THE BROWN-GOLASIŃSKI MODEL STRUCTURE ON STRICT ∞-GROUPOIDS REVISITED

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(*communicated by Ronald Brown*)

Abstract

We prove that the folk model structure on strict ∞-categories transfers to the category of strict ∞-groupoids (and more generally to the category of strict (∞, n)-categories), and that the resulting model structure on strict ∞-groupoids coincides with the one defined by Brown and Golasiński via crossed complexes.

Introduction

In [**4**], Brown and Golasiński build a model structure on the category of crossed complexes. In an earlier work [**5**], Brown and Higgins established an equivalence of categories between crossed complexes and strict ∞-groupoids, whence a model structure on the latter category.

On the other hand, there is a "folk" model structure on strict ∞-categories recently discovered by Lafont, Worytkiewicz and the second author [**10**], which extends previously known model structures on categories [**8**] and 2-categories [**9**]. Note that this model structure is also known as the "natural" or the "categorical" model structure by various authors.

This immediately raised the questions whether the folk model structure on ∞-categories may be transferred to ∞-groupoids by inclusion, and in that case whether the Brown-Golasiński model structure may be recovered this way.

The purpose of the present work is to show that both questions have affirmative answers.

Our paper is organized as follows: In Section 1, we recall the basic definitions of strict ∞-categories and strict ∞-groupoids. Then, in Section 2, we describe the Brown-Golasiński model structure on crossed complexes and ∞-groupoids. In Section 3, we prove the transfer theorem leading to the definition of the folk model structure on strict ∞-groupoids. Our proof uses neither crossed complexes nor the existence of the Brown-Golasiński model structure. Moreover, it also applies to the category of strict (∞, n)-categories for a fixed n (that is ∞-categories such that every m-arrow is invertible for $m > n$). Finally, in Section 4, we show that the two model structures previously defined on ∞-groupoids are in fact the same.

Acknowledgements

1. Strict ∞-groupoids

The purpose of this section is to introduce the definitions and notations about strict ∞-groupoids and their weak equivalences that we will use in the sequel of this paper. Our presentation is essentially the same as the one given in [**2**].

1.1. Globular sets

Let us denote by **O** the *globular category*, that is, the category generated by the graph

$$0 \mathrel{\substack{\sigma_0 \\ \longrightarrow \\ \longrightarrow \\ \tau_0}} 1 \mathrel{\substack{\sigma_1 \\ \longrightarrow \\ \longrightarrow \\ \tau_1}} \cdots \mathrel{\substack{\sigma_{i-1} \\ \longrightarrow \\ \longrightarrow \\ \tau_{i-1}}} i \mathrel{\substack{\sigma_i \\ \longrightarrow \\ \longrightarrow \\ \tau_i}} i+1 \mathrel{\substack{\sigma_{i+1} \\ \longrightarrow \\ \longrightarrow \\ \tau_{i+1}}} \cdots$$

under the coglobular relations

$$\sigma_{i+1}\sigma_i = \tau_{i+1}\sigma_i \quad \text{and} \quad \sigma_{i+1}\tau_i = \tau_{i+1}\tau_i, \qquad i \geqslant 0.$$

A *globular set* or ∞-*graph* is a presheaf on **O**. A globular set X amounts to a diagram of sets

$$\cdots \mathrel{\substack{s_{n+1} \\ \longrightarrow \\ \longrightarrow \\ t_{n+1}}} X_{n+1} \mathrel{\substack{s_n \\ \longrightarrow \\ \longrightarrow \\ t_n}} X_n \mathrel{\substack{s_{n-1} \\ \longrightarrow \\ \longrightarrow \\ t_{n-1}}} \cdots \mathrel{\substack{s_1 \\ \longrightarrow \\ \longrightarrow \\ t_1}} X_1 \mathrel{\substack{s_0 \\ \longrightarrow \\ \longrightarrow \\ t_0}} X_0$$

satisfying the globular relations

$$s_i s_{i+1} = s_i t_{i+1} \quad \text{and} \quad t_i s_{i+1} = t_i t_{i+1}, \qquad i \geqslant 0.$$

For $i \geqslant j \geqslant 0$, we will denote by s_j^i and t_j^i the maps from X_i to X_j defined by

$$s_j^i = s_j \cdots s_{i-2} s_{i-1} \quad \text{and} \quad t_j^i = t_j \cdots t_{i-2} t_{i-1}.$$

If X is a globular set, we will call X_0 the set of *objects* of X and X_n for $n \geqslant 0$ the set of *n-arrows* or *n-cells*. The notation $u \colon x \to y$ will mean that u is an n-arrow for $n \geqslant 1$ whose source is an $(n-1)$-arrow x (that is $s_{n-1}(u) = x$) and whose target is an $(n-1)$-arrow y (that is $t_{n-1}(u) = y$). We will say that two n-arrows u and v are *parallel* if either $n = 0$, or $n \geqslant 1$ and u, v have same source and same target. For $i \geqslant j \geqslant 0$, if u is an i-arrow, we will often write $s_j(u)$ for $s_j^i(u)$ and similarly $t_j(u)$ for $t_j^i(u)$.

If u and v are n-arrows, $X(u,v)$ will denote the globular set whose k-arrows are the $(n+k+1)$-arrows a of G such that $s_n(a) = u$ and $t_n(a) = v$. In particular, $X(u,v)_0$ is the set of $(n+1)$-arrows $a \colon u \to v$ in X.

1.2. Strict ∞-categories

An ∞-*precategory* is a globular set C endowed with maps

$$*_j^i \colon (X_i, s_j^i) \times_{X_j} (t_j^i, X_i) \to X_i, \quad i > j \geqslant 0,$$
$$k_i \colon X_{i-1} \to X_i, \quad i \geqslant 1,$$

such that

1. For every (u,v) in $(X_i, s_j^i) \times_{X_j} (t_j^i, X_i)$ with $i > j \geqslant 0$, we have

$$s_{i-1}(u *_j^i v) = \begin{cases} s_{i-1}(v), & j = i-1 \\ s_{i-1}(u) *_j^{i-1} s_{i-1}(v), & j < i-1 \end{cases};$$

2. For every (u,v) in $(X_i, s_j^i) \times_{X_j} (t_j^i, X_i)$ with $i > j \geqslant 0$, we have

$$t_{i-1}(u *_j^i v) = \begin{cases} t_{i-1}(u), & j = i-1 \\ t_{i-1}(u) *_j^{i-1} t_{i-1}(v), & j < i-1 \end{cases};$$

3. For every u in X_i with $i \geqslant 0$, we have

$$s_i k_{i+1}(u) = u = t_i k_{i+1}(u).$$

For $i \geqslant j \geqslant 0$, we will denote by k_i^j the map from $X_j \to X_i$ defined by

$$k_i^j = k_i \cdots k_{j+2} k_{j+1}.$$

If u and v are n-arrows for $n \geqslant 1$ of an ∞-precategory, we will often write $u *_k v$ for $u *_k^n v$. If u is an n-arrow, we will often write 1_u for the iterated identity $k_m^n(u)$ in a dimension $m \geqslant n$ clear by the context.

Definition 1.1. A *strict ∞-category* is an ∞-precategory X such that the following axioms are satisfied:

1. Associativity. For every (u,v,w) in $(X_i, s_j^i) \times_{X_j} (t_j^i, X_i, s_j^i) \times_{X_j} (t_j^i, X_i)$ with $i > j \geqslant 0$, we have

$$(u *_j v) *_j w = u *_j (v *_j w);$$

2. Exchange law. For every (u, u', v, v') in

$$(X_i, s_j^i) \times_{X_j} (t_j^i, X_i, s_k^i) \times_{X_k} (t_k^i, X_i, s_j^i) \times_{X_j} (t_j^i, X_i),$$

with $i > j > k \geqslant 0$, we have

$$(u *_j u') *_k (v *_j v') = (u *_k v) *_j (u' *_k v');$$

3. Units. For every u in X_i with $i \geqslant 1$ and every j such that $i > j \geqslant 0$, we have

$$u *_j 1_{s_j(u)} = u = 1_{t_j(u)} *_j u;$$

4. Functoriality of units. For every (u,v) in $(X_i, s_j^i) \times_{X_j} (t_j^i, X_i)$ with $i > j \geqslant 0$, we have the following equality between $(i+1)$-arrows:

$$1_{u *_j v} = 1_u *_j 1_v.$$

A *morphism of strict ∞-categories* or *∞-functor* is a morphism of globular sets compatible with the maps $*_j^i$ and k_i.

We will denote by ∞-**Cat** the category of ∞-categories. This category is a full reflexive subcategory of the presheaf category of globular sets. Moreover, it is stable under filtered colimits. Hence, by Theorem 1.46 of [1], ∞-**Cat** is locally presentable.

Note that if u and v are two n-arrows of a strict ∞-category C, the globular set $C(u,v)$ inherits a structure of strict ∞-category.

1.3. Strict ∞-groupoids

Let C be a strict ∞-category and u an i-arrow for $i \geqslant 1$. For j such that $0 \leqslant j < i$, a $*_j^i$-*inverse* v of u is an i-arrow such that $s_j(v) = t_j(u)$ and $t_j(v) = s_j(u)$, satisfying

$$u *_j v = 1_{t_j(u)} \quad \text{and} \quad v *_j u = 1_{s_j(u)}.$$

It is easy to see that if it exists, such an inverse is unique. For $i > j \geqslant 0$, we will say that C *admits* $*_j^i$-*inverses* if every i-arrow of C admits a $*_j^i$-inverse.

Definition 1.2. A *strict ∞-groupoid* is a strict ∞-category which admits $*_j^i$-inverses for every $i > j \geqslant 0$. We will denote by ∞-**Grp** the full subcategory of ∞-**Cat** whose objects are strict ∞-groupoids.

Let $n \geqslant 0$. A *strict (∞, n)-category* is a strict ∞-category which admits $*_j^i$-inverses for every $i > j \geqslant n$. We will denote by (∞, n)-**Cat** the full subcategory of ∞-**Cat** whose objects are strict (∞, n)-categories. Note that for $n = 0$ we recover the category of strict ∞-groupoids.

The same argument as for ∞-**Cat** shows that ∞-**Grp** is a locally presentable category.

If G is a strict ∞-groupoid and u is an i-arrow of G for $i \geqslant 1$, we will denote by $w_j^i(u)$ or simply by $w_j(u)$ the $*_j^i$-inverse of u and by u^{-1} the $*_{i-1}^i$-inverse. Note that if u and v are two n-arrows of a strict ∞-groupoid G, the strict ∞-category $G(u, v)$ is a strict ∞-groupoid.

Proposition 1.3. *Let C be a strict ∞-category. Then the following assertions are equivalent:*

1. *C is a strict ∞-groupoid;*

2. *C admits $*_{i-1}^i$-inverses for every $i \geqslant 1$;*

3. *C admits $*_0^i$-inverses for every $i \geqslant 1$;*

4. *For all $i \geqslant 1$, there exists j satisfying $0 \leqslant j < i$ such that C admits $*_j^i$-inverses.*

Proof. By induction, it suffices to show that for every $i > j > k \geqslant 0$, if C admits $*_k^j$-inverses, then C admits $*_k^i$-inverses if and only if it admits $*_j^i$-inverses. By using the fact that the 2-graph

$$C_i \mathrel{\substack{\xrightarrow{s_j^i} \\[-0.3em] \xrightarrow[t_j^i]{}}} C_j \mathrel{\substack{\xrightarrow{s_k^j} \\[-0.3em] \xrightarrow[t_k^j]{}}} C_k$$

has a natural structure of 2-category, one can assume that $k = 0$, $j = 1$ and $i = 2$. The result is thus a consequence of the following lemma. $\qquad\square$

Lemma 1.4. *Let C be a 2-category whose 1-arrows are invertible. Then a 2-arrow is invertible for horizontal composition (i.e., $*_0^2$) if and only if it is invertible for vertical composition (i.e., $*_1^2$).*

Proof. Let $a\colon u \to v$ be a 2-arrow. Suppose a admits a horizontal inverse a^*. Then $v *_0 a^* *_0 u$ is a vertical inverse. Conversely, suppose a admits a vertical inverse a^{-1}. Then $v^{-1} *_0 a^{-1} *_0 u^{-1}$ is a horizontal inverse. $\qquad\square$

1.4. Weak equivalences of strict ∞-groupoids

Let G be a strict ∞-groupoid. An n-arrow u of G is *homotopic* to another n-arrow v if there exists an $(n+1)$-arrow from u to v. This implies that the arrows u and v are parallel. If u is homotopic to v, we will write $u \sim v$. The relation $\sim$ is an equivalence relation on G_n: the properties with respect to source and target of the maps k_{n+1}, w_{n-1}^n and $*_{n-1}^n$ imply respectively that $\sim$ is reflexive, symmetric and transitive.

Let us denote by $\overline{G_n}$ the quotient of G_n by $\sim$. The composition

$$*_{n-1}^n \colon G_n \times_{G_{n-1}} G_n \to G_n$$

induces a map

$$*_{n-1}^n \colon \overline{G_n} \times_{G_{n-1}} \overline{G_n} \to \overline{G_n},$$

thanks to the properties with respect to source and target of the composition $*_{n-1}^{n+1}$. For $n \geqslant 1$, we can thus define a groupoid $\varpi_n(G)$ whose objects are $(n-1)$-arrows of G and whose morphisms are elements of $\overline{G_n}$. It is clear that ϖ_n defines a functor from the category of strict ∞-groupoids to the category of groupoids.

Definition 1.5. The set of *connected components* of G is

$$\pi_0(G) = \pi_0(\varpi_1(G)) = \overline{G_0}.$$

For $n \geqslant 1$ and x an object of G, the *n-th homotopy group of G at x* is

$$\pi_n(G, x) = \pi_1(\varpi_n(G), 1_x) = \mathsf{Aut}_{\varpi_n(G)}(1_x).$$

By functoriality of the ϖ_n's, π_0 induces a functor from the category of strict ∞-groupoids to the category of sets, and π_n, for $n \geqslant 1$, induces a functor from the category of pointed strict ∞-groupoids to the category of groups. By the Eckmann-Hilton argument, the groups $\pi_n(G, x)$ are abelian for $n \geqslant 2$. More generally, if u and v are two $(n-1)$-arrows for $n \geqslant 1$ we set

$$\pi_n(G, u, v) = \mathsf{Hom}_{\varpi_n(G)}(u, v) \quad \text{and} \quad \pi_n(G, u) = \pi_n(G, u, u).$$

Definition 1.6. A morphism $f \colon G \to H$ of strict ∞-groupoids is a *weak equivalence of strict ∞-groupoids* if the map $\pi_0(f) \colon \pi_0(G) \to \pi_0(H)$ is a bijection and if, for all $n \geqslant 1$ and all object x of G, the morphism $\pi_n(f, x) \colon \pi_n(G, x) \to \pi_n(H, f(x))$ is a group isomorphism. We will denote by $\mathcal{W}_{\mathrm{gr}}$ the class of such weak equivalences.

Proposition 1.7. *Let $f \colon G \to H$ be a morphism of strict ∞-groupoids. The following conditions are equivalent:*

1. *f is a weak equivalence of strict ∞-groupoids;*

2. *$\pi_0(f) \colon \pi_0(G) \to \pi_0(H)$ is a bijection, and for all $n \geqslant 1$ and every $(n-1)$-arrow u of G, f induces a bijection*

$$\pi_n(G, u) \to \pi_n(H, f(u));$$

3. *$\varpi_1(f) \colon \varpi_1(G) \to \varpi_1(H)$ is an equivalence of categories, and for all $n \geqslant 2$ and every pair (u, v) of parallel $(n-1)$-arrows of G, f induces a bijection*
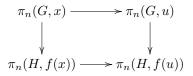
$$\pi_n(G, u, v) \to \pi_n(H, f(u), f(v));$$

4. $\varpi_1(f)\colon \varpi_1(G) \to \varpi_1(H)$ *is full and essentially surjective, and for all* $n \geqslant 2$ *and every pair* (u, v) *of parallel* $(n-1)$-*arrows of* G, f *induces a surjection*

$$\pi_n(G, u, v) \to \pi_n(H, f(u), f(v)).$$

Proof of $1 \Rightarrow 2$. The case $n = 1$ is obvious. Let $n \geqslant 2$ and let u be an $(n-1)$-arrow of G. Set $x = s_0(u)$. The map

$$\pi_n(G, x) \to \pi_n(G, u),$$

which sends an n-arrow $a\colon 1_x \to 1_x$ to the n-arrow $1_u *_0 a\colon u \to u$, is an isomorphism. Moreover, f commutes with this isomorphism, that is, the square

$$
\begin{array}{ccc}
\pi_n(G, x) & \longrightarrow & \pi_n(G, u) \\
\downarrow & & \downarrow \\
\pi_n(H, f(x)) & \longrightarrow & \pi_n(H, f(u))
\end{array}
$$

is commutative. The map $\pi_n(G, u) \to \pi_n(H, f(u))$ is thus a bijection for $n \geqslant 2$.

Proof of $2 \Rightarrow 3$. Let $n \geqslant 1$ and let u, v be two parallel $(n-1)$-arrows of G. Suppose there exists an n-arrow $a\colon u \to v$ in G. The map

$$\pi_n(G, u) \to \pi_n(G, u, v),$$

which sends an n-arrow $b\colon u \to u$ to the n-arrow $a *_{n-1} b\colon u \to v$, is a bijection. Moreover, f commutes with this bijection, that is, the square

$$
\begin{array}{ccc}
\pi_n(G, u) & \longrightarrow & \pi_n(G, u, v) \\
\downarrow & & \downarrow \\
\pi_n(H, f(u)) & \longrightarrow & \pi_n(H, f(u), f(v))
\end{array}
$$

is commutative.

Thus to conclude it suffices to show that if there exists an n-arrow $b\colon f(u) \to f(v)$ in H, then there exists an n-arrow $a\colon u \to v$ in G. It is clear when $n = 1$ by injectivity of $\pi_0(f)$. Let $n \geqslant 2$ and let $b\colon f(u) \to f(v)$ be an n-arrow of H. Set $x = s_{n-2}(u)$. The arrow $1_{f(u)^{-1}} *_{n-2} b$ is an n-arrow of H from $1_{f(x)}\colon f(x) \to f(x)$ to $f(u)^{-1} *_{n-2} f(v)\colon f(x) \to f(x)$. Since the map

$$\pi_{n-1}(G, x) \to \pi_{n-1}(H, f(x))$$

is injective, there exists an n-arrow a' of G from 1_x to $u^{-1} *_{n-2} v$. Then $a = 1_u *_{n-2} a'$ is an n-arrow of G from u to v.

Proof of $3 \Rightarrow 1$. Obvious.

Proof of $4 \Rightarrow 3$. Let $n \geqslant 1$, let u, v be two parallel $(n-1)$-arrows of G and let a, b be two n-arrows from u to v. Suppose we have $f(a) = f(b)$ in $\pi_n(H, f(u), f(v))$. Then there exists an $(n+1)$-arrow of H from $f(a)$ to $f(b)$. By surjectivity of the map

$$\pi_{n+1}(G, a, b) \to \pi_{n+1}(H, f(a), f(b)),$$

there exists an $(n+1)$-arrow in G from a to b. Thus $a = b$ in $\pi_n(G, u, v)$.

Proof of $3 \Rightarrow 4$. Obvious. $\square$

2. The Brown-Golasiński model structure

In [**4**], Brown and Golasiński introduce a model category structure on the category of crossed complexes. By the equivalence of categories between crossed complexes and strict ∞-groupoids constructed in [**5**], this model structure induces a model structure on strict ∞-groupoids. The purpose of this section is to describe this model structure.

2.1. Crossed complexes

Let us denote by $\mathbf{Grp}$ the category of groups and by $\mathbf{CGrp}_{\geqslant 2}$ the category of homological complexes of (not necessarily commutative) groups in dimension greater or equal to 2, that is of sequences of morphisms of groups

$$\cdots \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_3 \xrightarrow{d_3} C_2$$

such that for every $n \geqslant 4$, we have $d_{n-1}d_n = 1$, where 1 denotes the unit element of C_{n-2}. We have an inclusion functor $i_2 \colon \mathbf{Grp} \to \mathbf{CGrp}_{\geqslant 2}$ which sends a group G to the complex concentrated in degree 2 on G.

Let $C_{\leqslant 1}$ be a groupoid. We will denote by C_0 its set of objects and by $C_1(x, y)$ the set of morphisms from an object x to an object y in $C_{\leqslant 1}$. Let $C_1 \colon C_{\leqslant 1} \to \mathbf{Grp}$ be the functor defined in the following way: an object x of $C_{\leqslant 1}$ is sent to the group $C_1(x) = C_1(x, x)$; a morphism $u \colon x \to y$ of $C_{\leqslant 1}$ is sent to the morphism of conjugation by u, i.e.,

$$C_1(x) \to C_1(y)$$
$$v \mapsto uvu^{-1}.$$

Definition 2.1. A *precrossed complex* consists of

- A groupoid $C_{\leqslant 1}$;
- A functor $C_{\geqslant 2} \colon C_{\leqslant 1} \to \mathbf{CGrp}_{\geqslant 2}$;
- An augmentation of $C_{\geqslant 2}$ over C_1, that is a natural transformation $d_2 \colon C_{\geqslant 2} \to i_2 C_1$.

Explicitly, a precrossed complex is given by

- For every x in C_0, a complex

$$C_{\geqslant 2}(x) = \cdots \to C_n(x) \xrightarrow{d_n} C_{n-1}(x) \to \cdots \to C_3(x) \xrightarrow{d_3} C_2(x);$$

- For every x in C_0, a morphism $d_2 \colon C_2(x) \to C_1(x)$;
- For every $n \geqslant 2$ and every morphism $u \colon x \to y$ of $C_{\leqslant 1}$, a morphism $C_n(x) \to C_n(y)$ of groups functorial in u,

such that for every morphism $u \colon x \to y$ of $C_{\leqslant 1}$, the diagram

$$
\begin{array}{ccccccccc}
\longrightarrow & C_n(x) & \xrightarrow{d_n} & C_{n-1}(x) & \longrightarrow & \cdots & \longrightarrow & C_2(x) & \xrightarrow{d_2} & C_1(x) \\
& \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\longrightarrow & C_n(y) & \xrightarrow{d_n} & C_{n-1}(y) & \longrightarrow & \cdots & \longrightarrow & C_2(y) & \xrightarrow{d_2} & C_1(y),
\end{array}
$$

where $C_1(x) \to C_1(y)$ is the conjugation by u, is commutative.

If C is a precrossed complex we will denote by $C(x)$ the augmented complex

$$\cdots \to C_n(x) \xrightarrow{d_n} C_{n-1}(x) \to \cdots \to C_3(x) \xrightarrow{d_3} C_2(x) \xrightarrow{d_2} C_1(x).$$

If $u\colon x \to y$ is a morphism of $C_{\leqslant 1}$ and $n \geqslant 1$, we will call the map $C_n(x) \to C_n(y)$ the *action* of u on $C_n(x)$. In particular, an element of $C_1(x)$ *acts* on $C_n(x)$ for all $n \geqslant 1$.

Definition 2.2. A precrossed complex C is a *crossed complex* if for all x in C_0 the following conditions are satisfied:

- For every $n \geqslant 3$, the group $C_n(x)$ is abelian;
- For every u in $C_2(x)$, the element $d_2(u)$ of $C_1(x)$ acts
  - By conjugation by u on $C_2(x)$;
  - Trivially on $C_n(x)$ for $n \geqslant 3$.

Definition 2.3. Let C and D be two crossed complexes. A *morphism of crossed complexes* $f\colon C \to D$ consists of

- A functor $f_{\leqslant 1}\colon C_{\leqslant 1} \to D_{\leqslant 1}$;
- A natural transformation $f_{\geqslant 2}\colon C_{\geqslant 2} \to D_{\geqslant 2}f_{\leqslant 1}$,

such that $f_{\geqslant 2}$ is compatible with the augmentation, i.e., the square

$$
\begin{array}{ccc}
C_{\geqslant 2} & \longrightarrow & D_{\geqslant 2}f_{\leqslant 1} \\
\downarrow{\scriptstyle d_2} & & \downarrow{\scriptstyle d_2 f_{\leqslant 1}} \\
i_2 C_1 & \longrightarrow & i_2 D_1 f_{\leqslant 1}
\end{array}
$$

is commutative.

Explicitly, a morphism $f\colon C \to D$ is given by

- A functor $f_{\leqslant 1}\colon C_{\leqslant 1} \to D_{\leqslant 1}$;
- For every $n \geqslant 2$ and every x in C_0, a group morphism $f_n\colon C_n(x) \to D_n(f_0(x))$, where f_0 denotes the restriction of $f_{\leqslant 1}$ to objects,

such that for every x in C_0, the diagram

$$
\begin{array}{ccccccc}
\longrightarrow & C_n(x) & \xrightarrow{d_n} & C_{n-1}(x) & \longrightarrow \cdots \longrightarrow & C_1(x) \\
& \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_1} \\
\longrightarrow & D_n(f_0(x)) & \xrightarrow{d_n} & D_{n-1}(f_0(x)) & \longrightarrow \cdots \longrightarrow & D_1(f_0(x)),
\end{array}
$$

where f_1 denotes the restriction of $f_{\leqslant 1}$ to $C_1(x)$, is commutative. We will often simply write f for f_n.

We will denote the category of crossed complexes by **CrC**.

2.2. The equivalence with strict ∞-groupoids

Let G be a strict ∞-groupoid. One can associate to G a precrossed complex C in the following way:

- The groupoid $C_{\leqslant 1}$ is the 1-truncation of G obtained from G by throwing out the n-arrows for $n \geqslant 2$;

- $C_n(x)$ is the set of n-arrows u of G whose source is an iterated unit of an object, that is such that there exists an object x of G such that $s_{n-1}(u) = 1_x$;
- The group law on $C_n(x)$ is induced by the composition $*_0^n \colon G_n \times_{G_0} G_n \to G_n$;
- $d_n \colon C_n(x) \to C_{n-1}(x)$ is induced by the target map $t_{n-1} \colon G_n \to G_{n-1}$;
- For $u \colon x \to y$ a morphism in $C_{\leqslant 1}$ and v a morphism in $C_n(x)$ for $n \geqslant 2$, the action of u on v is

$$u *_0 v *_0 w_0(u).$$

This precrossed complex is a crossed complex (see paragraph 3 of [5]). Moreover, this construction defines a functor $\mathrm{A} \colon \infty\text{-}\mathbf{Grp} \to \mathbf{CrC}$.

Theorem 2.4 (Brown-Higgins). *The functor* $\mathrm{A} \colon \infty\text{-}\mathbf{Grp} \to \mathbf{CrC}$ *is an equivalence of categories.*

Proof. See Theorem 4.1 of [5]. $\qquad\qquad\square$

2.3.   The model structure on crossed complexes

Definition 2.5. Let C be a crossed complex. The set of *connected components* of C is

$$\pi_0(C) = \pi_0(C_{\leqslant 1}).$$

For x in C_0, the *fundamental group of C at x* is

$$\pi_1(C, x) = \mathrm{Coker}(d_2 \colon C_2(x) \to C_1(x)),$$

and for $n \geqslant 2$, the *n-th homotopy group of C at x* is

$$\pi_n(C, x) = H_n(C(x)).$$

It is clear that π_0 defines a functor from the category of crossed complexes to the category of sets and that for $n \geqslant 1$, π_n defines a functor from the category of pointed crossed complexes to the category of groups.

Definition 2.6. A morphism $f \colon C \to D$ of crossed complexes is a *weak equivalence of crossed complexes* (see [4]) if the map $\pi_0(f) \colon \pi_0(C) \to \pi_0(D)$ is a bijection and if for every x in C_0 and every $n \geqslant 1$, the morphism $\pi_n(f, x) \colon \pi_n(C, x) \to \pi_n(D, f(x))$ is an isomorphism.

Definition 2.7. Let $f \colon C \to D$ be a morphism of crossed complexes. The morphism f is a *trivial fibration of crossed complexes* (see Proposition 2.2 (ii) of [4]) if the following conditions are satisfied:

- For every y in D_0, there exists x in C_0 such that $f(x) = y$;
- For every x, y in C_0 and every morphism $v \colon f(x) \to f(y)$ in $D_{\leqslant 1}$, there exists a morphism $u \colon x \to y$ in $C_{\leqslant 1}$ such that $f(u) = v$;
- For every $n \geqslant 2$, x in C_0, t in $C_{n-1}(x)$ and every v in $D_n(f(x))$ such that $d_n(v) = f(t)$, there exists u in $C_n(x)$ such that $d_n(u) = t$ and $f(u) = v$.

Theorem 2.8 (Brown-Golasiński). *The weak equivalences and trivial fibrations of crossed complexes define a model category structure on the category of crossed complexes.*

Proof. See Theorem 2.12 of [4]. $\qquad\qquad\square$

2.4. The model structure on strict ∞-groupoids

One obtains a model category structure on strict ∞-groupoids by transferring the model structure on crossed complexes defined in the previous section via the equivalence of categories A: ∞-**Grp** $\to$ **CrC**. We will call this model structure the Brown-Golasiński model structure. A morphism f of strict ∞-groupoids is a weak equivalence (respectively a trivial fibration) for the Brown-Golasiński model structure if and only if A(f) is a weak equivalence (respectively a trivial fibration) of crossed complexes. We will denote these classes by $\mathcal{W}_{cc}$ and $\mathcal{TF}_{cc}$ respectively.

We now describe these two classes more explicitly.

Proposition 2.9. *We have $\mathcal{W}_{cc} = \mathcal{W}_{gr}$. In other words, a morphism $f\colon G \to H$ of strict ∞-groupoids is a weak equivalence of strict ∞-groupoids if and only if the morphism $\mathrm{A}(f)\colon \mathrm{A}(G) \to \mathrm{A}(H)$ is a weak equivalence of crossed complexes.*

Proof. Since the two notions of weak equivalences are defined in terms of homotopy groups, it suffices to show that the two notions of homotopy groups coincide.

Let G be a groupoid. By definition, $\pi_0(G) = \pi_0(\mathrm{A}(G))$.

Let x be an object of G. By definition, $\pi_1(\mathrm{A}(G), x)$ is the cokernel of the morphism $\mathrm{A}(G)_2(x) \to \mathrm{A}(G)_1(x)$. The set $\mathrm{A}(G)_1(x)$ is the set of 1-arrows $u\colon x \to x$ in G, and two such arrows u, u' are identified in the cokernel if and only if there exists a 2-arrow from 1_x to $u *_0 u'^{-1}$ in G. This condition is equivalent to the existence of a 2-arrow from u to u'. Hence $\pi_1(G, x) = \pi_1(\mathrm{A}(G), x)$.

Let $n \geqslant 2$. The kernel of the map $d_n\colon \mathrm{A}(G)_n(x) \to \mathrm{A}(G)_{n-1}(x)$ is the set $G_n(x, x)$. Thus the same argument as in dimension 1 shows that $\pi_n(G, x) = \pi_n(\mathrm{A}(G), x)$. $\square$

Proposition 2.10. *A morphism $f\colon G \to H$ of strict ∞-groupoids is in $\mathcal{TF}_{cc}$ if and only if it satisfies the following conditions:*

- *For every object y of H, there exists an object x of G such that $f(x) = y$;*
- *For every pair (x, y) of objects of G, the map*

$$G(x, y)_0 \to G(f(x), f(y))_0$$

  induced by f is a surjection;
- *For all $n \geqslant 2$, every object x of G and every $(n-1)$-arrow $u\colon 1_x \to 1_x$, the map*

$$G(1_x, u)_0 \to H(1_{f(x)}, f(u))_0$$

  is surjective.

Proof. By definition, f is in $\mathcal{TF}_{cc}$ if and only if A(f) is a trivial fibration of crossed complexes. This proposition is then just a matter of translation using the definition of the functor A. $\square$

3. The folk model structure on ∞-**Grp**

This section shows that the folk model structure on ∞-**Cat** defined in [**10**] transfers to ∞-**Grp** via the inclusion functor

$$U\colon \infty\text{-}\mathbf{Grp} \to \infty\text{-}\mathbf{Cat}.$$

We first give a brief review of the main results of [**10**] and introduce the material needed to prove the transfer theorem.

3.1. The folk model structure on ∞-Cat

Given an ∞-category C, we define *reversible* cells in C and the relation of ω-*equivalence* between cells of C by mutual coinduction on $n \geqslant 0$.

Definition 3.1. Let $n \in \mathbb{N}$:

- An n-cell x is ω-*equivalent* to an n-cell y if there is a reversible $(n+1)$-cell $u \colon x \to y$;

- An $(n+1)$-cell $u \colon x \to y$ is *reversible* if there is an $(n+1)$-cell $\bar{u} \colon y \to x$ such that $\bar{u} *_n u$ is ω-equivalent to 1_x and $u *_n \bar{u}$ is ω-equivalent to 1_y.

Note that, for each $r \geqslant 0$, if two cells are r-equivalent in the sense of [**12**], then they are ω-equivalent, with the converse being false. We also refer to [**7**] for a gentle introduction to coinductive methods. Remark also that if G is an ∞-groupoid, any $(n+1)$-cell u of G is reversible and the cell $\bar{u}$ whose existence is stated in Definition 3.1 is of course just u^{-1}.

Let $\mathcal{W}$ denote the class of those morphisms $f \colon C \to D$ satisfying the following two conditions:

1. For each 0-cell y in D, there is a 0-cell x in C such that fx is ω-equivalent to y;

2. For each pair (x, x') of parallel n-cells in C, where $n \geqslant 0$ and each $(n+1)$-cell $v \colon fx \to fx'$, there is an $(n+1)$-cell $u \colon x \to x'$ such that fu is ω-equivalent to v.

Now for each $n \geqslant 0$, we define the n-*globe* $\mathbf{O}^n$ as the free ∞-category generated by the representable globular set $Y(n) = \mathbf{O}(n, -)$. Thus $Y(n)$ has exactly one n-cell, two k-cells for each $k < n$ and no k-cell for $k > n$. Let also $\partial Y(n)$ be the globular set having the same cells as $Y(n)$ except in dimension n where $(\partial Y(n))_n = \emptyset$. We denote by $\partial \mathbf{O}^n$ the free ∞-category on $\partial Y(n)$. We finally have, for each $n \geqslant 0$, an inclusion morphism

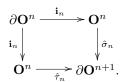$$\mathbf{i}_n \colon \partial \mathbf{O}^n \to \mathbf{O}^n.$$

The set $\{\mathbf{i}_n \mid n \in \mathbb{N}\}$ is denoted by I.

A map is a *trivial fibration* if it has the right-lifting property with respect to I and a *cofibration* if it has the left-lifting property with respect to all trivial fibrations.

Proposition 3.2. *Any ∞-functor f factors as $f = p \circ k$ where p is a trivial fibration and k is a cofibration.*

Proof. By a standard application of the small object argument, using the fact that ∞-**Cat** is locally presentable. □

On the other hand the maps $\sigma_n, \tau_n \colon n \to n+1$ in the globular category $\mathbf{O}$ (see Section 1) induce corresponding maps from $\mathbf{O}^n$ to $\mathbf{O}^{n+1}$, of the form $\mathbf{i}_{n+1} \circ \hat{\sigma}_n$ and

$\mathbf{i}_{n+1} \circ \hat{\tau}_n$ respectively, where $\hat{\sigma}_n, \hat{\tau}_n \colon \mathbf{O}^n \to \partial\mathbf{O}^{n+1}$. Moreover, we get a pushout diagram

$$
\begin{array}{ccc}
\partial\mathbf{O}^n & \xrightarrow{\ \mathbf{i}_n\ } & \mathbf{O}^n \\
{\scriptstyle \mathbf{i}_n}\big\downarrow & & \big\downarrow{\scriptstyle \hat{\sigma}_n} \\
\mathbf{O}^n & \xrightarrow[\hat{\tau}_n]{} & \partial\mathbf{O}^{n+1}.
\end{array}
$$

Now the above pushout determines a canonical map

$$\mathbf{o}_n \colon \partial\mathbf{O}^{n+1} \to \mathbf{O}^n$$

such that $\mathbf{o}_n \circ \hat{\sigma}_n = \mathbf{o}_n \circ \hat{\tau}_n = \mathrm{id}_{\mathbf{O}^n}$. Proposition 3.2 applies to $\mathbf{o}_n$, yielding an object $\mathbf{P}^n$ together with a trivial fibration $\mathbf{p}_n \colon \mathbf{P}^n \to \mathbf{O}^n$ and a cofibration $\mathbf{k}_n \colon \partial\mathbf{O}^{n+1} \to \mathbf{P}^n$ satisfying $\mathbf{o}_n = \mathbf{p}_n \circ \mathbf{k}_n$. We finally define $\mathbf{j}_n \colon \mathbf{O}^n \to \mathbf{P}^n$ as $\mathbf{k}_n \circ \hat{\sigma}_n$ and

$$J = \{\mathbf{j}_n \mid n \in \mathbb{N}\}.$$

Theorem 3.3. *There is a cofibrantly generated model structure on ∞-$\mathbf{Cat}$ where $\mathcal{W}$ is the class of weak equivalences, I a set of generating cofibrations and J a set of generating trivial cofibrations.*

This statement is in fact [**10**, Theorem 4.39] and the main result of that article.

3.2. Path object

Let C be an object in a model category and $\Delta_C \colon C \to C \times C$ be the diagonal map. A *path object* for C consists in an object P_C together with a factorization of Δ_C of the form

$$
C \xrightarrow{\ j\ } P_C \xrightarrow{\ p\ } C \times C,
$$
$$\Delta_C$$

where p is a fibration and j is a weak equivalence. Such a P_C is not unique: in the case of ∞-$\mathbf{Cat}$, one particular choice is given by the functor Γ we now describe. We first define, by induction on n, the notion of *n-cylinder between n-cells x, y of an ∞-category* C. Some notation will be useful: for each n-cell x, we denote by $x^\flat$ its 0-source $s_0 x$ and by $x^\sharp$ its 0-target $t_0 x$. Now let C be an ∞-category and x, y two 0-cells in it. There is an ∞-category $C(x, y)$ whose n-cells are the $(n+1)$-cells u of C such that $u^\flat = x$ and $u^\sharp = y$. Whenever u is such an $(n+1)$-cell of C, we denote by $[u]$ the corresponding n-cell of $C(x, y)$. Finally, let x, y, z be 0-cells of C. Each 1-cell $u \colon x \to y$ determines an ∞-functor $- \cdot u \colon C(y, z) \to C(x, z)$ given by $[v] \cdot u = [v *_0 u]$. Likewise, $u \colon y \to z$ determines an ∞-functor $u \cdot - \colon C(x, y) \to C(x, z)$ by $u \cdot [v] = [u *_0 v]$.

Definition 3.4.

1. A 0-*cylinder* $U \colon x \curvearrowright y$ in C is given by a reversible 1-cell $U^\sharp \colon x \to y$;

2. If $n > 0$, an n-*cylinder* $U \colon x \curvearrowright y$ in C is given by two reversible 1-cells $U^\flat \colon x^\flat \to y^\flat$ and $U^\sharp \colon x^\sharp \to y^\sharp$, together with some $(n-1)$-cylinder $[U] \colon U^\sharp \cdot [x] \curvearrowright [y] \cdot U^\flat$ in the ∞-category $[x^\flat, y^\sharp] = C(x^\flat, y^\sharp)$.

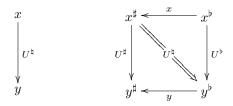Figure 1: n-cylinders for $n = 0, 1$

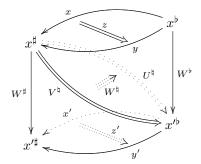Figure 2: Source and target of a 2-cylinder

If $U \colon x \curvearrowright y$ is an n-cylinder in C, we write $\pi^1_C U$ and $\pi^2_C U$ for the n-cells x and y, or simply $\pi^1 U$ and $\pi^2 U$. Figure 1 represents n-cylinders for $n = 0$ and $n = 1$. For each $n \in \mathbb{N}$, any $(n + 1)$-cylinder $W \colon z \curvearrowright z'$ in an ∞-category C determines a pair of n-cylinders in C:

Definition 3.5. The *source n-cylinder* $U \colon x \curvearrowright x'$ and the *target n-cylinder* $V \colon y \curvearrowright y'$ of the $(n + 1)$-cylinder $W \colon z \curvearrowright z'$ between $(n+1)$-cells $z \colon x \to y$ and $z' \colon x' \to y'$ are defined inductively by:

- If $n = 0$, then $U^\natural = W^\flat$ and $V^\natural = W^\sharp$;
- If $n > 0$, then $U^\flat = V^\flat = W^\flat$ and $U^\sharp = V^\sharp = W^\sharp$, whereas the two $(n-1)$-cylinders $[U]$ and $[V]$ are respectively defined as the source and the target of the n-cylinder $[W]$ in the ∞-category $[z^\flat, z'^\sharp]$.

If W has source U and target V we write $W \colon U \to V$ or $W \colon U \to V \mid z \curvearrowright z'$ (see Figure 2). It turns out that the source and target maps so defined satisfy the globular relations, so that the correspondence

$$n \mapsto \{U \mid U \text{ is an } n\text{-cylinder in } C\}$$

determines a globular set $\Gamma(C)$. We now turn to *trivial n-cylinders*:

Definition 3.6. The *trivial n-cylinder* $\tau\, x \colon x \curvearrowright x$ on the n-cell x is defined inductively by:

- If $n = 0$, then $(\tau\, x)^\natural = 1_x$;
- If $n > 0$, then $(\tau\, x)^\flat = 1_{x^\flat}$ and $(\tau\, x)^\sharp = 1_{x^\sharp}$, whereas $[\tau\, x]$ is the trivial cylinder $\tau[x]$ in $[x^\flat, x^\sharp]$.

We write $\tau_C\, x$ in case we need to mention the ambient ∞-category C.

Let us finally recall from [**10**, Appendix A] that $\Gamma(C)$ becomes a strict ∞-category when defining units and compositions as follows:

Definition 3.7. Let $U\colon x \curvearrowright y$ be an n-cylinder. We define the $(n+1)$-cylinder $1_U\colon U \to U \mid 1_x \curvearrowright 1_y$ by induction on n:

- If $n = 0$, then $(1_U)^\flat = (1_U)^\sharp = U^\natural$, whereas $[1_U] = \tau[1_{U^\natural}]$;
- If $n > 0$, then $(1_U)^\flat = U^\flat$ and $(1_U)^\sharp = U^\sharp$, whereas $[1_U] = 1_{[U]}$.

In order to define composition, we first introduce the operation of *concatenation*:

Definition 3.8. Let $U \mid x \curvearrowright y$ and $V \mid y \curvearrowright z$ be two n-cylinders. The *concatenation* $V * U \mid x \curvearrowright z$ of U and V is defined by induction on n:

- If $n = 0$, then $(V * U)^\natural = V^\natural *_0 U^\natural$;
- If $n > 0$, then $(V * U)^\flat = V^\flat *_0 U^\flat$, $(V * U)^\sharp = V^\sharp *_0 U^\sharp$ and $[V * U] = [V] \cdot U^\flat * V^\sharp \cdot [U]$.

Definition 3.9. Let $m \geqslant 1$, $0 \leqslant n < m$ and $U \mid x \curvearrowright x'$, $V \mid y \curvearrowright y'$ two m-cylinders such that $t_n(U) = s_n(V)$. The *composition* $V *_n U \mid y *_n x \curvearrowright y' *_n x'$ is defined by induction on n as follows:

- $(V *_0 U)^\flat = U^\flat$, $(V *_0 U)^\sharp = V^\sharp$ and $[V *_0 U] = y' \cdot [U] * [V] \cdot x$;
- If $n > 0$, then $(V *_n U)^\flat = U^\flat = V^\flat$, $(U *_n V)^\sharp = U^\sharp = V^\sharp$ and $[V *_n U] = [V] *_{n-1} [U]$.

Note that explicit formulas may be found in [**11**].

For example, Figure 3 below shows the composition $V *_0 U$ of two 1-cylinders $U\colon x \curvearrowright x'$ and $V\colon y \curvearrowright y'$ such that $U^\sharp = V^\flat$. Precisely, the composite $V *_0 U$ is the 1-cylinder $W\colon z \curvearrowright z'$ where $z = y *_0 x$, $z' = y' *_0 x'$, $W^\flat = U^\flat$, $W^\sharp = V^\sharp$, and the 0-cylinder $[W]$ of $C(z^\flat, z'^\sharp)$ is the reversible 1-cell of $C(z^\flat, z'^\sharp)$ given by the following corresponding reversible 2-cell of C:

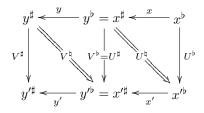$$W^\natural = (y' *_0 U^\natural) *_1 (V^\natural *_0 x).$$

Figure 3: Composition of 1-cylinders

The following result summarizes the main properties of Γ:

Theorem 3.10. *The correspondence $C \mapsto \Gamma(C)$ induces an endofunctor on ∞-**Cat**, and $\pi^1, \pi^2\colon \Gamma \to \mathrm{id}$, $\tau\colon \mathrm{id} \to \Gamma$ are natural transformations.*

An additional property, of particular importance here, is that the functor Γ preserves ∞-groupoids:

Lemma 3.11. *If G is an ∞-groupoid, so is $\Gamma(G)$.*

Proof. We show, by induction on $n \geqslant 1$, that if G is an ∞-groupoid and $W \colon U \to V \mid x \curvearrowright y$ is an n-cylinder of G, there is an n-cylinder $W' \colon V \to U \mid x^{-1} \curvearrowright y^{-1}$ such that $W' *_{n-1} W = 1_U$ and $W *_{n-1} W' = 1_V$.

Let G be an ∞-groupoid and $W \colon U \to V \mid x \curvearrowright y$ a 1-cylinder of G. By definition, we get two 1-cells $U^\natural \colon x^\flat \to y^\flat$, $V^\natural \colon x^\sharp \to y^\sharp$ and a 2-cell $W^\natural \colon V^\natural *_0 x \to y *_0 U^\natural$ in G. Consider $W^{\natural^{-1}} \colon y *_0 U^\natural \to V^\natural *_0 x$ the $*_1$-inverse of $W^\natural$ and build

$$y^{-1} *_0 W^{\natural^{-1}} *_0 x^{-1} \colon U^\natural *_0 x^{-1} \to y^{-1} *_0 V^\natural.$$

If $W' \colon V \to U \mid x^{-1} \curvearrowright y^{-1}$ is the 1-cylinder of G defined by

$$W'^\natural = y^{-1} *_0 W^{\natural^{-1}} *_0 x^{-1},$$

we get

$$W' *_0 W = 1_U \quad \text{and} \quad W *_0 W' = 1_V,$$

which proves the case $n = 1$.

Let $n > 1$ and suppose that the property holds for $n - 1$. Let G be an ∞-groupoid and $W \colon U \to V \mid x \curvearrowright y$ an n-cylinder of G. We get 1-cells $W^\flat \colon x^\flat \to y^\flat$, $W^\sharp \colon x^\sharp \to y^\sharp$ and an $(n-1)$-cylinder $[W] \colon [U] \to [V] \mid W^\sharp \cdot [x] \curvearrowright [y] \cdot W^\flat$ in $H = [x^\flat, y^\sharp]$. Now H is an ∞-groupoid, so that the induction hypothesis applies and there is an $(n-1)$-cylinder in H

$$[W]' \colon [V] \to [U] \mid W^\sharp \cdot [x^{-1}] \curvearrowright [y^{-1}] \cdot W^\flat$$

such that $[W]' *_{n-2} [W] = 1_{[U]}$ and $[W] *_{n-2} [W]' = 1_{[V]}$. Hence we may define an n-cylinder W' of G by $W'^\flat = W^\flat$, $W'^\sharp = W^\sharp$ and $[W'] = [W]'$. By construction

$$W' *_{n-1} W = 1_U \quad \text{and} \quad W *_{n-1} W' = 1_V. \qquad \square$$

Remark 3.12. Let $n \geqslant 0$. The proof of the previous lemma actually shows that if G is a strict (∞, n)-category, then so is $\Gamma(G)$.

3.3.   Immersions

We now introduce a class of morphisms which plays an important part in the proof of the transfer theorem.

Definition 3.13. An ∞-functor $f \colon C \to D$ belongs to the class $\mathcal{Z}$ of *immersions* if and only if there exist ∞-functors $g \colon D \to C$ and $h \colon D \to \Gamma(D)$ satisfying the following properties:

1. g is a retraction of f, that is $g \circ f = \mathrm{id}_C$;
2. $\pi_D^1 \circ h = f \circ g$ and $\pi_D^2 \circ h = \mathrm{id}_D$;
3. $h \circ f = \tau_D \circ f$. In other words, h is trivial on $f(C)$.

The following proposition summarizes the properties of immersions we need here.
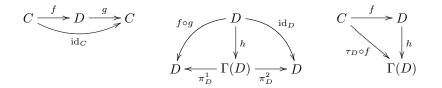
Figure 4: Immersions

Proposition 3.14. *The class $\mathcal{Z}$ of immersions satisfies the following properties:*

(i) *$\mathcal{Z}$ is closed by pushout;*

(ii) *All trivial cofibrations are immersions;*

(iii) *All immersions are weak equivalences.*

We refer to [**10**, Section 4.6] for the proofs of these statements.

3.4. Transfer

Let $\mathbf{C}$, $\mathbf{D}$ be two categories and $L\colon \mathbf{C} \to \mathbf{D}$, $R\colon \mathbf{D} \to \mathbf{C}$ be a pair of functors with L left adjoint to R and suppose that $\mathbf{C}$ is equipped with a model structure. We may define three classes of maps of $\mathbf{D}$ as follows:

- For each morphism f of $\mathbf{D}$, $f \in \mathcal{W}_{\mathbf{D}}$ if and only if $R(f)$ is a weak equivalence in $\mathbf{C}$;

- For each morphism f of $\mathbf{D}$, $f \in \mathcal{F}_{\mathbf{D}}$ if and only if $R(f)$ is a weak equivalence in $\mathbf{C}$;

- $\mathcal{C}_{\mathbf{D}}$ is the class of maps having the left-lifting property with respect to $\mathcal{W}_{\mathbf{D}} \cap \mathcal{F}_{\mathbf{D}}$.

We say that R *creates* a model structure on $\mathbf{D}$ if $\mathcal{W}_{\mathbf{D}}$, $\mathcal{F}_{\mathbf{D}}$ and $\mathcal{C}_{\mathbf{D}}$ are respectively the weak equivalences, fibrations and cofibrations of a model structure on $\mathbf{D}$. Sufficient conditions for this transfer to hold are given by [**3**, Proposition 2.3] or [**6**, Theorem 3.3]. The latter result immediately specializes to the following statement:

Proposition 3.15. *Let $\mathbf{C}$ a cofibrantly generated model structure, with I a set of generating cofibrations and J a set of generating trivial cofibrations. If $\mathbf{D}$ is locally presentable, then the following conditions are sufficient for R to create a model structure on $\mathbf{D}$:*

(C1) *The weak equivalences of $\mathbf{C}$ are closed under filtered colimits;*

(C2) *R preserves filtered colimits;*

(C3) *For each generating trivial cofibration j of $\mathbf{C}$, if g is a pushout of $L(j)$ in $\mathbf{D}$, then $R(g)$ is a weak equivalence of $\mathbf{C}$.*

Moreover, if these conditions hold, the model structure so defined is cofibrantly generated and has $L(I)$ as a set of generating cofibrations and $L(J)$ as a set of generating trivial cofibrations.

We now turn to the particular case where $\mathbf{C} = \infty\text{-}\mathbf{Cat}$, $\mathbf{D} = \infty\text{-}\mathbf{Grp}$ and R is the inclusion functor $U\colon \infty\text{-}\mathbf{Grp} \to \infty\text{-}\mathbf{Cat}$. Note that U has a left-adjoint $F\colon \infty\text{-}\mathbf{Cat} \to$

∞-**Grp** building the free ∞-groupoid on an ∞-category, as well as a right-adjoint M, building to the maximal ∞-groupoid in an ∞-category. Let us first establish a few properties about the adjunction $F \dashv U$.

Let T be the monad UF on ∞-**Cat**. Remark that, for any ∞-groupoid G, the free ∞-groupoid on the underlying ∞-category $U(G)$ is naturally isomorphic to G itself. In other words, the counit $\varepsilon\colon FU \to 1$ is a natural isomorphism. It follows that, for any ∞-groupoid G, we get an isomorphism

$$\eta_{U(G)}\colon U(G) \to UFU(G), \tag{1}$$

where η denotes the unit of the adjunction. Now, for each ∞-category C, $T(C)$ is of the form $U(G)$ where G is an ∞-groupoid and so is $\Gamma T(C)$ by Lemma 3.11, so that

$$\eta_{\Gamma T(C)}\colon \Gamma T(C) \to T\Gamma T(C)$$

is an isomorphism, as a special case of (1). Thus, we may define a natural transformation

$$\lambda\colon T\Gamma \to \Gamma T$$

by

$$\lambda_C = \eta_{\Gamma T(C)}^{-1} \circ T\Gamma(\eta_C).$$

Note also that the monad multiplication $\mu\colon T^2 \to T$ is also a natural isomorphism, and we get

$$T(\eta_C) = \eta_{T(C)} = \mu_C^{-1}.$$

We may now state the following result:

Lemma 3.16. *The monad T preserves immersions.*

Proof. Let $f\colon C \to D$ be an immersion, and $f' = T(f)$. By Definition 3.13, there are $g\colon D \to C$ and $h\colon D \to \Gamma(D)$ such that

$$\begin{aligned}
g \circ f &= \mathrm{id}_C, \\
\pi_D^1 \circ h &= f \circ g, \\
\pi_D^2 \circ h &= \mathrm{id}_D, \\
h \circ f &= \tau_D \circ f.
\end{aligned} \tag{2}$$

Let $g' = T(g)$ and $h' = \lambda_D \circ T(h)$. We need to establish the following equations:

$$g' \circ f' = \mathrm{id}_{T(C)}, \tag{3}$$

$$\pi_{T(D)}^1 \circ h' = f' \circ g', \tag{4}$$

$$\pi_{T(D)}^2 \circ h' = \mathrm{id}_{T(D)}, \tag{5}$$

$$h' \circ f' = \tau_{T(D)} \circ f'. \tag{6}$$

Equation (3) is just functoriality. Let us prove (4). First remark that π^1 is a natural

transformation, so that the following diagram commutes:

$$
\begin{array}{ccc}
\Gamma(D) & \xrightarrow{\pi_D^1} & D \\
{\scriptstyle \Gamma(\eta_D)}\downarrow & & \downarrow{\scriptstyle \eta_D} \\
\Gamma T(D) & \xrightarrow[\pi_{T(D)}^1]{} & T(D).
\end{array}
\tag{7}
$$

We may now build the following commutative diagram:

$$
\tag{8}
$$

$$
\begin{array}{ccc}
T\Gamma(D) & \xrightarrow{T(\pi_D^1)} & T(D) \\
\Big\downarrow{\scriptstyle T\Gamma(\eta_D)} & & \Big\downarrow{\scriptstyle T(\eta_D)} \\
T\Gamma T(D) & \xrightarrow{T(\pi_{T(D)}^1)} & T^2(D) \\
\Big\downarrow{\scriptstyle \eta_{\Gamma T(D)}^{-1}} & & \Big\downarrow{\scriptstyle \eta_{T(D)}^{-1}} \\
\Gamma T(D) & \xrightarrow[\pi_{T(D)}^1]{} & T(D).
\end{array}
$$

with λ_D on the left and $\mathrm{id}_{T(D)}$ on the right.

In fact the upper square is the image of (7) by T and the lower square commutes by naturality of η. Hence

$$
\begin{aligned}
\pi_{T(D)}^1 \circ h' &= \pi_{T(D)}^1 \circ \lambda_D \circ T(h) \\
&= T(\pi_D^1) \circ T(h) \\
&= T(\pi_D^1 \circ h) \\
&= T(f \circ g) \\
&= f' \circ g',
\end{aligned}
$$

which gives (4). Likewise, we get the following commutative diagram:

$$
\begin{array}{ccccc}
 & & \xrightarrow{h'} & & \\
T(D) & \xrightarrow{T(h)} & T\Gamma(D) & \xrightarrow{\lambda_D} & \Gamma T(D), \\
 & {\scriptstyle \mathrm{id}_{T(D)}}\searrow & \Big\downarrow{\scriptstyle T(\pi_D^2)} & \swarrow{\scriptstyle \pi_{T(D)}^2} & \\
 & & T(D) & &
\end{array}
$$

where the left-hand triangle commutes by applying T to (2), and the right-hand triangle commutes by replacing π^1 with π^2 in (8). Hence $\pi_{T(D)}^2 \circ h' = \mathrm{id}_{T(D)}$ and (5) is proved. Finally, by using the naturality of τ instead of π^1, we get a commutative

diagram analogue to (8):

$$T(D) \xrightarrow{T(\tau_D)} T\Gamma(D)$$

$$\mathrm{id}_{T(D)} \left(\begin{array}{c} T(\eta_D) \quad T\Gamma(\eta_D) \\ T^2(D) \xrightarrow{T(\tau_{T(D)})} T\Gamma T(D) \\ \eta_{T(D)}^{-1} \quad \eta_{\Gamma T(D)}^{-1} \end{array} \right) \lambda_D$$

$$T(D) \xrightarrow[\tau_{T(D)}]{} \Gamma T(D).$$

Hence

$$\begin{aligned}
h' \circ f' &= \lambda_D \circ T(h) \circ T(f) \\
&= \lambda_D \circ T(h \circ f) \\
&= \lambda_D \circ T(\tau_D \circ f) \\
&= \lambda_D \circ T(\tau_D) \circ f' \\
&= \tau_{T(D)} \circ f'
\end{aligned}$$

which gives (6) and ends the proof. $\square$

Lemma 3.17. *Let $f\colon C \to D$ be an immersion and suppose that the following square is a pushout in ∞-**Grp**:*

$$\begin{array}{ccc}
FC & \xrightarrow{u} & G \\
{\scriptstyle F(f)} \downarrow & & \downarrow {\scriptstyle g} \\
FD & \xrightarrow[v]{} & H.
\end{array}$$

Then $U(g)$ is an immersion.

Proof. As U is left adjoint to M, it preserves pushouts, so that the following square is a pushout in ∞-**Cat**:

$$\begin{array}{ccc}
TC & \xrightarrow{U(u)} & UG \\
{\scriptstyle T(f)} \downarrow & & \downarrow {\scriptstyle U(g)} \\
TD & \xrightarrow[U(v)]{} & UH.
\end{array}$$

By Lemma 3.16, $T(f)$ is an immersion and so is its pushout $U(g)$, by Proposition 3.14(i). $\square$

Lemma 3.18. *If j is a generating trivial cofibration of ∞-**Cat**, and g is a pushout of $F(j)$ in ∞-**Grp**, then $U(g)$ is a weak equivalence of ∞-**Cat**.*

Proof. Let j be a generating trivial cofibration of ∞-**Cat**, and g be a pushout of $F(j)$ in ∞-**Grp**. By Proposition 3.14(ii), j is an immersion and so is $U(g)$, by Lemma 3.17. By Proposition 3.14(iii), $U(g)$ is a weak equivalence. $\square$

We may finally state the main result of this section:

Theorem 3.19. *The forgetful functor* $U\colon \infty\text{-}\mathbf{Grp} \to \infty\text{-}\mathbf{Cat}$ *creates a model struc-ture on* $\infty\text{-}\mathbf{Grp}$ *in which the weak equivalences are the morphisms* f *such that* $U(f)$ *belongs to* $\mathcal{W}$. *Moreover, the model structure so defined has* $(F(\mathbf{i}_k))_{k \in \mathbb{N}}$ *as a family of generating cofibrations and* $(F(\mathbf{j}_k))_{k \in \mathbb{N}}$ *as a family of generating trivial cofibrations.*

Proof. As the model structure on $\infty\text{-}\mathbf{Cat}$ is cofibrantly generated and $\infty\text{-}\mathbf{Grp}$ is locally presentable, Proposition 3.15 applies, and it suffices to check conditions (C1), (C2) and (C3). Condition (C1) is proved in [**10**], and condition (C2) follows from the fact that U has a right-adjoint M, hence preserves colimits, and in particular filtered ones. Condition (C3) is Lemma 3.18. The statement about generating families follows from Proposition 3.15. $\qquad\square$

Remark 3.20. Using Remark 3.12, one can easily adapt the proof of the previous the-orem to show that a similar theorem holds for strict (∞, n)-categories. In particular, the inclusion functor $(\infty, n)\text{-}\mathbf{Cat} \to \infty\text{-}\mathbf{Cat}$ creates a model structure on $(\infty, n)\text{-}\mathbf{Cat}$.

We call the model structure just defined the *folk model structure* on ∞-groupoids. We denote its weak equivalences by $\mathcal{W}_{\mathrm{folk}}$ and its trivial fibrations by $\mathcal{TF}_{\mathrm{folk}}$. Note that a morphism f is in $\mathcal{TF}_{\mathrm{folk}}$ if and only if $U(f)$ is a trivial fibration of $\infty\text{-}\mathbf{Cat}$.

Proposition 3.21. *A morphism* $f\colon G \to H$ *of* ∞-*groupoids belongs to* $\mathcal{TF}_{\mathrm{folk}}$ *if and only if the following conditions are satisfied:*

1. *For every object* y *of* H, *there exists an object* x *of* G *such that* $f(x) = y$;

2. *For all* $n \geqslant 1$ *and every pair* (u, v) *of parallel* $(n-1)$-*arrows of* G, *the map*

$$G(u, v)_0 \to H(f(u), f(v))_0$$

   is surjective.

Proof. By definition, f belongs to $\mathcal{TF}_{\mathrm{folk}}$ if and only if $U(f)$ has the right lifting property with respect to I. This proposition is then just a matter of translation. $\qquad\square$

4.   Comparison

In this section, we show that the folk model structure on strict ∞-groupoids defined in the previous section coincides with the Brown-Golasiński model structure. To see this, it suffices to prove that they have the same weak equivalences and the same trivial fibrations.

Proposition 4.1. *We have* $\mathcal{W}_{\mathrm{gr}} = \mathcal{W}_{\mathrm{folk}} = \mathcal{W}_{\mathrm{cc}}$.

Proof. We first show that $\mathcal{W}_{\mathrm{folk}} = \mathcal{W}_{\mathrm{gr}}$. In a strict ∞-groupoid, two n-arrows f and g are ω-equivalent if and only if there exists an $(n+1)$-arrow $a\colon f \to g$, that is if and only if f and g are homotopic. Therefore a morphism of strict ∞-groupoids is in $\mathcal{W}_{\mathrm{folk}}$ if and only if it satisfies condition 4 of Proposition 1.7. The statement is thus exactly the equivalence between conditions 1 and 4 of this very proposition.

By Proposition 2.9, we have $\mathcal{W}_{\mathrm{cc}} = \mathcal{W}_{\mathrm{gr}}$, hence the result. $\qquad\square$

Proposition 4.2. *We have* $\mathcal{TF}_{\mathrm{folk}} = \mathcal{TF}_{\mathrm{cc}}$.

Proof. To prove the equivalence between the two notions of trivial fibrations, we will use the descriptions of these notions provided by Propositions 2.10 and 3.21. The conditions for being in $\mathcal{TF}_{\text{folk}}$ are *a priori* stronger. Let $f\colon G \to H$ be a in $\mathcal{TF}_{\text{cc}}$. Let us prove it is actually in $\mathcal{TF}_{\text{folk}}$. There is nothing to prove for the conditions in dimension 0 and 1. Let $n \geqslant 2$ and let u, v be two parallel $(n-1)$-arrows. We want to show that the map

$$G(u,v)_0 \to H(f(u), f(v))_0$$

is surjective. Let b be an n-arrow from $f(u)$ to $f(v)$ in H. Set $x = s_0(u)$. Then $b' = 1_{w_0(f(u))} *_0 b$ is an n-arrow of H from $1_{f(x)}$ to $w_0(f(u)) *_0 f(v)$. Since the map

$$G(1_x, w_0(u) *_0 v)_0 \to H(f(1_x), f(w_0(u) *_0 v))_0$$

is surjective, there exists an n-arrow a' of G from 1_x to $w_0(u) *_0 v$ such that $f(a') = b'$. Then, the n-arrow $a = 1_u *_0 a'$ is from u to v, and we have

$$\begin{aligned}
f(a) &= f(1_u *_0 a') \\
&= 1_{f(u)} *_0 b' \\
&= 1_{f(u)} *_0 1_{w_0(f(u))} *_0 b \\
&= b. \qquad\qquad \square
\end{aligned}$$

Theorem 4.3. *The Brown-Golasiński model structure and the folk model structure on strict ∞-groupoids coincide.*

Proof. By the two previous propositions, these model structures have the same weak equivalences and the same trivial fibrations. $\qquad\qquad\square$

References

[1] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, London Math. Soc. Lecture Note Ser. **189**, Cambridge University Press, Cambridge, 1994.

[2] D. Ara, Sur les ∞-groupoïdes de Grothendieck et une variante ∞-catégorique, Ph.D. thesis, Université Paris 7, 2010.

[3] T. Beke, Sheafifiable homotopy model categories. II. *Journal of Pure and Applied Algebra* **164** (2001), no. 3, 307–324.

[4] R. Brown and M. Golasiński, A model structure for the homotopy theory of crossed complexes, *Cahiers Topologie et Géométrie Différentielle Catégoriques* **30** (1989), no. 1, 61–82.

[5] R. Brown and P.J. Higgins, The equivalence of ∞-groupoids and crossed complexes, *Cahiers Topologie Géom. Différentielle* **22** (1981), no. 4, 371–386.

[6] S. Crans, Quillen closed model structures for sheaves, *Journal of Pure and Applied Algebra* **101** (1995), no. 1, 35–57.

[7] B. Jacobs and J. Rutten, A tutorial on (co)algebras and (co)induction. *Bulletin of the EATCS* **62** (1997), 222–259.

[8] A. Joyal and M. Tierney, Strong stacks and classifying spaces, in *Category theory*, Proc. Int. Conf. (Como/Italy 1990), Lecture Notes in Math. **1488** (1991), 213–236, Springer-Verlag, New York.

[9]   S. Lack, A Quillen model structure for 2-categories, *K-Theory* **26** (2002), no. 2, 171–205.

[10] Y. Lafont, F. Métayer, and K. Worytkiewicz, A folk model structure on omega-cat, *Advances in Mathematics* **224** (2010), no. 3, 1183–1231.

[11] F. Métayer, Resolutions by polygraphs, *Theory and Applications of Categories* **11** (2003), no. 7, 148–184, `http://www.tac.mta.ca/tac/`.

[12] R. Street, The algebra of oriented simplexes, *Journal of Pure and Applied Algebra* **49** (1987), no. 3, 283–335.

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