f\cup g\in S^{n+m}_G(X;M)$ is the cochain whose value on σ is given by the formula

$$(f\cup g)(\sigma)=(-1)^{mn}f(\sigma\rfloor_n)(M(\sigma_{n+1})g(\lfloor_m\sigma)),$$

where $\sigma_{n+1}=[\mathrm{id},\sigma\circ(\mathrm{id}_{G/H}\times\delta_{(1,\dots,n,n+2,\dots,n+m)})]$ is a morphism in ΠX from $(\sigma\rfloor_n)_H$

to $(\lfloor_m \sigma)_H$. A routine verification shows that $f \cup g$ belongs to $S_G^{n+m}(X;M)$, and

$$d(f \cup g) = df \cup g + (-1)^{\deg(f)} f \cup dg.$$

Therefore it induces a product in cohomology which is associative and graded commutative. Thus $H_G^*(X;M)$ is an associative graded algebra.

Suppose M is equivariant local coefficients on a G-simplicial set X with a G-fixed 0-simplex v. Then M determines an O_G-Λ-algebra M_0 equipped with an action of the O_G-group $\underline{\pi}X$ as described below.

For any subgroup H of G, let v_H be the object of type H in ΠX defined by

$$v_H \colon G/H \times \Delta[0] \to X, v_H(eH, \Delta_0) = v.$$

Then for any morphism $\hat{a} \colon G/H \to G/K$ in O_G given by a subconjugacy relation $a^{-1}Ha \subseteq K$, we have a morphism $[\hat{a}, k] \colon v_H \to v_K$ in ΠX, where $k \colon G/H \times \Delta[1] \to X$ is given by $k(eH, \Delta_1) = s_0 v$. Define an O_G-Λ-algebra M_0 by

$$M_0(G/H) := M(v_H), H \subseteq G$$

and $M_0(\hat{a}) = M[\widehat{a}, k]$ for a morphism $\hat{a}$ in O_G.

We now describe the action of the O_G-group $\underline{\pi}X$ on M_0. Let $\alpha = [\overline{\phi}] \in \underline{\pi}X(G/H) = \pi_1(X^H, v)$. Then the morphism $[\mathrm{id}, \phi] \colon v_H \to v_H$, determined by $\phi(eH, \Delta_1) = \overline{\phi}(\Delta_1)$, is an equivalence in the category ΠX. This yields a group homomorphism

$$b \colon \pi_1(X^H, v) \to \mathrm{Aut}_{\Pi X}(v_H), \alpha = [\overline{\phi}] \mapsto b(\alpha) = [\mathrm{id}, \phi].$$

The composition of the map b with the group homomorphism

$$\mathrm{Aut}_{\Pi X}(v_H) \to \mathrm{Aut}_{\Lambda\text{-alg}}(M(v_H)),$$

which sends $\alpha \in \mathrm{Aut}_{\Pi X}(v_H)$ to $[M(\alpha)]^{-1}$, defines the action of $\pi_1(X^H, v)$ on $M_0(G/H)$. It is routine to check that this action is natural with respect to morphisms of O_G.

Conversely, an O_G-Λ-algebra M_0, equipped with an action of the O_G-group $\underline{\pi}X$, defines equivariant local coefficients M on X, where X is G-connected and $v \in X^G$ a fixed 0-simplex [**17**].

4. The Eilenberg theorem

In this section we recall a version of the Eilenberg theorem [**18**] for simplicial Bredon-Illman cohomology with local coefficients.

Let $\mathcal{A}_\Lambda$ denote the category with objects the triples (T, M_0, ρ), where T is an O_G-simplicial set, M_0 an O_G-Λ-algebra and ρ is an O_G-group which operates on both T and M_0. A morphism from (T, M_0, ρ) to (T', M_0', ρ') is a triple (f_0, f_1, f_2), where $f_0 \colon T \to T'$, $f_1 \colon M_0' \to M_0$ and $f_2 \colon \rho \to \rho'$ are maps in the appropriate categories such that

$$\begin{aligned} f_0(G/H)(\alpha x) &= f_2(G/H)(\alpha) f_0(G/H)(x), f_1(G/H)[f_2(G/H)(\alpha) m_0']i \\ &= \alpha f_1(G/H)(m_0'), H \subseteq G, x \in T(G/H), \alpha \in \rho(G/H), m_0' \in M_0'(G/H). \end{aligned}$$

The ρ-equivariant cohomology of T with coefficients M_0 is defined as follows: We have an O_G-chain complex $\{\underline{C}_*(T), \partial_*\}$ defined by

$$\underline{C}_n(T)\colon O_G \to \Lambda\text{-mod}, G/H \mapsto C_n(T(G/H);\Lambda),$$

where $C_n(T(G/H);\Lambda)$ is the free Λ-module generated by the non-degenerate n-simplices of $T(G/H)$. For any morphism $\hat{a}\colon G/H \to G/K$ in O_G,

$$\underline{C}_n(T)(\hat{a}) = a_{\#}\colon C_n(T(G/K);\Lambda) \to C_n(T(G/H);\Lambda)$$

is induced by the simplicial map $T(\hat{a})\colon T(G/K) \to T(G/H)$. The boundary map

$$\partial_n\colon \underline{C}_n(T) \to \underline{C}_{n-1}(T)$$

is a natural transformation defined by

$$\partial_n(G/H)\colon C_n(T(G/H);\Lambda) \to C_{n-1}(T(G/H);\Lambda),$$

where $\partial_n(G/H)$ is the ordinary boundary map of the simplicial set $T(G/H)$. The action of ρ on T induces an action of ρ on the O_G-chain complex $\{\underline{C}_*(T), \partial_*\}$. We form the cochain complex

$$\{C^*_\rho(T;M_0) = \mathrm{Hom}_\rho(\underline{C}_*(T), M_0), \delta^*\},$$

where $\mathrm{Hom}_\rho(\underline{C}_n(T), M_0)$ consists of all natural transformations $\underline{C}_n(T) \xrightarrow{f} M_0$ respecting the action of ρ and $\delta^n f$ is given by $f \circ \partial_{n+1}$. Then the n-th ρ-equivariant cohomology of T with coefficients M_0 is given by

$$H^n_\rho(X;M_0) := H_n(C^*_\rho(T;M_0)).$$

Remark 4.1. It is easy to observe that a morphism in $\mathcal{A}_\Lambda$ $(f_0, f_1, f_2)\colon (T, M_0, \rho) \to (T', M_0', \rho')$ induces a cochain map $C^*(f_0,f_1,f_2)\colon C^*_\rho(T;M_0) \to C^*_{\rho'}(T';M_0')$.

The cochain complex $C^*_\rho(T;M_0)$ is equipped with a cup product, defined as follows: We have a natural transformation

$$\underline{\xi}\colon \underline{C}_*(T\times T) \to \underline{C}_*(T) \otimes \underline{C}_*(T),$$

where $\underline{\xi}(G/H)$ is the Alexander-Whitney map for the simplicial set $T(G/H)$, $H \subseteq G$ [**13**]. We have a ρ-action on $\underline{C}_*(T)$ induced by the ρ-action on T and hence diagonal actions of ρ on $T\times T$ and on $\underline{C}_*(T)\otimes\underline{C}_*(T)$. Since the Alexander-Whitney map of simplicial sets is a natural map, $\underline{\xi}$ is equivariant with the induced actions of ρ on $\underline{C}_*(T\times T)$ and $\underline{C}_*(T)\otimes\underline{C}_*(T)$. Then the cup product is defined as the composition of the maps

$$C^*_\rho(T;M_0)\otimes C^*_\rho(T;M_0) \xrightarrow{\alpha} \mathrm{Hom}_\rho(\underline{C}_*(T)\otimes\underline{C}_*(T), M_0) \xrightarrow{\underline{\xi}^*} C^*_\rho(T\times T;M_0)$$

with the map

$$C^*_\rho(T\times T;M_0) \xrightarrow{D^*} C^*_\rho(T;M_0).$$

Here $\alpha\colon C^*_\rho(T;M_0)\otimes C^*_\rho(T;M_0) \to \mathrm{Hom}_\rho(\underline{C}_*(T\times T), M_0)$ is defined by

$$\alpha(f\otimes g)(G/H)(x\otimes y) = (-1)^{\deg(x)\deg(y)} f(G/H)(x) g(G/H)(y),$$

where $f, g \in C^*_\rho(T;M_0)$ and $x, y \in \underline{C}_*(T)(G/H)$ and $D\colon T \to T\times T$ is the diagonal map.

Remark 4.2. The cochain complex $C^*_\rho(T;M_0)$, equipped with the above cup product, is an associative differential Λ-algebra, and the induced product in the cohomology is associative and graded commutative.

We now relate the simplicial Bredon-Illman cohomology with local coefficients of a one vertex G-Kan complex with the equivariant cohomology of its universal O_G-covering complex [**18**].

Let X be a one vertex G-Kan complex. We denote the G-fixed vertex by v. Let M be equivariant local coefficients on X and M_0 be the associated O_G-Λ-algebra, as described at the end of the previous section. For any subgroup H of G, let

$$p_H\colon \widetilde{X^H} \to X^H$$

be the universal cover [**11, 18**] of X^H. The left translation $a\colon X^K \to X^H$, corresponding to a G-map $\hat{a}\colon G/H \to G/K,\ a^{-1}Ha \subseteq K$, induces a simplicial map $\tilde{a}\colon \widetilde{X^K} \to \widetilde{X^H}$ such that $p_H \circ \tilde{a} = a \circ p_K$. This defines an O_G-Kan complex $\widetilde{X}$ by setting $\widetilde{X}(G/H) = \widetilde{X^H}$ and $\widetilde{X}(\hat{a}) = \tilde{a}$. This is called the universal O_G-covering complex of X. This is the simplicial analogue of the O_G-covering space as introduced in [**15**]. We refer to [**12**] for a more general version, called the 'universal covering functor'. The natural actions of $\underline{\pi}X(G/H) = \pi_1(X^H, v)$ on $\widetilde{X}(G/H) = \widetilde{X^H}$ as H varies over subgroups of G, define an action of the O_G-group $\underline{\pi}X$ on $\widetilde{X}$. Thus $(\tilde{X}, M_0, \underline{\pi}X)$ is an object of $\mathcal{A}_\Lambda$.

Theorem 4.3 ([**18**]). *Let X be a one vertex G-Kan complex with equivariant local coefficients M on it. Then, with notations as above, there exists an isomorphism of graded algebras*

$$H^*_G(X; M) \cong H^*_{\underline{\pi}X}(\tilde{X}; M_0).$$

The proof is obtained by constructing isomorphism at the cochain level. The explicit isomorphism is described as follows [**18**]: Define

$$\mu\colon S^n_G(X; M) \to \mathrm{Hom}_{\underline{\pi}X}(\underline{C}_n(\widetilde{X}), M_0)$$

as follows: Let $f \in S^n_G(X; M)$ and y be a non-degenerate n-simplex in $\widetilde{X^H}$. Let σ be the equivariant n-simplex of type H in X such that $\overline{\sigma} = p_H \circ \overline{y}$, where $\overline{y}\colon \Delta[n] \to \widetilde{X^H}$ is the simplicial map with $\overline{y}(\Delta_n) = y$. Then $\mu(f) \in \mathrm{Hom}_{\underline{\pi}X}(\underline{C}_n(\widetilde{X}), M_0)$ is given by

$$\mu(f)(G/H)(y) = M(b\xi_H(\partial_{(1,2,\ldots,n)}y))f(\sigma),$$

where $\partial_{(1,2,\ldots,n)}y = \partial_1\partial_2\cdots\partial_n y$.

The inverse of μ,

$$\mu^{-1}\colon \mathrm{Hom}_{\underline{\pi}X}(\underline{C}_n(\tilde{X}), M_0) \to C^n_G(X; M),$$

is described as follows: Let $f \in \mathrm{Hom}_{\underline{\pi}X}(\underline{C}_n(\tilde{X}), M_0)$ and σ be a non-degenerate equivariant n-simplex of type H in X. Choose an n-simplex y in $\widetilde{X^H}$ such that $p_H(y) = \sigma(eH, \Delta_n)$. Then $\mu^{-1}(f)$ is given by

$$\mu^{-1}(f)(\sigma) = M(b\xi_H(\partial_{(1,2,\ldots,n)}y))^{-1}f(G/H)(y).$$

It is easy to check that $\mu(f \cup g) = \mu(f) \cup \mu(g)$ for $f, g \in S^*_G(X; M)$. Hence we have an isomorphism

$$\mu^*\colon H^*_G(X; M) \cong H^*_{\underline{\pi}X}(\tilde{X}; M_0)$$

of graded Λ-algebras.

5. Steenrod reduced power operations

In this section we briefly recall the relevant part of the general algebraic approach to Steenrod operations by P. May [**14**], necessary for our purpose. We apply this method to construct Steenrod power operations in equivariant cohomology of O_G-simplicial sets in general. In particular, for a one vertex G-Kan complex X, we have reduced power operations defined for $\underline{\pi}X$-equivariant cohomology of the universal O_G-covering complex $\tilde{X}$. We then apply Theorem 4.3 to deduce the Steenrod power operations in simplicial Bredon-Illman cohomology with local coefficients.

Let Λ be a commutative ring. By a Λ-complex K, we will mean a $\mathbb{Z}$-graded cochain complex of Λ-modules with differential of degree 1. A morphism of Λ-complexes is a degree zero map commuting with the differential. If π is a group, then we let $\Lambda\pi$ denote its group ring over Λ.

Let p be an odd prime and Σ_p denote the symmetric group on p-letters. For the rest of this section, unless otherwise stated, Λ will be the commutative ring $\mathbb{Z}_p$ and π will be the cyclic subgroup of Σ_p, generated by the permutation $\alpha = (p, 1, 2, \ldots, p-1)$. If not mentioned explicitly, all tensor products are over the ring Λ.

Let V, W be free resolutions of Λ over $\Lambda\Sigma_p, \Lambda\pi$ respectively. We shall use the following canonical model of W: Let W_i be $\Lambda\pi$-free module on one generator $e_i, i \geqslant 0$. Let $N = 1 + \alpha + \cdots + \alpha^{p-1}$ and $T = \alpha - 1$ in $\Lambda\pi$. Define differential d, augmentation $\epsilon\colon W_0 \to \Lambda$, and coproduct ψ on W, respectively by the formulas

$$\begin{aligned} d(e_{2i+1}) &= Te_{2i},\ d(e_{2i}) = Ne_{2i-1}, \epsilon(\alpha^j e_0) = 1, \\ \psi(e_{2i+1}) &= \sum_{j+k=i} e_{2j} \otimes e_{2k+1} + \sum_{j+k=i} e_{2j+1} \otimes \alpha e_{2k}, \\ \psi(e_{2i}) &= \sum_{j+k=i} e_{2j} \otimes e_{2k} + \sum_{j+k=(i-1)} \sum_{0 \leqslant r < s < p} \alpha^r e_{2j+1} \otimes \alpha^s e_{2k}. \end{aligned}$$

Thus W is a differential $\Lambda\pi$-coalgebra and a $\Lambda\pi$-free resolution of Λ.

We denote the p-fold tensor product $K \otimes \cdots \otimes K$ by K^p. Then K^p becomes a $\Lambda\pi$-complex by the following π operation:

$$\tau(u_1 \otimes \cdots \otimes u_p) = \gamma(\tau) u_1 \otimes \cdots u_{i-1} \otimes u_{i+1} \otimes u_i \otimes u_{i+2} \cdots \otimes u_p,$$

where $\gamma(\tau) = (-1)^{\deg(u_i)\deg(u_{i+1})}$ if τ is the interchange of the i-th and $(i+1)$-th factor. We consider W as a non-positively graded $\Lambda\pi$-complex. The inclusion of π in Σ_p induces a morphism $j\colon W \to V$ of $\Lambda\pi$-complexes.

We have the following algebraic category $\mathfrak{C}(p)$ on which the Steenrod operations are defined: The objects of this category are pairs (K, θ), where K is a Λ-complex, equipped with a homotopy associative multiplication $K \otimes K \to K$, and $\theta\colon W \otimes K^p \to K$ is a morphism of $\Lambda\pi$-complexes, satisfying the following two conditions:

1. The restriction of θ to $e_0 \otimes K^p$ is Λ-homotopic to the iterated product $K^p \to K$, associative in some order.
2. The morphism θ is $\Lambda\pi$-homotopic to a composite $W \otimes K^p \xrightarrow{j \otimes 1} V \otimes K^p \xrightarrow{\varnothing} K$, where $\varnothing$ is a morphism of $\Lambda\Sigma_p$-complexes.

A morphism $f\colon (K, \theta) \to (K', \theta')$ is a morphism of Λ-complexes $f\colon K \to K'$ such that the following diagram is $\Lambda\pi$-homotopy commutative:

$$\begin{tikzcd} W \otimes K^p \arrow[r, "\theta"] \arrow[d, "\mathrm{id} \otimes f^p"'] & K \arrow[d, "f"] \\ W \otimes (K')^p \arrow[r, "\theta'"] & K'. \end{tikzcd}$$

The tensor product of two objects (K, θ) and (K', θ') is the pair $(K \otimes K', \tilde{\theta})$, where $\tilde{\theta}$ is the composite

$$W \otimes (K \otimes K')^p \xrightarrow{\psi \otimes \tilde{U}} W \otimes W \otimes K^p \otimes K'^p \xrightarrow{\mathrm{id} \otimes \tilde{t} \otimes \mathrm{id}} W \otimes K^p \otimes W \otimes K'^p \xrightarrow{\theta \otimes \theta'} K \otimes K'.$$

Here $\psi \colon W \to W \otimes W$ is the coproduct, $\tilde{U} \colon (K \otimes K')^p \to K^p \otimes K'^p$ is the shuffling isomorphism and $\tilde{t}(x \otimes y) = (-1)^{\deg(x)\deg(y)} y \otimes x$.

Definition 5.1. An object $(K, \theta) \in \mathfrak{C}(p)$ is said to be a Cartan object if the product $K \otimes K \to K$ is a morphism from $(K \otimes K, \tilde{\theta})$ to (K, θ).

For an object (K, θ) of $\mathfrak{C}(p)$, there are maps $D_i \colon H^q(K) \to H^{pq-i}(K), i \geqslant 0$, defined as follows: For $x \in H^q(K)$, $e_i \otimes x^p$ is a well-defined element of $H^{pq-i}(W \otimes_{\Lambda\pi} K^p)$ [**14**] and define $D_i(x) = \theta_*(e_i \otimes x^p)$, where $\theta_* \colon H^{pq-i}(W \otimes_{\Lambda\pi} K^p) \to H^{pq-i}(K)$ is induced by θ. We make the convention that $D_i = 0$ for $i < 0$. Then the Steenrod power operations

$$\mathcal{P}^s \colon H^q(K) \to H^{q+2s(p-1)}(K), \beta\mathcal{P}^s \colon H^q(K) \to H^{q+2s(p-1)+1}(K)$$

are defined by the following formulas:

$$\mathcal{P}^s(x) = (-1)^r (m!)^q D_{(q-2s)(p-1)}(x), \beta\mathcal{P}^s(x) = (-1)^r (m!)^q D_{(q-2s)(p-1)-1}(x),$$

where $m = (p-1)/2$ and $r = s + m(q + q^2)/2$.

Proposition 5.2. *The power operations satisfy the following properties:*

1. $\mathcal{P}^s$ *and* $\beta\mathcal{P}^s$ *are natural homomorphisms.*
2. $\mathcal{P}^s(x) = 0$ *if* $2s > q$, $\beta\mathcal{P}^s = 0$ *if* $2s \geqslant q$, *and* $\mathcal{P}^s(x) = x^p$ *if* $2s = q$.
3. *If* (K, θ) *is a Cartan object, then* $\mathcal{P}^s$ *satisfies the Cartan formulas*

$$\mathcal{P}^s(xy) = \sum_{i+j=s} \mathcal{P}^i(x)\mathcal{P}^j(y),$$

$$\beta\mathcal{P}^{s+1}(xy) = \sum_{i+j=s} [\beta\mathcal{P}^{i+1}(x)\mathcal{P}^j(y) + (-1)^{\deg(x)}\mathcal{P}^i(x)\beta\mathcal{P}^{j+1}(y)].$$

Remark 5.3. In general, $\beta\mathcal{P}^s$ is single notation. But if (K, θ) is reduced mod p ([**14**]), then the Bockstein homomorphism

$$\beta \colon H^n(K) \to H^{n+1}(K)$$

can be defined, and $\beta\mathcal{P}^s$ is the composition of $\mathcal{P}^s$ with the Bockstein.

Next we recall the definition of an 'Adem object' in $\mathfrak{C}(p)$ [**14**]. We need the following notations for the definition: Consider Σ_{p^2} as permutations on the p^2 symbols $\{(i,j) \mid 1 \leqslant i,j \leqslant p\}$. Embed $\pi = \langle\alpha\rangle$ $(\subseteq \Sigma_p)$ in Σ_{p^2} by letting $\alpha(i,j) = (i+1,j)$. Let $\alpha_i \in \Sigma_{p^2}, 1 \leqslant i \leqslant p$, be defined by $\alpha_i(i,j) = (i,j+1)$ and $\alpha_i(k,j) = (k,j)$ for $k \neq i$. Let

$$\beta = \alpha_1 \cdots \alpha_p, \nu = \langle\beta\rangle, \sigma = \pi\nu, \tau = \langle\alpha_1,\ldots,\alpha_p,\alpha\rangle.$$

Note that β and α_i are of order p and the following relations hold:

$$\alpha\alpha_i = \alpha_{i+1}\alpha; \alpha_i\alpha_j = \alpha_j\alpha_i; \alpha\beta = \beta\alpha.$$

Let $W_1 = W$ and $W_2 = W$ regarded as $\Lambda\pi$-free and $\Lambda\nu$-free resolutions of Λ respectively. Let ν,π operate trivially on W_1, W_2 respectively. Then $W_1 \otimes W_2$ is a $\Lambda\sigma$-free resolution of Λ with the diagonal action of σ on $W_1 \otimes W_2$.

For any ν-module M, let τ operate on M^p by letting α operate by cyclic permutation and by letting α_i operate on the i-th factor as does β. Let α_i operate trivially on W_1. Then τ operates on W_1 and hence τ operates diagonally on $W_1 \otimes M^p$. In particular, $W_1 \otimes W_2^p$ is then a $\Lambda\tau$-free resolution of Λ.

Let $(K,\theta) \in \mathfrak{C}(p)$. We let Σ_{p^2} operate on K^{p^2} by permutations, where we consider K^{p^2} as $\otimes_{i=1}^{p}(\otimes_{j=1}^{p} K_{i,j}), K_{i,j} = K$. We let ν operate on $W_2 \otimes K^p$ by letting β act as a cyclic permutation on K^p. By the previous paragraph this fixes an action of τ on $W_1 \otimes (W_2 \otimes K^p)^p$.

Let Y be any $\Lambda\Sigma_{p^2}$-free resolution of Λ with $Y_0 = \Lambda\Sigma_{p^2}$ and let $w\colon W_1 \otimes W_2^p \to Y$ be any morphism of $\Lambda\tau$-complexes. Observe that w exists since Y is acyclic and any two choices of w are $\Lambda\tau$-equivariantly homotopic.

With these notations, we have the following definition:

Definition 5.4. Let $(K,\theta) \in \mathfrak{C}(p)$. We say that (K,θ) is an Adem object if there exists a morphism of $\Lambda\Sigma_{p^2}$-complexes $\eta\colon Y \otimes K^{p^2} \to K$, such that the following diagram is $\Lambda\tau$-equivariant homotopy commutative:

$$\begin{tikzcd}
(W_1 \otimes W_2^p) \otimes K^{p^2} \arrow[r, "w\otimes \mathrm{id}"] & Y \otimes K^{p^2} \arrow[r, "\eta"] & K \\
W_1 \otimes (W_2 \otimes K^p)^p \arrow[u, "\mathrm{id}\times\tilde{U}"] \arrow[r, "\mathrm{id}\otimes\theta^p"] & W_1 \otimes K^p \arrow[r, "\theta"] & K. \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Here $\tilde{U}$ is the shuffle map and Σ_{p^2} acts trivially on K.

The following relations among the operations $\mathcal{P}^s$ and $\beta\mathcal{P}^s$ are valid on all cohomology classes of Adem objects in $\mathfrak{C}(p), p > 2$ a prime [**14**]:

- If $a < pb$, then

$$\beta^e\mathcal{P}^a\mathcal{P}^b = \sum_i (-1)^{a+i}(a-pi,(p-1)b-a+i-1)\beta^e\mathcal{P}^{a+b-i}\mathcal{P}^i.$$

- If $a \leqslant pb$, then

$$\beta^e\mathcal{P}^a\beta\mathcal{P}^b = (1-e)\sum_i(-1)^{a+i}(a-pi,(p-1)b-a+i-1)\beta\mathcal{P}^{a+b-i}\mathcal{P}^i - \sum_i(-1)^{a+i}(a-pi-1,(p-1)b-a+i)\beta^e\mathcal{P}^{a+b-i}\beta\mathcal{P}^i,$$

where $e = 0, 1$ and $\beta^0\mathcal{P}^s = \mathcal{P}^s$ and $\beta^1\mathcal{P}^s = \beta\mathcal{P}^s$.

We apply the above algebraic construction to define Steenrod reduced power operations in equivariant simplicial cohomology of an O_G-simplicial set, as defined in the Section 4. This is done by constructing a functor Γ from $\mathcal{A}_\Lambda$ to $\mathfrak{C}(p)$.

Let (T, M_0, ρ) be an object of $\mathcal{A}_\Lambda$. Recall that the cochain complex $C^*_\rho(T; M_0)$, equipped with the cup product, is an associative differential graded Λ-algebra (cf. Remark 4.2). We now construct a morphism of $\Lambda\pi$-complexes

$$\theta\colon W \otimes C^*_\rho(T; M_0)^p \to C^*_\rho(T; M_0),$$

so that $(C^*_\rho(T; M_0), \theta)$ becomes an object of the category $\mathfrak{C}(p)$.

For a simplicial set L, let $C_*(L)$ denote the normalized chain complex of L with coefficients Λ. We recall the following lemma from [**14**]:

Lemma 5.5. *Let π be a subgroup of Σ_p (π not necessarily cyclic of order p) and W be a $\Lambda\pi$-free resolution of Λ such that $W_0 = \Lambda\pi$ with generator e_0. For simplicial sets $L_1, \dots, L_p$, there exists a chain map*

$$\Phi\colon W \otimes C_*(L_1 \times \cdots \times L_p) \to W \otimes C_*(L_1) \otimes \cdots \otimes C_*(L_p),$$

which is natural in the L_i and satisfies the following properties:

1. *For $\sigma \in \pi$, the following diagram is commutative:*

$$\begin{tikzcd} W \otimes C_*(L_1 \times \cdots \times L_p) \arrow[r, "\Phi"] \arrow[d, "\sigma"] & W \otimes C_*(L_1) \otimes \cdots \otimes C_*(L_p) \arrow[d, "\sigma"] \\ W \otimes C_*(L_{\sigma(1)} \times \cdots \times L_{\sigma(p)}) \arrow[r, "\Phi"] & W \otimes C_*(L_{\sigma(1)}) \otimes \cdots \otimes C_*(L_{\sigma(p)}). \end{tikzcd}$$

2. *Φ is the identity homomorphism on $W \otimes C_0(L_1 \times \cdots \times L_p)$.*
3. *$\Phi(e_0 \otimes (x_1, \dots, x_p)) = e_0 \otimes \xi(x_1, \dots, x_p)$, where $x_i \in L_j$ for $1 \leqslant i \leqslant p$ and*

$$\xi\colon C_*(L_1 \times \cdots \times L_p) \to C_*(L_1) \otimes \cdots \otimes C_*(L_p)$$

is the Alexander-Whitney map.

4. *$\Phi(W \otimes C_j(L_1 \times \cdots \times L_p)) \subseteq \sum_{k \leqslant pj} W \otimes [C_*(L_1) \otimes \cdots \otimes C_*(L_p)]_k$.*
5. *Any two such Φ are naturally equivariantly homotopic.*

In the special case $L_1 = \cdots = L_p = L$, we obtain a natural morphism of chain complexes of $\Lambda\pi$-modules

$$\Phi\colon W \otimes C_*(L^p) \to W \otimes C_*(L)^p,$$

which satisfies the last four conditions of Lemma 5.5.

Let $T \in O_G\mathcal{S}$. Applying the above special case of Lemma 5.5 to each simplicial set $T(G/H)$, we obtain the chain map $\Phi_H\colon W \otimes C_*(T(G/H)^p) \to W \otimes C_*(T(G/H))^p$ which is π-equivariant. Since Φ_H is natural with respect to maps of simplicial sets, we see that $\Phi_H \circ (\mathrm{id}_W \otimes \underline{C}_*(T(\hat{a})^p)) = (\mathrm{id}_W \otimes \underline{C}_*(T(\hat{a}))^p) \circ \Phi_K$, where $a^{-1}Ha \subseteq K$. Thus we have a morphism $\underline{\Phi}$ of O_G-chain complexes

$$\underline{\Phi}\colon W \otimes \underline{C}_*(T^p) \to W \otimes \underline{C}_*(T)^p, \text{ defined by } \underline{\Phi}(G/H) = \Phi_H, G/H \in O_G.$$

Now suppose that an O_G-group ρ operates on T. The diagonal action of ρ on T^p induces a ρ-action on $\underline{C}_*(T^p)$. Also we have an induced ρ-action on $\underline{C}_*(T)$. We let ρ operate diagonally on $\underline{C}_*(T)^p$ and trivially on W. The naturality of Φ_H with respect to maps from $T(G/H)$ into itself shows that Φ_H is $\rho(G/H)$-equivariant. Thus the map $\underline{\Phi}$ is $(\pi \times \rho)$-equivariant. Hence we obtain the following corollary:

Corollary 5.6. *Let $T \in O_G\mathcal{S}$ and an O_G-group ρ operates on T. For a subgroup π of Σ_p (π not necessarily cyclic of order p), let W be a $\Lambda\pi$-free resolution of Λ such that $W_0 = \Lambda\pi$ with generator e_0. Then there is a natural transformation*

$$\underline{\Phi}\colon W \otimes \underline{C}_*(T^p) \to W \otimes \underline{C}_*(T)^p$$

such that

1. *The map $\underline{\Phi}$ is $(\pi \times \rho)$-equivariant;*
2. *The map $\underline{\Phi}$ is the identity homomorphism on $W \otimes \underline{C}_0(T^p)$;*
3. *For each object G/H of O_G,*

   $$\underline{\Phi}(G/H)(e_0 \otimes (x_1, \ldots, x_p)) = e_0 \otimes \underline{\xi}(G/H)(x_1, \ldots, x_p),$$

   where $x_i \in T(G/H)$ for $1 \leqslant i \leqslant p$ and $\underline{\xi}(G/H)\colon C_*(T(G/H)^p) \to C_*(T(G/H))^p$ is the Alexander-Whitney map of the simplicial set $T(G/H)$;
4. *$\underline{\Phi}(G/H)(W \otimes C_j(T(G/H)^p)) \subseteq \sum_{k \leqslant pj} W \otimes (C_*(T(G/H))^p)_k$;*
5. *The map $\underline{\Phi}$ is natural with respect to equivariant maps of O_G-simplicial sets and any two such $\underline{\Phi}$ are naturally equivariantly homotopic.*

Next we construct the map $\theta\colon W \otimes C^*_\rho(T; M_0)^p \to C^*_\rho(T; M_0)$.

For an object $(T, M_0, \rho) \in \mathcal{A}_\Lambda$, let $D\colon T \to T^p$ be the diagonal map

$$D(G/H)(x) = (x, \ldots, x), x \in T(G/H),$$

which induces a map $D_*\colon \underline{C}_*(T) \to \underline{C}_*(T^p)$. Define $\underline{\Delta}\colon W \otimes \underline{C}_*(T) \to \underline{C}_*(T)^p$ to be the composite

$$\underline{\Delta}\colon W \otimes \underline{C}_*(T) \xrightarrow{\mathrm{id} \otimes D_*} W \otimes \underline{C}_*(T^p) \xrightarrow{\underline{\Phi}} W \otimes \underline{C}_*(T)^p \to \underline{C}_*(T)^p,$$

where the last map is the augmentation. Observe that the map $\underline{\Delta}$ is $(\pi \times \rho)$-equivariant. Moreover, we have a natural map

$$\alpha\colon [C^*_\rho(T; M_0)]^p \to \mathrm{Hom}_\rho(\underline{C}_*(T)^p, M_0)$$

defined by

$$\alpha(f_1 \otimes \cdots \otimes f_p)(G/H)(x_1 \otimes \cdots \otimes x_p) = (-1)^a f_1(G/H)(x_1) \cdots f_p(G/H)(x_p),$$

where $f_i \in C^*_\rho(T;M_0), x_i \in \underline{C}_*(T)(G/H), i = 1,\dots,p$ and $a = \prod_{k=1}^p \deg(x_k)$. Hence dualising $\underline{\Delta}$, we get a natural morphism of $\Lambda\pi$-complexes,

$$\theta\colon W \otimes C^*_\rho(T;M_0)^p \to C^*_\rho(T;M_0),$$

given by

$$\theta(w\otimes f)(G/H)(x) = (-1)^{\deg(w)\deg(x)}\alpha(f)(G/H)(\underline{\Delta}(G/H)(w\otimes x)),$$

where $w \in W, f \in C^*_\rho(T;M_0)^p, x \in C_*(T(G/H))$.

Remark 5.7. Note that $\theta(e_0 \otimes f) = D^*\underline{\xi}^*\alpha(f)$ for any $f \in C^*_\rho(T;M_0)^p$. As before, let V denote a $\Lambda\Sigma_p$-free resolution of Λ and $j\colon W \to V$ be the map induced by the inclusion $\pi \hookrightarrow \Sigma_p$. We apply Corollary 5.6 for the (sub)group Σ_p to get $\tilde{\Phi}\colon V \otimes \underline{C}_*(T^p) \to W \otimes \underline{C}_*(T)^p$. Then $\tilde{\Phi}\circ(j\otimes \mathrm{id})$ satisfies the first four conditions of Corollary 5.6 for the subgroup π and hence must be equivariantly homotopic to $\underline{\Phi}$. Therefore, $\tilde{\theta}\colon V \otimes C^*_\rho(T;M_0)^p \to C^*_\rho(T;M_0)$ can be defined such that $\tilde{\theta}\circ(j\otimes\mathrm{id})$ is $\Lambda\pi$-equivariantly homotopic to θ. Therefore $(C^*_\rho(T;M_0),\theta)$ is an object of the category $\mathfrak{C}(p)$. Thus we obtain a contravariant functor $\Gamma\colon \mathcal{A}_\Lambda \to \mathfrak{C}(p)$ by letting $\Gamma(T,M_0,\rho) = (C^*_\rho(T;M_0),\theta)$ and $\Gamma(f_0,f_1,f_2) = C^*(f_0,f_1,f_2)$ on morphisms (cf. Remark 4.1).

The next lemma is the key to show that $(C^*_\rho(T;M_0),\theta)$ is a Cartan object of $\mathfrak{C}(p)$. Let $\phi = (\epsilon\otimes\mathrm{id})\Phi$ where Φ is obtained from Lemma 5.5 and $\epsilon\colon W\to\Lambda$ is the augmentation.

Lemma 5.8. *Let $L_i, S_i, i = 1,\dots,p$ be simplicial sets. Let*

$$u\colon \Big(\prod_{i=1}^p L_i \times \prod_{i=1}^p S_i\Big) \to \prod_{i=1}^p (L_i\times S_i)$$

and

$$U\colon (\otimes_{i=1}^p C_*(L_i)) \otimes (\otimes_{i=1}^p C_*(S_i)) \to \otimes_{i=1}^p [C_*(L_i)\otimes C_*(S_i)]$$

be shuffle maps. Let t denote the flip map, that is, $t(x\otimes y) = y\otimes x$. Then there exists a homotopy

$$\mathcal{H}\colon W\otimes C_*\Big(\prod_{i=1}^p L_i \times \prod_{i=1}^p S_i\Big) \to \bigotimes_{i=1}^p [C_*(L_i)\otimes C_*(S_i)]$$

of the chain maps $\xi^p\phi(\mathrm{id}\otimes u)$ and $U(\phi\otimes\phi)(\mathrm{id}\otimes t\otimes\mathrm{id})(\psi\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{id}\times\xi)$, so that the following diagram is homotopy commutative:

$$\begin{tikzcd}
W\otimes C_*\Big(\prod_{i=1}^p L_i \times \prod_{i=1}^p S_i\Big) \arrow[r, "\mathrm{id}\times u"] \arrow[d, "\mathrm{id}\otimes\xi"'] & W\otimes C_*\Big(\prod_{i=1}^p (L_i\times S_i)\Big) \arrow[r, "\phi"] & \bigotimes_{i=1}^p [C_*(L_i\times S_i)] \arrow[d, "\xi^p"] \\
W\otimes C_*\Big(\prod_{i=1}^p L_i\Big)\otimes C_*\Big(\prod_{i=1}^p S_i\Big) \arrow[rr, "U(\phi\otimes\phi)(\mathrm{id}\otimes t\otimes\mathrm{id})(\psi\otimes\mathrm{id}\otimes\mathrm{id})"'] & & \bigotimes_{i=1}^p [C_*(L_i)\otimes C_*(S_i)].
\end{tikzcd}$$

Moreover, the homotopy $\mathcal{H}$ is natural in the L_i, S_i and the following diagram commutes for $\sigma \in \pi$:

$$\begin{tikzcd}
W \otimes C_*\Big(\prod_{i=1}^p L_i \times \prod_{i=1}^p S_i\Big) \arrow[r, "\mathcal{H}"] \arrow[d, "\sigma\otimes\sigma"'] & \bigotimes_{i=1}^p [C_*(L_i) \otimes C_*(S_i)] \arrow[d, "\sigma"] \\
W \otimes C_*\Big(\prod_{i=1}^p L_{\sigma(i)} \times \prod_{i=1}^p S_{\sigma(i)}\Big) \arrow[r, "\mathcal{H}"'] & \bigotimes_{i=1}^p [C_*(L_{\sigma(i)}) \otimes C_*(S_{\sigma(i)})].
\end{tikzcd}$$

Proof. The proof is similar to the proof of Lemma 7.1 of [**14**]. Let us use the notation $A_j = C_j(\prod_{i=1}^p L_i \times \prod_{i=1}^p S_i)$ and $B_j = [\otimes_{i=1}^p C_*(L_i) \otimes C_*(S_i)]_j$. We construct $\mathcal{H}$ on $W_i \otimes A_j$ by induction on i and for fixed i by induction on j. Note that the two maps agree on $W \otimes A_0$, so H is the zero map on $W \otimes A_0$. To define $\mathcal{H}$ on $W_0 \otimes A_j$, $j \geqslant 0$, it suffices to define on $e_0 \otimes A_j$, since $\mathcal{H}$ can then be uniquely extended to all of $W_0 \otimes A_j$ using the commutativity of the second diagram. The functor $e_0 \otimes A_j$ is represented by the model $\Delta[j]^p \times \Delta[j]^p$ and $W \otimes B_j$ is acyclic on this model. Therefore, by the acyclic model argument, $\mathcal{H}$ can be defined on $e_0 \otimes A_j$, provided $\mathcal{H}$ is known on $e_0 \otimes A_{j-1}$. But $\mathcal{H}$ has already been defined on $W_0 \otimes A_0$. Hence by induction on j, we can define $\mathcal{H}$ on $e_0 \otimes A_j$, $j \geqslant 0$. To define $\mathcal{H}$ on $W_i \otimes A_j$, assume that it has already been defined on $W_{i'} \otimes A_j$, $i' < i$, $j \geqslant 0$ and on $W_i \otimes A_{j'}$, $j' < j$. Choose a $\Lambda\pi$-basis $\{w_k\}$ for W_i. As before, it suffices to define $\mathcal{H}$ on $w \otimes A_j$, $w \in \{w_k\}$. We can repeat the acyclic model argument replacing e_0 by w, and hence we are through by induction. $\square$

In the special case $L_1 = \cdots = L_p = L$, $S_1 = \cdots = S_p = S$, we obtain the following corollary:

Corollary 5.9. *For simplicial sets L and S, the two chain maps $\xi^p\phi(\mathrm{id} \otimes u)$ and $U(\phi \otimes \phi)(\mathrm{id} \otimes t \otimes \mathrm{id})(\psi \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \times \xi)$ from $W \otimes C_*(L^p \times S^p)$ to $[C_*(L) \otimes C_*(S)]^p$ are $\Lambda\pi$-equivariantly homotopic and the homotopy is natural in L and S.*

Suppose (T, M_0, ρ) and (T', M_0', ρ') are objects of $\mathcal{A}_\Lambda$. With the product actions of $\rho \times \rho'$ on $T \times T'$ and $M_0 \otimes M_0'$, we have an object $(T \times T', M_0 \otimes M_0', \rho \times \rho') \in \mathcal{A}_\Lambda$. The lemma below relates $\Gamma(T \times T', M_0 \otimes M_0', \rho \times \rho') = (C^*_{\rho\times\rho'}(T \times T'; M_0 \otimes M_0'), \theta)$ to $\Gamma(T, M_0, \rho) \otimes \Gamma(T', M_0', \rho') = (C^*_\rho(T; M_0) \otimes C^*_{\rho'}(T'; M_0'), \tilde{\theta})$.

Let

$$\tilde{\alpha}\colon C^*_\rho(T; M_0) \otimes C^*_{\rho'}(T'; M_0') \to \mathrm{Hom}_{\rho\times\rho'}(\underline{C}_*(T) \otimes \underline{C}_*(T'), M_0 \otimes M_0')$$

be defined by

$$\tilde{\alpha}(f \otimes g)(G/H)(x \otimes y) = (-1)^{\deg(x)\deg(y)} f(G/H)(x) \otimes g(G/H)(y), H \subseteq G,$$

where $f \in C^*_\rho(T; M_0), g \in C^*_{\rho'}(T'; M_0'), x \in \underline{C}_*(T)(G/H), y \in \underline{C}_*(T')(G/H)$.

Lemma 5.10. *With the notations as above, the following diagram is $\Lambda\pi$-homotopy commutative:*

$$\begin{tikzcd}
W \otimes C^*_{\rho\times\rho'}(T\times T'; M_0\otimes M'_0)^p \arrow[r, "\theta"] & C^*_{\rho\times\rho'}(T\times T'; M_0\otimes M'_0) \\
W \otimes [C^*_\rho(T; M_0)\otimes C^*_{\rho'}(T'; M'_0)]^p \arrow[r, "\tilde\theta"] \arrow[u, "\mathrm{id}\otimes(\underline{\xi}^*\tilde\alpha)^p"] & C^*_\rho(T; M_0)\otimes C^*_{\rho'}(T'; M'_0). \arrow[u, "\underline{\xi}^*\tilde\alpha"']
\end{tikzcd}$$

Proof. Let D, D', and $\tilde{D}$ be the diagonals for T, T', and $T\times T'$ respectively. Let

$$\underline{u}\colon T^p\times T'^p\to (T\times T')^p \quad \text{and} \quad \underline{U}\colon \underline{C}_*(T)^p\otimes \underline{C}_*(T')^p\to [\underline{C}_*(T)\otimes\underline{C}_*(T')]^p$$

be the shuffle maps. Let t be the switch map.

By the definitions of θ and $\tilde\theta$, it suffices to prove that the following diagram of O_G-chain complexes is $\Lambda(\pi\times\rho\times\rho')$-equivariant homotopy commutative:

$$(1)\qquad \begin{tikzcd}
W\otimes \underline{C}_*(T\times T') \arrow[r, "\underline{\Delta}"] \arrow[d, "\mathrm{id}\times\underline{\xi}"'] & \underline{C}_*(T\times T')^p \arrow[d, "\underline{\xi}^p"] \\
W\otimes \underline{C}_*(T)\otimes\underline{C}_*(T') \arrow[r, "\zeta"] & [\underline{C}_*(T)\otimes\underline{C}_*(T')]^p.
\end{tikzcd}$$

Here

$$\underline{\Delta} = (\epsilon\otimes\mathrm{id})\underline{\Phi}(\mathrm{id}\otimes\tilde{D}), \zeta = \underline{U}(\underline{\Delta}\otimes\underline{\Delta})(\mathrm{id}\otimes t\otimes\mathrm{id})(\psi\otimes\mathrm{id}\otimes\mathrm{id}).$$

Let $\underline{\phi} = (\epsilon\otimes\mathrm{id})\underline{\Phi}$. Observe that $\tilde{D} = \underline{u}(D\times D')$ and

$$(\mathrm{id}\otimes D\otimes\mathrm{id}\otimes D')(\mathrm{id}\otimes t\otimes\mathrm{id})(\psi\otimes\mathrm{id}\otimes\mathrm{id}) = (\mathrm{id}\otimes t\otimes\mathrm{id})(\psi\otimes\mathrm{id}\otimes\mathrm{id})(\mathrm{id}\otimes D\otimes D').$$

Observe that the following diagram commutes by naturality of $\underline{\xi}$:

$$(2)\qquad \begin{tikzcd}
W\otimes \underline{C}_*(T\times T') \arrow[r, "\mathrm{id}\otimes(D\times D')"] \arrow[d, "\mathrm{id}\otimes\underline{\xi}"'] & W\otimes\underline{C}_*(T^p\times T'^p) \arrow[d, "\mathrm{id}\otimes\underline{\xi}"] \\
W\otimes \underline{C}_*(T)\otimes\underline{C}_*(T') \arrow[r, "\mathrm{id}\otimes D\otimes D'"] & W\otimes\underline{C}_*(T^p)\otimes\underline{C}_*(T'^p).
\end{tikzcd}$$

Let $\mathcal{F}$ denote the following diagram of O_G-chain complexes of Λ-modules:

$$(3)\qquad \begin{tikzcd}
W\otimes \underline{C}_*(T^p\times T'^p) \arrow[r, "\mathrm{id}\otimes\underline{u}"] \arrow[d, "\mathrm{id}\otimes\underline{\xi}"'] & W\otimes\underline{C}_*([T\times T']^p) \arrow[r, "\underline{\phi}"] & \underline{C}_*(T\times T')^p \arrow[d, "\underline{\xi}^p"] \\
W\otimes \underline{C}_*(T^p)\otimes\underline{C}_*(T'^p) \arrow[rr, "\underline{U}(\underline{\phi}\otimes\underline{\phi})(\mathrm{id}\otimes t\otimes\mathrm{id})(\psi\otimes\mathrm{id}\otimes\mathrm{id})"] & & [\underline{C}_*(T)\otimes\underline{C}_*(T')]^p.
\end{tikzcd}$$

Then $\mathcal{F}(G/H)$ is $\Lambda\pi$-equivariant homotopy commutative, by Corollary 5.9. The naturality of this homotopy with respect to maps from $T(G/H)$ into itself implies that the homotopy is equivariant for the $\rho(G/H)$-action on $T(G/H)$. Similarly, the homotopy is $\rho'(G/H)$-equivariant. These natural equivariant homotopies of chain complexes combine together to form $\Lambda(\pi \times \rho \times \rho')$-equivariant homotopy, which makes diagram (3) $\Lambda(\pi \times \rho \times \rho')$-equivariant homotopy commutative.

Now observe that diagram (1) is juxtaposition of diagrams (2) and (3). Hence diagram (1) is $\Lambda(\pi \times \rho \times \rho')$-equivariant homotopy commutative. □

Proposition 5.11. *For an object* (T, M_0, ρ) *of* $\mathcal{A}_\Lambda$, $\Gamma(T, M_0, \rho) = (C^*_\rho(T; M_0), \theta)$ *is a Cartan object of* $\mathfrak{C}(p)$.

Proof. Recall that $(C^*_\rho(T; M_0), \theta)$ is called a Cartan object if the cup product is a morphism of $\mathfrak{C}(p)$. Now observe that

$$(T, M_0, \rho) \xrightarrow{(D,\mathrm{id},\mathrm{id})} (T \times T, M_0, \rho) \xrightarrow{(\mathrm{id},m,D)} (T \times T, M_0 \otimes M_0, \rho \times \rho)$$

are morphisms in $\mathcal{A}_\Lambda$, where $m\colon M_0 \otimes M_0 \to M_0$ is the multiplication, D denotes the diagonal map, and we let ρ to operate diagonally on $T \times T$.

Applying Lemma 5.10 with $(T, M_0, \rho) = (T', M'_0, \rho')$, and composing with the morphism $C^*(\mathrm{id}, m, D)$, we see that the composite $\underline{\xi}^*\alpha$

$$C^*_\rho(T; M_0) \otimes C^*_\rho(T; M_0) \xrightarrow{\alpha} \mathrm{Hom}_\rho(\underline{C}_*(T) \otimes \underline{C}_*(T), M_0) \xrightarrow{\underline{\xi}^*} C^*_\rho(T \times T; M_0)$$

is a morphism in $\mathfrak{C}(p)$. Also note that $C^*(D, \mathrm{id}, \mathrm{id})\colon C^*_\rho(T \times T; M_0) \to C^*_\rho(T; M_0)$ is a morphism in $\mathfrak{C}(p)$. Hence the cup product is a morphism in $\mathfrak{C}(p)$. □

Next we show that $(C^*_\rho(T; M_0)$ is an 'Adem object' in $\mathfrak{C}(p)$.

Proposition 5.12. *For an object* (T, M_0, ρ) *of* $\mathcal{A}_\Lambda$, $\Gamma(T, M_0, \rho) = (C^*_\rho(T; M_0), \theta)$ *is an Adem object of* $\mathfrak{C}(p)$.

Proof. With the notations of Definition 5.4, we first construct the map

$$\eta\colon Y \otimes C^*_\rho(T; M_0)^{p^2} \to C^*_\rho(T; M_0).$$

The procedure is similar to the construction of θ. We remark that the proof of Lemma 5.5 works for any subgroup π of Σ_r, r being any positive integer. Thus we have a chain map

$$\Phi\colon Y \otimes C_*(L_1 \times \cdots \times L_r) \to Y \otimes C_*(L_1) \otimes \cdots \otimes C_*(L_r),$$

satisfying properties of Lemma 5.5. As before, we specialize to $L_1 = \cdots = L_r = L$ and take $\pi = \Sigma_r$. The naturality of Φ with respect to maps of a simplicial set into itself allows us to pass to an O_G-simplicial set T, equipped with an action of an O_G-group ρ, so that we get $\Lambda(\Sigma_r \times \rho)$-equivariant map of O_G-chain complexes $\underline{\Phi}\colon Y \otimes \underline{C}_*(T^r) \to Y \otimes \underline{C}_*(T)^r$. As a consequence, we obtain a map of O_G-chain complexes $\underline{\Delta}\colon Y \otimes \underline{C}(T) \to \underline{C}(T)^{p^2}$ which is $(\Sigma_{p^2} \times \rho)$-equivariant. Next, following the construction of the map θ, we obtain η.

Note that, dualising the diagram in Definition 5.4, it suffices to prove that the following diagram is $\Lambda(\tau\times\rho)$-homotopy commutative.

$$\begin{tikzcd} W_1\otimes W_2^p\otimes \underline{C}_*(T) \arrow[r, "w\otimes\mathrm{id}"] \arrow[d, "t\otimes\mathrm{id}"'] & Y\otimes \underline{C}_*(T) \arrow[r, "\underline{\Delta}"] & \underline{C}_*(T)^{p^2} \\ W_2^p\otimes W_1\otimes \underline{C}_*(T) \arrow[r, "\mathrm{id}\otimes\underline{\Delta}"] & W_2^p\otimes \underline{C}_*(T)^p \arrow[r, "\underline{U}"] & {[W_2\otimes \underline{C}_*(T)]^p.} \arrow[u, "\underline{\Delta}^p"'] \end{tikzcd}$$

Here the notations are as in Lemma 5.10. Define the maps of O_G-chain complexes $\chi,\Omega\colon W_1\otimes W_2^p\otimes \underline{C}_*(T^{p^2})\to \underline{C}_*(T)^{p^2}$ by

$$\chi=\underline{\phi}(w\otimes \mathrm{id}_{\underline{C}_*(T^{p^2})})\mathrm{and}\Omega=\underline{\phi}^p\underline{U}(\mathrm{id}_{W_1\otimes W_2^p}\otimes\underline{\phi})(t\otimes \mathrm{id}_{\underline{C}_*(T^{p^2})}).$$

Let $D\colon \underline{C}_*(T)\to \underline{C}_*(T^{p^2})$ be induced by diagonal. Following [**13**], we observe that

$$\underline{\Delta}(w\otimes\mathrm{id})=\chi(\mathrm{id}\otimes\mathrm{id}\otimes D)$$

and

$$\underline{\Delta}^p\underline{U}(\mathrm{id}\otimes\underline{\Delta})(t\otimes\mathrm{id})=\Omega(\mathrm{id}\otimes\mathrm{id}\otimes D).$$

Therefore it suffices to show that the maps of the O_G-chain complex χ,Ω are $\Lambda(\tau\times\rho)$-equivariantly homotopic. Here τ operates by permutation of factors, and the O_G-group ρ operates diagonally on T^{p^2} and on $\underline{C}_*(T)^{p^2}$. We replace $\underline{C}_*(T^{p^2})$ by $C_*(\prod_{i,j=1}^p L_{i,j})$ and $\underline{C}_*(T)^{p^2}$ by $\bigotimes_{i,j=1}^p C_*(L_{i,j})$ in the definitions of the maps χ and Ω, where $L_{i,j}$s are simplicial sets. Then the chain maps, corresponding to χ and Ω, can be shown to be τ-equivariantly homotopic, and the homotopy is natural with respect to maps of simplicial sets. In the special case $L_{i,j}=L, 1\leqslant i,j\leqslant p$, the naturality of this homotopy for maps of a simplicial set into itself implies that the chain maps $\chi(G/H)$ and $\Omega(G/H)$ are $\Lambda(\tau\times\rho(G/H))$-equivariantly homotopic, $H\subseteq G$ being a subgroup. Again the naturality of homotopy shows that the maps of O_G-chain complexes χ,Ω are $\Lambda(\tau\times\rho)$-equivariantly homotopic. □

Thus we have the following theorem.

Theorem 5.13. *Let $(T,M_0,\rho)\in\mathcal{A}_\Lambda$, $\Lambda=\mathbb{Z}_p, p>2$ a prime. Then there exist functions*

$$\mathcal{P}^s\colon H_\rho^q(T;M_0)\to H_\rho^{q+2s(p-1)}(T;M_0),$$
$$\beta\mathcal{P}^s\colon H_\rho^q(T;M_0)\to H_\rho^{q+2s(p-1)+1}(T;M_0),$$

which satisfy the following properties:

1. *$\mathcal{P}^s$ and $\beta\mathcal{P}^s$ are natural homomorphisms.*
2. *$\mathcal{P}^s=\beta\mathcal{P}^s=0$ if $s<0$. Also $\mathcal{P}^s(x)=0$ if $2s>q$, $\beta\mathcal{P}^s=0$ if $2s\geqslant q$.*
3. *$\mathcal{P}^s(x)=x^p$ if $2s=q$.*

4. *(Cartan formula). For* $x, y \in H^q_\rho(T; M_0)$,

$$\mathcal{P}^s(x \cup y) = \sum_{i+j=s} \mathcal{P}^i(x) \cup \mathcal{P}^j(y),$$

$$\beta\mathcal{P}^{s+1}(x \cup y) = \sum_{i+j=s} [\beta\mathcal{P}^{i+1}(x) \cup \mathcal{P}^j(y) + (-1)^{\deg(x)}\mathcal{P}^i(x) \cup \beta\mathcal{P}^{j+1}(y)].$$

5. *(Adem relation). If* $a < pb$, *then*

$$\beta^e\mathcal{P}^a\mathcal{P}^b = \sum_i (-1)^{a+i}(a - pi, (p-1)b - a + i - 1)\beta^e\mathcal{P}^{a+b-i}\mathcal{P}^i.$$

If $a \leqslant pb$, *then*

$$\beta^e\mathcal{P}^a\beta\mathcal{P}^b = (1-e)\sum_i (-1)^{a+i}(a - pi, (p-1)b - a + i - 1)\beta\mathcal{P}^{a+b-i}\mathcal{P}^i \\ - \sum_i (-1)^{a+i}(a - pi - 1, (p-1)b - a + i)\beta^e\mathcal{P}^{a+b-i}\beta\mathcal{P}^i,$$

where $e = 0, 1$ *and* $\beta^0\mathcal{P}^s = \mathcal{P}^s$ *and* $\beta^1\mathcal{P}^s = \beta\mathcal{P}^s$.

Proof. We only need to prove that $\mathcal{P}^s = \beta\mathcal{P}^s = 0$ for $s < 0$. By definition of the power operations, it suffices to show that $D_i(x) = 0$ for $i > pq - q$, $\deg(x) = q$. Recall that $\underline{\Delta} = (\epsilon \otimes \mathrm{id})\underline{\Phi}(\mathrm{id} \times D)$ and

$$\underline{\Phi}(e_i \otimes D(x)) \in \sum_{j<pq} W_{pq-j} \otimes [\underline{C}_*(T)]^p_j \subseteq \mathrm{Ker}(\epsilon \otimes \mathrm{id}) \qquad \text{for} \quad i > pq - q.$$

Hence $\underline{\Delta}(e_i \otimes x) = 0$ for $x \in \underline{C}_{pq-i}(T)$. $\square$

Let X be a one vertex G-Kan complex and M be equivariant local coefficients of Λ-algebras on X, where $\Lambda = \mathbb{Z}_p$, $p > 2$ a prime. We define the Steenrod reduced power operations in simplicial Bredon-Illman cohomology with local coefficients by

$$\mathcal{P}^s = \mu^{*-1}\mathcal{P}^s\mu^* \quad \text{and} \quad \beta\mathcal{P}^s = \mu^{*-1}(\beta\mathcal{P}^s)\mu^*,$$

where the symbols $\mathcal{P}^s$ and $\beta\mathcal{P}^s$ on the right side of the above equalities denote the power operations as constructed in the category $\mathcal{A}_\Lambda$, and μ^* is the isomorphism $\mu^*\colon H^*_G(X; M) \cong H^*_{\underline{\pi}X}(\widetilde{X}; M_0)$, as obtained in Theorem 4.3. Thus we have the following theorem.

Theorem 5.14. *Let X be a one vertex G-Kan complex and M be equivariant local coefficients of Λ-algebras on X, $\Lambda = \mathbb{Z}_p, p > 2$ a prime. Then there exist natural homomorphisms*

$$\mathcal{P}^s\colon H^q_G(X; M) \to H^{q+2s(p-1)}_G(X; M),$$

$$\beta\mathcal{P}^s\colon H^q_G(X; M) \to H^{q+2s(p-1)+1}_G(X; M),$$

which satisfy properties (1)–(5) of Theorem 5.13.

*If G is trivial, then $\mathcal{P}^s$ can be naturally identified with the reduced power operations in local coefficients [**9**].*

Proof. Since the isomorphism μ^* of the Eilenberg theorem, Theorem 4.3 is natural and respects the cup product, and the first part follows from Theorem 5.13.

For the second part, we just remark that when G is trivial, the map

$$\underline{\Delta}\colon W \otimes \underline{C}_*(T) \to \underline{C}_*(T)^p$$

reduces to the $(\pi \times \rho)$-equivariant chain mapping $\phi'\colon W \otimes C_*(X) \to C_*(X)^p$, as constructed by Gitler in Section 4.2 of [**9**]. □

Remark 5.15. Let $p = 2$, $\Lambda = \mathbb{Z}_2$. For an object $(K, \theta) \in \mathfrak{C}(2)$, we have the maps

$$D_i\colon H^q(K) \to H^{2q-i}(K), i \geqslant 0,$$

defined as before by $D_i(x) = \theta_*(e_i \otimes x^2)$ with $D_i = 0$ for $i < 0$, $x \in H^q(K)$. Then the Steenrod's square operations are defined by

$$Sq^i(x) := D_{q-i}(x).$$

It may be mentioned that in this general setup cup-i products $\cup_i\colon K \otimes K \to K$ are also defined and are given by

$$x \cup_i y := \theta(e_i \otimes x \otimes y), x \in K_q, y \in K_q.$$

(See [**14**, §6].) In terms of these $\cup_i$ products, Steenrod's squares are given by

$$Sq^i(x) = \begin{cases} x \cup_{q-i} x, & 0 \leqslant i \leqslant q \\ 0 & \text{if } i > q. \end{cases}$$

In our situation $K = C^*_\rho(T; M_0)$, where $(T, M_0, \rho) \in \mathcal{A}_\Lambda, \Lambda = \mathbb{Z}_2$, and we obtain $Sq^i\colon H^q_\rho(T; M_0) \to H^{q+i}_\rho(T; M_0)$ by the above formula. As in the case $p > 2$, we use the equivariant Eilenberg theorem to define Steenrod square operations

$$Sq^i\colon H^q_G(X; M) \to H^{q+i}_G(X; M),$$

where X is a one vertex G-Kan complex and M is equivariant local coefficients on X, taking values in $\mathbb{Z}_2$-algebras.

Our approach is simplicial and the motivation comes from Gitler's work [**9**]. The key points of our construction are the use of general algebraic approach to Steenrod operations due to Peter May [**14**] and that of the equivariant Eilenberg theorem in the present context.

In contrast, for a topological space X equipped with an action of a topological group G, Ginot [**8**] gave a direct construction of Steenrod's squares on the Bredon-Illman cohomology of X with local coefficients M that take values in $\mathbb{Z}_2$-algebras. Ginot's idea was to deduce cup-i products on the Bredon-Illman cochain complex of X using a brace (or homotopy Gerstenhaber) algebra structure on this complex [**16**]. For $p = 2$ and a discrete group action, our construction leads to the same operations as defined in [**8**] via the geometric realization functor of simplicial sets.

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