

OPERATIONS ON THE HOPF-HOCHSCHILD COMPLEX FOR MODULE-ALGEBRAS

DONALD YAU

(communicated by Claude Cibils)

Abstract

It is proved that Kaygun’s Hopf-Hochschild cochain complex for a module-algebra is a brace algebra with multiplication. As a result, an analogue of Deligne’s Conjecture holds for module-algebras, and the Hopf-Hochschild cohomology of a module-algebra has a Gerstenhaber algebra structure.

1. Introduction

Let H be a bialgebra, and let A be an associative algebra. The algebra A is said to be an H -module-algebra if there is an H -module structure on A such that the multiplication on A becomes an H -module morphism. For example, if S denotes the Landweber-Novikov algebra [15, 22], then the complex cobordism $MU^*(X)$ of a topological space X is an S -module-algebra. Likewise, the singular mod p cohomology $H^*(X; \mathbf{F}_p)$ of a topological space X is an \mathcal{A}_p -module-algebra, where \mathcal{A}_p denotes the Steenrod algebra associated to the prime p [7, 20]. Other similar examples from algebraic topology can be found in [4]. Important examples of module-algebras from Lie and Hopf algebras theory can be found in, e.g., [12, V.6].

In [14] Kaygun defined a Hochschild-like cochain complex $\mathrm{CH}_{\mathrm{Hopf}}^*(A, A)$ associated to an H -module-algebra A , called the *Hopf-Hochschild cochain complex*, that takes into account the H -linearity. In particular, if H is the ground field, then Kaygun’s Hopf-Hochschild cochain complex reduces to the usual Hochschild cochain complex $C^*(A, A)$ of A [11]. Kaygun [14] showed that the Hopf-Hochschild cohomology of A shares many properties with the usual Hochschild cohomology. For example, it can be described in terms of derived functors, and it satisfies Morita invariance.

The usual Hochschild cochain complex $C^*(A, A)$ has a very rich structure. Namely, it is a brace algebra with multiplication [10]. Combined with a result of McClure and Smith [19] concerning the singular chain operad associated to the little squares operad \mathcal{C}_2 , the brace algebra with multiplication structure on $C^*(A, A)$ leads to a positive solution of Deligne’s Conjecture [6]. Also, passing to cohomology, the brace algebra with multiplication structure implies that the Hochschild cohomology modules $HH^*(A, A)$ form a Gerstenhaber algebra, which is a graded version of a Poisson algebra. This fact was first observed by Gerstenhaber [8].

Received March 8, 2011, revised April 10, 2011; published on May 22, 2011.

2000 Mathematics Subject Classification: 16E40, 16W30.

Key words and phrases: module-algebra, Hopf-Hochschild cohomology, Gerstenhaber algebra, brace algebra, Deligne’s conjecture.

Article available at <http://intlpress.com/HHA/v13/n1/a9> and doi:10.4310/HHA.2011.v13.n1.a9

Copyright © 2011, International Press. Permission to copy for private use granted.

The purpose of this note is to observe that Kaygun's Hopf-Hochschild cochain complex $\mathrm{CH}_{\mathrm{Hopf}}^*(A, A)$ of a module-algebra A also admits the structure of a brace algebra with multiplication. In fact, the Hopf-Hochschild complex is a sub-complex of the usual Hochschild complex, and the former inherits the latter's structure. As in the classical case, this leads to a version of Deligne's Conjecture for module-algebras. Also, the Hopf-Hochschild cohomology modules $HH_{\mathrm{Hopf}}^*(A, A)$ form a Gerstenhaber algebra. When the bialgebra H is the ground field, these structures reduce to the ones in Hochschild cohomology.

A couple of remarks are in order. First, there is another cochain complex $C_{MA}^*(A)$ that can be associated to an H -module-algebra A [24]. The cochain complex $C_{MA}^*(A)$ is different from the Hopf-Hochschild complex $\mathrm{CH}_{\mathrm{Hopf}}^*(A, A)$. The former controls the deformations of A , in the sense of Gerstenhaber [9], with respect to both the H -module structure and the algebra structure on A . It is not known if there is any relationship between the two complexes.

Second, the results and arguments here can be adapted to module-coalgebras, comodule-algebras, and comodule-coalgebras. To do that, one replaces the crossed product algebra X (section 2.3) associated to an H -module-algebra A by a suitable crossed product (co)algebra [1, 2, 3] and replaces Kaygun's Hopf-Hochschild cochain complex by a suitable variant.

1.1. Organization

The rest of this paper is organized as follows.

In the following section, we recall the construction of the Hopf-Hochschild cochain complex $\mathrm{CH}_{\mathrm{Hopf}}^*(A, A)$ from Kaygun [14].

In Section 3, it is observed that $\mathrm{CH}_{\mathrm{Hopf}}^*(A, A)$ has the structure of an operad with multiplication (Theorem 3.1). This leads in Section 4 to the desired brace algebra with multiplication structure on $\mathrm{CH}_{\mathrm{Hopf}}^*(A, A)$ (Corollary 4.1). Explicit formulas of these brace operations are given (11).

In Section 5, it is observed that the brace algebra with multiplication structure on $\mathrm{CH}_{\mathrm{Hopf}}^*(A, A)$ leads to a homotopy G -algebra structure (Corollary 5.1). The differential from this homotopy G -algebra and the Hopf-Hochschild differential are then identified, up to a sign (Theorem 5.2).

Passing to cohomology, this leads in Section 6 to a Gerstenhaber algebra structure on the Hopf-Hochschild cohomology modules $HH_{\mathrm{Hopf}}^*(A, A)$ (Corollary 6.1). The graded associative product and the graded Lie bracket on $HH_{\mathrm{Hopf}}^*(A, A)$ are explicitly described (16).

In the final section, combining our results with a result of McClure and Smith [19], a version of Deligne's Conjecture for module-algebras is obtained (Corollary 7.1). This section can be read immediately after Section 4 and is independent of Sections 5 and 6.

Acknowledgements

The author thanks the anonymous referee for his/her helpful suggestions.

2. Hopf-Hochschild cohomology

In this section, we fix some notations and recall from [14, Section 3] the Hopf-Hochschild cochain complex associated to a module-algebra.

2.1. Notations

Fix a ground field K once and for all. Tensor product and vector space are all meant over K .

Let $H = (H, \mu_H, \Delta_H)$ denote a K -bialgebra with associative multiplication μ_H and coassociative comultiplication Δ_H . It is assumed to be unital and counital, with its unit and counit denoted by 1_H and $\varepsilon: H \rightarrow K$, respectively.

Let $A = (A, \mu_A)$ denote an associative, unital K -algebra with unit 1_A (or simply 1).

In a coalgebra (C, Δ) , we use Sweedler's notation [23] for comultiplication:

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}, \Delta^2(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}, \text{etc.}$$

These notations will be used throughout the rest of this paper.

2.2. Module-algebra

Recall that the algebra A is said to be an H -module-algebra [5, 12, 21, 23] if and only if there exists a left H -module structure on A such that μ_A and the unit map $1_A: K \rightarrow A$ are both H -module morphisms, i.e.,

$$b(a_1 a_2) = \sum (b_{(1)} a_1)(b_{(2)} a_2) \tag{1}$$

for $b \in H$ and $a_1, a_2 \in A$, and

$$b(1_A) = \varepsilon(b)1_A$$

for $b \in H$.

We will assume that A is an H -module-algebra for the rest of this paper.

2.3. Crossed product algebra

Define the vector space

$$X \stackrel{\text{def}}{=} A \otimes A \otimes H.$$

Define a multiplication on X [14, Definition 3.1] by setting

$$(a_1 \otimes a'_1 \otimes b^1)(a_2 \otimes a'_2 \otimes b^2) = \sum a_1 \left(b_{(1)}^1 a_2 \right) \otimes \left(b_{(3)}^1 a'_2 \right) a'_1 \otimes b_{(2)}^1 b^2$$

for $a_1 \otimes a'_1 \otimes b^1$ and $a_2 \otimes a'_2 \otimes b^2$ in X . It is shown in [14, Lemma 3.2] that X is an associative, unital K -algebra, called the *crossed product algebra*.

Note that if $H = K$ (= the trivial group bialgebra $K[\{e\}]$), then X is just the enveloping algebra $A \otimes A^{\text{op}}$, where A^{op} is the opposite algebra of A .

The algebra A is a left X -module via the action

$$(a \otimes a' \otimes b)a_0 = a(ba_0)a'$$

for $a \otimes a' \otimes b \in X$ and $a_0 \in A$. Likewise, the vector space $A^{\otimes(n+2)}$ is a left X -module

via the action

$$(a \otimes a' \otimes b)(a_0 \otimes \cdots \otimes a_{n+1}) = \sum ab_{(1)}a_0 \otimes b_{(2)}a_1 \otimes \cdots \otimes b_{(n+1)}a_n \otimes b_{(n+2)}a_{n+1}a'$$

for $a_0 \otimes \cdots \otimes a_{n+1} \in A^{\otimes(n+2)}$.

2.4. Bar complex

Consider the chain complex $CB_*(A)$ of vector spaces with

$$CB_n(A) = A^{\otimes(n+2)},$$

and with differential

$$d_n^{CB} = \sum_{j=0}^n (-1)^j \partial_j : CB_n(A) \rightarrow CB_{n-1}(A),$$

where

$$\partial_j(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes (a_j a_{j+1}) \otimes \cdots \otimes a_{n+1}.$$

It is mentioned above that each vector space $CB_n(A) = A^{\otimes(n+2)}$ is a left X -module. Using the module-algebra condition (1), it is straightforward to see that each ∂_j is X -linear. Therefore, $CB_*(A)$ can be regarded as a chain complex of left X -modules.

Note that in the case $H = K$, the chain complex $CB_*(A)$ of $A \otimes A^{op}$ -modules is the usual bar complex of A .

2.5. Hopf-Hochschild cochain complex

The *Hopf-Hochschild cochain complex of A with coefficients in A* is the cochain complex of vector spaces:

$$(CH_{Hopf}^*(A, A), d_{CH}) \stackrel{\text{def}}{=} \text{Hom}_X((CB_*(A), d^{CB}), A). \tag{2}$$

Its n th cohomology module is denoted by $HH_{Hopf}^n(A, A)$ and is called the *n th Hopf-Hochschild cohomology of A with coefficients in A* .

When $H = K$, the cochain complex $(CH_{Hopf}^*(A, A), d_{CH})$ is the usual Hochschild cochain complex of A with coefficients in itself [11], and $HH_{Hopf}^n(A, A)$ is the usual Hochschild cohomology module.

In what follows, we will use the notation $CH_{Hopf}^*(A, A)$ to denote (i) the Hopf-Hochschild cochain complex $(CH_{Hopf}^*(A, A), d_{CH})$, (ii) the sequence $\{CH_{Hopf}^n(A, A)\}$ of vector spaces, or (iii) the graded vector space $\oplus_n CH_{Hopf}^n(A, A)$. It should be clear from the context what $CH_{Hopf}^*(A, A)$ means.

3. Algebraic operad

The purpose of this section is to show that the vector spaces $CH_{Hopf}^*(A, A)$ in the Hopf-Hochschild cochain complex of an H -module-algebra A with self coefficients form an operad with multiplication (Theorem 3.1).

3.1. Operads

Recall from [16, 17, 18] that a *non-Σ operad* $\mathcal{O} = \{\mathcal{O}(n), \gamma, \text{Id}\}$ consists of a sequence of vector spaces $\mathcal{O}(n)$ ($n \geq 1$) together with structure maps

$$\gamma: \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \rightarrow \mathcal{O}(n_1 + \cdots + n_k),$$

for $k, n_1, \dots, n_k \geq 1$ and an *identity element* $\text{Id} \in \mathcal{O}(1)$, satisfying the following two axioms.

1. The structure maps γ are required to be *associative*, in the sense that

$$\begin{aligned} \gamma(\gamma(f; g_{1,k}); h_{1,N}) &= \gamma(f; \gamma(g_1; h_{1,N_1}), \dots, \\ &\quad \gamma(g_i; h_{N_{i-1}+1, N_i}), \dots, \gamma(g_k; h_{N_{k-1}+1, N_k})). \end{aligned}$$

Here $f \in \mathcal{O}(k)$, $g_i \in \mathcal{O}(n_i)$, $N = n_1 + \cdots + n_k$, and $N_i = n_1 + \cdots + n_i$. Given elements x_i, x_{i+1}, \dots , the symbol $x_{i,j}$ is the abbreviation for the sequence x_i, x_{i+1}, \dots, x_j or $x_i \otimes \cdots \otimes x_j$ whenever $i \leq j$.

2. The maps

$$\gamma(-; \text{Id}, \dots, \text{Id}): \mathcal{O}(k) \rightarrow \mathcal{O}(k) \quad \text{and} \quad \gamma(\text{Id}; -): \mathcal{O}(k) \rightarrow \mathcal{O}(k)$$

are both equal to the identity map on $\mathcal{O}(k)$ for each $k \geq 1$.

For the rest of this paper, we will refer to non-Σ operads simply as operads.

3.2. Operad morphism

Let \mathcal{O} and \mathcal{P} be operads. An *operad morphism* $\phi: \mathcal{O} \rightarrow \mathcal{P}$ consists of a sequence of linear maps $\phi_n: \mathcal{O}(n) \rightarrow \mathcal{P}(n)$ such that

$$\phi_1(\text{Id}_{\mathcal{O}}) = \text{Id}_{\mathcal{P}}$$

and the diagram

$$\begin{array}{ccc} \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) & \xrightarrow{\gamma} & \mathcal{O}(N) \\ \phi_k \otimes \phi_{n_1} \otimes \cdots \otimes \phi_{n_k} \downarrow & & \downarrow \phi_N \\ \mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) & \xrightarrow{\gamma} & \mathcal{P}(N) \end{array} \tag{3}$$

commutes for all $k, n_1, \dots, n_k \geq 1$, where $N = n_1 + \cdots + n_k$.

3.3. Operad with multiplication

Let \mathcal{O} be an operad. A *multiplication* on \mathcal{O} [10, Section 1.2] is an element $m \in \mathcal{O}(2)$ that satisfies

$$\gamma(m; m, \text{Id}) = \gamma(m; \text{Id}, m). \tag{4}$$

In this case, (\mathcal{O}, m) is called an *operad with multiplication*. A multiplication on \mathcal{O} is equivalent to an operad morphism $\mathcal{A}s \rightarrow \mathcal{O}$, where $\mathcal{A}s$ is the operad for associative algebras. The associative operad $\mathcal{A}s$ is generated by an element $\mu \in \mathcal{A}s(2)$, whose image under an operad morphism $\mathcal{A}s \rightarrow \mathcal{O}$ is a multiplication on \mathcal{O} .

3.4. Operad with multiplication structure on $\text{CH}_{\text{Hopf}}^*(A, A)$

In what follows, in order to simplify the typography, we will sometimes write $\mathcal{C}(n)$ for the vector space $\text{CH}_{\text{Hopf}}^n(A, A)$. To show that the vector spaces $\text{CH}_{\text{Hopf}}^*(A, A)$

form an operad with multiplication, we first define the structure maps, the identity element, and the multiplication.

Structure maps For $k, n_1, \dots, n_k \geq 1$, define a map

$$\gamma: \mathcal{C}(k) \otimes \mathcal{C}(n_1) \otimes \cdots \otimes \mathcal{C}(n_k) \rightarrow \mathcal{C}(N)$$

by setting

$$\begin{aligned} & \gamma(f; g_{1,k})(a_{0,N+1}) \\ &= f(a_0 \otimes g_1(1 \otimes a_{1,n_1} \otimes 1) \otimes \cdots \otimes g_i(1 \otimes a_{N_{i-1}+1, N_i} \otimes 1) \otimes \cdots \otimes a_{N+1}). \end{aligned} \quad (5)$$

Here the notation is as in the definition of an operad above, and each $a_i \in A$.

Identity element Let $\text{Id} \in \mathcal{C}(1)$ be the element such that

$$\text{Id}(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 a_2. \quad (6)$$

This is indeed an element of $\mathcal{C}(1)$, since the identity map on A is H -linear.

Multiplication Let $\pi \in \mathcal{C}(2)$ be the element such that

$$\pi(a_0 \otimes a_1 \otimes a_2 \otimes a_3) = a_0 a_1 a_2 a_3. \quad (7)$$

This is indeed an element of $\mathcal{C}(2)$, since the multiplication map $A^{\otimes 2} \rightarrow A$ on A is H -linear.

Theorem 3.1. *The data*

$$\mathcal{C} = \{\mathcal{C}(n), \gamma, \text{Id}\}$$

forms an operad. Moreover, $\pi \in \mathcal{C}(2)$ is a multiplication on the operad \mathcal{C} .

Proof. It is immediate from (5) and (6) that $\gamma(-; \text{Id}^{\otimes k})$ and $\gamma(\text{Id}; -)$ are the identity map on $\mathcal{C}(k)$ for each $k \geq 1$.

To prove associativity of γ , we use the notations in the definition of an operad and compute as follows:

$$\begin{aligned} & \gamma(\gamma(f; g_{1,k}); h_{1,N})(a_0 \otimes \cdots \otimes a_{M+1}) \\ &= \gamma(f; g_{1,k})(a_0 \otimes \cdots \otimes h_j(1 \otimes a_{M_{j-1}+1, M_j} \otimes 1) \otimes \cdots \otimes a_{M+1}) \\ &= f(a_0 \otimes \cdots \otimes g_i(1 \otimes z_i \otimes 1) \otimes \cdots \otimes a_{M+1}) \\ &= \gamma(f; \dots, \gamma(g_i; h_{N_{i-1}+1, N_i}), \dots)(a_0 \otimes \cdots \otimes a_{M+1}). \end{aligned}$$

Here the element z_i ($1 \leq i \leq k$) is given by

$$\begin{aligned} z_i &= \bigotimes_{l=N_{i-1}+1}^{N_i} h_l(1 \otimes a_{M_{l-1}+1, M_l} \otimes 1) \\ &= h_{N_{i-1}+1}(1 \otimes a_{M_{N_{i-1}+1}, M_{N_{i-1}+1}} \otimes 1) \otimes \cdots \otimes h_{N_i}(1 \otimes a_{M_{N_{i-1}+1}, M_{N_i}} \otimes 1). \end{aligned}$$

This shows that γ is associative and that $\mathcal{C} = \{\mathcal{C}(n), \gamma, \text{Id}\}$ is an operad.

To see that $\pi \in \mathcal{C}(2)$ is a multiplication on \mathcal{C} , one observes that both

$$\gamma(\pi; \pi, \text{Id})(a_0 \otimes \cdots \otimes a_4) \quad \text{and} \quad \gamma(\pi; \text{Id}, \pi)(a_0 \otimes \cdots \otimes a_4)$$

are equal to the product $a_0 a_1 a_2 a_3 a_4$.

This finishes the proof of Theorem 3.1. \square

3.5. Relationship with the Hochschild complex

Here we observe that the operad with multiplication (\mathcal{C}, π) induces the usual operad with multiplication structure on the Hochschild complex.

Recall that the Hochschild complex

$$C^*(A, A) = \{C^n(A, A) = \text{Hom}_K(A^{\otimes n}, A)\}$$

is an operad with multiplication μ_A [10]. Indeed, $C^*(A, A)$ is the endomorphism operad of A . Its structure maps γ are given by

$$\gamma(f; g_{1,k})(a_{1,N}) = f(\cdots \otimes g_i(a_{N_{i-1}+1, N_i}) \otimes \cdots)$$

for $f \in C^n(A, A)$, $g_i \in C^{n_i}(A, A)$, and $a_j \in A$. The notations are as in section 3.1 with $N_0 = 0$. The identity element in $C^1(A, A)$ is the identity map on A .

Denote by

$$\phi_{\text{Hopf}}: \mathcal{A}s \rightarrow \text{CH}_{\text{Hopf}}^*(A, A) \quad \text{and} \quad \phi: \mathcal{A}s \rightarrow C^*(A, A)$$

the operad morphisms corresponding to the operads with multiplication $\text{CH}_{\text{Hopf}}^*(A, A)$ and $C^*(A, A)$, respectively. The following observation says that ϕ is induced by ϕ_{Hopf} .

Corollary 3.2. *The operad with multiplication $C^*(A, A)$ is induced by $\text{CH}_{\text{Hopf}}^*(A, A)$ in the sense that there is an operad morphism $\rho: \text{CH}_{\text{Hopf}}^*(A, A) \rightarrow C^*(A, A)$ such that the diagram*

$$\begin{array}{ccc} \mathcal{A}s & \xrightarrow{\phi_{\text{Hopf}}} & \text{CH}_{\text{Hopf}}^*(A, A) \\ \phi \downarrow & & \swarrow \rho \\ C^*(A, A) & & \end{array} \quad (8)$$

commutes.

Proof. Let us abbreviate $C^n(A, A)$ to C^n and $\text{CH}_{\text{Hopf}}^n(A, A)$ to $\mathcal{C}(n)$. The component map $\rho: \mathcal{C}(n) \rightarrow C^n$ is defined as

$$\rho(g)(a_{1,n}) = g(1 \otimes a_{1,n} \otimes 1)$$

for $a_i \in A$. To see that the map ρ is an operad morphism, first observe that it preserves the identity elements because

$$\rho(\text{Id})(a) = \text{Id}(1 \otimes a \otimes 1) = a$$

by (6). Moreover, the commutativity of the diagram (3) for ρ follows from the fact that both $\rho(\gamma(f; g_{1,k}))(a_{1,N})$ and $\gamma(\rho(f); \rho(g_{1,k}))(a_{1,N})$ are equal to

$$f(1 \otimes \cdots \otimes g_i(1 \otimes a_{N_{i-1}+1, N_i} \otimes 1) \otimes \cdots \otimes 1).$$

Here $f \in \mathcal{C}(k)$, $g_i \in \mathcal{C}(n_i)$, $N = n_1 + \cdots + n_k$, $N_0 = 0$, and $a_j \in A$.

Now we show that the diagram (8) commutes. Note that the operad morphism ϕ is uniquely determined by the property $\phi(\mu) = \mu_A$, where $\mu \in \mathcal{A}s(2)$ is the generator for the associative operad. Likewise, we have $\phi_{\text{Hopf}}(\mu) = \pi$. Since we have

$$\rho(\pi)(a_1 \otimes a_2) = \pi(1 \otimes a_1 \otimes a_2 \otimes 1) = a_1 a_2$$

by (7), we conclude that $\rho(\pi) = \mu_A$ and $\rho(\phi_{\text{Hopf}}(\mu)) = \mu_A$. This implies by uniqueness that $\phi = \rho \circ \phi_{\text{Hopf}}$. \square

4. Brace algebra

The purpose of this section is to show that the graded vector space $\text{CH}_{\text{Hopf}}^*(A, A)$ admits the structure of a brace algebra with multiplication. Explicit formulas of the brace operations are given in (11).

4.1. Brace algebra

For a graded vector space $V = \bigoplus_{n=1}^{\infty} V^n$ and an element $x \in V^n$, set $\deg x = n$ and $|x| = n - 1$. Elements in V^n are said to have *degree* n .

Recall from [10, Definition 1] that a *brace algebra* is a graded vector space $V = \bigoplus V^n$ together with a collection of brace operations $x\{x_1, \dots, x_n\}$ of degree $-n$, satisfying the associativity axiom:

$$x\{x_{1,m}\}\{y_{1,n}\} = \sum_{0 \leq i_1 \leq \dots \leq i_m \leq n} (-1)^\varepsilon x\{y_{1,i_1}, x_1\{y_{i_1+1,j_1}\}, y_{j_1+1}, \dots, y_{i_m}, x_m\{y_{i_m+1,j_m}\}, y_{j_m+1}, n\}.$$

Here the sign is given by $\varepsilon = \sum_{p=1}^m (|x_p| \sum_{q=1}^{i_p} |y_q|)$.

4.2. Brace algebra with multiplication

Let $V = \bigoplus V^n$ be a brace algebra. A *multiplication* on V [10, Section 1.2] is an element $m \in V^2$ such that

$$m\{m\} = 0. \tag{9}$$

In this case, we call $V = (V, m)$ a *brace algebra with multiplication*.

4.3. Brace algebra from operad

Suppose that $\mathcal{O} = \{\mathcal{O}(n), \gamma, \text{Id}\}$ is an operad. Define the following operations on the graded vector space $\mathcal{O} = \bigoplus \mathcal{O}(n)$:

$$x\{x_1, \dots, x_n\} \stackrel{\text{def}}{=} \sum (-1)^\varepsilon \gamma(x; \text{Id}, \dots, \text{Id}, x_1, \text{Id}, \dots, \text{Id}, x_n, \text{Id}, \dots, \text{Id}). \tag{10}$$

Here the sum runs over all possible substitutions of x_1, \dots, x_n into $\gamma(x; \dots)$ in the given order. The sign is determined by $\varepsilon = \sum_{p=1}^n |x_p| i_p$, where i_p is the total number of inputs in front of x_p . Note that

$$\deg x\{x_1, \dots, x_n\} = \deg x - n + \sum_{p=1}^n \deg x_p,$$

so the operation (10) is of degree $-n$.

Proposition 1 in [10] establishes that the operations (10) make the graded vector space $\bigoplus \mathcal{O}(n)$ into a brace algebra. Moreover, a multiplication on the operad \mathcal{O} in the sense of § 3.3 is equivalent to a multiplication on the brace algebra $\bigoplus \mathcal{O}(n)$. In fact, for an element $m \in \mathcal{O}(2)$, one has that

$$m\{m\} = \gamma(m; m, \text{Id}) - \gamma(m; \text{Id}, m).$$

It follows that the condition (4) is equivalent to (9). In other words, an operad with multiplication (\mathcal{O}, m) gives rise to a brace algebra with multiplication $(\bigoplus \mathcal{O}(n), m)$. Combining this discussion with Theorem 3.1, we obtain the following result.

Corollary 4.1. *The graded vector space $\text{CH}_{\text{Hopf}}^*(A, A)$ is a brace algebra with brace operations as in (10) and multiplication π (7).*

The brace operations on $\text{CH}_{\text{Hopf}}^*(A, A)$ can be described more explicitly as follows. For $f \in \mathcal{C}(k)$ and $g_i \in \mathcal{C}(m_i)$ ($1 \leq i \leq n$), we have

$$f\{g_1, \dots, g_n\} = \sum (-1)^\varepsilon \gamma(f; \text{Id}^{r_1}, g_1, \text{Id}^{r_2}, g_2, \dots, \text{Id}^{r_n}, g_n, \text{Id}^{r_{n+1}}),$$

where $\text{Id}^r = \text{Id}^{\otimes r}$. Here the r_j are given by

$$r_j = \begin{cases} i_1 & \text{if } j = 1, \\ i_j - i_{j-1} - 1 & \text{if } 2 \leq j \leq n, \\ k - i_n - 1 & \text{if } j = n + 1, \end{cases}$$

and

$$\varepsilon = \sum_{p=1}^n (m_p - 1) i_p.$$

Write $M = \sum_{i=1}^n m_i$ and $M_j = \sum_{i=1}^j m_i$. For an element $a_{0, k+M-n+1}$ in $A^{\otimes(k+M-n)}$, we have

$$\begin{aligned} f\{g_{1,n}\}(a_{0, k+M-n+1}) &= \sum (-1)^\varepsilon f(a_{0, i_1} \otimes g_1(1 \otimes a_{i_1+1, i_1+m_1} \otimes 1) \otimes \dots \\ &\quad \otimes a_{i_{j-1}+M_{j-1}-(j-1)+2, i_j+M_{j-1}-j+1} \\ &\quad \otimes g_j(1 \otimes a_{i_j+M_{j-1}-j+2, i_j+M_j-j+1} \otimes 1) \otimes \dots \\ &\quad \otimes a_{i_n+M-n+2, k+M-n+1}). \end{aligned} \tag{11}$$

5. Homotopy Gerstenhaber algebra

The purpose of this section is to observe that the brace algebra with multiplication structure on $\text{CH}_{\text{Hopf}}^*(A, A)$ induces a homotopy Gerstenhaber algebra structure (Corollary 5.1). The underlying cochain complex of this homotopy Gerstenhaber algebra is canonically isomorphic to the Hopf-Hochschild cochain complex of A (Corollary 5.3).

5.1. Homotopy G -algebra

Recall from [10, Definition 2] that a *homotopy G -algebra* (V, d, \cup) consists of a brace algebra $V = \oplus V^n$, a degree +1 differential d , and a degree 0 associative \cup -product that makes V into a differential graded algebra, satisfying the following two conditions.

1. The \cup -product is required to satisfy the condition

$$(x_1 \cup x_2)\{y_{1,n}\} = \sum_{k=0}^n (-1)^\varepsilon x_1\{y_{1,k}\} \cup x_2\{y_{k+1,n}\},$$

where $\varepsilon = |x_2| \sum_{p=1}^k |y_p|$, for $x_i, y_j \in V$.

2. The differential is required to satisfy the condition

$$\begin{aligned} & d(x\{x_{1,n+1}\}) - (dx)\{x_{1,n+1}\} \\ & - (-1)^{|x|} \sum_{i=1}^{n+1} (-1)^{|x_1|+\dots+|x_{i-1}|} x\{x_1, \dots, dx_i, \dots, x_{n+1}\} \\ & = (-1)^{|x||x_1|+1} x_1 \cup x\{x_{2,n+1}\} - x\{x_{1,n}\} \cup x_{n+1} \\ & + (-1)^{|x|} \sum_{i=1}^n (-1)^{|x_1|+\dots+|x_{i-1}|} x\{x_1, \dots, x_i \cup x_{i+1}, \dots, x_{n+1}\}. \end{aligned}$$

5.2. Homotopy G -algebra from brace algebra with multiplication

A brace algebra with multiplication $V = (V, m)$ gives rise to a homotopy G -algebra (V, d, \cup) [10, Theorem 3], where the \cup -product and the differential are defined as:

$$\begin{aligned} x \cup y &= (-1)^{\deg x} m\{x, y\}, \\ d(x) &= m\{x\} - (-1)^{|x|} x\{m\}. \end{aligned} \tag{12}$$

In particular, this applies to the brace algebra $\text{CH}_{\text{Hopf}}^*(A, A)$ with multiplication π (Corollary 4.1).

Corollary 5.1. *For an H -module-algebra A , $\mathcal{C} = (\text{CH}_{\text{Hopf}}^*(A, A), d, \cup)$ is a homotopy G -algebra.*

5.3. Comparing differentials

There are two differentials on the graded vector space $\text{CH}_{\text{Hopf}}^*(A, A)$, namely, the differential d^n (12) induced by the multiplication π and the Hopf-Hochschild differential d_{CH}^n (2). The following result ensures that the cohomology modules defined by these two differentials are the same.

Theorem 5.2. *The equality*

$$d_{\text{CH}}^n = (-1)^{n+1} d^n$$

holds for each n .

Proof. Pick $f \in \text{CH}_{\text{Hopf}}^n(A, A)$. Then we have

$$\begin{aligned} d^n f &= \pi\{f\} + (-1)^n f\{\pi\} \\ &= (-1)^{n-1} \gamma(\pi; \text{Id}, f) + \gamma(\pi; f, \text{Id}) + (-1)^n \sum_{i=1}^n (-1)^{i-1} \gamma(f; \text{Id}^{i-1}, \pi, \text{Id}^{n-i}). \end{aligned}$$

It follows that

$$(-1)^{n+1} d^n f = \gamma(\pi; \text{Id}, f) + (-1)^{n+1} \gamma(\pi; f, \text{Id}) + \sum_{i=1}^n (-1)^i \gamma(f; \text{Id}^{i-1}, \pi, \text{Id}^{n-i}). \tag{13}$$

Observe that [14]

$$g(a_{0,n+1}) = a_0 g(1 \otimes a_{1,n} \otimes 1) a_{n+1} \tag{14}$$

for $g \in \text{CH}_{\text{Hopf}}^n(A, A)$. Using (14) and applying the various terms in (13) to an element

$a_{0,n+2} \in \text{CB}_{n+1}(A) = A^{\otimes(n+3)}$, we obtain

$$\begin{aligned} \gamma(\pi; \text{Id}, f)(a_{0,n+2}) &= f(a_0 a_1 \otimes a_{2,n+2}), \\ \gamma(\pi; f, \text{Id})(a_{0,n+2}) &= f(a_{0,n} \otimes a_{n+1} a_{n+2}), \\ \gamma(f; \text{Id}^{i-1}, \pi, \text{Id}^{n-i})(a_{0,n+2}) &= f(a_{0,i-1} \otimes a_i a_{i+1} \otimes a_{i+2,n+2}). \end{aligned} \tag{15}$$

The Theorem now follows immediately from (13) and (15). □

Corollary 5.3. *There is an isomorphism of cochain complexes,*

$$\begin{aligned} (\text{CH}_{\text{Hopf}}^*(A, A), d_{\text{CH}}) &\cong (\text{CH}_{\text{Hopf}}^*(A, A), d) \\ x &\mapsto (-1)^{\frac{n(n+1)}{2}} x, \end{aligned}$$

for $x \in \text{CH}_{\text{Hopf}}^n(A, A)$. In particular, the cohomology modules on $\text{CH}_{\text{Hopf}}^*(A, A)$ defined by the differentials d_{CH} and d are equal.

6. Gerstenhaber algebra

The purpose of this section is to observe that the homotopy G -algebra structure on $\text{CH}_{\text{Hopf}}^*(A, A)$ gives rise to a G -algebra structure on the Hopf-Hochschild cohomology modules $HH_{\text{Hopf}}^*(A, A)$. Explicit formulas of these G -algebra operations are given in (16).

6.1. Gerstenhaber algebra

Recall from [10, Section 2.2] that a G -algebra $(V, \cup, [-, -])$ consists of a graded vector space $V = \bigoplus V^n$, a degree 0 associative \cup -product, and a degree -1 graded Lie bracket

$$[-, -]: V^m \otimes V^n \rightarrow V^{m+n-1},$$

satisfying the following two conditions:

$$\begin{aligned} x \cup y &= (-1)^{\deg x \deg y} y \cup x, \\ [x, y \cup z] &= [x, y] \cup z + (-1)^{|x| \deg y} y \cup [x, z]. \end{aligned}$$

In other words, the \cup -product is graded commutative, and the Lie bracket is a graded derivation for the \cup -product. In particular, a G -algebra is a graded version of a Poisson algebra. This algebraic structure was first studied by Gerstenhaber [8].

6.2. G -algebra from homotopy G -algebra

If (V, d, \cup) is a homotopy G -algebra, one can define a degree -1 operation on V as

$$[x, y] \stackrel{\text{def}}{=} x\{y\} - (-1)^{|x||y|} y\{x\}.$$

Passing to cohomology, $(H^*(V, d), \cup, [-, -])$ becomes a G -algebra ([10] Corollary 5 and its proof).

Combining the previous paragraph with Corollary 5.1 and Corollary 5.3, we obtain the following result.

Corollary 6.1. *The Hopf-Hochschild cohomology modules $HH_{\text{Hopf}}^*(A, A)$ of an H -module-algebra A admits the structure of a G -algebra.*

This G -algebra can be described on the cochain level more explicitly as follows. Pick $\varphi \in \text{CH}_{\text{Hopf}}^n(A, A)$ and $\psi \in \text{CH}_{\text{Hopf}}^m(A, A)$. Then

$$\begin{aligned} (\psi \cup \varphi)(a_{0,m+n+1}) &= (-1)^{m+n-1} \psi(a_{0,m} \otimes 1) \varphi(1 \otimes a_{m+1,m+n+1}), \\ [\psi, \varphi] &= \psi\{\varphi\} - (-1)^{(m-1)(n-1)} \varphi\{\psi\}. \end{aligned} \tag{16}$$

Writing $a = a_{0,m+n}$, we have

$$\begin{aligned} \psi\{\varphi\}(a) &= \sum_{i=1}^m (-1)^{(i-1)(n-1)} \psi(a_{0,i-1} \otimes \varphi(1 \otimes a_{i,i+n-1} \otimes 1) \otimes a_{i+n,m+n}), \\ \varphi\{\psi\}(a) &= \sum_{j=1}^n (-1)^{(j-1)(m-1)} \varphi(a_{0,j-1} \otimes \psi(1 \otimes a_{j,j+m-1} \otimes 1) \otimes a_{j+m,m+n}). \end{aligned}$$

In particular, if $m = n = 1$, then the bracket operation

$$[\psi, \varphi](a_{0,2}) = \psi(a_0 \otimes \varphi(1 \otimes a_1 \otimes 1) \otimes a_2) - \varphi(a_0 \otimes \psi(1 \otimes a_1 \otimes 1) \otimes a_2)$$

gives $HH_{\text{Hopf}}^1(A, A)$ a Lie algebra structure. There is another description of this Lie algebra in terms of (inner) derivations in [14, Proposition 3.9].

7. Deligne’s Conjecture for module-algebras

The purpose of this section is to observe that a version of Deligne’s Conjecture holds for the Hopf-Hochschild cochain complex of a module-algebra. The original Deligne’s Conjecture for Hochschild cohomology is as follows.

Deligne’s Conjecture ([6]). The Hochschild cochain complex $C^*(R, R)$ of an associative algebra R is an algebra over a suitable chain model of May’s little squares operad \mathcal{C}_2 [17].

A positive answer to Deligne’s conjecture was given by, among others, McClure and Smith [19, Theorem 1.1] and Kaufmann [13, Theorem 4.2.2]. There is an operad \mathcal{H} whose algebras are the brace algebras with multiplication (section 4.2). For an associative algebra R , the Hochschild cochain complex $C^*(R, R)$ is a brace algebra with multiplication and hence an \mathcal{H} -algebra. McClure and Smith showed that \mathcal{H} is quasi-isomorphic to the chain operad \mathcal{S} obtained from the little squares operad \mathcal{C}_2 by applying the singular chain functor, thereby proving Deligne’s Conjecture.

It has been observed that the Hopf-Hochschild cochain complex $\text{CH}_{\text{Hopf}}^*(A, A)$ is a brace algebra with multiplication (Corollary 4.1). Therefore, we can use the result of McClure and Smith [19, Theorem 1.1] to obtain the following version of Deligne’s Conjecture for module-algebras.

Corollary 7.1 (Deligne’s Conjecture for module-algebras). *The Hopf-Hochschild complex $\text{CH}_{\text{Hopf}}^*(A, A)$ of an H -module-algebra A is an algebra over the McClure-Smith operad \mathcal{H} that is a chain model for the little squares operad \mathcal{C}_2 .*

References

- [1] R. Akbarpour and M. Khalkhali, Hopf algebra equivariant cyclic homology and cyclic homology of crossed product algebras, *J. reine angew. Math.* **559** (2003), 137–152.
- [2] R. Akbarpour and M. Khalkhali, Cyclic cohomology of crossed coproduct coalgebras, preprint, [arXiv:math.KT/0107166](https://arxiv.org/abs/math/0107166).
- [3] R. Akbarpour and M. Khalkhali, Cyclic homology of Hopf comodule algebras and Hopf module coalgebras, *Comm. Alg.* **31** (2003), 5653–5671.
- [4] J.M. Boardman, Stable operations in generalized cohomology, in: *Handbook of algebraic topology*, 585–686, North-Holland, Amsterdam, 1995.
- [5] S. Dăscălescu, C. Năstăsescu and Ş. Raianu, *Hopf algebra: An introduction*, Marcel Dekker, New York, 2001.
- [6] P. Deligne, Letter to Stasheff, Gerstenhaber, May, Schechtman, Drinfeld, May 17, 1993.
- [7] D.B.A. Epstein and N.E. Steenrod, *Cohomology operations*, Princeton Univ. Press, Princeton, NJ, 1962.
- [8] M. Gerstenhaber, The cohomology structure of an associative ring, *Ann. Math.* **78** (1963), 267–288.
- [9] M. Gerstenhaber, On the deformation of rings and algebras, *Ann. Math.* **79** (1964), 59–103.
- [10] M. Gerstenhaber and A.A. Voronov, Homotopy G -algebras and moduli space operad, *Int. Math. Res. Notices* **1995**, 141–153.
- [11] G. Hochschild, On the cohomology groups of an associative algebra, *Ann. Math.* **46** (1945), 58–67.
- [12] C. Kassel, *Quantum groups*, Springer-Verlag, New York, 1995.
- [13] R.M. Kaufmann, On spineless cacti, Deligne’s conjecture and Connes-Kreimer’s Hopf algebra, *Topology* **46** (2007), 39–88.
- [14] A. Kaygun, Hopf-Hochschild (co)homology of module algebras, *Homology Homotopy Appl.* **9** (2007), 451–472.
- [15] P.S. Landweber, Cobordism operations and Hopf algebras, *Trans. Amer. Math. Soc.* **129** (1967), 94–110.
- [16] M. Markl, S. Shnider and J. Stasheff, *Operads in Algebra, Topology and Physics*, Amer. Math. Soc., Providence, RI, 2002.
- [17] J.P. May, *The geometry of iterated loop spaces*, Lectures Notes in Math. **271**, Springer-Verlag, Berlin-New York, 1972.
- [18] J.P. May, Definitions: operads, algebras and modules, *Contemp. Math.* **202** (1997), 1–7.
- [19] J.E. McClure and J.H. Smith, A solution of Deligne’s Hochschild cohomology conjecture, *Contemp. Math.* **293** (2002), 153–193.
- [20] J. Milnor, The Steenrod algebra and its dual, *Ann. Math.* **67** (1958), 150–171.
- [21] S. Montgomery, *Hopf algebras and their actions on rings*, Amer. Math. Soc., Providence, RI, 1993.

- [22] S.P. Novikov, Methods of algebraic topology from the point of view of cobordism theory, *Izv. Akad. Nauk SSSR Ser. Mat.* **31** (1967), 855–951.
- [23] M. Sweedler, *Hopf algebras*, W.A. Benjamin, New York, 1969.
- [24] D. Yau, Deformation bicomplex of module-algebras, *Homology Homotopy Appl.* **10** (2008), 97–128.

Donald Yau dyau@math.ohio-state.edu

Department of Mathematics, The Ohio State University Newark, 1179 University Drive, Newark, OH 43055, USA