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MATRADS, BIASSOCIAHEDRA, AND A_∞-BIALGEBRAS

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(*communicated by James Stasheff*)

Abstract

We introduce the notion of a matrad $M = \{M_{n,m}\}$ whose submodules $M_{*,1}$ and $M_{1,*}$ are non-Σ operads. We define the free matrad $\mathcal{H}_\infty$ generated by a singleton θ_m^n in each bidegree (m, n) and realize $\mathcal{H}_\infty$ as the cellular chains on a new family of polytopes $\{KK_{n,m} = KK_{m,n}\}$, called *biassociahedra*, of which $KK_{n,1} = KK_{1,n}$ is the associahedron K_n. We construct the universal enveloping functor from matrads to PROPs and define an A_∞-bialgebra as an algebra over $\mathcal{H}_\infty$.

1. Introduction

Let H be a DG module over a commutative ring R with identity. In [**11**], we defined an A_∞-bialgebra structure on H in terms of a square-zero $\circledcirc$-product on the universal PROP $U_H = End(TH)$. In this paper we take an alternative point-of-view motivated by three classical constructions: First, chain maps $Ass \to U_H$ and $\mathcal{A}_\infty \to U_H$ in the category of non-Σ operads define strictly (co)associative and A_∞-(co)algebra structures on H; second, there is a minimal resolution of operads $\mathcal{A}_\infty \to Ass$; and third, $\mathcal{A}_\infty$ is realized by the cellular chains on the Stasheff associahedra $K = \sqcup K_n$ [**7**], [**6**]. It is natural, therefore, to envision a category in which analogs of Ass and $\mathcal{A}_\infty$ define strictly biassociative and A_∞-bialgebra structures on H.

In this paper we introduce the notion of a *matrad* whose distinguished objects $\mathcal{H}$ and $\mathcal{H}_\infty$ play the role of Ass and $\mathcal{A}_\infty$. But unlike the operadic case, freeness considerations are subtle since biassociative bialgebras cannot be simultaneously free and cofree. Although $\mathcal{H}$ and $\mathcal{H}_\infty$ are generated by singletons in each bidegree, those in $\mathcal{H}$ are module generators while those in $\mathcal{H}_\infty$ are matrad generators. Indeed, as a non-free matrad, $\mathcal{H}$ has two matrad generators and $\mathcal{H}_\infty$ is its minimal resolution. Thus $\mathcal{H}$ and $\mathcal{H}_\infty$ are the smallest possible constructions that control biassociative bialgebras structures and their homotopy versions (cf. [**5**], [**12**], [**8**]).

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Given a finite sequence $\mathbf{x}$ in $\mathbb{N}$, let $|\mathbf{x}| = \sum x_i$. A matrad (M, γ) is a bigraded module $M = \{M_{n,m}\}_{m,n \geqslant 1}$ together with a family of structure maps

$$\gamma = \left\{ \gamma_{\mathbf{x}}^{\mathbf{y}} \colon \Gamma_p^{\mathbf{y}}(M) \otimes \Gamma_{\mathbf{x}}^q(M) \to M_{|\mathbf{y}|,|\mathbf{x}|} \right\}_{\mathbf{x} \times \mathbf{y} \in \mathbb{N}^{1 \times p} \times \mathbb{N}^{q \times 1}}$$

defined on certain submodules

$$\Gamma_p^{\mathbf{y}}(M) \otimes \Gamma_{\mathbf{x}}^q(M) \subseteq \bigotimes_{j=1}^{q} M_{y_j,p} \otimes \bigotimes_{i=1}^{p} M_{q,x_i}$$

and generated by certain components of the S-U diagonal Δ_P on permutahedra [**10**]; its substructures $(\Gamma_1^{\mathbf{y}}(M), \gamma)$ and $\left(\Gamma_{\mathbf{x}}^1(M), \gamma\right)$ are non-Σ operads. We think of monomials in $\Gamma_{\mathbf{x}}^q(M)$ as p-fold tensor products of multilinear operations, each with q outputs, and monomials in $\Gamma_p^{\mathbf{y}}(M)$ as q-fold tensor products of multilinear operations, each with p inputs.

A general PROP, and U_H in particular, admits a canonical matrad structure and chain maps $\mathcal{H} \to U_H$ and $\mathcal{H}_\infty \to U_H$ in the category of matrads define biassociative bialgebra and A_∞-bialgebra structures on H. Furthermore, $\mathcal{H}_\infty$ is realized by the cellular chains on a new family of polytopes $KK = \bigsqcup_{m,n \geqslant 1} KK_{n,m}$, called *biassociahedra,* of which $KK_{m,n} = KK_{n,m}$, and $KK_{1,n}$ is the Stasheff associahedron K_n. We identify the top dimensional cell of $KK_{n,m}$ with the indecomposable matrad generator θ_m^n represented graphically by a "double corolla" with data flowing upward through m input and n output channels (see Figure 1). The action of the matrad product γ on the submodule $\mathbf{\Theta} = \langle \theta_m^n \rangle_{m,n \geqslant 1}$ generates $\mathcal{H}_\infty$; we define a differential ∂ of degree -1 on θ_m^n and extend it as a derivation of γ (as in Example 6.11).

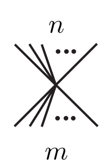
$\theta_m^n \leftrightarrow \curlyvee_m^n :=$ 

Figure 1.

Among the various attempts to construct homotopy versions of bialgebras, recent independent results of Markl and Shoikhet are related to ours through the theory of PROPs: In characteristic 0, the low-order relations in Markl's version of a homotopy bialgebra [**5**] agree with our A_∞-bialgebra relations and Shoikhet's composition product on the universal preCROC [**12**] agrees with our prematrad operation on U_H. Thus we construct a functor from matrads to PROPs called the *universal enveloping functor.*

The paper is organized as follows: In Section 2 we construct the biassociahedra $KK_{n,m}$ in the range $m + n \leqslant 6$. These polytopes have a simple description in terms of the S-U diagonal Δ_K on associahedra [**10**] and demonstrate the general construction while avoiding the complicating subtleties. In Section 3 we discuss various submodules of TTM (the tensor module of TM), which model the geometry of our construction. We introduce the notion of a prematrad in Section 4, the notion of a k-approximation in Section 5, and the notion of a matrad in Section 6. We construct the posets $\mathcal{PP}$

and $\mathcal{KK}$ in Section 7, introduce the notion of the combinatorial join of permutahedra in Section 8, and construct PP and KK as geometric realizations of $\mathcal{PP}$ and $\mathcal{KK}$ in Section 9. We identify the cellular chains of KK with the A_∞-bialgebra matrad $\mathcal{H}_\infty$ and prove that the restriction of the free resolution of prematrads $\rho^{\text{pre}}: F^{\text{pre}}(\Theta) \to \mathcal{H}$ to $\mathcal{H}_\infty$ is a free resolution in the category of matrads.

2. Low Dimensional Biassociahedra

Our construction of the biassociahedra $\{KK_{n,m}\}$ in the range $1 \leqslant m, n \leqslant 4$ and $m + n \leqslant 6$ is controlled by the S-U diagonal on associahedra K; the polytope $KK_{n,m}$ is identical to B^n_m constructed by M. Markl in [**5**]. In the course of his construction, Markl makes arbitrary choices, which correspond to choices we made when constructing the S-U diagonal Δ_K. So for us, all choices in our construction were made a priori once and for all.

2.1. The Fraction Product

Let $\Theta = \left\langle \theta^n_m \neq 0 \mid \theta^1_1 = \mathbf{1} \right\rangle_{m,n\geqslant 1}$ and let $M = \{M_{n,m}\}_{m,n\geqslant 1}$ be the free PROP generated by Θ. For simplicity, we assume that M is a free bigraded $\mathbb{Z}_2$-module; sign considerations that arise over a general ground ring will be addressed in subsequent sections. For $p, q \geqslant 1$, let $\mathbf{x} \times \mathbf{y} = (x_1, \ldots, x_p) \times (y_1, \ldots, y_q) \in \mathbb{N}^p \times \mathbb{N}^q$. In [**5**], M. Markl defined the submodule S of *special elements* in M whose additive generators are monomials $\alpha^{\mathbf{y}}_{\mathbf{x}}$ expressed as "elementary fractions" of the form

$$\alpha^{\mathbf{y}}_{\mathbf{x}} = (\alpha^{y_1}_p \cdots \alpha^{y_q}_p)/(\alpha^q_{x_1} \cdots \alpha^q_{x_p}), \tag{1}$$

where $\alpha^q_{x_i}$ and $\alpha^{y_j}_p$ are additive generators of S and the j^{th} output of $\alpha^q_{x_i}$ is linked to the i^{th} input of $\alpha^{y_j}_p$ (juxtaposition in the numerator and denominator denotes tensor product). Thus $\dim \alpha^{\mathbf{y}}_{\mathbf{x}} = \sum_{i,j} \dim \alpha^q_{x_i} + \dim \alpha^{y_j}_p$, and $\alpha^{\mathbf{y}}_{\mathbf{x}}$ is represented graphically by a connected non-planar graph under the identification $\theta^n_m \leftrightarrow \curlyvee^n_m$ (see Example 2.1). We refer to $\mathbf{x}$ and $\mathbf{y}$ as the *leaf sequences* of $\alpha^{\mathbf{y}}_{\mathbf{x}}$.

Let TM denote the tensor module of M. All elementary fractions define a non-associative *fraction product* $/ : TM \otimes TM \to S$. For example, in the iterated fraction

$$A/B/C = \curlywedge / (\curlywedge\ \mathbf{1}) / (\curlywedge\ \mathbf{1}\ \mathbf{1})$$

with $\curlywedge \leftrightarrow \theta^1_2$, we have $(A/B)/C \neq 0$ and $A/(B/C) = 0$. Notationally, let $\mathbf{M}^q_{\mathbf{x}} = M_{q,x_1} \otimes \cdots \otimes M_{q,x_p}$ and $\mathbf{M}^{\mathbf{y}}_p = M_{y_1,p} \otimes \cdots \otimes M_{y_q,p}$; then the fraction product (1) can be expressed in terms of our *prematrad product* $\gamma^{\mathbf{y}}_{\mathbf{x}} : \mathbf{M}^{\mathbf{y}}_p \otimes \mathbf{M}^q_{\mathbf{x}} \to M_{|\mathbf{y}|,|\mathbf{x}|}$ as

$$\left(\alpha^{y_1}_p \cdots \alpha^{y_q}_p\right) / \left(\alpha^q_{x_1} \cdots \alpha^q_{x_p}\right) = \gamma^{\mathbf{y}}_{\mathbf{x}} \left(\alpha^{y_1}_p \cdots \alpha^{y_q}_p ; \alpha^q_{x_1} \cdots \alpha^q_{x_p}\right).$$

Note that the free action of γ on up-rooted m-leaf corollas $\curlywedge_m$ (or on down-rooted n-leaf corollas $\curlyvee^n$) generates all planar rooted trees with levels (PLTs), and the projection to planar rooted trees (PRTs) by forgetting levels induces the standard operadic product.

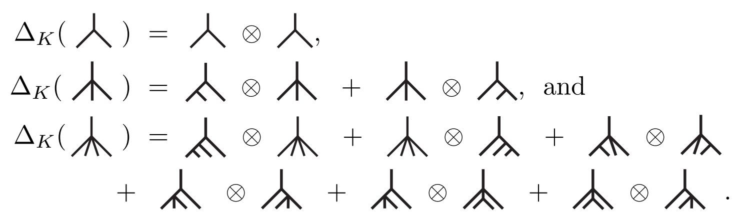
Although the matrad $\mathcal{H}_\infty$ is not defined in terms of PROP, one can identify $\mathcal{H}_\infty$ with a proper submodule of S and think of the matrad product γ as the fraction product restricted to $\mathcal{H}_\infty$. Thus we may regard the A_∞-operad $\mathcal{A}_\infty$ as either

$$\mathcal{H}_{1,*} = \left(\left\langle \theta_m^1 \mid m \geqslant 1\right\rangle / \sim,\ \partial\right),\ \text{where } \partial\left(\theta_m^1\right) = \sum_{\substack{\alpha_m^1 \in \mathcal{H}_{1,*}\cdot\mathcal{H}_{1,*} \\ \dim \alpha_m^1 = m-3}} \alpha_m^1 \quad \text{or} \tag{2}$$

$$\mathcal{H}_{*,1} = \left(\langle \theta_1^n \mid n \geqslant 1\rangle / \sim,\ \partial\right),\ \text{where } \partial\left(\theta_1^n\right) = \sum_{\substack{\alpha_1^n \in \mathcal{H}_{*,1}\cdot\mathcal{H}_{*,1} \\ \dim \alpha_1^n = n-3}} \alpha_1^n.$$

2.2. Low Dimensional Matrad Products

Let us construct $\{\mathcal{H}_{n,m}\}_{m+n\leqslant 6}$ inductively as stage $\mathcal{F}_6$ of the increasing filtration $\mathcal{F}_k = \bigoplus_{m+n\leqslant k}\mathcal{H}_{n,m}$. Our construction is controlled by the S-U diagonal Δ_K on cellular chains of the associahedra K (see Subsection 5.1 and [**10**]), which in the range of dimensions considered here is given by



Note that Δ_K agrees with the Alexander-Whitney diagonal on $K_2 = *$ and $K_3 = I$. Define $\Delta_K^{(0)} = \mathbf{1}$; for each $k \geqslant 1$, define the *(left) k-fold iterate of* Δ_K by

$$\Delta_K^{(k)} = \left(\Delta_K \otimes \mathbf{1}^{\otimes k-1}\right)\Delta_K^{(k-1)}$$

and view $\Delta_K^{(k)}(\curlywedge_p)$ as a $(p-2)$-dimensional subcomplex of $K_p^{\times k+1}$, and dually for $\Delta_K^{(k)}(\curlyvee^p)$.

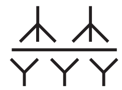
Referring to (2) above, define $\mathcal{F}_3 = \mathcal{H}_{1,1}\oplus\mathcal{H}_{1,2}\oplus\mathcal{H}_{2,1}$. To construct $\mathcal{F}_4$, use the generators of $\mathcal{F}_3$ to construct all possible elementary fractions with two inputs and two outputs. There are exactly two such elementary fractions, namely,

$$\alpha_2^2 \leftrightarrow \curlyvee / \curlywedge \quad \text{and} \quad \alpha_{11}^{11} \leftrightarrow (\curlywedge\curlywedge)/(\curlyvee\curlyvee),$$

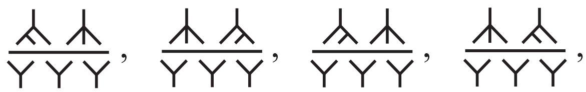
each of dimension zero. Let $\mathcal{H}_{2,2} = \left\langle \theta_2^2, \alpha_2^2, \alpha_{11}^{11}\right\rangle$ and define $\partial\left(\theta_2^2\right) = \alpha_2^2 + \alpha_{11}^{11}$. Then $\mathcal{H}_{2,2}$ is a proper submodule of $M_{2,2}$ and $KK_{2,2}$ is an interval I whose edge is identified with θ_2^2 and whose vertices are identified with α_2^2 and α_{11}^{11}. Define $\mathcal{F}_4 = \mathcal{F}_3\oplus\mathcal{H}_{1,3}\oplus\mathcal{H}_{2,2}\oplus\mathcal{H}_{3,1}$.

Although all fraction products are used to construct $\mathcal{F}_4$, more fractions than we need appear at the next stage of the construction and beyond. Note that each numerator or denominator of α_2^2 and α_{11}^{11} is an iterated S-U diagonal $\Delta_K^{(k)}(\curlyvee)$ or $\Delta_K^{(k)}(\curlywedge)$ for some $k = 0, 1$. Indeed, the components of $\Delta_K^{(k)}(\curlywedge_m)$ and $\Delta_K^{(k)}(\curlyvee^n)$ will determine which fraction products to admit and which to discard.

Continuing with the construction of $\mathcal{F}_5$, use the generators of $\mathcal{F}_4$ to construct all possible fractions with three inputs and two outputs. There are 18 of these: one in dimension 2, nine in dimension 1 and eight in dimension 0. Since $KK_{2,3}$ is to have a single top dimensional (indecomposable) 2-cell, we must discard the 2-dimensional (decomposable) generator

$$e \ =$$ 

as well as the appropriate components of its boundary. Note that e represents a square given by the Cartesian product of the three points in the denominator with the two intervals in the numerator. Thus the boundary of e consists of the four edges

 (3)

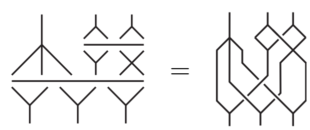
the first two of which contain components of $\Delta_K^{(1)}(\curlywedge_3)$ and $\Delta_K^{(2)}(\curlyvee)$ in their numerators and denominators. Our selection rule admits the first two edges and their vertices.

Express each of the factors $\alpha_{x_i}^{q}$ and $\alpha_{p}^{y_j}$ in $\alpha_{\mathbf{x}}^{\mathbf{y}} = \gamma_{\mathbf{x}}^{\mathbf{y}}\left(\alpha_p^{y_1}\cdots\alpha_p^{y_q};\alpha_{x_1}^{q}\cdots\alpha_{x_p}^{q}\right)$ in terms of their respective *leaf sequences* $\mathbf{x}_i$, $\mathbf{q}_i$, $\mathbf{y}_j$ and $\mathbf{p}_j$ so that

$$\alpha_{\mathbf{x}}^{\mathbf{y}} = \frac{\alpha_{\mathbf{p}_1}^{\mathbf{y}_1}\cdots\alpha_{\mathbf{p}_q}^{\mathbf{y}_q}}{\alpha_{\mathbf{x}_1}^{\mathbf{q}_1}\cdots\alpha_{\mathbf{x}_p}^{\mathbf{q}_p}}.$$

Then $(\mathbf{p}_1,\ldots,\mathbf{p}_q)$ and $(\mathbf{q}_1,\ldots,\mathbf{q}_p)$ define the *upper and lower contact sequences of* $\alpha_{\mathbf{x}}^{\mathbf{y}}$, respectively.

Example 2.1. The upper and lower contact sequences of

$$\alpha_{111}^{12} \quad = \quad \frac{\theta_3^1\alpha_{12}^{11}}{\theta_1^2\theta_1^2\theta_1^2} \quad \leftrightarrow$$  (4)

are $((3),(1,2))$ and $((2),(2),(2))$, respectively.

A non-vanishing *matrad monomial of codimension* 1

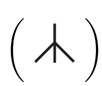
$$\alpha_{\mathbf{x}}^{\mathbf{y}} = \frac{\alpha_{\mathbf{p}_1}^{\mathbf{y}_1}\cdots\alpha_{\mathbf{p}_q}^{\mathbf{y}_q}}{\alpha_{\mathbf{x}_1}^{\mathbf{q}_1}\cdots\alpha_{\mathbf{x}_p}^{\mathbf{q}_p}} \in M_{|\mathbf{y}|,|\mathbf{x}|} \text{ with } |\mathbf{x}|+|\mathbf{y}|\leqslant 6$$

satisfies the following two conditions:

(i) The upper contact sequence $(\mathbf{p}_1,\ldots,\mathbf{p}_q)$ is the list of leaf sequences in some component of $\Delta_K^{(q-1)}(\curlywedge_p)$.

(ii) The lower contact sequence $(\mathbf{q}_1,\ldots,\mathbf{q}_p)$ is the list of leaf sequences in some component of $\Delta_K^{(p-1)}(\curlyvee^q)$.

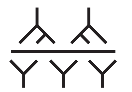
Example 2.2. The elementary fraction $\alpha_{111}^{12} = \left(\theta_3^1\alpha_{12}^{11}\right)/\left(\theta_1^2\theta_1^2\theta_1^2\right)$ in Example 2.1 is a non-vanishing 2-dimensional matrad generator since its upper contact sequence $((3),(1,2))$ is the list of leaf sequences in the component

 $\otimes$  of $\Delta_K^{(1)}$ 

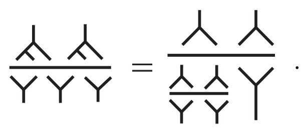
and its lower contact sequence $((2),(2),(2))$ is the list of leaf sequences in

$$\Delta_K^{(2)}(\mathsf{Y}) = \mathsf{Y}\otimes\mathsf{Y}\otimes\mathsf{Y}.$$

Having discarded the last two fractions in (3), our selection rule admits seven 1-dimensional generators labeling the edges of $KK_{2,3}$. Now linearly extend the boundary map ∂ to these matrad generators and compute the seven admissible 0-dimensional generators labeling the vertices of $KK_{2,3}$ (see Figure 12). Then in addition to the 2-dimensional generator e and the last two 1-dimensional generators in (3), our selection rule discards the common vertex

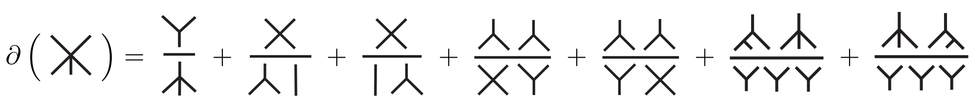


Different elementary fractions may represent the same element. For example,


(5)

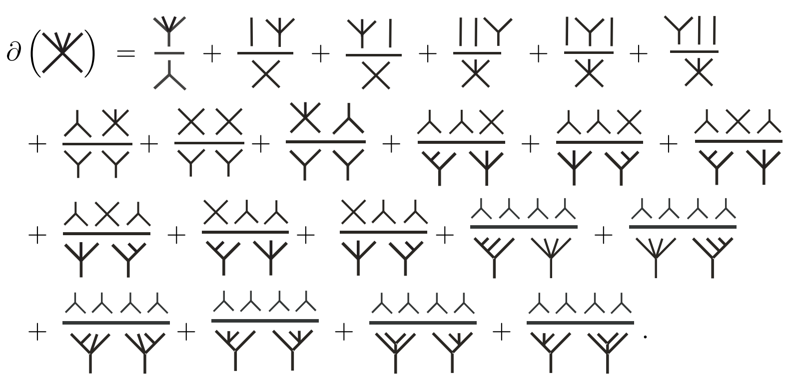
The associativity and unit axioms in the definition of a prematrad (see Definition 4.4 below) identify various representations such as these.

Finally, $\mathcal{H}_{2,3}$ is the proper submodule of $M_{2,3}$ generated by θ_3^2 and the 14 admissible fractions α_3^2 given by the selection rule above. Define



Then $KK_{2,3}$ is the heptagon pictured in Figure 2. Since $KK_{3,2}$ is homeomorphic to $KK_{2,3}$ (see Figure 19), we simultaneously obtain $\mathcal{H}_{3,2}$. Define $\mathcal{F}_5 = \mathcal{F}_4 \oplus \mathcal{H}_{1,4} \oplus \mathcal{H}_{2,3} \oplus \mathcal{H}_{3,2} \oplus \mathcal{H}_{4,1}$.

We continue with the construction of $\mathcal{F}_6$. Use the generators of $\mathcal{F}_5$ to construct all possible fractions with two inputs and four outputs. Using the selection rule defined above, admit all elementary fractions in dimension 2 whose upper and lower contact sequences agree with lists of leaf sequences of components in $\Delta_K^{(k)}(\curlywedge_p)$ or $\Delta_K^{(k)}(\curlyvee^q)$; these represent the 2-faces of $KK_{4,2}$. Linearly extend the boundary map ∂ and compute the admissible generators in dimensions 0 and 1. Let $\mathcal{H}_{4,2}$ be the proper submodule of $M_{4,2}$ generated by θ_2^4 and all admissible fractions α_2^4, and define



Then $KK_{4,2}$ is the 3-dimensional polytope pictured in Figures 3 and 21.

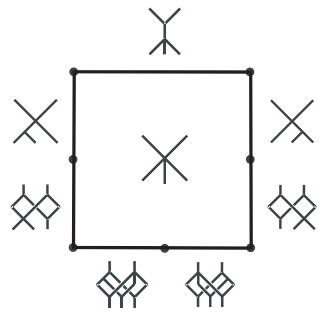
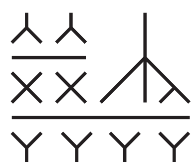


Figure 2: The biassociahedron $KK_{2,3}$.

Again, $KK_{2,4}$ is homeomorphic to $KK_{4,2}$ (see Figure 22) and we simultaneously obtain $\mathcal{H}_{2,4}$. While similar, the calculations for $KK_{3,3}$ (see Figure 20) also involve elements such as (4); we leave this case to the reader. Define $\mathcal{F}_6 = \mathcal{F}_5 \oplus \mathcal{H}_{1,5} \oplus \mathcal{H}_{2,4} \oplus \mathcal{H}_{3,3} \oplus \mathcal{H}_{4,2} \oplus \mathcal{H}_{5,1}$.

Note that all fractions in $\mathcal{F}_6$ are "operadic" in the sense that each contact sequence is identified with some component of $\Delta_K^{(p)}(\curlywedge_q)$ or $\Delta_K^{(p)}(\curlyvee^q)$. When $n > 6$, however, $\mathcal{F}_n$ contains "matradic" fractions whose contact sequences are identified with components of $\Delta_P^{(p)}(\curlywedge_q)$ or $\Delta_P^{(p)}(\curlyvee^q)$, the iterated diagonal on the permutahedron P_{q-1}. In $\mathcal{F}_7$, for example, there is the fraction

 .

Nevertheless, the low dimensional examples discussed here demonstrate the general principle, and with this in mind we proceed with the general construction.

3. Submodules of TTM

Let $M = \{M_{n,m}\}_{m,n\geqslant 1}$ be a bigraded module over a commutative ring R with identity 1_R. Various submodules of TTM will be important in our work, the most basic of which is the $q \times p$ *matrix submodule* $(M^{\otimes p})^{\otimes q}$. The name "matrix submodule" is motivated by the fact that each pair of $q \times p$ matrices $X = (x_{ij})$, $Y = (y_{ij}) \in \mathbb{N}^{q\times p}$ with $p, q \geqslant 1$ uniquely determines the submodule

$$M_{Y,X} = \left(M_{y_{1,1},x_{1,1}} \otimes \cdots \otimes M_{y_{1,p},x_{1,p}}\right) \otimes \cdots \otimes \left(M_{y_{q,1},x_{q,1}} \otimes \cdots \otimes M_{y_{q,p},x_{q,p}}\right) \subset TTM.$$

Fix a set of bihomogeneous module generators $G \subset M$. A *monomial in TM* is an element of $G^{\otimes p}$ and a *monomial in TTM* is an element of $(G^{\otimes p})^{\otimes q}$. Thus $A \in (G^{\otimes p})^{\otimes q}$ is naturally represented by the $q \times p$ matrix

$$[A] = \begin{bmatrix} g_{x_{1,1}}^{y_{1,1}} & \cdots & g_{x_{1,p}}^{y_{1,p}} \\ \vdots & & \vdots \\ g_{x_{q,1}}^{y_{q,1}} & \cdots & g_{x_{q,p}}^{y_{q,p}} \end{bmatrix}$$

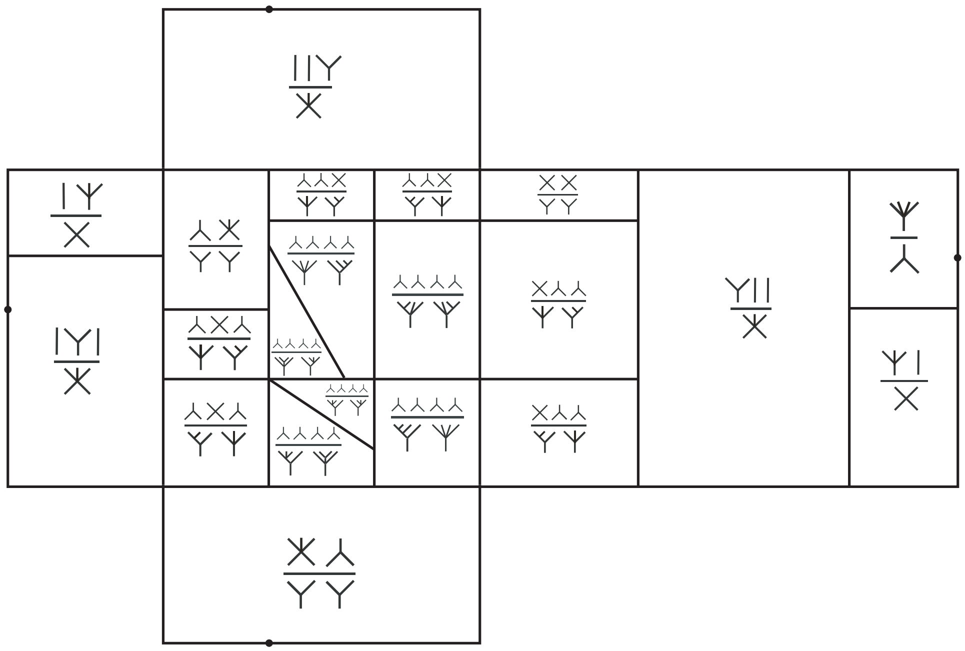


Figure 3: The 2-skeleton of $KK_{4,2}$.

with entries in G and rows identified with elements of $G^{\otimes p}$. Thus A is the q-fold tensor product of the rows of $[A]$, and we refer to A as a $q\times p$ *monomial* (we use the symbols A and $[A]$ interchangeably). Consequently,

$$\left(M^{\otimes p}\right)^{\otimes q}=\bigoplus_{X,Y\in\mathbb{N}^{q\times p}} M_{Y,X}$$

and we refer to

$$\overline{\mathbf{M}}=\bigoplus_{\substack{X,Y\in\mathbb{N}^{q\times p}\\ p,q\geqslant 1}} M_{Y,X} \quad\text{and}\quad \overline{\mathbf{V}}=\bigoplus_{\substack{X,Y\in\mathbb{N}^{1\times p}\cup\mathbb{N}^{q\times 1}\\ p,q\geqslant 1}} M_{Y,X}$$

as the *matrix submodule* and the *vector submodule,* respectively. The matrix transpose $A\mapsto A^T$ induces the permutation of tensor factors $\sigma_{p,q}\colon\ \left(M^{\otimes p}\right)^{\otimes q}\overset{\approx}{\to}\left(M^{\otimes q}\right)^{\otimes p}$ given by

$$\left(\alpha^{y_{1,1}}_{x_{1,1}}\otimes\cdots\otimes\alpha^{y_{1,p}}_{x_{1,p}}\right)\otimes\cdots\otimes\left(\alpha^{y_{q,1}}_{x_{q,1}}\otimes\cdots\otimes\alpha^{y_{q,p}}_{x_{q,p}}\right)\mapsto \tag{6}$$

$$\left(\alpha^{y_{1,1}}_{x_{1,1}}\otimes\cdots\otimes\alpha^{y_{q,1}}_{x_{q,1}}\right)\otimes\cdots\otimes\left(\alpha^{y_{1,p}}_{x_{1,p}}\otimes\cdots\otimes\alpha^{y_{q,p}}_{x_{q,p}}\right).$$

Throughout this paper, $\mathbf{x}$ and $\mathbf{y}$ denote matrices in $\mathbb{N}^{q\times p}$ with constant columns and constant rows, respectively; $\mathbf{x}$ and $\mathbf{y}$ will often be row and column matrices.

Define

$$\overline{\mathbf{M}}_{\text{row}} = \bigoplus_{\mathbf{x},Y\in\mathbb{N}^{q\times p};\ p,q\geqslant 1} M_{Y,\mathbf{x}} \quad\text{and}\quad \overline{\mathbf{M}}^{\text{col}} = \bigoplus_{X,\mathbf{y}\in\mathbb{N}^{q\times p};\ p,q\geqslant 1} M_{\mathbf{y},X}.$$

Consider $q\times p$ monomials

$$A = \begin{bmatrix} \alpha_{x_1}^{y_{1,1}} & \cdots & \alpha_{x_p}^{y_{1,p}} \\ \vdots & & \vdots \\ \alpha_{x_1}^{y_{q,1}} & \cdots & \alpha_{x_p}^{y_{q,p}} \end{bmatrix} \in M_{Y,\mathbf{x}} \ \text{ and } \ B = \begin{bmatrix} \beta_{x_{1,1}}^{y_1} & \cdots & \beta_{x_{1,p}}^{y_1} \\ \vdots & & \vdots \\ \beta_{x_{q,1}}^{y_q} & \cdots & \beta_{x_{q,p}}^{y_q} \end{bmatrix} \in M_{\mathbf{y},X}.$$

The *row (or coderivation) leaf sequence of* A is the p-tuple of lower (input) indices $\text{rls}A = (x_1,\ldots,x_p)$ along each row of A. Dually, the *column (or derivation) leaf sequence of* B is the q-tuple of upper (output) indices $\text{cls}B = (y_1,\ldots,y_q)^T$ along each column of B. Pictorially, each graph in the j^{th} column of A has x_j inputs and each graph in the i^{th} row of B has y_i outputs (see Figure 4).

The *bisequence submodule* of TTM is the intersection

$$\mathbf{M} = \overline{\mathbf{M}}_{\text{row}} \cap \overline{\mathbf{M}}^{\text{col}} = \bigoplus_{\mathbf{x},\mathbf{y}\in\mathbb{N}^{q\times p};\ p,q\geqslant 1} M_{\mathbf{y},\mathbf{x}}.$$

A $q\times p$ monomial $A\in\mathbf{M}$ is represented as a *bisequence matrix*

$$A = \begin{bmatrix} \alpha_{x_1}^{y_1} & \cdots & \alpha_{x_p}^{y_1} \\ \vdots & & \vdots \\ \alpha_{x_1}^{y_q} & \cdots & \alpha_{x_p}^{y_q} \end{bmatrix}; \tag{7}$$

in this case $\text{rls}A = (x_1,\ldots,x_p)$ and $\text{cls}A = (y_1,\ldots,y_q)^T$. Let

$$\mathbf{M}_{\mathbf{x}}^{\mathbf{y}} = \langle A\in\mathbf{M} \mid \mathbf{x} = \text{rls}A \text{ and } \mathbf{y} = \text{cls}A\rangle;$$

then

$$\mathbf{M} = \bigoplus_{\substack{\mathbf{x}\times\mathbf{y}\in\mathbb{N}^{1\times p}\times\mathbb{N}^{q\times 1}\\ p,q\geqslant 1}} \mathbf{M}_{\mathbf{x}}^{\mathbf{y}}.$$

Given a finite sequence of positive integers $\mathbf{u} = (u_1,\ldots,u_k)$, let $|\mathbf{u}| = \Sigma u_i$. By identifying $(H^{\otimes q})^{\otimes p} \approx (H^{\otimes p})^{\otimes q}$ with $(q,p)\in\mathbb{N}^2$, we can think of a $q\times p$ monomial $A\in\mathbf{M}_{\mathbf{x}}^{\mathbf{y}}$ as an operator on the positive integer lattice $\mathbb{N}^2$, pictured as an arrow $(|\mathbf{x}|, q)\mapsto(p,|\mathbf{y}|)$ (see Figure 4 and Example 4.11). While this representation is helpful conceptually, it is unfortunately not faithful.

Unless explicitly indicated otherwise, we henceforth assume that

$$\mathbf{x}\times\mathbf{y} = (x_1,\ldots,x_p)\times(y_1,\ldots,y_q)^T \in \mathbb{N}^{1\times p}\times\mathbb{N}^{q\times 1}$$

with $p,q\geqslant 1$. When the context is clear, we will often write $\mathbf{y}$ as a row vector. The *bisequence vector submodule* is the intersection

$$\mathbf{V} = \overline{\mathbf{V}}\cap\mathbf{M} = \bigoplus_{s,t\in\mathbb{N}} \mathbf{M}_s^{\mathbf{y}} \oplus \mathbf{M}_{\mathbf{x}}^t.$$

$A \in \mathbf{M}^{\substack{2\\4}}_{23} \quad \leftrightarrow$

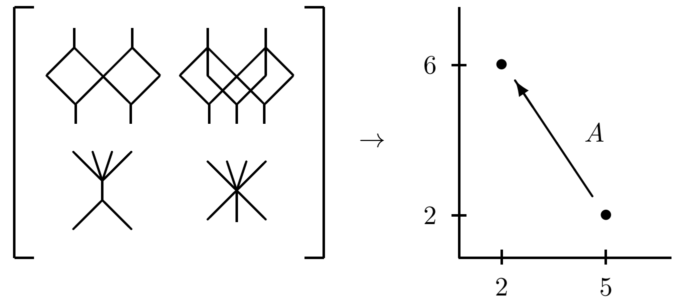


Figure 4: A 2×2 monomial in $\mathbf{M}$ and its arrow representation.

Example 3.1. Let $M = M_{1,1} \oplus M_{2,2} = \langle \theta_1^1 \rangle \oplus \langle \theta_2^2 \rangle$. Then the bisequence vector submodule of TTM is

$$\begin{aligned}\mathbf{V} = \mathbf{M}_1^1 \oplus \mathbf{M}_{11}^1 \oplus \mathbf{M}_{111}^1 \oplus \cdots \oplus \mathbf{M}_1^{11} \oplus \mathbf{M}_1^{111} \oplus \cdots \oplus \\ \mathbf{M}_2^2 \oplus \mathbf{M}_{22}^2 \oplus \mathbf{M}_{222}^2 \oplus \cdots \oplus \mathbf{M}_2^{22} \oplus \mathbf{M}_2^{222} \oplus \cdots \\ \approx T^+(M_{1,1}) \oplus T^+(M_{1,1}) \oplus T^+(M_{2,2}) \oplus T^+(M_{2,2}).\end{aligned}$$

A submodule

$$\mathbf{W} = M \oplus \bigoplus_{\mathbf{x},\mathbf{y} \notin \mathbb{N};\ s,t \in \mathbb{N}} \mathbf{W}_s^{\mathbf{y}} \oplus \mathbf{W}_{\mathbf{x}}^t \subseteq \mathbf{V}$$

is *telescoping* if for all $\mathbf{x}, \mathbf{y}, s, t$

(i) $\mathbf{W}_s^{\mathbf{y}} \subseteq \mathbf{M}_s^{\mathbf{y}}$ and $\mathbf{W}_{\mathbf{x}}^t \subseteq \mathbf{M}_{\mathbf{x}}^t$;

(ii) $\alpha_s^{y_1} \otimes \cdots \otimes \alpha_s^{y_q} \in \mathbf{W}_s^{\mathbf{y}}$ implies $\alpha_s^{y_1} \otimes \cdots \otimes \alpha_s^{y_j} \in \mathbf{W}_s^{y_1 \cdots y_j}$ for all $j < q$;

(iii) $\beta_{x_1}^t \otimes \cdots \otimes \beta_{x_p}^t \in \mathbf{W}_{\mathbf{x}}^t$ implies $\beta_{x_1}^t \otimes \cdots \otimes \beta_{x_i}^t \in \mathbf{W}_{x_1 \cdots x_i}^t$ for all $i < p$.

Thus the truncation maps $\tau \colon \mathbf{W}_s^{y_1 \cdots y_j} \to \mathbf{W}_s^{y_1 \cdots y_{j-1}}$ and $\tau \colon \mathbf{W}_{x_1 \cdots x_i}^t \to \mathbf{W}_{x_1 \cdots x_{i-1}}^t$ determine the following "telescoping" sequences of submodules:

$$\begin{aligned}\tau(\mathbf{W}_s^{\mathbf{y}}) \subseteq \tau^2(\mathbf{W}_s^{\mathbf{y}}) \subseteq \cdots \subseteq \tau^{q-1}(\mathbf{W}_s^{\mathbf{y}}) = \mathbf{W}_s^{y_1} \\ \tau(\mathbf{W}_{\mathbf{x}}^t) \subseteq \tau^2(\mathbf{W}_{\mathbf{x}}^t) \subseteq \cdots \subseteq \tau^{p-1}(\mathbf{W}_{\mathbf{x}}^t) = \mathbf{W}_{x_1}^t.\end{aligned}$$

In general, $\mathbf{W}_{\mathbf{x}}^t$ is an *additive* submodule of $\mathbf{M}_{x_1}^t \otimes \cdots \otimes \mathbf{M}_{x_p}^t$ and does *not* necessarily decompose as $B_1 \otimes \cdots \otimes B_p$ with $B_i \subseteq \mathbf{M}_{x_i}^t$.

The *telescopic extension* of a telescoping submodule $\mathbf{W} \subseteq \mathbf{V}$ is the submodule of matrices $\mathcal{W} \subseteq \overline{\mathbf{M}}$ with the following properties: If $A = \left[\alpha_{x_{i,j}}^{y_{i,j}}\right]$ is a $q \times p$ monomial in $\mathcal{W}$ and

(i) $\left[\alpha_{x_{i,j}}^{y_{i,j}} \ \cdots \ \alpha_{x_{i,j+m}}^{y_{i,j+m}}\right]$ is a string in the i^{th} row of A such that $y_{i,j} = \cdots = y_{i,j+m} = t$, then $\alpha_{x_{i,j}}^t \otimes \cdots \otimes \alpha_{x_{i,j+m}}^t \in \mathbf{W}_{x_{i,j},\ldots,x_{i,j+m}}^t$.

(ii) $\left[\alpha_{x_{i,j}}^{y_{i,j}} \ \cdots \ \alpha_{x_{i+r,j}}^{y_{i+r,j}}\right]^T$ is a string in the j^{th} column of A such that $x_{i,j} = \cdots = x_{i+r,j} = s$, then $\alpha_s^{y_{i,j}} \otimes \cdots \otimes \alpha_s^{y_{i+r,j}} \in \mathbf{W}_s^{y_{i,j},\ldots,y_{i+r,j}}$.

Thus if $A \in \mathbf{M}_{\mathbf{x}}^{\mathbf{y}} \cap \mathcal{W}$, the i^{th} row of A lies in $\mathbf{W}_{\mathbf{x}}^{y_i}$ and j^{th} column of A lies in $\mathbf{W}_{x_j}^{\mathbf{y}}$.

4. Premătrads

4.1. Υ-products on $\overline{\mathbf{M}}$

Given a family of maps $\bar{\gamma} = \{M^{\otimes q} \otimes M^{\otimes p} \to M\}_{p,q \geqslant 1}$, there is a canonical extension of the component $\gamma = \{\gamma^{\mathbf{y}}_{\mathbf{x}} \colon \mathbf{M}^{\mathbf{y}}_{p} \otimes \mathbf{M}^{q}_{\mathbf{x}} \to \mathbf{M}^{|\mathbf{y}|}_{|\mathbf{x}|}\}$ to a global product $\Upsilon \colon \overline{\mathbf{M}} \otimes \overline{\mathbf{M}} \to \overline{\mathbf{M}}$. Pairs of bisequence matrices in $\mathbf{M}^{\mathbf{y}}_{p} \otimes \mathbf{M}^{q}_{\mathbf{x}}$ are called "transverse pairs."

Definition 4.1. A pair $A \otimes B = \left[\alpha^{y_{kl}}_{v_{kl}}\right] \otimes \left[\beta^{u_{ij}}_{x_{ij}}\right]$ of $(q \times s, t \times p)$ monomials in $\overline{\mathbf{M}} \otimes \overline{\mathbf{M}}$ is a

(i) **Transverse Pair** (TP) if $s = t = 1$, $u_{1,j} = q$, and $v_{k,1} = p$ for all j and k, i.e., setting $x_j = x_{1,j}$ and $y_k = y_{k,1}$ gives
$$A \otimes B = \begin{bmatrix} \alpha^{y_1}_{p} \\ \vdots \\ \alpha^{y_q}_{p} \end{bmatrix} \otimes \begin{bmatrix} \beta^{q}_{x_1} & \cdots & \beta^{q}_{x_p} \end{bmatrix} \in \mathbf{M}^{\mathbf{y}}_{p} \otimes \mathbf{M}^{q}_{\mathbf{x}}.$$

(ii) **Block Transverse Pair** (BTP) if there exist $t \times s$ block decompositions $A = \left[A'_{k',l}\right]$ and $B = \left[B'_{i,j'}\right]$ such that $A'_{il} \otimes B'_{il}$ is a TP for *all* i and l.

The block sizes in a BTP decomposition are uniquely determined. Unlike the blocks in a standard block matrix, the blocks A'_{il} (or B'_{il}) of a BTP $A \otimes B \in \overline{\mathbf{M}} \otimes \overline{\mathbf{M}}$ may vary in length within a given row (or column). However, if $A \otimes B \in \mathbf{M}^{\mathbf{y}_1 \cdots \mathbf{y}_t}_{p_1 \cdots p_s} \otimes \mathbf{M}^{q_1 \cdots q_t}_{\mathbf{x}_1 \cdots \mathbf{x}_s}$ is a BTP, each TP $A'_{il} \otimes B'_{il} \in \mathbf{M}^{\mathbf{y}_i}_{p_l} \otimes \mathbf{M}^{q_i}_{\mathbf{x}_l}$ so that for fixed i (or l) the blocks A'_{il} (or B'_{il}) have constant length q_i (or p_l). Furthermore, $A \otimes B \in \mathbf{M}^{\mathbf{y}}_{\mathbf{v}} \otimes \mathbf{M}^{\mathbf{u}}_{\mathbf{x}}$ is a BTP if and only if $\mathbf{x} \times \mathbf{y} \in \mathbb{N}^{1 \times |\mathbf{v}|} \times \mathbb{N}^{|\mathbf{u}| \times 1}$ if and only if the initial point of arrow A coincides with the terminal point of arrow B in $\mathbb{N}^2$.

Example 4.2. A $(4 \times 2, 2 \times 3)$ monomial pair $A \otimes B \in \mathbf{M}^{1543}_{21} \otimes \mathbf{M}^{31}_{123}$ is a 2×2 BTP per the block decompositions

$$A = \left[\begin{array}{c|c} \alpha^1_2 & \alpha^1_1 \\ \alpha^5_2 & \alpha^5_1 \\ \alpha^4_2 & \alpha^4_1 \\ \hline \alpha^3_2 & \alpha^3_1 \end{array}\right] \qquad B = \left[\begin{array}{cc|c} \beta^3_1 & \beta^3_2 & \beta^3_3 \\ \hline \beta^1_1 & \beta^1_2 & \beta^1_3 \end{array}\right].$$

Given a family of maps $\bar{\gamma} = \{M^{\otimes q} \otimes M^{\otimes p} \to M\}_{p,q \geqslant 1}$, extend the component $\gamma = \{\gamma^{\mathbf{y}}_{\mathbf{x}} \colon \mathbf{M}^{\mathbf{y}}_{p} \otimes \mathbf{M}^{q}_{\mathbf{x}} \to \mathbf{M}^{|\mathbf{y}|}_{|\mathbf{x}|}\}$ to a global product $\Upsilon \colon \overline{\mathbf{M}} \otimes \overline{\mathbf{M}} \to \overline{\mathbf{M}}$ by defining
$$\Upsilon(A \otimes B) = \begin{cases} \left[\gamma\left(A'_{ij} \otimes B'_{ij}\right)\right], & A \otimes B \text{ is a BTP} \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$
where $A'_{ij} \otimes B'_{ij}$ is the $(i,j)^{th}$ TP in the BTP decomposition of $A \otimes B$.

We denote the Υ-product by "$\cdot$" or juxtaposition. When $A \otimes B = \left[\alpha_p^{y_j}\right]^T \otimes \left[\beta_{x_i}^q\right]$ is a TP, we write

$$AB = \gamma(\alpha_p^{y_1}, \ldots, \alpha_p^{y_q}; \beta_{x_1}^q, \ldots, \beta_{x_p}^q).$$

As an arrow, AB "transgresses" from the x-axis to the y-axis $\mathbb{N}^2$. When $A^{q\times s} \otimes B^{t\times p} \in \mathbf{M}_{\mathbf{v}}^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^{\mathbf{u}}$ is a BTP and $A'_{ij} \otimes B'_{ij} \in \mathbf{M}_{p_j}^{\mathbf{y}_i} \otimes \mathbf{M}_{\mathbf{x}_j}^{q_i}$ is its BTP decomposition, AB is a $t \times s$ matrix in $\mathbf{M}_{|\mathbf{x}_1|\cdots|\mathbf{x}_s|}^{|\mathbf{y}_1|\cdots|\mathbf{y}_t|}$. As an arrow, AB runs from the initial point of B to the terminal point of A.

Note that Υ-products always restrict to the submodules $\overline{\mathbf{M}}_{\text{row}}$ and $\overline{\mathbf{M}}^{\text{col}}$, and consequently to $\mathbf{M}$. To see this, consider a BTP $A \otimes B \in \overline{\mathbf{M}}^{\text{col}} \otimes \overline{\mathbf{M}}^{\text{col}}$ with block decomposition $\left[A'_{ij}\right] \otimes \left[B'_{ij}\right]$. Since each entry along the i^{th} row of B has q outputs, each block A'_{ij} is a column of length q. Since all entries along a row of A have the same number of outputs, the total number of outputs from each block A'_{ij} is the same for all j. Thus $AB \in \overline{\mathbf{M}}^{\text{col}}$ and Υ is closed in $\overline{\mathbf{M}}^{\text{col}}$. Dually, Υ is closed in $\overline{\mathbf{M}}_{\text{row}}$, and consequently, Υ is closed in $\mathbf{M}$.

Example 4.3. Continuing Example 4.2, the action of Υ on the $(4 \times 2, 2 \times 3)$ monomial pair $A \otimes B \in \mathbf{M}_{21}^{1543} \otimes \mathbf{M}_{123}^{31}$ produces a 2×2 monomial in $\mathbf{M}_{33}^{10,3}$:

$$\begin{bmatrix} \alpha_2^1 & \alpha_1^1 \\ \alpha_2^5 & \alpha_1^5 \\ \alpha_2^4 & \alpha_1^4 \\ \alpha_2^3 & \alpha_1^3 \end{bmatrix} \begin{bmatrix} \beta_1^3 & \beta_2^3 & \beta_3^3 \\ \beta_1^1 & \beta_2^1 & \beta_3^1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \alpha_2^1 \\ \alpha_2^5 \\ \alpha_2^4 \end{bmatrix} \begin{bmatrix} \beta_1^3 & \beta_2^3 \end{bmatrix} & \begin{bmatrix} \alpha_1^1 \\ \alpha_1^5 \\ \alpha_1^4 \end{bmatrix} \begin{bmatrix} \beta_3^3 \end{bmatrix} \\ \begin{bmatrix} \alpha_2^3 \end{bmatrix} \begin{bmatrix} \beta_1^1 & \beta_2^1 \end{bmatrix} & \begin{bmatrix} \alpha_1^3 \end{bmatrix} \begin{bmatrix} \beta_3^1 \end{bmatrix} \end{bmatrix}$$

In the target, $(|\mathbf{x}_1|, |\mathbf{x}_2|) = (1 + 2, 3)$ since $(p_1, p_2) = (2, 1)$; and $(|\mathbf{y}_1|, |\mathbf{y}_2|) = (1 + 5 + 4, 3)$ since $(q_1, q_2) = (3, 1)$. As an arrow in $\mathbb{N}^2$, AB initializes at $(6, 2)$ and terminates at $(2, 13)$.

4.2. Prematrads Defined

Let $1^{1\times p} = (1, \ldots, 1) \in \mathbb{N}^{1\times p}$ and $1^{q\times 1} = (1, \ldots, 1)^T \in \mathbb{N}^{q\times 1}$; we often suppress the exponents when the context is clear.

Definition 4.4. A **prematrad** (M, γ, η) is a bigraded R-module $M = \{M_{n,m}\}_{m,n\geqslant 1}$ together with a family of structure maps $\gamma = \{\gamma_{\mathbf{x}}^{\mathbf{y}} \colon \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^q \to \mathbf{M}_{|\mathbf{x}|}^{|\mathbf{y}|}\}$ and a unit $\eta\colon R \to \mathbf{M}_1^1$ such that

(i) $\Upsilon(\Upsilon(A; B); C) = \Upsilon(A; \Upsilon(B; C))$ whenever $A \otimes B$ and $B \otimes C$ are BTPs in $\overline{\mathbf{M}} \otimes \overline{\mathbf{M}}$;

(ii) the following compositions are the canonical isomorphisms:

$$R^{\otimes b} \otimes \mathbf{M}_a^b \overset{\eta^{\otimes b}\otimes\mathbf{1}}{\longrightarrow} \mathbf{M}_1^{\mathbf{1}^{b\times 1}} \otimes \mathbf{M}_a^b \overset{\gamma_a^{\mathbf{1}^{b\times 1}}}{\longrightarrow} \mathbf{M}_a^b;$$

$$\mathbf{M}_a^b \otimes R^{\otimes a} \overset{\mathbf{1}\otimes\eta^{\otimes a}}{\longrightarrow} \mathbf{M}_a^b \otimes \mathbf{M}_{\mathbf{1}^{1\times a}}^1 \overset{\gamma_{\mathbf{1}^{1\times a}}^b}{\longrightarrow} \mathbf{M}_a^b.$$

We denote the element $\eta(1_R)$ by $\mathbf{1_M}$. A **morphism of prematrads** (M,γ) and (M',γ') is a map $f\colon M\to M'$ such that $f\gamma_{\mathbf{x}}^{\mathbf{y}}=\gamma'^{\mathbf{y}}_{\mathbf{x}}(f^{\otimes q}\otimes f^{\otimes p})$ for all $\mathbf{x}\times\mathbf{y}$.

Although Υ fails to act associatively on $\overline{\mathbf{M}}$, Axiom (i) implies that $(AB)\,C=A\,(BC)$ whenever $A,B,C\in\overline{\mathbf{M}}$, $AB\neq 0$, and $BC\neq 0$. On the other hand, Υ acts associatively on $\mathbf{M}$, which is the content of Proposition 4.7 below. Given a bisequence matrix $A^{q\times p}\in\mathbf{M}_{\mathbf{x}}^{\mathbf{y}}$, let $\mathbf{1}^{q\times|\mathbf{x}|}$ and $\mathbf{1}^{|\mathbf{y}|\times p}$ denote the (bisequence) matrices whose entries are constantly $\mathbf{1_M}$. Then Axiom (ii) implies $\Upsilon\left(\mathbf{1}^{|\mathbf{y}|\times p};A\right)=A=\Upsilon\left(A;\mathbf{1}^{q\times|\mathbf{x}|}\right)$.

In the discussion that follows, we think of a string of matrices as a composition of operators and index the matrices in the order applied (from right-to-left).

Definition 4.5. Let $M=\{M_{n,m}\}$ be a bigraded R-module, and let $m,n\geqslant 1$. A string of matrices $A_s\cdots A_1$ is a **basic string of bidegree** (m,n) if

(i) $A_1\in\mathbf{M}_{\mathbf{x}}^{b},\ |\mathbf{x}|=m,$

(ii) $A_i\in\overline{\mathbf{M}}\smallsetminus\{\mathbf{1}^{q\times p}\mid p,q\in\mathbb{N}\}$ for all i,

(iii) $A_s\in\mathbf{M}_{a}^{\mathbf{y}}$, $|\mathbf{y}|=n$, and

(iv) some association of $A_s\cdots A_1$ defines a sequence of BTPs.

In particular, if M is a prematrad, $A_s\cdots A_1$ is a basic string of bidegree (m,n), and each BTP in Axiom (iv) defines a non-zero Υ-product, then $A_s\cdots A_1$ defines a non-zero element of $\mathbf{M}_m^n=M_{n,m}$. Indeed, this is exactly the situation when a prematrad (M,γ,η) is "free" (see Definition 4.13 below).

Lemma 4.6. *Let (M,γ,η) be a prematrad. Then Υ acts associatively on a basic string $A_s\cdots A_1$ if and only if $A_i\in\mathbf{M}$ for all i.*

Proof. Suppose Υ acts associatively on a basic string $A_s\cdots A_1$. If $s=1,2$ there is nothing to prove, so assume that $s>2$ and $1<i<s$. Since Υ acts associatively, every association of $A_s\cdots A_1$ defines a sequence of BTPs. Hence

$$BA_iC=(A_s\cdots A_{i+1})\,A_i\,(A_{i-1}\cdots A_1)$$

is a basic string and $B\otimes A_i$ and $A_i\otimes C$ are BTPs. So write $B=\left[b_p^{y_1}\cdots b_p^{y_q}\right]^T$ and $C=\left[c_{x_1}^r\cdots c_{x_t}^r\right]$; then the i^{th} row of A_i has the form $\left[a_{u_1}^{w_i}\cdots a_{u_p}^{w_i}\right]$ and the j^{th} column of A_i has the form $\left[a_{z_j}^{v_1}\cdots a_{z_j}^{v_r}\right]^T$. Thus $A_i\in\overline{\mathbf{M}}_{\text{row}}\cap\overline{\mathbf{M}}^{\text{col}}=\mathbf{M}$.

Conversely, we proceed by induction on string length s. Consider a basic string ABC with $B\in\mathbf{M}$, and suppose that $A\otimes B$ and $AB\otimes C$ are BTPs. Write

$$A\otimes B=\begin{bmatrix}a_p^{y_1}\\ \vdots\\ a_p^{y_r}\end{bmatrix}\otimes\begin{bmatrix}b_{u_1}^{v_1}&\cdots&b_{u_p}^{v_1}\\ \vdots&&\vdots\\ b_{u_1}^{v_q}&\cdots&b_{u_p}^{v_q}\end{bmatrix};$$

let B^i and B_j denote the i^{th} row and j^{th} column of B. Set $v_0=0$ and express A as the block matrix $\left[A^1 \cdots\ A^q\right]^T$ where $A^i = \left[a_p^{y_{v_1+\cdots+v_{i-1}+1}} \cdots\ a_p^{y_{v_1+\cdots+v_i}}\right]^T$. Then

$$AB = \begin{bmatrix} A^1B^1 \\ \vdots \\ A^qB^q \end{bmatrix}, \quad \text{where } A^iB^i = \begin{bmatrix} a_p^{y_{v_1+\cdots+v_{i-1}+1}} \\ \vdots \\ a_p^{y_{v_1+\cdots+v_i}} \end{bmatrix} \left[b_{u_1}^{v_i}\ \cdots\ b_{u_p}^{v_i}\right].$$

Since $AB \otimes C$ is a BTP, C has the form $\left[c_{x_1}^q \cdots c_{x_s}^q\right]$, where $s = u_1 + \cdots + u_p$. Set $u_0 = 0$ and express C as the block matrix $[C_1 \cdots C_p]$, where

$$C_j = \left[c^q_{x_{u_1+\cdots+u_{j-1}+1}} \cdots\ c^q_{x_{u_1+\cdots+u_j}}\right].$$

Then

$$B_j \otimes C_j = \begin{bmatrix} b_{u_j}^{v_1} \\ \vdots \\ b_{u_j}^{v_q} \end{bmatrix} \otimes \left[c^q_{x_{u_1+\cdots+u_{j-1}+1}} \cdots\ c^q_{x_{u_1+\cdots+u_j}}\right]$$

is a TP and $B \otimes C$ is a BTP. Therefore $(AB)\,C = A\,(BC)$ by Definition 4.4, Axiom (i). Similarly, if $B \otimes C$ and $A \otimes BC$ are BTPs, then $A \otimes B$ is a BTP.

Next consider a basic string $ABCD$ with $B, C \in \mathbf{M}$. If $B \otimes C$, $A \otimes BC$, and $A\,(BC) \otimes D$ are BTPs, $A\,(BC)$ is a column matrix whose entries are basic strings of the form $A_i\,([B_{i1} \cdots B_{ip}]\,[C_{i1} \cdots C_{ip}])$. Hence $A\,(BC) = (AB)\,C$ by the calculations above; and dually, $(BC)\,D = B\,(CD)$. Furthermore, the equalities

$$(A\,(BC))\,D = ((AB)\,C)\,D \qquad \text{and} \qquad A\,((BC)\,D) = A\,(B\,(CD))$$

imply $((AB)\,C)\,D = (AB)\,(CD) = A\,(B\,(CD))$.

Inductively, let $k \geqslant 4$, and assume that Υ acts associatively on the basic strings $A_s \cdots A_1$ of length $s \leqslant k$ with $A_i \in \mathbf{M}$ for all i, and consider a basic string $A_{k+1} \cdots A_1$ with $A_j \in \mathbf{M}$ for all j. Since some association of $A_{k+1} \cdots A_1$ defines a sequence of BTPs, there is an innermost BTP $A_{j+1} \otimes A_j$. Let $B = A_{j+1}A_j$; then B is a bisequence matrix since Υ is closed in $\mathbf{M}$, and Υ acts associatively on $A_{k+1} \cdots A_{j+2}BA_{j-1} \cdots A_1$. If $1 < j < k$, let $C = A_{k+1} \cdots A_{j+2}$ and $D = A_{j-1} \cdots A_1$; then Υ acts associatively on $CA_{j+1}A_jD$, completing the proof. □

Proposition 4.7. *Let (M, γ, η) be a prematrad; then Υ acts associatively on $\mathbf{M}$.*

Proof. If $A, B, C \in \mathbf{M}$ such that $A \otimes B$ and $AB \otimes C$ are BTPs, the entries of $(AB)\,C$ are basic strings on which Υ acts associatively by Lemma 4.6. Hence $B \otimes C$ is a BTP and $(AB)\,C = A\,(BC)$. If $A \otimes B$ is a BTP and $B \otimes C$ is not, neither is $AB \otimes C$. Dually, if $B \otimes C$ is a BTP and $A \otimes B$ is not, neither is $A \otimes BC$. In either case, $(AB)\,C = A\,(BC) = 0$. □

Some examples of prematrads now follow.

Remark 4.8. If (M, γ, η) is a prematrad, the restrictions $\left(M_{1,*}, \gamma_*^1, \eta\right)$ and $(M_{*,1}, \gamma_1^*, \eta)$ are non-Σ operads in the sense of May (see [**6**]).

Example 4.9. A non-Σ operad (K, γ_*) with

$$\gamma_{\mathbf{x}} \colon K(p) \otimes K(x_1) \otimes \cdots \otimes K(x_p) \to K(|\mathbf{x}|)$$

is a prematrad via

$$M_{n,m} = \begin{cases} K(m), & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \gamma_{\mathbf{x}}^{\mathbf{y}} = \begin{cases} \gamma_{\mathbf{x}}, & \text{if } \mathbf{y} = 1 \\ 0, & \text{otherwise} \end{cases}$$

(cf. Remark 4.8). For a discussion of the differential in the special case $K = \mathcal{A}_\infty$, see Example 6.11.

Example 4.10. Let $(K = \bigoplus_{n \geqslant 1} K(n), \gamma_*)$ and $(L = \bigoplus_{m \geqslant 1} L(m), \gamma^*)$ be non-Σ operads with $K(1) = L(1)$ and the same unit η. Set

$$M_{n,m} = \begin{cases} K(n), & \text{if } m = 1 \\ L(m), & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \gamma_{\mathbf{x}}^{\mathbf{y}} = \begin{cases} \gamma_{\mathbf{x}}, & \text{if } \mathbf{y} = 1 \\ \gamma^{\mathbf{y}}, & \text{if } \mathbf{x} = 1 \\ 0, & \text{otherwise,} \end{cases}$$

then (M, γ) is a prematrad.

Example 4.11 (The Prematrad PROP M). The free PROP M, with its horizontal and vertical products $\times \colon M_{n,m} \otimes M_{n',m'} \to M_{n+n',m+m'}$ and $\circ \colon M_{r,q} \otimes M_{q,p} \to M_{r,p}$ (cf. [**1**], [**6**]), is endowed with *a canonical prematrad* structure $(M^{\mathrm{pre}}, \gamma, \eta)$, with η determined by $\eta\,(1_R) = \{\text{unit of the PROP } M\}$. To define the structure map γ, define $\times^0 = \mathbf{1}$ and iterate $\times$ to obtain

$$\times^{q-1} \otimes \times^{p-1} \colon \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^q \longrightarrow \mathbf{M}_{pq}^{|\mathbf{y}|} \otimes \mathbf{M}_{|\mathbf{x}|}^{qp}.$$

View $\alpha_{|\mathbf{x}|}^{qp} \in \mathbf{M}_{|\mathbf{x}|}^{qp}$ as a graph with p groups of q outputs $(y_{1,1} \cdots y_{1,q}) \cdots (y_{p,1} \cdots y_{p,q})$ labeled from left-to-right. The leaf permutation

$$\sigma_{q,p} \colon (y_{1,1} \cdots y_{1,q}) \cdots (y_{p,1} \cdots y_{p,q}) \mapsto (y_{1,1} \cdots y_{p,1}) \cdots (y_{1,q} \cdots y_{p,q})$$

induces a map $\sigma_{q,p}^* \colon \mathbf{M}_{|\mathbf{x}|}^{qp} \to \mathbf{M}_{|\mathbf{x}|}^{pq}$. Then γ is the sum of the compositions

$$\gamma_{\mathbf{x}}^{\mathbf{y}} \colon \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^q \overset{\times^{q-1} \otimes \times^{p-1}}{\longrightarrow} \mathbf{M}_{pq}^{|\mathbf{y}|} \otimes \mathbf{M}_{|\mathbf{x}|}^{qp} \overset{\mathbf{1} \otimes \sigma_{q,p}^*}{\longrightarrow} \mathbf{M}_{pq}^{|\mathbf{y}|} \otimes \mathbf{M}_{|\mathbf{x}|}^{pq} \overset{\circ}{\longrightarrow} \mathbf{M}_{|\mathbf{x}|}^{|\mathbf{y}|}. \tag{9}$$

The induced associative product Υ on $\mathbf{M}$ takes values in matrices of (typically) non-planar graphs as in Figure 4. In particular, let H be a free DG R-module of finite type and view the universal PROP $U_H = End(TH)$ as the bigraded R-module $M = \{Hom\,(H^{\otimes m}, H^{\otimes n})\}_{n,m \geqslant 1}$. Then a $q \times p$ monomial $A \in \mathbf{M}_{\mathbf{x}}^{\mathbf{y}}$ admits a representation as an operator on $\mathbb{N}^2$ via the identification $(H^{\otimes p})^{\otimes q} \leftrightarrow (p, q)$ with the action of A given by the composition

$$\begin{aligned} \left(H^{\otimes|\mathbf{x}|}\right)^{\otimes q} &\approx \left(H^{\otimes x_1} \otimes \cdots \otimes H^{\otimes x_p}\right)^{\otimes q} \overset{A}{\longrightarrow} \left(H^{\otimes y_1}\right)^{\otimes p} \otimes \cdots \otimes \left(H^{\otimes y_q}\right)^{\otimes p} \\ &\overset{\sigma_{y_1,p} \otimes \cdots \otimes \sigma_{y_q,p}}{\longrightarrow} \left(H^{\otimes p}\right)^{\otimes y_1} \otimes \cdots \otimes \left(H^{\otimes p}\right)^{\otimes y_q} \approx \left(H^{\otimes p}\right)^{\otimes|\mathbf{y}|}. \end{aligned}$$

This motivates the representation of a general A as an arrow $(|\mathbf{x}|, q) \mapsto (p, |\mathbf{y}|)$ in $\mathbb{N}^2$ (see Figure 4). The map $\times^{q-1} \otimes \times^{p-1}$ in (9) is the canonical isomorphism and γ agrees with the composition product on the universal preCROC [**12**].

Example 4.12 (The Universal Enveloping PROP U). Recall that the structure map γ_{FP} in the free PROP $FP(M)$ is the sum of all possible (iterated) "horizontal" and "vertical" products $\times \colon M_{b,a} \otimes M_{b',a'} \to M_{b+b',a+a'}$ and $\circ \colon M_{c,b} \otimes M_{b,a} \to M_{c,a}$. Furthermore, the tensor product induces left S_n- and right S_m-actions

$$M_{b,b} \otimes M_{b,a} \overset{\sigma \otimes \mathbf{1}}{\to} M_{b,b} \otimes M_{b,a} \overset{\gamma_{FP}}{\to} M_{b,a} \text{ and}$$

$$M_{b,a} \otimes M_{a,a} \overset{\mathbf{1} \otimes \sigma}{\to} M_{b,a} \otimes M_{a,a} \overset{\gamma_{FP}}{\to} M_{b,a}.$$

Note that FP is functorial: Given bigraded modules M and N, a map $f = \{f_{b,a}\colon M_{b,a} \to N_{b,a}\}$ extends to a map $FP(f)\colon FP(M) \to FP(N)$ preserving horizontal and vertical products, i.e.,

$$FP(f)(M_{b,a} \times M_{b',a'}) = f(M_{b,a}) \times f(M_{b',a'})$$

and $FP(f)(M_{c,b} \circ M_{b,a}) := f(M_{c,b}) \circ f(M_{b,a})$. Now if (M, γ_M) is a prematrad, γ_M is a component of γ_{FP} on $\mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^q$. Let J be the two-sided ideal generated by $\bigoplus_{\mathbf{x} \times \mathbf{y}} (\gamma_{FP} - \gamma_M) \left(\mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^q \right)$. The **universal enveloping PROP of** M is the quotient

$$U(M) = FP(M) \big/ J.$$

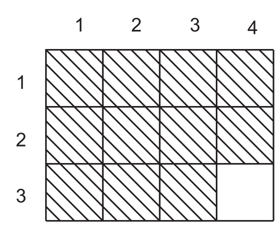
Note that the restriction of U to operads is the standard functor from operads to PROPs [**1**].

4.3. Free Prematrads

"Free prematrads" are fundamentally important. Our definition of a free prematrad (below) involves an inductive definition of the intermediate set $G^{\text{pre}} = G^{\text{pre}}_{*,*}$ in which $G^{\text{pre}}_{n,m}$ is defined in terms of the set

$$G_{[n,m]} = \bigcup_{\substack{i \leqslant m,\ j \leqslant n, \\ i+j<m+n}} G^{\text{pre}}_{j,i}.$$

We think of $G_{[n,m]}$ as an $n \times m$ array whose $(j,i)^{th}$ cell contains $G^{\text{pre}}_{j,i}$ when $i + j < m + n$ and whose $(n,m)^{th}$ cell is empty. For example, we picture $G_{[3,4]}$ as:



Borrowing our notation for the matrix and bisequence submodules of TTM, we denote the set of matrices over $G_{[n,m]}$ by $\overline{\mathbf{G}}_{[n,m]}$, the set of matrices over G^{pre} by $\overline{\mathbf{G}}$, and the subset of bisequence matrices in $\overline{\mathbf{G}}$ by $\mathbf{G}$.

Definition 4.13. Let $\Theta = \left\langle \theta_m^n \mid \theta_1^1 = \mathbf{1} \neq 0 \right\rangle_{m,n \geqslant 1}$ be a free bigraded R-module generated by singletons θ_m^n and set $G^{\text{pre}}_{1,1} = \mathbf{1}$. Inductively, if $m + n \geqslant 3$ and $\overline{\mathbf{G}}_{[n,m]}$ has been constructed, define

$$G^{\mathrm{pre}}_{n,m} = \theta^n_m \cup \left\{ \text{basic strings } A_s \cdots A_1 \text{ of bidegree } (m,n),\ A_i \in \overline{\mathbf{G}}_{[n,m]},\ s \geqslant 2 \right\}.$$

Let $\sim$ be the equivalence relation on G^{pre} generated by $[A_{ij}B_{ij}] \sim [A_{ij}]\,[B_{ij}]$ if and only if $[A_{ij}] \times [B_{ij}] \in \overline{\mathbf{G}} \times \overline{\mathbf{G}}$ is a BTP, and let $F^{\mathrm{pre}}(\Theta) = \langle G^{\mathrm{pre}} \diagup \sim \rangle$. The **free prematrad generated by** Θ is the prematrad $(F^{\mathrm{pre}}(\Theta), \gamma, \eta)$, where γ is juxtaposition and $\eta \colon R \to F^{\mathrm{pre}}_{1,1}(\Theta)$ is given by $\eta(1_R) = \mathbf{1}$.

Example 4.14 (The $\mathcal{A}_\infty$-operad). Let $\theta_m = \theta^1_m \neq 0$ and $\theta^n = \theta^n_1 \neq 0$ for all $m, n \geqslant 1$. The non-sigma operads $K = F^{\mathrm{pre}}(\theta_*)$ and $L = F^{\mathrm{pre}}(\theta^*)$ are isomorphic to the $\mathcal{A}_\infty$-operad and encode the combinatorial structure of an A_∞-algebra and an A_∞-coalgebra, respectively. Let $\theta^p_{m,i}$ denote the $1 \times p$ matrix $[\theta_1 \cdots \theta_m \cdots \theta_1]$ with θ_m in the i^{th} position, and let $\theta^{n,j}_q$ denote the $q \times 1$ matrix $\left[\theta^1 \cdots \theta^n \cdots \theta^1\right]^T$ with θ^n in the j^{th} position. Then modulo prematrad axioms (i) and (ii), the bases for K and L given by Definition 4.13 are

$$\left\{ \theta_{p_k} \theta^{p_k}_{m_k,i_k} \cdots \theta^{p_1}_{m_1,i_1} \in K\,(m_1 + p_1 - 1) \mid m_r = p_{r-1} - p_r + 1 \right\}$$

and

$$\left\{ \theta^{n_l,j_l}_{q_l} \cdots \theta^{n_1,j_1}_{q_1} \theta^{q_1} \in L\,(n_l + q_l - 1) \mid n_r = q_{r+1} - q_r + 1 \right\}.$$

Given $A \in G^{\mathrm{pre}}_{n,m} \diagup \sim$, choose a representative $A_s \cdots A_1 \in G^{\mathrm{pre}}_{n,m}$. In view of Definition 4.5, Axiom (iv), some association of $A_s \cdots A_1$ defines a sequence of BTPs; thus $s-1$ successive applications of Υ produces a 1×1 (bisequence) representative B.

Definition 4.15. Let $A \in G^{\mathrm{pre}}_{n,m} \diagup \sim$. A **factorization of** A is a representative $A_s \cdots A_1 \in A$. A factorization $A_s \cdots A_1 \in A$ is a Θ**-factorization** if the entries of A_i are elements of Θ for all i. A factorization $B_t \cdots B_1 \in A$ is a **bisequence factorization** if $B_i \in \mathbf{G}$ for all i. A bisequence factorization $A_s \cdots A_1 \in A$ is **bisequence decomposable** if there is a bisequence factorization $B_t \cdots B_1 \in A$ such that $t > s$.

Since bisequence factorizations are basic strings of matrices, bisequence factorizations are characterized by Lemma 4.6: $A_s \cdots A_1 \in A$ is a bisequence factorization if and only if Υ acts associatively on $A_s \cdots A_1$.

Given $A \in G^{\mathrm{pre}} \diagup \sim$, consider a factorization $A_s \cdots A_1 \in A$. Each association of $A_s \cdots A_1$ determines a fraction, any two of which will look quite different. For example, the two associations of the bisequence Θ-factorization

$$A_3 A_2 A_1 = \begin{bmatrix} \theta^2_2 \\ \theta^2_2 \\ \theta^1_2 \end{bmatrix} \begin{bmatrix} \theta^2_2 & \theta^2_1 \\ \theta^1_2 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \theta^2_2 & \theta^2_2 & \theta^2_2 \end{bmatrix} \in \mathbf{G}^6_6, \tag{10}$$

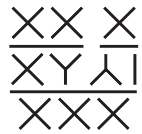
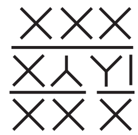
which are

$$(A_3 A_2)\, A_1 = \begin{bmatrix} \begin{bmatrix} \theta^2_2 \\ \theta^2_2 \end{bmatrix} \begin{bmatrix} \theta^2_2 & \theta^2_1 \end{bmatrix} \\ \\ \begin{bmatrix} \theta^2_2 \end{bmatrix} \begin{bmatrix} \theta^1_2 & \mathbf{1} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \theta^2_2 & \theta^2_2 & \theta^2_2 \end{bmatrix}$$

and

$$A_3\,(A_2 A_1) = \begin{bmatrix} \theta^2_2 \\ \theta^2_2 \\ \theta^2_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \theta^2_2 \\ \theta^1_2 \end{bmatrix} \begin{bmatrix} \theta^2_2 & \theta^2_2 \end{bmatrix} & \quad \begin{bmatrix} \theta^2_1 \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} \theta^2_2 \end{bmatrix} \end{bmatrix},$$

determine the fractions

 and  .

Define the *graph of* A to be the graph of the 1×1 representative $A'\in A$. If a fraction f represents an association of $A_s\cdots A_1\in A$, the graph of A can be obtained from f by removing fraction bars and making the prescribed connections.

The arrows representing the factors of $A_s\cdots A_1\in A$ form a polygonal path in $\mathbb{N}^2$ from the x-axis to the y-axis, and evaluating subproducts in an association changes the path. For example, the 3-step path in Figure 5 represents the factorization $A_3A_2A_1$ in (10) thought of as the composition

$$\overset{\xrightarrow{\qquad\text{XXX}\qquad}}{H^{\otimes 6}\longrightarrow \left(H^{\otimes 2}\right)^{\otimes 3} \overset{\sigma_{2,3}}{\longrightarrow}} \left(H^{\otimes 3}\right)^{\otimes 2} \overset{\xrightarrow{\qquad\frac{\text{XY}}{\lambda\text{I}}\qquad}}{\longrightarrow \left(H^{\otimes 2}\right)^{\otimes 3} \overset{\sigma_{2,2}\otimes\mathbf{1}}{\longrightarrow}} \left(H^{\otimes 2}\right)^{\otimes 3} \overset{\text{X}/\text{X}/\text{X}}{\longrightarrow} H^{\otimes 6},$$

and the 2-step paths represent the associations $(A_3A_2)\,A_1$ and $A_3\,(A_2A_1)$, where the subproducts in parentheses have been evaluated and the remaining unevaluated products are thought of as the compositions

$$\overset{\xrightarrow{\qquad\text{XXX}\qquad}}{H^{\otimes 6}\longrightarrow \left(H^{\otimes 2}\right)^{\otimes 3} \overset{\sigma_{2,3}}{\longrightarrow}} \left(H^{\otimes 3}\right)^{\otimes 2} \overset{\frac{\text{XX}}{\text{XY}}\,\frac{\text{X}}{\lambda\text{I}}}{\longrightarrow} H^{\otimes 6}$$

and

$$\overset{\xrightarrow{\qquad\frac{\text{X}\lambda}{\text{YX}}\,\frac{\text{YI}}{\text{X}}\qquad}}{H^{\otimes 6}\longrightarrow \left(H^{\otimes 3}\right)^{\otimes 2} \overset{\sigma_{3,2}}{\longrightarrow}} \left(H^{\otimes 2}\right)^{\otimes 3} \overset{\text{X}/\text{X}/\text{X}}{\longrightarrow} H^{\otimes 6}.$$

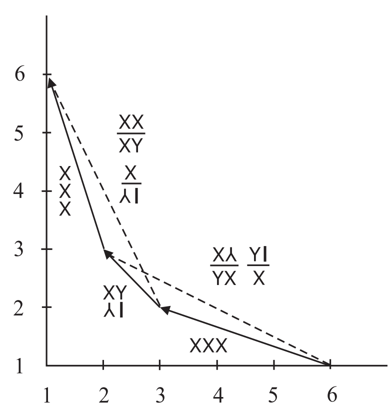


Figure 5: Polygonal paths of $A_3A_2A_1$.

In the special operadic situations of Example 4.14, every Θ-factorization is a bisequence factorization, but this is not true, in general. For example, the Θ-factorization

$$B_3B_2B_1 = \begin{bmatrix} \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \theta_2^1 & \mathbf{1} \\ \mathbf{1} & \theta_2^1 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \theta_1^2 & \theta_1^2 \end{bmatrix} \in G_{2,3}^{\mathrm{pre}}(\Theta)$$

is not a bisequence factorization since B_2 is not a bisequence matrix. Furthermore, $B_3B_2B_1$ only associates on the left since $B_2 \otimes B_1$ is *not* a BTP. However, $C_2 = B_3B_2$, $C_1 = B_1$, and $C = C_2C_1$ are bisequence matrices; hence C_2C_1 and C are bisequence factorizations of which

$$C_2C_1 = \begin{bmatrix} \theta_2^1 \begin{bmatrix} \theta_2^1 & \mathbf{1} \end{bmatrix} \\ \\ \theta_2^1 \begin{bmatrix} \mathbf{1} & \theta_2^1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \theta_1^2 & \theta_1^2 & \theta_1^2 \end{bmatrix} \tag{11}$$

is bisequence indecomposable (see Figure 6).

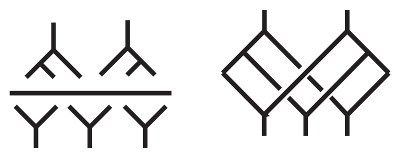


Figure 6: Graphical representations of C_2C_1 and C.

Bisequence indecomposables are especially important, and we wish to identify a canonical bisequence indecomposable representative of a given class $A \in G_{n,m}^{\mathrm{pre}}/\sim$. First consider a bisequence indecomposable $A_k \cdots A_1 \in A$ with some $A_i \in \left\{\theta_*^1, \theta_1^*\right\}$. If $A_i = \theta_s^1$ for some s, then $A_i \cdots A_1$ is identified with an up-rooted PLT with i levels and m leaves; dually, if $A_i = \theta_1^t$ for some t, then $A_k \cdots A_i$ is identified with a down-rooted PLT with $k-i+1$ levels and n leaves. Note that both situations occur in factorizations of the form $A_k \cdots \theta_1^t \theta_s^1 \cdots A_1$. But in either case, the indicated PLT represents an isomorphism class of PLTs whose elements determine distinct bisequence indecomposable factorizations of A. For example, the isomorphic PLTs

 $: \bullet\bullet\bullet\bullet \to (\bullet\bullet)\bullet\bullet \to (\bullet\bullet)(\bullet\bullet)$

and

 $: \bullet\bullet\bullet\bullet \to \bullet\bullet(\bullet\bullet) \to (\bullet\bullet)(\bullet\bullet)$

respectively determine the distinct equivalent bisequence indecomposables

$$A_4A_3A_2A_1 = \left[\theta_1^2\ \mathbf{1}\ \mathbf{1}\right]^T \left[\mathbf{1}\ \theta_1^2\right]^T \left[\theta_1^2\right] \left[\theta_3^1\right]$$

and

$$A_4'A_3'A_2'A_1' = \left[\mathbf{1}\ \mathbf{1}\ \theta_1^2\right]^T \left[\theta_1^2\ \mathbf{1}\right]^T \left[\theta_1^2\right] \left[\theta_3^1\right]$$

in $G_{4,3}^{\mathrm{pre}}$.

Now recall that a PLT with s levels and t leaves specifies the order in which s pairs of parentheses are inserted into a string of t indeterminates. When faced with the situations described above, *choose the factorization indexed by the PLT that successively inserts parentheses as far to the left as possible.* Such factorizations have *preferred operadic substrings.*

Definition 4.16. A **preferred factorization** is a factorization with preferred operadic substrings (if any) and factors whose entries have this property. A bisequence matrix $B \in G^{\text{pre}}$ is **balanced** if each entry of B is a preferred bisequence indecomposable factorization. A **balanced factorization** of $A \in G^{\text{pre}} \diagup \sim$ is a preferred bisequence indecomposable factorization $A_s \cdots A_1$ in which A_i is balanced for all i.

In the example above, $A_4A_3A_2A_1$ is the (obviously unique) balanced factorization of $A \in G^{\text{pre}}_{4,3} \diagup \sim$. In view of Definition 4.4, Axiom (i), two associations of a basic string $A_s \cdots A_1$ that define sequences of BTPs are equal. Hence each class $A \in G^{\text{pre}}_{n,m} \diagup \sim$ has a unique preferred factorization of maximal length. In fact:

Proposition 4.17. *Each class $A \in G^{\text{pre}}_{n,m} \diagup \sim$ has a unique balanced factorization.*

Proof. If $m + n \leqslant 4$, balanced factorizations of A are Θ-factorizations, and obviously unique. Inductively, assume the statement holds for all $B \in G^{\text{pre}}_{l,k} \diagup \sim$ with $k \leqslant m$, $l \leqslant n$ and $k + l < m + n$, and consider a class $A \in G^{\text{pre}}_{n,m} \diagup \sim$. If $\theta^n_m \in A$, then A is a balanced singleton class. Otherwise, consider the unique preferred class representative $A_s \cdots A_1$ of maximal length. If $A_s \cdots A_1$ is a bisequence factorization, it is balanced. If not, evaluate Υ-products and obtain a bisequence factorization B_1B_2 of A. Either $B_1 = A_s$ or $B_1 = A_sB'_1$. In either case, there is a bisequence factorization A_sB of A. If B is bisequence indecomposable, A_sB is the unique balanced factorization of A. If not, there is a bisequence factorization C_1C_2 of B. If C_1 is bisequence indecomposable, consider C_2; otherwise, decompose C_1. Continue in this manner until the process terminates. □

We now construct a set that conveniently indexes the module generators $G^{\text{pre}} \diagup \sim$ for $F^{\text{pre}}(\Theta)$. Define a map ϕ that splits the projection $G^{\text{pre}} \to G^{\text{pre}} \diagup \sim$ as follows: For each pair (m, n) with $m + n \leqslant 3$, set $\phi(\operatorname{cls} \theta^n_m) = \theta^n_m$. Inductively, for each pair (m, n) with $m + n \geqslant 4$, assume that ϕ has been defined on $G^{\text{pre}}_{j,i} \diagup \sim$ for $i + j < m + n$. Then given a class $g \in G^{\text{pre}}_{n,m} \diagup \sim$, define $\phi_{n,m}(g)$ to be the balanced factorization of g. Let $\mathcal{B}^{\text{pre}}_{n,m} = \operatorname{Im} \phi_{n,m}$ and $\mathcal{B}^{\text{pre}} = \bigcup \mathcal{B}^{\text{pre}}_{n,m}$.

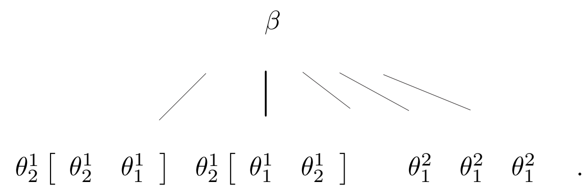
Although a balanced factorization $\beta \in \mathcal{B}^{\text{pre}}$ is not a Θ-factorization, there is a related PLT $\Psi(\beta)$ whose leaves are Θ-factorizations. Let $\beta = B_s \cdots B_1 \in \mathcal{B}^{\text{pre}}$ and set $\Psi_1(\beta) = \beta$. If β is a Θ-factorization, set $\Psi(\beta) = \Psi_1(\beta)$. Otherwise, let (β_k) denote the tuple of entries of B_k listed in row order and replace each entry β_k of (β_k) with its balanced factorization $\beta'_k \in \mathcal{B}^{\text{pre}}$. Form the 2-level tree $\Psi_2(\beta)$ with root β and leaves labeled by the entries of (β'_k) (see Example 4.18). If each β'_k is a Θ-factorization, set $\Psi(\beta) = \Psi_2(\beta)$. Otherwise, repeat this process for each β'_k, i.e., let $\Psi_1(\beta'_k) = \beta'_k$; if β'_k is not a Θ-factorization, construct $\Psi_2(\beta'_k)$. Now construct 3-level tree $\Psi_3(\beta)$ either by appending the tree $\Psi_2(\beta'_k)$ to the leaf β'_k if β'_k is not a $Theta$-factorization, or by extending the branch of β'_k otherwise. If each level 3 entry of $\Psi_3(\beta)$ is a Θ-factorization, set $\Psi(\beta) = \Psi_3(\beta)$; otherwise continue inductively. This process terminates after r steps and uniquely determines an r-level tree $\Psi(\beta)$ whose leaves are balanced Θ-factorizations.

Example 4.18. The balanced factorization

$$\beta=\begin{bmatrix}\theta_2^1\begin{bmatrix}\theta_2^1 & \theta_1^1\end{bmatrix}\\ \\ \theta_2^1\begin{bmatrix}\theta_1^1 & \theta_2^1\end{bmatrix}\end{bmatrix}\begin{bmatrix}\theta_1^2 & \theta_1^2 & \theta_1^2\end{bmatrix}$$

is associated with the 2-level tree

$\Psi(\beta) =$



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Finally, let $\mathcal{C}^{\text{pre}}_{n,m}=\left\{\Psi(\beta)\mid\beta\in\mathcal{B}^{\text{pre}}_{n,m}\right\}$; then $\mathcal{C}^{\text{pre}}=\bigcup_{m,n}\mathcal{C}^{\text{pre}}_{n,m}$ indexes the set of module generators for $F^{\text{pre}}(\Theta)$. In Subsection 6.2 we identify a subset $\mathcal{C}\subset\mathcal{C}^{\text{pre}}$ whose elements simultaneously index module generators of the "free matrad" $F(\Theta)$ and cells of the biassociahedra KK. This identification relates the module structure of $F(\Theta)$ to the combinatorics of permutahedra.

4.4. The Bialgebra Prematrad

As is the case for operads and PROPs, prematrads can be described by generators and relations.

Definition 4.19. Let $\Theta=\langle\theta_1^1=\mathbf{1},\theta_2^1,\theta_1^2\mid\theta_i^j\neq 0\rangle$. For A and $B\in F^{\text{pre}}(\Theta)$, define $A\sim B$ if $\text{bideg}(A)=\text{bideg}(B)$. Let γ be the structure map induced by projection and let $\eta(1)=\mathbf{1}$. Then the **bialgebra permatrad** $(\mathcal{H}^{\text{pre}}:=F^{\text{pre}}(\Theta)\diagup\!\sim,\gamma,\eta)$ satisfies the following axioms:

(i) Associativity: $\gamma(\theta_2^1;\theta_2^1,\mathbf{1})=\gamma(\theta_2^1;\mathbf{1},\theta_2^1)$.

(ii) Coassociativity: $\gamma(\theta_1^2,\mathbf{1};\theta_1^2)=\gamma(\mathbf{1},\theta_1^2;\theta_1^2)$.

(iii) Hopf compatibility: $\gamma(\theta_1^2;\theta_2^1)=\gamma(\theta_2^1,\theta_2^1;\theta_1^2,\theta_1^2)$.

Each bigraded component $\mathcal{H}^{\text{pre}}_{n,m}$ is generated by a singleton $c_{n,m}$; for example, $c_{1,2}=\theta_2^1$, $c_{2,1}=\theta_1^2$, $c_{1,3}=\gamma(\theta_2^1;\theta_2^1,\mathbf{1})$, $c_{3,1}=\gamma(\theta_1^2,\mathbf{1};\theta_1^2)$, $c_{2,2}=\gamma(\theta_1^2;\theta_2^1)$, and so on. Note that $\mathcal{H}^{\text{pre}}_{1,*}$ and $\mathcal{H}^{\text{pre}}_{*,1}$ are operads; the first is generated by θ_2^1 subject to (i) while the second is generated by θ_1^2 subject to (ii). Both are isomorphic to the associativity operad $\underline{\mathcal{A}ss}$ [**6**]. And furthermore, for the bialgebra PROP $\mathcal{B}$ we have $\mathcal{B}^{\text{pre}}=\mathcal{H}^{\text{pre}}$ and $U(\mathcal{H}^{\text{pre}})=\mathcal{B}$.

Given a graded R-module H, a map of prematrads $\mathcal{H}^{\text{pre}}\to U_H$ defines a bialgebra structure on H and vise versa. Since each path of arrows from $(m,1)$ to $(1,n)$ in $\mathbb{N}^2$ represents some Υ-factorization of $c_{n,m}$ (see Figure 5), we think of all such paths as equal. Therefore $\mathcal{H}^{\text{pre}}$ is the smallest module among existing general constructions that describe bialgebras (cf. [**5**], [**12**]). Although the symmetric groups do not act on prematrads, the permutation $\sigma_{n,m}$ built into the associativity axiom minimizes the modules involved.

4.5. Local Prematrads

Let $M=\{M_{n,m}\}_{m,n\geqslant 1}$ be a bigraded R-module, let $\mathbf{W}$ be a telescoping submodule of TTM, let $\mathcal{W}$ be its telescopic extension, and let $\gamma_{\mathbf{W}}=\left\{\gamma_{\mathbf{x}}^{\mathbf{y}}:\mathbf{W}_p^{\mathbf{y}}\otimes\mathbf{W}_{\mathbf{x}}^{q}\to\mathbf{W}_{|\mathbf{x}|}^{|\mathbf{y}|}\right\}$

be a structure map. If $A \otimes B$ is a BTP in $\mathcal{W} \otimes \mathcal{W}$, each TP $A' \otimes B'$ in $A \otimes B$ lies in $\mathbf{W}_p^{\mathbf{y}} \otimes \mathbf{W}_{\mathbf{x}}^q$ for some $\mathbf{x}, \mathbf{y}, p, q$. Consequently, $\gamma_{\mathbf{W}}$ extends to a global product $\Upsilon \colon \mathcal{W} \otimes \mathcal{W} \to \mathcal{W}$ as in (8). In fact, $\mathcal{W}$ is the smallest matrix submodule containing $\mathbf{W}$ on which Υ is well-defined.

Definition 4.20. Let $\mathbf{W}$ be a telescoping submodule of TTM, let

$$\gamma_{\mathbf{W}} = \left\{ \gamma_{\mathbf{x}}^{\mathbf{y}} \colon \mathbf{W}_p^{\mathbf{y}} \otimes \mathbf{W}_{\mathbf{x}}^q \to \mathbf{W}_{|\mathbf{x}|}^{|\mathbf{y}|} \right\}$$

be a structure map, and let $\eta \colon R \to \mathbf{M}$. The triple $(M, \gamma_{\mathbf{W}}, \eta)$ is a **local prematrad (with domain W)** if the following axioms are satisfied:

(i) $\mathbf{W}_{\mathbf{x}}^1 = \mathbf{M}_{\mathbf{x}}^1$ and $\mathbf{W}_1^{\mathbf{y}} = \mathbf{M}_1^{\mathbf{y}}$ for all $\mathbf{x}, \mathbf{y}$;

(ii) Υ is associative on $\mathcal{W} \cap \mathbf{M}$;

(iii) the prematrad unit axiom holds for $\gamma_{\mathbf{W}}$.

Example 4.21. Let $\mathbf{V}$ be the bisequence vector submodule of TTM. If (M, γ) is a prematrad, then $\gamma = \gamma_{\mathbf{V}}$ and (M, γ) is a local prematrad with domain $\mathbf{V}$. Local subprematrads $(M, \gamma_{\mathbf{W}}) \subset (M, \gamma)$ are obtained by restricting the domain to submodules such as $\mathbf{M}_1^*$, $\mathbf{M}_*^1$, and $\mathbf{M}_*^1 \cup \mathbf{M}_1^*$. Note that $\left(M, \gamma_{\mathbf{M}_1^*}\right)$ and $\left(M, \gamma_{\mathbf{M}_*^1}\right)$ are operads.

5. Diagonal Approximations

A diagonal approximation Δ_X on a cellular complex X determines a "k-subdivision" $X^{(k)}$ of X and a cellular inclusion $\Delta^{(k)} \colon X^{(k)} \hookrightarrow X^{\times k+1}$ whose image is the subcomplex of $X^{\times k+1}$ we shall denote by $\Delta^{(k)}(X)$. In particular, the subcomplex $\Delta^{(k)}(P_n) \subset P_n^{\times k+1}$ determined by the S-U diagonal Δ_P defines the selection rule in Subsection 2.2 and, more generally, in the next section.

Recall that a map $f \colon X \to Y$ of CW-complexes is homotopic to a cellular map $g \colon X \to Y$, which in turn induces a chain map $g \colon C_*(X) \to C_*(Y)$. Given a geometric diagonal $\Delta \colon X \to X \times X$, a cellular map $\Delta_X \colon X \to X \times X$ homotopic to Δ is called a *diagonal approximation.* A diagonal approximation Δ_X induces a chain map $\Delta_X \colon C_*(X) \to C_*(X) \otimes C_*(X)$, called a *diagonal.* A brief review of the S-U diagonals Δ_P and Δ_K on cellular chains of permutahedra $P = \sqcup_{n \geqslant 1} P_n$ and associahedra $K = \sqcup_{n \geqslant 2} K_n$ (up to sign) now follows (see [**10**] for details).

5.1. The S-U Diagonals Δ_P and Δ_K

Let $\underline{n} = \{1, 2, \dots, n\}$, $n \geqslant 1$. A matrix E with entries from $\{0\} \cup \underline{n}$ is a *step matrix* if the following conditions hold:

(i) Each element of $\underline{n}$ appears as an entry of E exactly once.

(ii) Elements of $\underline{n}$ in each row and column of E form an increasing contiguous block.

(iii) Each diagonal parallel to the main diagonal of E contains exactly one element of $\underline{n}$.

The non-zero entries in a step matrix form a continuous staircase connecting the lower-left and upper-right most entries. There is a bijective correspondence between step matrices with non-zero entries in $\underline{n}$ and permutations of $\underline{n}$.

Given a $q \times p$ integer matrix $M = (m_{ij})$, choose proper subsets

$$S_i \subset \{\text{non-zero entries in } \operatorname{row}(i)\} \quad \text{and} \quad T_j \subset \{\text{non-zero entries in } \operatorname{col}(j)\},$$

and define *down-shift* and *right-shift* operations D_{S_i} and R_{T_j} on M as follows:

(i) If $S_i \neq \varnothing$, $\max \operatorname{row}(i+1) < \min S_i = m_{ij}$, and $m_{i+1,k} = 0$ for all $k \geqslant j$, then $D_{S_i}M$ is the matrix obtained from M by interchanging each $m_{ik} \in S_i$ with $m_{i+1,k}$; otherwise $D_{S_i}M = M$.

(ii) If $T_j \neq \varnothing$, $\max \operatorname{col}(j+1) < \min T_j = m_{ij}$, and $m_{k,j+1} = 0$ for all $k \geqslant i$, then $R_{T_j}M$ is the matrix obtained from M by interchanging each $m_{k,j} \in T_j$ with $m_{k,j+1}$; otherwise $R_{Tj}M = M$.

Given a $q \times p$ step matrix E together with subsets $S_1, \ldots, S_q$ and $T_1, \ldots, T_p$ as above, there is the *derived matrix*

$$R_{T_p} \cdots R_{T_2} R_{T_1} D_{S_q} \cdots D_{S_2} D_{S_1} E.$$

In particular, step matrices are derived matrices under the trivial action with $S_i = T_j = \varnothing$ for all i, j.

Let $a = A_1|A_2|\cdots|A_p$ and $b = B_q|B_{q-1}|\cdots|B_1$ be partitions of $\underline{n}$. Then $a \times b$ is a (p,q)-*complementary pair* (CP) if there is a $q \times p$ derived matrix $M = (m_{ij})$ such that $A_j = \{m_{ij} \neq 0 \mid 1 \leqslant i \leqslant q\}$ and $B_i = \{m_{ij} \neq 0 \mid 1 \leqslant j \leqslant p\}$. Thus (p,q)-CPs, which are in one-to-one correspondence with derived matrices, identify a particular set of product cells in $P_n \times P_n$.

Definition 5.1. Define $\Delta_P \colon C_0(P_1) \to C_0(P_1) \otimes C_0(P_1)$ by $\Delta_P(\underline{1}) = \underline{1} \otimes \underline{1}$. Inductively, having defined $\Delta_P \colon C_*(P_k) \to C_*(P_k) \otimes C_*(P_k)$ for all $k \leqslant n$, define Δ_P on $\underline{n+1} \in C_n(P_{n+1})$ by

$$\Delta_P(\underline{n+1}) = \sum_{\substack{(p,q)\text{-CPs } a\times b \\ p+q=n+2}} \pm\, a \otimes b$$

and extend Δ_P multiplicatively, i.e., $\Delta_P(u_1|\cdots|u_r) = \Delta_P(u_1)|\cdots|\Delta_P(u_r)$.

The diagonal Δ_p induces a diagonal Δ_K on $C_*(K)$. Recall that faces of P_n in codimension k are indexed by PLTs with $n+1$ leaves and $k+1$ levels, and forgetting levels defines the cellular projection $\vartheta_0 \colon P_n \to K_{n+1}$ given by A. Tonks [**15**]. Thus faces of P_n indexed by PLTs with multiple nodes in the same level degenerate under ϑ_0, and corresponding generators span the kernel of the induced map $\vartheta_0 \colon C_*(P_n) \to C_*(K_{n+1})$. The diagonal Δ_K is given by

$$\Delta_K \vartheta_0 = (\vartheta_0 \otimes \vartheta_0)\Delta_P.$$

5.2. k-Subdivisions and k-Approximations

When X is a polytope one can choose a diagonal approximation $\Delta_X \colon X \to X \times X$ such that

(i) Δ_X acts on each face $e \subset X$ as a (topological) inclusion $\Delta_X(e) \subset e \times e$, and

(ii) there is an induced (cellular)1-*subdivision* $X^{(1)}$ of X that converts Δ_X into a cellular inclusion $\Delta_X^{(1)} \colon X^{(1)} \to X \times X$.

If $e=\bigcup_{i=1}^{m}e_i$ and $e_i\subseteq X^{(1)}$, there are faces $u_i,v_i\subseteq e_i$ such that $\Delta_X^{(1)}(e_i)=u_i\times v_i$. Thus $\Delta_X(e)=\bigcup_{i=1}^{m}\Delta_X^{(1)}(e_i)=\bigcup_{i=1}^{m}u_i\times v_i$, and in particular, $\Delta_X^{(1)}$ agrees with the geometric diagonal Δ only on vertices of X.

The 1-subdivision $X^{(1)}$ arising from an explicit diagonal approximation Δ_X can be thought of as the cell complex obtained by gluing the cells in $\Delta_X(X)$ together along their common boundaries in the only way possible. For example, the A-W diagonal on the simplex Δ^n, the Serre diagonal on the cube I^n, and the S-U diagonals on the permutahedron P_{n+1} and the associahedron K_{n+2} (see [**10**]) induce explicit 1-subdivisions $(\Delta^n)^{(1)}$, $(I^n)^{(1)}$, $P_{n+1}^{(1)}$ and $K_{n+2}^{(1)}$ (see Figures 7–10), and it is a good exercise to determine how the vertices of $(I^n)^{(1)}$ resolve in $P_{n+1}^{(1)}$ and $K_{n+2}^{(1)}$.

Algebraically, the assignment

$$e\mapsto\{u_i\times v_i\}_{1\leqslant i\leqslant m}\tag{12}$$

determines a DG diagonal $\Delta_X\colon C_*(X)\to C_*(X)\otimes C_*(X)$ on cellular chains such that $\Delta_X(C_*(X))\subseteq C_*\left(\Delta_X^{(1)}(X^{(1)})\right)$, where equality holds in the (unique) case $X=*$. Conversely, if a DG diagonal approximation Δ_X on $C_*(X)$ is determined by a cellular decomposition as in (12), there is a corresponding 1-subdivision $X^{(1)}$ and a 1-*approximation* $\Delta_X^{(1)}\colon X^{(1)}\to X\times X$.

Furthermore, there is a 2-*subdivision* $X^{(2)}$ of X (see Figure 11) and a corresponding 2-*approximation* $\tilde\Delta_X^{(2)}\colon X^{(2)}\to X^{(1)}\times X$ that sends each cell of $X^{(2)}$ onto a single cell of $X^{(1)}\times X$; consequently, the composition $\Delta_X^{(2)}=\left(\Delta_X^{(1)}\times\mathbf{1}\right)\tilde\Delta_X^{(2)}$ sends each cell of $X^{(2)}$ onto a single cell of $X^{\times 3}$. Inductively, for each k, there is a k-subdivision $X^{(k)}$ and a k-approximation $\tilde\Delta_X^{(k)}\colon X^{(k)}\to X^{(k-1)}\times X$ such that $\Delta_X^{(k)}=\left(\Delta_X^{(k-1)}\times\mathbf{1}\right)\tilde\Delta_X^{(k)}$ sends each cell of $X^{(k)}$ onto a single cell of $X^{\times k+1}$. Thus, for each $k\geqslant 0$, a diagonal approximation Δ_X *fixes* the subcomplex $\Delta^{(k)}(X):=\Delta_X^{(k)}(X^{(k)})\subset X^{\times k+1}$, which says that Δ_X acts on $\Delta^{(k)}(X)$ as an inclusion.

Figure 7: The 1-subdivision of $P_2=K_3=I$.

The subcomplex $\Delta^{(k)}(P_n)\subset P_n^{\times k+1}$ defines the "configuration module" of a local prematrad in the next section.

6. Matrads

In this section we introduce the objects in the category of matrads; morphisms require a relative theory constructed in the sequel. As motivation, let $\Theta=\langle\theta_m^n\neq 0\mid\theta_1^1=\mathbf{1}\rangle_{m,n\geqslant 1}$, and consider the canonical projection $\rho^{\rm pre}\colon F^{\rm pre}(\Theta)\to\mathcal{H}^{\rm pre}$; then

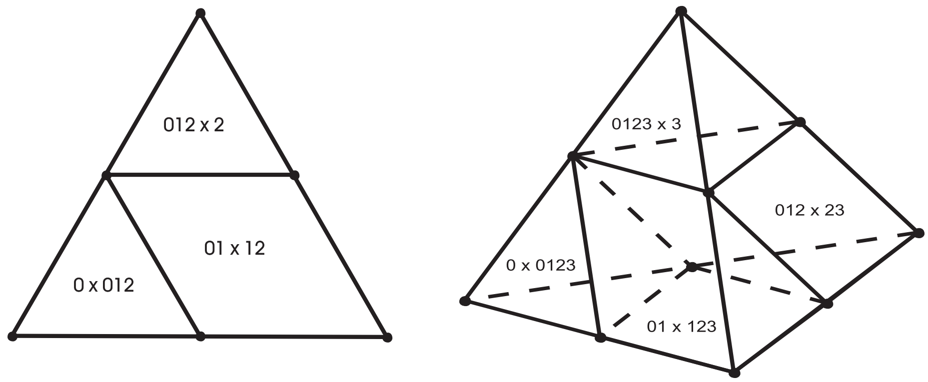


Figure 8: 1-subdivisions of Δ^2 and Δ^3 via the A-W diagonal.

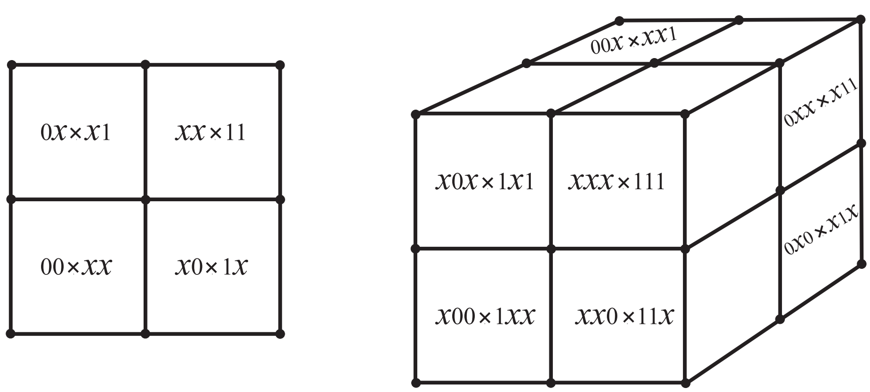


Figure 9: 1-subdivisions of I^2 and I^3 via the Serre diagonal.

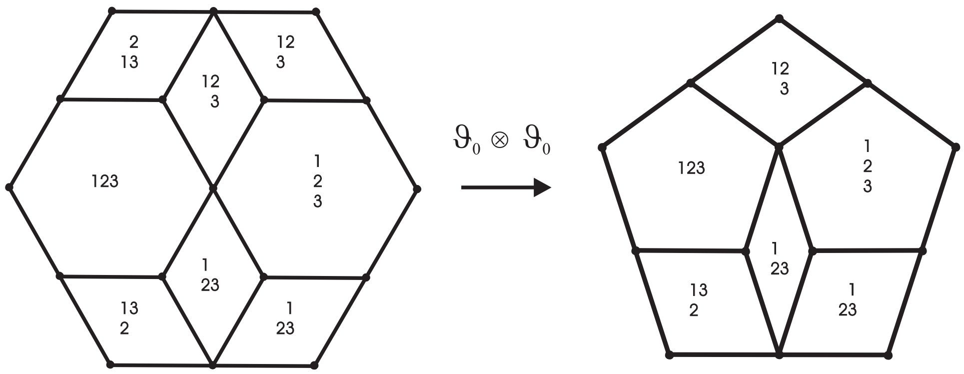


Figure 10: 1-subdivisions of P_3 and K_4 via S-U diagonals Δ_P and Δ_K.

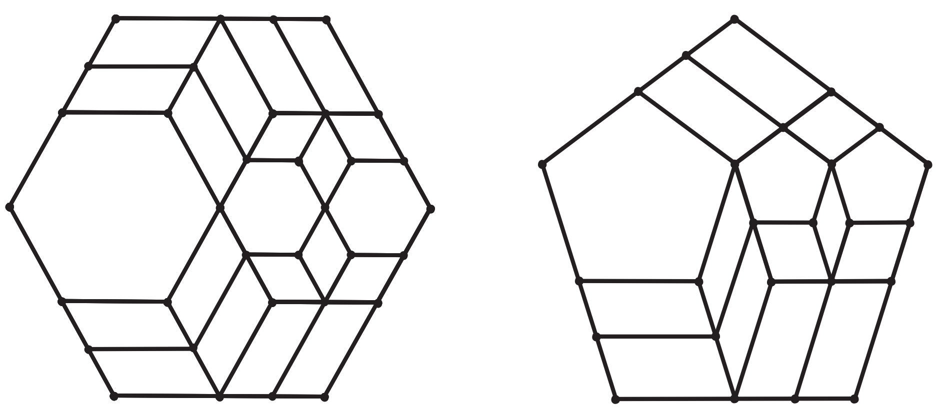


Figure 11: 2-subdivisions of P_3 and K_4.

$$\rho^{\text{pre}}(\theta_m^n) = \begin{cases} \theta_m^n, & m+n \leqslant 3 \\ 0, & \text{otherwise.} \end{cases}$$

Now consider a differential ∂^{pre} on $F^{\text{pre}}(\Theta)$ such that ρ^{pre} is a free resolution in the category of prematrads. Then the induced isomorphism $\varrho_*^{\text{pre}}\colon H_*\left(F^{\text{pre}}(\Theta), \partial^{\text{pre}}\right) \approx \mathcal{H}^{\text{pre}}$ implies

$$\partial^{\text{pre}}(\theta_2^1) = \partial^{\text{pre}}(\theta_1^2) = 0$$

$$\partial^{\text{pre}}(\theta_3^1) = \gamma(\theta_2^1; \mathbf{1}, \theta_2^1) - \gamma(\theta_2^1; \theta_2^1, \mathbf{1})$$

$$\partial^{\text{pre}}(\theta_2^2) = \gamma(\theta_1^2; \theta_2^1) - \gamma(\theta_2^1\theta_2^1; \theta_1^2\theta_1^2)$$

$$\partial^{\text{pre}}(\theta_1^3) = \gamma(\mathbf{1}, \theta_1^2; \theta_1^2) - \gamma(\theta_1^2, \mathbf{1}; \theta_1^2).$$

However, defining ∂^{pre} on all of Θ is quite subtle, and while it is possible to canonically extend ∂^{pre} to all of Θ, acyclicity is difficult to verify. Instead, there is a canonical proper submodule $\mathcal{H}_\infty \subset F^{\text{pre}}(\Theta)$ and a differential ∂ on $\mathcal{H}_\infty$ such that the canonical projection $\varrho\colon \mathcal{H}_\infty \to \mathcal{H}^{\text{pre}}$ is a free resolution in the category of "matrads." Furthermore, we conjecture that the minimal resolution of $\mathcal{H}$ in the category of PROPs (and consequently, in the category of prematrads) is recovered by the universal enveloping functor U discussed in Example 6.8 below, i.e., the minimal resolution of the bialgebra PROP $\mathcal{B}$ is $U(\rho)\colon U(\mathcal{H}_\infty) \to U(\mathcal{H}^{\text{pre}}) = \mathcal{B}$, in which case $\mathcal{H}_\infty$ is the smallest extension of $\mathcal{H}^{\text{pre}}$ in the category of modules.

The precise definition of $\mathcal{H}_\infty$ requires more machinery.

6.1. Matrads Defined

Consider a family of pairs $(\mathbf{W}_\alpha, \gamma_\alpha)$, where $\mathbf{W}_\alpha \subset TTM$ is a telescoping submodule, and the corresponding family of telescopic extensions $(\mathcal{W}_\alpha, \Upsilon_\alpha)$. To each pair $(\mathcal{W}_\alpha, \Upsilon_\alpha)$ the Υ_α-factorizations via Definition 6.1 below determine a unique "configuration module" $\Gamma(\mathbf{W}_\alpha) \subseteq \mathcal{W}_\alpha$ with the following property: If $\mathbf{W}_\alpha \subseteq \mathbf{W}_\beta$ and $\gamma_{\mathbf{W}_\alpha} = \gamma_{\mathbf{W}_\beta}\Big|_{\mathbf{W}_\alpha}$, then $\Gamma(\mathbf{W}_\alpha) \subseteq \Gamma(\mathbf{W}_\beta)$. The local prematrad $\left(M, \gamma_{\mathbf{W}_\alpha}\right)$ is a "matrad" if $\mathbf{W}_\alpha$ is "Γ-stable," i.e., $\mathbf{W}_\alpha$ is the smallest telescoping submodule such that $\Gamma(\mathbf{W}_\alpha) = \Gamma(\mathbf{W}_\beta)$ whenever $\mathbf{W}_\alpha \subset \mathbf{W}_\beta$ and $\gamma_{\mathbf{W}_\alpha} = \gamma_{\mathbf{W}_\beta}\Big|_{\mathbf{W}_\alpha}$.

Matrads are intimately related to the permutahedra. Recall that each codimension $k-1$ face of P_{m-1} is identified with two PLTs–an up-rooted PLT and its down-rooted mirror image–with m leaves and $k \geqslant 2$ levels (see [**4**], [**10**]). Define the $(m,1)$-*row descent sequence* of $\curlywedge_m$ to be $\mathbf{m} = (m)$. Given an up-rooted PLT $T = T^k$ with k levels, express $T^i = T^{i-1} / \curlywedge_{m_{i,1}} \cdots \curlywedge_{m_{i,r_i}}$ for $k \geqslant i > 1$ and $T^1 = \curlywedge_{m_{1,1}}$. Define the i^{th} *leaf sequence* of T to be the row matrix $\mathbf{m}_i = (m_{i,1}, \ldots, m_{i,r_i})$ and the (m,k)-*row descent sequence of* T to be the k-tuple of row matrices $(\mathbf{m}_1, \ldots, \mathbf{m}_k)$. Note that the vertices of P_{m-1} are identified with $(m, m-1)$-row descent sequences $(\mathbf{m}_1, \ldots, \mathbf{m}_{m-1})$, where $\mathbf{m}_i = (1, \ldots, 2, \ldots, 1) \in \mathbb{N}^{r_i}$ with exactly one 2 in position j for some $1 \leqslant j \leqslant r_i$, in which case $\mathbf{m}_1 = (2)$. Dually, define the $(n,1)$-*column descent sequence* of $\curlyvee^n$ to be $\mathbf{n} = (n)$. Given an down-rooted PLT $T = T^l$ with l levels, express $T^i = \curlyvee^{n_{i,1}} \cdots \curlyvee^{n_{i,s_i}} / T^{i-1}$ for $l \geqslant i > 1$ and $T^1 = \curlyvee^{n_{1,1}}$. Define the i^{th} *leaf sequence* of T to be the column matrix $\mathbf{n}_i = (n_{i,1}, \ldots, n_{i,s_i})^T$ and the (n,l)-*column descent sequence of* T to be the l-tuple of column matrices $(\mathbf{n}_l, \ldots, \mathbf{n}_1)$.

Given a telescoping submodule $\mathbf{W}$ and its telescopic extension $\mathcal{W}$, let $\mathcal{W}_{\text{row}} = \mathcal{W} \cap \overline{\mathbf{M}}_{\text{row}}$ and $\mathcal{W}^{\text{col}} = \mathcal{W} \cap \overline{\mathbf{M}}^{\text{col}}$.

Definition 6.1. Given a local prematrad $(M, \gamma_{\mathbf{W}})$ with domain $\mathbf{W}$, let $\zeta \in M_{*,m}$ and $\xi \in M_{n,*}$ be elements with $m, n \geqslant 2$.

(i) A **row factorization of ζ with respect to W** is an Υ-factorization $A_1 \cdots A_k = \zeta$ such that $A_j \in \mathcal{W}_{\text{row}}$ and $\text{rls} A_j \neq \mathbf{1}$ for all j. The sequence $(\text{rls}A_1, \ldots, \text{rls}A_k)$ is the related (m,k)-row **descent sequence of ζ**.

(ii) A **column factorization of ξ with respect to W** is an Υ-factorization $B_l \cdots B_1 = \xi$ such that $B_i \in \mathcal{W}^{\text{col}}$ and $\text{cls}B_i \neq \mathbf{1}$ for all i. The sequence $(\text{cls}B_l, \ldots, \text{cls}B_1)$ is the related (n,l)-**column descent sequence of ξ**.

Column and row factorizations are not unique. Note that an element $A \in M_{n,*}$ always has a trivial column factorization as the 1×1 matrix $[A]$. When matrix entries in a row factorization are pictured as graphs, terms of the row descent sequence are "lower (input) leaf sequences" of the graphs along any row, and dually for column factorizations.

Example 6.2. An Υ-product of bisequence matrices is simultaneously a row and column factorization. For example, consider the Υ-product

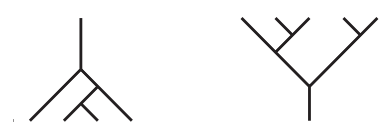
$$C = C_1 C_2 C_3 = \begin{bmatrix} \theta_2^1 \\ \alpha_2^2 \\ \beta_2^2 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \zeta_2^2 \\ \theta_1^1 & \theta_2^1 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \xi_2^2 & \theta_1^2 \end{bmatrix} \in M_{5,4}.$$

As a row factorization, the $(4,3)$-row descent sequence of C is

$$(\text{rls}C_1, \text{rls}C_2, \text{rls}C_3) = ((2), (12), (121)),$$

and as a column factorization, the $(5,3)$-column descent sequence of C is

$$(\text{cls}C_1, \text{cls}C_2, \text{cls}C_3) = \left(\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, (2) \right).$$



((2), (12), (121)) $\quad$ $((122)^T, (21)^T, (2))$

Given a local prematrad $(M, \gamma_{\mathbf{W}})$ and elements $A \in M_{*,s}$ and $B \in M_{t,*}$ with $s, t \geqslant 2$, choose a row factorization $A_1 \cdots A_k$ of A with respect to $\mathbf{W}$ and (s,k)-row descent sequence α, and a column factorization $B_l \cdots B_1$ of B with respect to $\mathbf{W}$ and (t,l)-column descent sequence β. Then α identifies A with an up-rooted s-leaf, k-level PLT and a codimension $k-1$ face $\hat{e}_A$ of P_{s-1}, and β identifies B with a down-rooted t-leaf, l-level PLT and a codimension $l-1$ face $\overset{\vee}{e}_B$ of P_{t-1}. Extending to Cartesian products, identify the monomials $A = A_1 \otimes \cdots \otimes A_q \in (M_{*,s})^{\otimes q}$ and $B = B_1 \otimes \cdots \otimes B_p \in (M_{t,*})^{\otimes p}$ with the product cells

$$\hat{e}_A = \hat{e}_{A_1} \times \cdots \times \hat{e}_{A_q} \subset P_{s-1}^{\times q} \text{ and } \overset{\vee}{e}_B = \check{e}_{B_1} \times \cdots \times \check{e}_{B_p} \subset P_{t-1}^{\times p}.$$

Now consider the S-U diagonal Δ_P and recall from Section 5.2 that there is a k-subdivision $P_r^{(k)}$ of P_r and a cellular inclusion $P_r^{(k)} \hookrightarrow \Delta^{(k)}(P_r) \subset P_r^{\times k+1}$ for each k and r. Thus for each $q \geqslant 2$, the product cell $\hat{e}_A$ either *is* or *is not* a subcomplex of $\Delta^{(q-1)}(P_{s-1}) \subset P_{s-1}^{\times q}$, and dually for $\overset{\vee}{e}_B$. This leads to the notion of "configuration module."

Let $\mathbf{x}_{m,i}^p = (1, \ldots, m, \ldots, 1) \in \mathbb{N}^{1\times p}$ with m in the i^{th} position and let $\mathbf{y}_q^{n,j} = (1, \ldots, n, \ldots, 1)^T \in \mathbb{N}^{q\times 1}$ with n in the j^{th} position.

Definition 6.3. The **(left) configuration module** of a local prematrad $(M, \gamma_{\mathbf{W}})$ is the R-module

$$\Gamma(M, \gamma_{\mathbf{W}}) = M \oplus \bigoplus_{\mathbf{x},\mathbf{y} \notin \mathbb{N};\ s,t \geqslant 1} \Gamma_s^{\mathbf{y}}(M) \oplus \Gamma_{\mathbf{x}}^t(M), \text{ where}$$

$$\Gamma_s^{\mathbf{y}}(M) = \begin{cases} \mathbf{M}_1^{\mathbf{y}}, & s = 1;\ \mathbf{y} = \mathbf{y}_q^{n,j} \text{ for some } n, j, q \\ \left\langle A \in \mathbf{M}_s^{\mathbf{y}} \mid \hat{e}_A \subset \Delta^{(q-1)}(P_{s-1}) \right\rangle, & s \geqslant 2 \\ 0, & \text{otherwise,} \end{cases}$$

$$\Gamma_{\mathbf{x}}^t(M) = \begin{cases} \mathbf{M}_{\mathbf{x}}^1, & t = 1;\ \mathbf{x} = \mathbf{x}_{m,i}^p \text{ for some } m, i, p \\ \left\langle B \in \mathbf{M}_{\mathbf{x}}^t \mid \check{e}_B \subset \Delta^{(p-1)}(P_{t-1}) \right\rangle, & t \geqslant 2 \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\Gamma_{\mathbf{x}}^t(M)$ is generated by those tensor monomials $B = \beta_{x_1}^t \otimes \cdots \otimes \beta_{x_p}^t \in M_{t,x_1} \otimes \cdots \otimes M_{t,x_p}$ whose tensor factor $\beta_{x_i}^t$ is identified with some factor of a product cell in $\Delta^{(p-1)}(P_{t-1})$ corresponding to some column factorization $\beta_{x_i}^t = B_{i,l} \cdots B_{i,1}$ with respect to $\mathbf{W}$, and dually for $\Gamma_s^{\mathbf{y}}(M)$.

Example 6.4. Referring to Example 3.1, let $M = M_{1,1} \oplus M_{2,2} = \langle \theta_1^1 \rangle \oplus \langle \theta_2^2 \rangle$, and consider the local prematrad $(M, \gamma_{\mathbf{V}})$. Note that the action of $\gamma_{\mathbf{V}}$ is trivial modulo unit (e.g., $\mathbf{M}_2^2 \cdot \mathbf{M}_{11}^1 = \mathbf{M}_2^2$ and $\mathbf{M}_2^{22} \cdot \mathbf{M}_{22}^2 \subseteq \mathbf{M}_4^4 = 0$). Then

$$\Gamma_*^1(M) \approx \Gamma_1^*(M) \approx T^+(M_{1,1}).$$

Since $A = (\theta_2^2)^{\otimes q}$ can be thought of as an element of either $\mathbf{M}_{2\cdots 2}^2$ or $\mathbf{M}_2^{2\cdots 2}$, we have either $\hat{e}_A = \Delta^{(q-1)}(P_1) = P_1^{\times q}$ or $\overset{\vee}{e}_A = \Delta^{(q-1)}(P_1) = P_1^{\times q}$ so that

$$\Gamma_*^2(M) \approx \Gamma_2^*(M) \approx T^+(M_{2,2}).$$

Thus

$$\Gamma(M, \gamma_{\mathbf{V}}) = \mathbf{V}.$$

Since $r=1$ is the only case in which the equality $\Delta^{(k)}(P_r)=P_r^{\times k+1}$ holds for each k, it follows immediately that if $A\in M_{n,m}$ is $\gamma_{\mathbf{W}}$-indecomposable (in which case its row and column factorizations with respect to $\mathbf{W}$ are trivial) and $\hat{e}_A\times\cdots\times\hat{e}_A$ $=e^{m-2}\times\cdots\times e^{m-2}$ is a subcomplex of $\Delta^{(n-1)}(P_{m-1})$, then either $m=2$ and n is arbitrary or $m>2$ and $n=1$; dually, if $\overset{\vee}{e}_A\times\cdots\times\overset{\vee}{e}_A=e^{n-2}\times\cdots\times e^{n-2}$ is a subcomplex of $\Delta^{(m-1)}(P_{n-1})$, then either m is arbitrary and $n=2$ or $m=1$ and $n>2$. Consequently,

$$\bigoplus_{\mathbf{x}}\Gamma^2_{\mathbf{x}}(M)\approx T^+(M_{2,*})\quad\text{and}\quad\bigoplus_{\mathbf{y}}\Gamma^{\mathbf{y}}_2(M)\approx T^+(M_{*,2}).$$

Furthermore, if $m+n\geqslant 4$, $(m,n)\neq(2,2)$, and $A^{\otimes r}\in\Gamma(M,\gamma_{\mathbf{W}})$, then $r=1$, and the inclusion $\Gamma^{\mathbf{y}}_s(M)\subseteq\mathbf{M}^{\mathbf{y}}_s$ is proper whenever $s\geqslant 3$, $\mathbf{y}\in\mathbb{N}^{q\times 1}$ with $q\geqslant 2$, and $M_{y_j,s}$ contains a $\gamma_{\mathbf{W}}$-indecomposable element for each $1\leqslant j\leqslant q$, and dually for $\Gamma^t_{\mathbf{x}}(M)\subseteq\mathbf{M}^t_{\mathbf{x}}$.

We are ready to define the notion of a matrad.

Definition 6.5. A local prematrad $(M,\gamma_{\mathbf{W}},\eta)$ is a **(left) matrad** if

$$\Gamma^{\mathbf{y}}_p(M,\gamma_{\mathbf{W}})\otimes\Gamma^q_{\mathbf{x}}(M,\gamma_{\mathbf{W}})=\mathbf{W}^{\mathbf{y}}_p\otimes\mathbf{W}^q_{\mathbf{x}}$$

for all $p,q\geqslant 2$. A **morphism of matrads** is a map of underlying local prematrads.

Example 6.6 (The Bialgebra Matrad $\mathcal{H}$). The bialgebra prematrad $\mathcal{H}^{\text{pre}}$ satisfies $\Gamma^{\mathbf{y}}_p(M)\otimes\Gamma^q_{\mathbf{x}}(M)=\mathbf{M}^{\mathbf{y}}_p\otimes\mathbf{M}^q_{\mathbf{x}}$ for $p,q\geqslant 2$. Hence $\mathcal{H}^{\text{pre}}$ is also a matrad, called the *bialgebra matrad* and henceforth denoted by $\mathcal{H}$.

Example 6.7. Continuing Example 4.21, the inclusions $(M_{1,*},\gamma_{\mathbf{M}^1_*})\subset(M,\gamma_{\mathbf{M}^1_*})$ and $(M_{*,1},\gamma_{\mathbf{M}^*_1})\subset(M,\gamma_{\mathbf{M}^*_1})$ are inclusions of matrads since $\Gamma^{\mathbf{y}}_p(M,\gamma_{\mathbf{M}^*_1})\otimes\Gamma^q_{\mathbf{x}}(M,\gamma_{\mathbf{M}^1_*})=\mathbf{M}^{\mathbf{y}}_p\otimes\mathbf{M}^q_{\mathbf{x}}=0$ for $p,q\geqslant 2$. In particular, when $M=F^{\text{pre}}(\Theta)$ and $\theta^n_m\neq 0$ for all $m,n\geqslant 1$, the free operad $(\mathcal{A}_\infty,\gamma_{\mathbf{M}^1_*})$ embeds in $(F^{\text{pre}}(\Theta),\gamma_{\mathbf{M}^1_*})$ as a submatrad (cf. Example 4.14 and Definition 6.9 below).

Example 6.8 (The Universal Enveloping Functor U). The universal enveloping PROP U discussed in Example 4.12 induces the **universal enveloping functor** U from the category of matrads to the category of PROPs. Given a matrad $(M,\gamma_{\mathbf{W}})$, let $FP(M)$ be the free PROP generated by M and let J be the two-sided ideal generated by the elements $\bigoplus_{\mathbf{x}\times\mathbf{y}}(\gamma_{\mathbf{W}}-\gamma_{_{FP}})\left(\Gamma^{\mathbf{y}}_p(M)\otimes\Gamma^q_{\mathbf{x}}(M)\right)$. Then $U(M)=FP(M)/J$.

6.2. Free Matrads

Recall that the domain of the free prematrad $(M=F^{\text{pre}}(\Theta),\gamma,\,\eta)$ generated by $\Theta=\langle\theta^n_m\rangle_{m,n\geqslant 1}$ is $\mathbf{V}=M\oplus\bigoplus_{\mathbf{x},\mathbf{y}\notin\mathbb{N};\ s,t\in\mathbb{N}}\mathbf{M}^{\mathbf{y}}_s\oplus\mathbf{M}^t_{\mathbf{x}}$ whose submodules M, $\mathbf{M}^1_{\mathbf{x}}$, and $\mathbf{M}^{\mathbf{y}}_1$ are contained in the configuration module $\Gamma(M)$. As above, the symbol "$\cdot$" denotes the γ product.

Definition 6.9. Let $(M=F^{\text{pre}}(\Theta),\gamma,\eta)$ be the free prematrad generated by $\Theta=\langle\theta^n_m\rangle_{m,n\geqslant 1}$, let $F(\Theta)=\Gamma(M)\cdot\Gamma(M)$, and let $\gamma_{_{F(\Theta)}}=\gamma|_{\Gamma(M)\otimes\Gamma(M)}$. The **free matrad generated by** Θ is the triple $\left(F(\Theta),\gamma_{_{F(\Theta)}},\eta\right)$.

Remark 6.10. Let $\omega = \sum \theta_m^n$ and consider the biderivative d_ω. Then $F(\Theta)$ is generated by the components of $d_\omega \circledcirc d_\omega$ in $F^{\mathrm{pre}}(\Theta)$ (the admissible fractions). Thus $\Gamma(F(\Theta), \gamma_{F(\Theta)}) \setminus F(\Theta) = \Gamma(F^{\mathrm{pre}}(\Theta), \gamma_{\mathbf{V}}) \setminus F^{\mathrm{pre}}(\Theta)$ (cf. Proposition 6.14).

Let $\beta \in \mathcal{B}^{\mathrm{pre}}$. In Subsection 4.3 we constructed the r-level tree $\Psi(\beta)$ whose leaves are balanced Θ-factorizations; the set $\mathcal{C}^{\mathrm{pre}} = \{\Psi(\beta) \mid \beta \in \mathcal{B}^{\mathrm{pre}}\}$ indexes the set $G^{\mathrm{pre}}/\!\sim$ of module generators of the free prematrad $F^{\mathrm{pre}}(\Theta)$. Let $\mathfrak{G} = F(\Theta) \cap G^{\mathrm{pre}}/\!\sim$ and let $\mathcal{B} = \phi(\mathfrak{G})$. Then

$$\mathcal{C} = \{\Psi(\beta)) \mid \beta \in \mathcal{B}\}$$

indexes the set $\mathfrak{G}$ of module generators of $F(\Theta)$.

To establish the relationship between elements of $\mathfrak{G}$ and cells of $KK_{n,m}$, let $\beta = A_s \cdots A_1 \in \mathcal{B}^{\mathrm{pre}}_{n,m}$ and observe that $\beta \in \mathcal{B}_{n,m}$ if and only if the tensor monomials along each row and column of A_k lie in $\Gamma(F^{\mathrm{pre}}(\Theta))$ for all k (see (23) below). Let

$$\mathcal{C}''_{n,m} = \{\Psi(\beta) \mid \beta = A_s \cdots A_1 \in \mathcal{B}_{n,m} \text{ and either } A_1 \text{ or } A_s \text{ is a } 1 \times 1\};$$

then in particular, $\mathcal{C}''_{1,m} = \mathcal{C}_{1,m}$ and $\mathcal{C}''_{n,1} = \mathcal{C}_{n,1}$. Let $\mathcal{C}'_{n,m} = \mathcal{C}_{n,m} \setminus \mathcal{C}''_{n,m}$; then $\mathcal{C}_{n,m} = \mathcal{C}'_{n,m} \sqcup \mathcal{C}''_{n,m}$ for each $m, n \geqslant 1$. Elements of $\mathcal{C}'$ are defined in terms of Δ_P; elements of $\mathcal{C}''$ are independent of Δ_P.

Define the dimension of $\theta_1^1 = \mathbf{1}$ to be zero and the dimension of θ_m^n to be $m + n - 3$; if $A \in \mathfrak{G}$, then the dimension of A, denoted by $|A|$, is the sum of the dimensions of the matrix entries in any representative monomial in G, and in particular, in its balanced factorization in $\mathcal{B}$. Clearly, given $\mathbf{x} \times \mathbf{y} \in \mathbb{N}^{1\times p} \times \mathbb{N}^{q \times 1}$, the set $\{|A| \mid A \in \mathbf{G}^{\mathbf{y}}_{\mathbf{x}}\}$ is bounded and, consequently, has a maximal element. For example, if A is a monomial in $\Gamma^{\mathbf{y}}_s(F(\Theta))$ with $s \geqslant 2$ and $\hat{e}_A$ is the corresponding subcomplex of $\Delta^{(q-1)}(P_{s-1})$, then $|A| = \left|\hat{e}_A\right| + |\mathbf{y}| - q$; and dually, if B is a monomial in $\Gamma^t_{\mathbf{x}}(F(\Theta))$ with $t \geqslant 2$ and $\check{e}_B$ is the corresponding subcomplex of $\Delta^{(p-1)}(P_{t-1})$, then $|B| = \left|\check{e}_B\right| + |\mathbf{x}| - p$. Consequently, $\max\{|A| \mid A \in \Gamma^{\mathbf{y}}_s(F(\Theta))\} = |\mathbf{y}| + s - q - 2$ and $\max\{|B| \mid B \in \Gamma^t_{\mathbf{x}}(F(\Theta))\} = |\mathbf{x}| + t - p - 2$. In particular, if $A \in \mathfrak{G}_{n+1,m+1}$ has balanced factorization $\beta = A_s \cdots A_1$, $1 \leqslant s \leqslant m + n$, then $\operatorname{codim} A \geqslant s - 1$ and $\operatorname{codim} A = s - 1$ if and only if the dimensions of each bisequence matrix A_k is maximal (see (28) below).

Given $m + n \geqslant 3$ and $\mathbf{x} \times \mathbf{y} \in \mathbb{N}^{1\times p} \times \mathbb{N}^{q\times 1}$ such that $|\mathbf{x}| = m$ and $|\mathbf{y}| = n$, define the codimension 1 face $e_{(\mathbf{y},\mathbf{x})} \subset P_{m+n-2}$ as follows: If $|\mathbf{x}| = m > p \geqslant 2$, let $A_{\mathbf{x}}|B_{\mathbf{x}}$ be the codimension 1 face of P_{m-1} with leaf sequence $\mathbf{x}$; dually, if $|\mathbf{y}| = n > q \geqslant 2$, let $A_{\mathbf{y}}|B_{\mathbf{y}}$ be the codimension 1 face of P_{n-1} with leaf sequence $\mathbf{y}$. If $A = \{a_1, \ldots, a_r\} \subset \mathbb{Z}$ and $z \in \mathbb{Z}$, define $-A = \{-a_1, \ldots, -a_n\}$ and $A + z = \{a_1 + z, \ldots, a_r + z\}$; then set

$$A_1 = \begin{cases} \underline{m-1}, & \text{if } \mathbf{x} = \mathbf{1}^m,\ m \geqslant 1 \\ \varnothing, & \text{if } \mathbf{x} = m \geqslant 2 \\ -A_{\mathbf{x}} + m, & \text{otherwise,} \end{cases} \qquad A_2 = \begin{cases} \varnothing, & \text{if } \mathbf{y} = \mathbf{1}^n,\ n \geqslant 1 \\ \underline{n-1}, & \text{if } \mathbf{y} = n \geqslant 2 \\ A_{\mathbf{y}}, & \text{otherwise,} \end{cases}$$

$$B_1 = \begin{cases} \varnothing, & \text{if } \mathbf{x} = \mathbf{1}^m,\ m \geqslant 1 \\ \underline{m-1}, & \text{if } \mathbf{x} = m \geqslant 2 \\ -B_{\mathbf{x}} + m, & \text{otherwise,} \end{cases} \qquad B_2 = \begin{cases} \underline{n-1}, & \text{if } \mathbf{y} = \mathbf{1}^n,\ n \geqslant 1 \\ \varnothing, & \text{if } \mathbf{y} = n \geqslant 2 \\ B_{\mathbf{y}}, & \text{otherwise,} \end{cases}$$

and define

$$e_{(\mathbf{y},\mathbf{x})} = A_1 \cup (A_2 + m - 1) | B_1 \cup (B_2 + m - 1). \tag{13}$$

For example, $e_{(\mathbf{1}^n,\mathbf{1}^m)} = \underline{m-1}\,|\,(\underline{n-1} + m - 1)$, $\quad e_{(n,m)} = (\underline{n-1} + m - 1)\,|\,\underline{m-1}$, and $e_{((21),(21))} = 13|24$.

Example 6.11 (The A_∞-bialgebra Matrad $\mathcal{H}_\infty$). Let $\Theta = \left\langle \theta_m^n \neq 0 \mid \theta_1^1 = \mathbf{1} \right\rangle_{m,n \geqslant 1}$. We say that $\beta \in \mathcal{B}_{n,m}$ has **word length** 2 if $\beta = C_2 C_1$ for some $C_2 \times C_1 \in \mathbf{G}_p^{\mathbf{y}} \times \mathbf{G}_{\mathbf{x}}^q$, where $\mathbf{x} \times \mathbf{y} \in \mathbb{N}^{1 \times p} \times \mathbb{N}^{q \times 1}$, $|\mathbf{x}| = m$, and $|\mathbf{y}| = n$. Let

$$\mathcal{A}_p^{\mathbf{y}} \times \mathcal{B}_{\mathbf{x}}^q = \{ \Psi(\beta) \mid \beta \in \mathcal{B}_{n,m} \text{ has word length 2 } \}.$$

Denote the corresponding bases of $\Gamma_{\mathbf{x}}^q(F(\Theta))$ and $\Gamma_p^{\mathbf{y}}(F(\Theta))$ by $\{(B_{\mathbf{x}}^q)_\beta\}_{\beta \in \mathcal{B}_{\mathbf{x}}^q}$ and $\{\left(A_p^{\mathbf{y}}\right)_\alpha\}_{\alpha \in \mathcal{A}_p^{\mathbf{y}}}$, respectively. Then $A_i^j = B_i^j = \theta_i^j$ with $\left|\theta_i^j\right| = i + j - 3$; $B_{\mathbf{x}}^1 = \theta_m^{p;i}$ with $\mathbf{x} = \mathbf{x}_m^{p;i}$ and $|B_{\mathbf{x}}^1| = m - 2$, and $A_1^{\mathbf{y}} = \theta_{q,j}^n$ with $\mathbf{y} = \mathbf{y}_{q,j}^n$ and $|A_1^{\mathbf{y}}| = n - 2$ (cf. Example 4.14). In general, for $p, q \geqslant 2$, $|(B_{\mathbf{x}}^q)_\beta| = |\mathbf{x}| + q - p - 2$ and $|(A_p^{\mathbf{y}})_\alpha| = |\mathbf{y}| + p - q - 2$. Then each $\hat{e}_{A_\alpha}$ is a subcomplex of $\Delta^{(q-1)}(P_{p-1})$ with the associated sign $(-1)^{\epsilon_\alpha}$ and each $\check{e}_{B_\beta}$ is a subcomplex of $\Delta^{(p-1)}(P_{q-1})$ with the associated sign $(-1)^{\epsilon_\beta}$. Define a differential $\partial\colon F(\Theta) \to F(\Theta)$ of degree -1 as follows: Define ∂ on generators by

$$\partial(\theta_m^n) = \sum_{(\alpha,\beta) \in \mathcal{AB}_{m,n}} (-1)^{\epsilon + \epsilon_\alpha + \epsilon_\beta} \gamma \left[(A_p^{\mathbf{y}})_\alpha ; (B_{\mathbf{x}}^q)_\beta \right], \tag{14}$$

where $(-1)^\epsilon$ is the standard sign of $e_{(\mathbf{y},\mathbf{x})} \subset P_{m+n-2}$. Extend ∂ as a derivation of γ; then $\partial^2 = 0$ follows from the associativity of γ. The DG matrad $(F(\Theta), \partial)$, denoted by $\mathcal{H}_\infty$ and called the A_∞-*bialgebra matrad*, is realized by the biassociahedra $\{KK_{n,m} \leftrightarrow \theta_m^n\}$ (see Theorem 9.13 below). One recovers $\mathcal{A}_\infty$ by restricting ∂ to $(\mathcal{H}_\infty)_{1,*}$ or $(\mathcal{H}_\infty)_{*,1}$. Note that $\epsilon = i(m-1)$ in (14) gives the sign of the cell $e_{(\mathbf{y}_{p,i}^m,\mathbf{1})} = e_{(\mathbf{1},\mathbf{x}_m^{p,i})} \subset P_{m+p-2}$ (see [**10**]). This simplifies the standard sign in the differential on $\mathcal{A}_\infty$ [**6**].

Example 6.12. For $\mathbf{x} = (2,1)$ and $\mathbf{y} = (1,1,1)^T$, we have $\mathcal{A}_2^{111} = \{\alpha\}$ and $\mathcal{B}_{21}^3 = \{\beta_1, \beta_2, \beta_3\}$. The corresponding bases are

$$A_\alpha = \begin{bmatrix} \theta_2^1 \\ \theta_2^1 \\ \theta_2^1 \end{bmatrix} \text{ and } B_{\beta_1} = \begin{bmatrix} \theta_2^3 & \begin{bmatrix} \theta_1^1 \\ \theta_1^2 \end{bmatrix} \theta_1^2 \end{bmatrix},$$

$$B_{\beta_2} = \begin{bmatrix} \begin{bmatrix} \theta_2^2 \\ \theta_2^1 \end{bmatrix} [\theta_1^2\, \theta_1^2] & \theta_1^3 \end{bmatrix}, \quad B_{\beta_3} = \begin{bmatrix} \begin{bmatrix} \theta_1^2 \\ \theta_1^1 \end{bmatrix} \theta_2^2 & \theta_1^3 \end{bmatrix}.$$

Thus, $\partial(\theta_3^3) = -\gamma(A_\alpha\, ;\, B_{\beta_1} + B_{\beta_2} + B_{\beta_3}) + \cdots$ (see Example 9.14).

6.3. The Biderivative

In [**11**] we used the canonical prematrad structure γ on the universal PROP $U_A = End(TA)$ to define the biderivative operator. By replacing U_A with an arbitrary prematrad (M, γ) we obtain the general biderivative operator $Bd_\gamma\colon \mathbf{M} \to \mathbf{M}$ having the property $Bd_\gamma \circ Bd_\gamma = Bd_\gamma$. An element $A \in \mathbf{M}$ is a γ-*biderivative* if $A = Bd_\gamma(A)$. Note that $Bd_\gamma(\mathbf{M}) \subseteq (\Gamma(M), \gamma)$; when M is generated by singletons in each bidegree, the image $Bd_\gamma(\mathbf{M})$ is the module of fixed points of Bd_γ and gives rise to an algorithmic construction of an additive basis for the A_∞-bialgebra matrad $\mathcal{H}_\infty$. More precisely:

Proposition 6.13. *Let (M,γ) be a prematrad generated by $\Theta = \{\theta_m^n \mid \theta_1^1 = \mathbf{1}\}_{m,n\geqslant 1}$, and let $Bd_\gamma\colon \mathbf{M}\to\mathbf{M}$ denote the associated biderivative operator. Then*
- *(i) $Bd_\gamma(\mathbf{M}) \subseteq \Gamma(M,\gamma)$ and $Bd_\gamma \circ Bd_\gamma = Bd_\gamma$.*
- *(ii) Each element $\theta\in\Theta$ has a unique γ-biderivative $d_\theta^\gamma \in \mathbf{M}$.*
- *(iii) $\Gamma(M,\gamma) = \langle d_\theta^\gamma \rangle$.*

Thus the γ-biderivative can be viewed as a non-linear map $d_\bullet^\gamma\colon M\to\mathbf{M}$. When $M = End\,(TA)$ we omit the symbol γ and denote the biderivative of θ by d_θ as in [**11**].

In particular, the modules $\langle Bd_\gamma(M_{*,2})\rangle \subseteq T^+(M_{*,2})$ and $\langle Bd_\gamma(M_{2,*})\rangle \subseteq T^+(M_{2,*})$ are spanned by symmetric tensors (cf. Example 6.4); furthermore, $Bd_\gamma(M_{n,m}) = M_{n,m}$ for $m,n>2$.

Finally, the algorithm that produces d_θ^γ for $(M,\gamma) = (F^{\mathrm{pre}}(\Theta),\gamma)$ simultaneously produces an additive basis for $\mathcal{H}_\infty$.

Proposition 6.14. *Let $\Theta = \left\langle \theta_m^n \neq 0 \mid \theta_1^1 = \mathbf{1}\right\rangle_{m,n\geqslant 1}$ as in Example 6.11. Elements of the bases $\{(A_p^{\mathbf{y}})_\alpha\}_{\alpha\in\mathcal{A}_p^{\mathbf{y}}}$ and $\{(B_{\mathbf{x}}^q)_\beta\}_{\beta\in\mathcal{B}_{\mathbf{x}}^q}$ are exactly the components of d_θ^γ in $\Gamma_p^{\mathbf{y}}(F(\Theta),\gamma)$ and $\Gamma_{\mathbf{x}}^q(F(\Theta),\gamma)$ with degrees $|\mathbf{y}|+p-q-2$ and $|\mathbf{x}|+q-p-2$, respectively. Thus*

$$\partial\left(\theta_m^n\right) = \sum_{\substack{|\mathbf{x}|=m;\ |\mathbf{y}|=n \\ A\times B\in (d_\theta^\gamma)_p^{\mathbf{y}}\times(d_\theta^\gamma)_{\mathbf{x}}^q}} \gamma\left(A;B\right).$$

Proof. The proof follows from the definition of d_θ^γ and is straightforward. □

6.3.1. The ⊚-Product
Given a prematrad (M,γ), define a (non-bilinear) operation

$$\circledcirc\colon M\times M \overset{d_\bullet^\gamma\times d_\bullet^\gamma}{\longrightarrow} \mathbf{M}\times\mathbf{M} \overset{\Upsilon}{\longrightarrow} \mathbf{M} \overset{proj}{\to} M, \tag{15}$$

where $proj$ is the canonical projection. The following facts are now obvious:

Proposition 6.15. *The ⊚ operation acts bilinearly on $M_{*,1}$ and $M_{1,*}$. In fact, when $M = End\,(TH)$, the ⊚ operation coincides with Gerstenhaber's $\circ_i$-operation on $M_{1.*}$ (see [2]) and dually on $M_{*,1}$.*

Remark 6.16. The bilinear part of the ⊚ operation, i.e., its restriction to either $M_{*,1}$ or $M_{1,*}$, is completely determined by the associahedra $K=\sqcup K_n$ (rather than permutahedra) and induces the cellular projection $\vartheta_0\colon P_n\to K_{n+1}$ due to A. Tonks [**15**].

7. The Posets $\mathcal{PP}$ and $\mathcal{KK}$

In this section we construct a poset $\mathcal{PP}$ and an appropriate quotient poset $\mathcal{KK}$. The elements of $\mathcal{KK}$ correspond with the 0-dimensional module generators of the free matrad $\mathcal{H}_\infty$. The geometric realization of $\mathcal{KK}$, constructed in Section 8, is the disjoint union of biassociahedra $KK = \{KK_{n,m}\}_{m,n\geqslant 1}$ whose cellular chains are identified with $\mathcal{H}_\infty$.

Let $\mathcal{V}_n$ denote the set of vertices of P_n and identify $\mathcal{V}_n$ with the set S_n of permutations of $\underline{n} = \{1, 2, \ldots, n\}$ via the standard bijection $\mathcal{V}_n \leftrightarrow S_n$. The Bruhat partial ordering on S_n generated by the relation $a_1|\cdots|a_n < a_1|\cdots|a_{i+1}|a_i|\cdots|a_n$ if and only if $a_i < a_{i+1}$ imposes a poset structure on $\mathcal{V}_n$. For $n \geqslant 1$, set $\mathcal{PP}_{n,0} = \mathcal{PP}_{0,n} = \mathcal{V}_n$ and define the *geometric realization* $PP_{n,0} = |\mathcal{PP}_{n,0}| = |\mathcal{PP}_{0,n}| = PP_{0,n}$ to be the permutahedron P_n. Then $KK_{n+1,1} = |\mathcal{KK}_{n+1,1}| = |\mathcal{KK}_{1,n+1}| = KK_{1,n+1}$ is the Stasheff associahedron K_{n+1} (see [**13**], [**14**], [**10**]). In the discussion that follows, we construct the posets $\mathcal{PP}_{n,m}$ and $\mathcal{KK}_{n+1,m+1}$ and their geometric realizations $PP_{n,m}$ and $KK_{n+1,m+1}$ for all $m, n \geqslant 1$.

Denote the sets of up-rooted and down-rooted binary trees with $n+1$ leaves and n levels by $\wedge_n$ and $\vee_n$, respectively; then each vertex of P_n is indexed by two binary PLTs, one the reflection of the other. These indexing sets have a poset structure induced by the standard bijections $\hat{\ell}\colon \wedge_n \to \mathcal{V}_n$ and $\check{\ell}\colon \vee_n \to \mathcal{V}_n$, and the products $\wedge_m^{\times n}$, $\vee_n^{\times m}$, and $\wedge_m^{\times n} \times {\vee_n}^{\times m}$ are posets with respect to lexicographic ordering. Now consider the subcomplexes $\Delta^{(n)}(P_m) \subseteq P_m^{\times n+1}$ and $\Delta^{(m)}(P_n) \subseteq P_n^{\times m+1}$ with faces of P_m and P_n indexed by up-rooted and down-rooted PLTs, respectively. Then the 0-skeletons $X_m^{n+1} \subseteq \Delta^{(n)}(P_m)$ and $Y_n^{m+1} \subseteq \Delta^{(m)}(P_n)$ are subposets of $\wedge_m^{\times n+1}$ and $\vee_n^{\times m+1}$ and there is the inclusion of posets

$$X_m^{n+1} \times Y_n^{m+1} \hookrightarrow \wedge_m^{\times n+1} \times \vee_n^{\times m+1}.$$

Express $x \in \wedge_m^{\times n}$ as an $n \times 1$ column matrix of up-rooted binary trees and replace x with its (unique) BTP Υ-factorization $x_1 \cdots x_m \in \overline{\mathbf{M}}$, where x_i is an $n \times i$ matrix over $\{\mathbf{1}, \curlywedge\}$ with $\curlywedge$ appearing in each row exactly once. Dually, express $y \in \vee_n^{\times m}$ as an $1 \times m$ row matrix of down-rooted binary trees and replace y with its (unique) BTP factorization as a Υ-product $y_n \cdots y_1 \in \overline{\mathbf{M}}$, where y_j is an $j \times m$ matrix over $\{\mathbf{1}, \curlyvee\}$ with $\curlyvee$ appearing in each column exactly once.

Example 7.1. Whereas the product $\Delta^{(1)}(P_1) = P_1^{\times 2}$ can be thought of as either $\curlywedge \times \curlywedge$ or $\curlyvee \times \curlyvee$, we have

$$X_1^2 = \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \quad \text{and} \quad Y_1^2 = [\curlyvee\curlyvee] \quad \text{so that } X_1^2 \times Y_1^2 = \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} [\curlyvee\curlyvee].$$

The poset of vertices in $\Delta^{(1)}(P_2) \subset P_2^{\times 2}$ expresses the following products of permutations and matrix sequences:

$$\begin{array}{ccccccc} a|b \times c|d & : & 1|2 \times 1|2 & < & 1|2 \times 2|1 & < & 2|1 \times 2|1 \\ X_2^2 & : & \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix}\begin{bmatrix} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{bmatrix} & < & \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix}\begin{bmatrix} \curlywedge & \mathbf{1} \\ \mathbf{1} & \curlywedge \end{bmatrix} & < & \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix}\begin{bmatrix} \mathbf{1} & \curlywedge \\ \mathbf{1} & \curlywedge \end{bmatrix} \\ Y_2^2 & : & \begin{bmatrix} \curlyvee & \curlyvee \\ \mathbf{1} & \mathbf{1} \end{bmatrix}[\curlyvee\curlyvee] & < & \begin{bmatrix} \curlyvee & \mathbf{1} \\ \mathbf{1} & \curlyvee \end{bmatrix}[\curlyvee\curlyvee] & < & \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \curlyvee & \curlyvee \end{bmatrix}[\curlyvee\curlyvee] \end{array}$$

Furthermore, thinking of the product $\Delta^{(2)}(P_1) = P_1^{\times 3}$ as $\curlyvee \times \curlyvee \times \curlyvee$, we have

$Y_1^3 = [\curlyvee \curlyvee \curlyvee]$; consequently $X_2^2 \times Y_1^3 =$

$$\left\{ \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{bmatrix} [\curlyvee \curlyvee \curlyvee] < \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \mathbf{1} & \curlywedge \end{bmatrix} [\curlyvee \curlyvee \curlyvee] < \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \mathbf{1} & \curlywedge \\ \mathbf{1} & \curlywedge \end{bmatrix} [\curlyvee \curlyvee \curlyvee] \right\}.$$

Definition 7.2. Let $A = [a_{ij}]$ be an $(n+1) \times m$ matrix over $\{\mathbf{1}, \curlywedge\}$, each row of which contains the entry $\curlywedge$ exactly once. Let $B = [b_{ij}]$ be an $n \times (m+1)$ matrix over $\{\mathbf{1}, \curlyvee\}$, each column of which contains the entry $\curlyvee$ exactly once. Then (A, B) is an (i,j)**-edge pair** if

(i) $A \otimes B$ is a BTP,

(ii) $a_{ij} = a_{i+1,j} = \curlywedge$ and $b_{ij} = b_{i,j+1} = \curlyvee$.

For $u = A_1 \cdots A_m B_n \cdots B_1 \in X_m^{n+1} \times Y_n^{m+1}$, the only possible edge pair in u is (A_m, B_n). In $X_2^3 \times Y_2^3$, for example, the respective matrix sequences

$$\begin{bmatrix} \curlywedge \\ \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{bmatrix} \begin{bmatrix} \curlyvee & \curlyvee & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \curlyvee \end{bmatrix} [\curlyvee \curlyvee \curlyvee] \text{ and } \begin{bmatrix} \curlywedge \\ \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{bmatrix} \begin{bmatrix} \curlyvee & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \curlyvee & \curlyvee \end{bmatrix} [\curlyvee \curlyvee \curlyvee]$$

do and *do not* contain an edge pair.

Definition 7.3. Let Q be a poset and let $x_1 \leqslant x_2 \in Q$. The pair (x_1, x_2) is an **edge of** Q if $x \in Q$ and $x_1 \leqslant x \leqslant x_2$ implies $x = x_1$ or $x = x_2$.

Edges of $X_m^{n+1} \times Y_n^{m+1}$ correspond to 1-dimensional elements of $\mathcal{H}_\infty$ generated by $\{\mathbf{1}, \theta_2^1, \theta_1^2, \ \theta_3^1, \theta_1^3\}$; 1-dimensional elements of $\mathcal{H}_\infty$ generated by $\{\mathbf{1}, \theta_2^1, \theta_1^2, \theta_2^2\}$ correspond to edges of a poset $Z_{n,m}$ related to but disjoint from $X_m^{n+1} \times Y_n^{m+1}$, which we now define.

Let A^{i*} and B^{*j} denote the matrices obtained by deleting the i^{th} row of A and the j^{th} column of B.

Definition 7.4. Let $c = C_1 \cdots C_r$ be a string of matrices in which (C_k, C_{k+1}) is an (i,j)-edge pair for some $k \leqslant r-1$, and suppose that some association of $C_1 \cdots (C_k C_{k+1}) \cdots C_r$ defines a sequence of BTPs. The (i,j)**-transposition of** c **in position** k is the string

$$\mathcal{T}_{ij}^k(c) = C_1 \cdots C_{k+1}^{*j} C_k^{i*} \cdots C_r.$$

The symbol $\mathcal{T}_{ij}^k(c)$ implies that the action of $\mathcal{T}_{ij}^k$ on c is defined.

Note that if $\mathcal{T}_{ij}^k$ acts on $u = A_1 \cdots A_m B_n \cdots B_1 \in X_m^{n+1} \times Y_n^{m+1}$, then $k = m$ and the potential edge pairs of consecutive matrices in

$$\mathcal{T}_{ij}^m(u) = A_1 \cdots A_{m-1} B_n^{*j} A_m^{i*} B_{n-1} \cdots B_1$$

are (A_{m-1}, B_n^{*j}) and (A_m^{i*}, B_{n-1}). If (A_{m-1}, B_n^{*j}) is an edge pair and $\mathcal{T}_{kl}^{m-1}$ is defined on $\mathcal{T}_{ij}^m(u)$, then

$$\mathcal{T}_{kl}^{m-1} \mathcal{T}_{ij}^m(u) = A_1 \cdots A_{m-2} B_n^{*j*l} A_{m-1}^{k*} A_m^{i*} B_{n-1} \cdots B_1,$$

and so on. In this manner, iterate $\mathcal{T}$ on each element $u \in X_m^{n+1} \times Y_n^{m+1}$ in all possible ways and obtain

$$Z_{n,m}=\left\{ \mathcal{T}_{i_tj_t}^{k_t}\cdots\mathcal{T}_{i_1j_1}^{k_1}\left(u\right) \;\middle|\; u\in X_m^{n+1}\times Y_n^{m+1},\ t\geqslant 1\right\}.$$

Then

$$\mathcal{PP}_{n,m}=X_m^{n+1}\times Y_n^{m+1}\cup Z_{n,m}.$$

To extend the partial ordering to $Z_{n,m}$, first define $c<\mathcal{T}_{ij}^{k}\left(c\right)$ for $c\in\mathcal{PP}_{n,m}$. To define a generating relation on $Z_{n,m}$, note that each composition $\mathcal{T}_{i_tj_t}^{k_t}\cdots\mathcal{T}_{i_1j_1}^{k_1}$ defined on $u\in X_m^{n+1}\times Y_n^{m+1}$ uniquely determines an (m,n)-shuffle σ, in which case we denote

$$\mathcal{T}_{\sigma}\left(u\right)=\mathcal{T}_{i_tj_t}^{k_t}\cdots\mathcal{T}_{i_1j_1}^{k_1}\left(u\right)$$

and define $\mathcal{T}_{\mathrm{Id}}=\mathrm{Id}$. When $\mathcal{T}_{\sigma}\left(u\right)$ is defined, multiple compositions of (i,j)-transpositions on u may determine the same σ; thus $\mathcal{T}_{\sigma}\left(u\right)$ is a set, in general. For $u_1\leqslant u_2\in X_m^{n+1}\times Y_n^{m+1}$, define $\mathcal{T}_{\sigma}\left(u_1\right)\leqslant\mathcal{T}_{\sigma}\left(u_2\right)$ if (u_1,u_2) is an edge of $X_m^{n+1}\times Y_n^{m+1}$ or u_2 is "σ -compatible" with u_1 in the following sense: Let $a=a_1|\cdots|a_m\in S_m$ and $b=b_1|\cdots|b_n\in S_n$. The action of σ on $(a;b)$ decomposes a and b into subsequences $\mathbf{m}_1,\ldots,\mathbf{m}_k$ and $\mathbf{n}_1,\ldots,\mathbf{n}_l$ in one of the following four ways:

$$\sigma\left(a;b\right)=\left\{\begin{array}{ll}\mathbf{m}_1,\mathbf{n}_1,\mathbf{m}_2,\mathbf{n}_2,\ldots,\mathbf{n}_{k-1},\mathbf{m}_k, & \sigma\left(a_1\right)=a_1,\ \sigma\left(b_n\right)\neq b_n\\ \mathbf{m}_1,\mathbf{n}_1,\mathbf{m}_2,\mathbf{n}_2,\ldots,\mathbf{m}_k,\mathbf{n}_k, & \sigma\left(a_1\right)=a_1,\ \sigma\left(b_n\right)=b_n\\ \mathbf{n}_1,\mathbf{m}_1,\mathbf{n}_2,\mathbf{m}_2,\ldots,\mathbf{n}_k,\mathbf{m}_k, & \sigma\left(a_1\right)\neq a_1,\ \sigma\left(b_n\right)\neq b_n\\ \mathbf{n}_1,\mathbf{m}_1,\mathbf{n}_2,\mathbf{m}_2,\ldots,\mathbf{m}_k,\mathbf{n}_{k+1}, & \sigma\left(a_1\right)\neq a_1,\ \sigma\left(b_n\right)=b_n.\end{array}\right.$$

Define $\mathbf{I}_{\sigma}=\{(\alpha_1,\ldots,\alpha_{n+1})\in S_m^{\times n+1}\mid\alpha_i\in S_{\#\mathbf{m}_1}\times\cdots\times S_{\#\mathbf{m}_k}\subset S_m\}$ for all i and $\mathbf{J}_{\sigma}=\{(\beta_1,\ldots,\beta_{m+1})\in S_n^{\times m+1}\mid\beta_j\in S_{\#\mathbf{n}_1}\times\cdots\times S_{\#\mathbf{n}_l}\subset S_n\}$ for all j. Let $\chi\colon\mathcal{V}_m\to\mathcal{V}_m$ be the involutory bijection defined by

$$\chi(a_1|\cdots|a_m)=(m+1-a_m)\,|\cdots|\,(m+1-a_1)$$

and fix the inclusion of posets

$$X_m^{n+1}\times Y_n^{m+1}\overset{\kappa}{\hookrightarrow}\mathcal{V}_m^{\times n+1}\times\mathcal{V}_n^{\times m+1}\leftrightarrow S_m^{\times n+1}\times S_n^{\times m+1},\tag{16}$$

where $\kappa=\left(\chi\circ\hat{\ell}\right)^{\times n+1}\times\left(\check{\ell}\right)^{\times m+1}$. Then u_2 is *σ-compatible* with u_1 if $u_2=(\alpha\times\beta)(u_1)$ for some $\alpha\times\beta\in\mathbf{I}_{\sigma}\times\mathbf{J}_{\sigma}$.

To view this geometrically, suppose $u_2=A_1'\cdots A_m'B_n'\cdots B_1'$ is σ-compatible with $u_1=A_1\cdots A_mB_n\cdots B_1$ in $X_m^{n+1}\times Y_n^{m+1}$. For each i, let $a_i=i_1|\cdots|i_m$ and $a_i'=i_1'|\cdots|i_m'$ be the permutations of $\underline{m}$ corresponding with the up-rooted trees given by γ-products $A_{i,1}\cdots A_{i,m}$ and $A_{i,1}'\cdots A_{i,m}'$ of i^{th} rows, respectively; dually, for each j, let $b_j=j_1|\cdots|j_n$ and $b_j'=j_1'|\cdots|j_n'$ be the permutations of $\underline{n}$ corresponding with the down-rooted trees given by the γ-products $B_{n,j}\cdots B_{1,j}$ and $B_{n,j}'\cdots B_{1,j}'$ of j^{th} columns, respectively. Then for each (i,j), the σ-partition of $(a_i;b_j)$ determines a product face $\mathbf{m}_1|\cdots|\mathbf{m}_k\times\mathbf{n}_1|\cdots|\mathbf{n}_l\subset P_m\times P_n$ containing the vertices $a_i\times b_j$ and $a_i'\times b_j'$ and an oriented path of edges from $a_i\times b_j$ to $a_i'\times b_j'$.

Remark 7.5.

(i) The map χ used to define the poset structure of $\mathcal{PP}_{n,m}$ is evoked to induce the correct orientation of the quotient poset $\mathcal{KK}_{n+1,m+1}$ (see below), and is

necessary to establish the bijection in Theorem 1 (see also item (iii) below). For geometric realizations of $\mathcal{KK}_{n+1,m+1}$ and $\mathcal{KK}_{m+1,n+1}$ compare Figures 21 and 22.

(ii) Note that if (u_1, u_2) is an edge of $X_m^{n+1} \times Y_n^{m+1}$, the partial ordering in $\mathcal{PP}_{n,m}$ implies that $(\mathcal{T}(u_1), \mathcal{T}(u_2))$ is an edge of $\mathcal{PP}_{n,m}$.

(iii) The transpose map $X_m^{n+1} \times Y_n^{m+1} \to X_n^{m+1} \times Y_m^{n+1}$ given by

$$A_1 \cdots A_m B_n \cdots B_1 \mapsto B_1^T \cdots B_n^T A_m^T \cdots A_1^T$$

induces a canonical order-preserving bijection $\mathcal{PP}_{n,m} \leftrightarrow \mathcal{PP}_{m,n}$.

Example 7.6. Using the notation of Example 7.1, let us determine those elements $u_i \in X_2^2 \times Y_1^3$ that are σ-compatible with u_1. Since all matrices in u_1 have constant columns or rows, $a_i \times b_j = 1|2 \times 1 \subset P_2 \times P_1$ for all i, j. The $(2,1)$-shuffles of $(1,2;3)$ are $\sigma_0 = 1|2|3$, $\sigma_1 = 1|3|2$ and $\sigma_2 = 3|1|2$. The σ_1-partition of $(1|2;1)$ determines the face $1|2 \times 1 \subset P_2 \times P_1$ whose only vertex is u_1. Hence the only element of $X_2^2 \times Y_1^3$ that is σ_1-compatible with u_1 is itself. If $\sigma \in \{\sigma_0, \sigma_2\}$, the σ-partition of $(1|2;1)$ determines the face $12 \times 1 \subset P_2 \times P_1$ with vertices are $1|2 \times 1$ and $2|1 \times 1$. Since all matrices in u_3 have constant columns or rows, $a_i' \times b_j' = 2|1 \times 1$ for all i, j implies that u_3 is σ-compatible with u_1. Furthermore, $a_1' \times b_j' = 1|2 \times 1$ and $a_2' \times b_j' = 2|1 \times 1$ for all j implies that u_2 is also σ-compatible with u_1. Since $u_1 < u_3$ we have $\mathcal{T}_{\sigma_2}(u_1) < \mathcal{T}_{\sigma_2}(u_3)$.

Example 7.7. Since $Z_{1,1} = [\curlyvee]\,[\curlywedge]$ we have

$$\mathcal{PP}_{1,1} = \left\{ \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} [\curlyvee\curlyvee] < [\curlyvee][\curlywedge] \right\}.$$

Using the notation of Example 7.6, the action of $\mathcal{T}$ on

$$u_1 = \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{bmatrix} [\curlyvee\ \curlyvee\ \curlyvee] \text{ and } u_3 = \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \mathbf{1} & \curlywedge \\ \mathbf{1} & \curlywedge \end{bmatrix} [\curlyvee\ \curlyvee\ \curlyvee]$$

produces the following four elements of $Z_{1,2}$:

$$u_1 \overset{\mathcal{T}_{\sigma_1}}{\longmapsto} \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} [\curlyvee\curlyvee]\,[\curlywedge\ \mathbf{1}] \overset{\mathcal{T}_{\sigma_2}}{\longmapsto} [\curlyvee][\curlywedge]\,[\curlywedge\ \mathbf{1}]\,;$$

$$u_3 \overset{\mathcal{T}_{\sigma_1}}{\longmapsto} \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} [\curlyvee\curlyvee]\,[\mathbf{1}\ \curlywedge] \overset{\mathcal{T}_{\sigma_2}}{\longmapsto} [\curlyvee][\curlywedge]\,[\mathbf{1}\ \curlywedge]\,.$$

Thus $\mathcal{PP}_{1,2} = \{u_1 < \mathcal{T}_{\sigma_1}(u_1) < \mathcal{T}_{\sigma_2}(u_1),\ u_2,\ u_3 < \mathcal{T}_{\sigma_1}(u_3) < \mathcal{T}_{\sigma_2}(u_3)\}$. Recall that the action of $\mathcal{T}$ on u_2 is undefined, and as mentioned in Examples 7.1 and 7.6, $u_1 < u_2 < u_3$ and $\mathcal{T}_{\sigma_2}(u_1) < \mathcal{T}_{\sigma_2}(u_3)$ (see Figure 12).

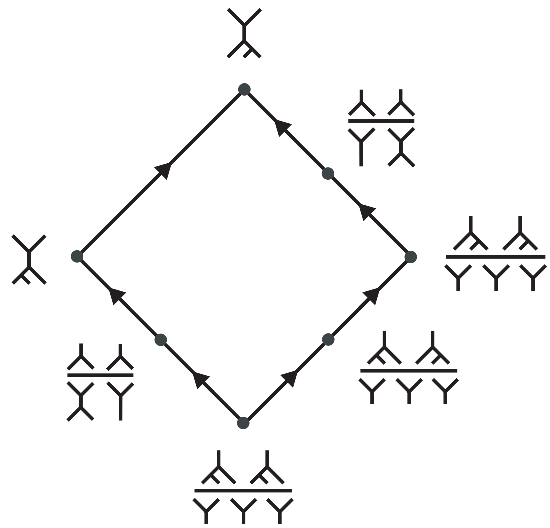


Figure 12: The digraph of $\mathcal{PP}_{1,2}$.

One can represent $u = A_1 \cdots A_m B_n \cdots B_1 \in X_m^{n+1} \times Y_n^{m+1}$ and $z = \mathcal{T}_\sigma(u) \in Z_{n,m}$ as piecewise linear paths of from $(m+1,1)$ to $(1,n+1)$ in the integer lattice $\mathbb{N}^2$ with $m+n$ horizontal and vertical directed components. The arrow $(i+1,n+1) \to (i,n+1)$ represents A_i, while the arrow $(m+1,j) \to (m+1,j+1)$ represents B_j. Consequently, u is represented by the path

$$(m+1,1) \to \cdots \to (m+1,n+1) \to \cdots \to (1,n+1)$$

and z is represented by some other path. In general, if the path $(r+1,s-1) \to (r+1,s) \to (r,s)$ represents the edge pair (A'_k, B'_l) in z, the path $(r+1,s-1) \to (r,s-1) \to (r,s)$ represents its transposition (B''_l, A''_k) in $\mathcal{T}(z)$ (see Figure 13).

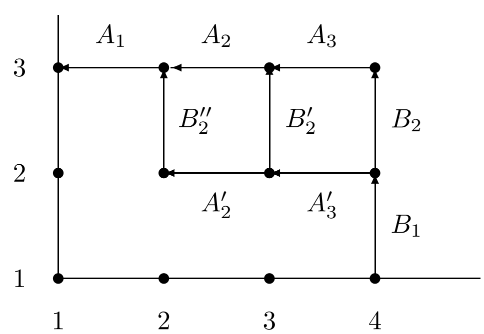


Figure 13: $A_1A_2A_3B_2B_1 < A_1A_2B'_2A'_3B_1 < A_1B''_2A'_2A'_3B_1$.

The poset $\mathcal{KK}$ is a quotient of $\mathcal{PP}$, which we now describe. Recall Tonks' projection $\vartheta_0 \colon P_m \to K_{m+1}$ [**15**]: If a and b are faces of P_m, then $\vartheta_0(a) = \vartheta_0(b)$ if and only if corresponding PLTs are isomorphic as planar rooted trees (forgetting levels). Define

$a \sim b$ if $\vartheta_0(a) = \vartheta_0(b)$. Then $\tilde{\mathcal{V}}_{m+1} = \vartheta_0(\mathcal{V}_m)$ is the set of vertices of K_{m+1}. For example, $3|1|2 = 1|3|2 \in \tilde{\mathcal{V}}_4$, since $3|1|2$ and $1|3|2$ are end points of the degenerate edge $13|2 \subset P_3$, and in terms matrix sequences we have $[\curlywedge][\curlywedge\ \mathbf{1}][\mathbf{1}\ \mathbf{1}\ \curlywedge] = [\curlywedge][\mathbf{1}\ \curlywedge][\curlywedge\ \mathbf{1}\ \mathbf{1}]$ (and dually $[\mathbf{1}\ \mathbf{1}\ \curlyvee]^T[\curlyvee\ \mathbf{1}]^T[\curlyvee] = [\curlyvee\ \mathbf{1}\ \mathbf{1}]^T[\mathbf{1}\ \curlyvee]^T[\curlyvee]$). Of course, $\mathcal{V}_1 = \tilde{\mathcal{V}}_2 = [\curlywedge]$ and $\mathcal{V}_2 = \tilde{\mathcal{V}}_3 = \{[\curlywedge][\curlywedge\ \mathbf{1}], [\curlywedge][\mathbf{1}\ \curlywedge]\}$.

For matrix sequences in X_m^{n+1}, define $x_1' \cdots x_m' \sim_X x_1 \cdots x_m$ if the trees produced by γ-products of i^{th} rows are equivalent for each i. Dually, for matrix sequences in Y_n^{m+1}, define $y_n' \cdots y_1' \sim_Y y_n \cdots y_1$ if the trees produced by γ-products of j^{th} columns are equivalent for each j. Define $a \times b \sim c \times d$ in $X_m^{n+1} \times Y_n^{m+1}$ if $a \sim_X c$ and $b \sim_Y d$. Finally, for $u_1 \leqslant u_2 \in X_m^{n+1} \times Y_n^{m+1}$ and $z_1 = \mathcal{T}_\sigma(u_1) \leqslant z_2 = \mathcal{T}_\sigma(u_2)$, define $z_1 \sim z_2$ if $u_1 \sim u_2$. Then

$$\mathcal{KK}_{n+1,m+1} = \mathcal{PP}_{n,m}/\sim$$

and $\vartheta\vartheta \colon \mathcal{PP}_{n,m} \to \mathcal{KK}_{n+1,m+1}$ denotes the projection.

8. The Combinatorial Join of Permutahedra

The combinatorial join of permutahedra, which resembles the standard join of spaces, plays an important role in our construction of the biassociahedra to follow. The *combinatorial join* $P_m *_c P_n$ of permutahedra P_m and P_n is the permutahedron P_{m+n} constructed as follows: Given faces $A_1|\cdots|A_k \subseteq P_m$ and $B_1|\cdots|B_l \subseteq P_n$, let s be an integer such that $\max\{k,l\} \leqslant s \leqslant k+l$, and let $(\mathbf{i};\mathbf{j}) = (i_1 < \cdots < i_k; j_1 < \cdots < j_l)$, where $\mathbf{i} \cup \mathbf{j} = \underline{s}$. Obtain $A_1'|\cdots|A_s'$ and $B_1'|\cdots|B_s'$ by setting $A_{i_r}' = A_r$, $B_{j_t}' = B_t$, and $A_i' = B_j' = \varnothing$ otherwise. Note that $(A_r', B_r') \neq (\varnothing, \varnothing)$ for all r. Given a set $B = \{b_1, \ldots, b_k\} \subset \mathbb{N}$ and $m \in \mathbb{N}$, define $B + m = \{b_1 + m, \ldots, b_k + m\}$ and consider the codimension $s-1$ face

$$A_1|\cdots|A_k *_{(\mathbf{i};\mathbf{j})} B_1|\cdots|B_l = A_1' \cup (B_1' + m) \mid \cdots \mid A_s' \cup (B_s' + m) \subset P_{m+n}.$$

When $s = m+n$, each pair of vertices $A_1|\cdots|A_m \times B_1|\cdots|B_n \subset P_m \times P_n$ generates $\binom{m+n}{m}$ vertices $A_1|\cdots|A_m *_{(\mathbf{i};\mathbf{j})} B_1|\cdots|B_n$ of P_{m+n} as $(\mathbf{i};\mathbf{j})$ ranges over all (m,n)-shuffles of $(A_1, \ldots, A_m; B_1 + m, \ldots, B_n + m)$. Define

$$P_m *_c P_n = \bigcup_{\substack{A_1|\cdots|A_k \times B_1|\cdots|B_l \subseteq P_m \times P_m \\ \mathbf{i}\cup\mathbf{j}=\underline{s};\ \max\{k,l\}\leqslant s\leqslant k+l}} A_1|\cdots|A_k *_{(\mathbf{i};\mathbf{j})} B_1|\cdots|B_l.$$

Thus, given $m, n \geqslant 1$ and a cell $e \subseteq P_{m+n}$, there is a unique decomposition $e = A_1|\cdots|A_k *_{(\mathbf{i};\mathbf{j})} B_1|\cdots|B_l$ with $A_1|\cdots|A_k \subset P_m$ and $B_1|\cdots|B_l \subset P_n$.

Example 8.1. Setting $s = 2$ produces the 14 codimension 1 faces of $P_2 *_c P_2 = P_4$:

$(\mathbf{i};\mathbf{j})$	A	B	$A *_{(\mathbf{i};\mathbf{j})} B$
$(1,2;1,2)$	1|2	1|2	13|24
	1|2	2|1	14|23
	2|1	1|2	23|14
	2|1	2|1	24|13
$(1;1,2)$	12	1|2	123|4
	12	2|1	124|3
$(1;2)$	12	12	12|34

$(\mathbf{i};\mathbf{j})$	A	B	$A *_{(\mathbf{i};\mathbf{j})} B$
$(2;1,2)$	12	1|2	3|124
	12	2|1	4|123
$(1,2;1)$	1|2	12	134|2
	2|1	12	234|1
$(1,2;2)$	1|2	12	1|234
	2|1	12	2|134
$(2;1)$	12	12	34|12

Fraction products a/b reappear as combinatorial joins $a *_c b$ in Step 2 of the construction that follows in the next section.

9. Constructions of PP and KK

We conclude the paper with constructions of the geometric realizations $PP = |\mathcal{PP}|$ and $KK = |\mathcal{KK}|$. While the edges of PP and KK realize the edges of $\mathcal{PP}$ and $\mathcal{KK}$, it is difficult to imagine their higher dimensional faces. Fortunately, $PP_{n,m}$ is a subdivision of the permutahedron P_{m+n}, which is a subdivision of I^{n+m-1}. Thus the higher dimensional combinatorics of $PP_{n,m}$ are determined by the orientation on the faces of I^{m+n-1}.

Our construction of $PP_{n,m}$ has two steps: (1) Perform an "(m,n)-subdivision" of the codimension 1 cell $\underline{m}\,|\,(\underline{n}+m) \subset P_{m+n}$ and (2) use the (m,n)-subdivision to subdivide certain other cells of P_{m+n}. We emphasize that Δ_P is used only in step (1) and only in terms of its geometrical definition. Thus the non-coassociativity and non-cocommutativity of Δ_P are not in play here (see also Remark 9.4 below). We begin with some preliminaries.

9.1. Matrices with constant rows or columns

Given a set Q of matrix sequences, let

$$\operatorname{con} Q = \{C_1 \cdots C_s \in Q \mid C_k \text{ has constant rows or constant columns}\}.$$

Note that if $A_1 \cdots A_m \in \operatorname{con} X_m^{n+1}$, each A_i has constant columns; dually, if $B_n \cdots B_1 \in \operatorname{con} Y_n^{m+1}$, each B_j has constant rows. Consequently, the inclusion of posets $\kappa\colon X_m^{n+1} \times Y_n^{m+1} \hookrightarrow \mathcal{V}_m^{\times n+1} \times \mathcal{V}_n^{\times m+1}$ given in (16) restricts to an order-preserving bijection

$$\begin{aligned}\left(\chi \circ \hat{\ell}\right) \times \check{\ell}\colon \operatorname{con}\left(X_m^{n+1} \times Y_n^{m+1}\right) &\leftrightarrow \operatorname{con} X_m^{n+1} \times \operatorname{con} Y_n^{m+1} \\ &\leftrightarrow \Delta\left(\mathcal{V}_m^{\times n+1}\right) \times \Delta\left(\mathcal{V}_n^{\times m+1}\right) \leftrightarrow \mathcal{V}_m \times \mathcal{V}_n, \quad (17)\end{aligned}$$

where $\Delta\left(\mathcal{V}_m^{\times n+1}\right) \leftrightarrow \mathcal{V}_m$ is given by the embedding $\mathcal{V}_m \hookrightarrow \mathcal{V}_m^{\times n+1}$ along the diagonal subposet $\Delta\left(\mathcal{V}_m^{\times n+1}\right) = \{(v, \ldots, v) \mid v \in \mathcal{V}_m\}$. Thus elements of $\mathcal{V}_m \times \mathcal{V}_n$ may be represented as matrix strings in $\operatorname{con}\left(X_m^{n+1} \times Y_n^{m+1}\right)$.

Note that (i,j)-transpositions preserve constant rows and columns, i.e., $u \in \operatorname{con}\mathcal{PP}_{n,m}$ if and only if $\mathcal{T}_{ij}^k(u) \in \operatorname{con}\mathcal{PP}_{n.m}$. And furthermore, if $u = A_1 \cdots A_m B_n \cdots B_1 \in \operatorname{con}\left(X_m^{n+1} \times Y_n^{m+1}\right)$ and σ is an (m,n)-shuffle, $\mathcal{T}_\sigma(u)$ is defined since each A_i has a constant column of $\curlywedge$'s and each B_j has a constant row of $\curlyvee$'s. Thus

$$\operatorname{con}\mathcal{PP}_{n,m} = \bigcup_{(m,n)\text{-shuffles } \sigma} \mathcal{T}_\sigma\left(\operatorname{con}\left(X_m^{n+1} \times Y_n^{m+1}\right)\right). \quad (18)$$

The order-preserving bijection

$$\operatorname{con}\mathcal{PP}_{1,2} = \mathcal{PP}_{1,2} \setminus \{u_2\} \leftrightarrow \mathcal{V}_3$$

discussed in Example 7.7 illustrates the following remarkable fact:

Proposition 9.1. *The bijection*

$$\left(\chi \circ \hat{\ell}\right) \times \check{\ell} \colon \operatorname{con}\left(X_m^{n+1} \times Y_n^{m+1}\right) \to \mathcal{V}_m \times \mathcal{V}_n$$

extends to a canonical order-preserving bijection

$$\kappa_\# : \operatorname{con}\mathcal{PP}_{n,m} \to \mathcal{V}_{m+n}.$$

Thus $\left|\kappa_\#\right| : \left|\operatorname{con}\mathcal{PP}_{n,m}\right| \xrightarrow{\approx} P_{m+n}$.

Proof. There is the order-preserving bijection $\left(\chi \circ \hat{\ell}\right) \times \check{\ell} \colon \operatorname{con}\left(X_m^{n+1} \times Y_n^{m+1}\right) \leftrightarrow S_m \times S_n$ via the identification $\mathcal{V}_m \times \mathcal{V}_n \leftrightarrow S_m \times S_n$. Thus

$$\operatorname{con}\mathcal{PP}_{n,m} \leftrightarrow \{\sigma \circ (\sigma_m \times \sigma_n) \mid \sigma \text{ is an } (m,n)\text{-shuffle; } \sigma_m \times \sigma_n \in S_m \times S_n\}$$

by formula (18). But each permutation in S_{m+n} factors as $\sigma \circ (\sigma_m \times \sigma_n)$ for some (m,n)-shuffle σ and some $\sigma_m \times \sigma_n \in S_m \times S_n$. Therefore $\left(\chi \circ \hat{\ell}\right) \times \check{\ell}$ extends to $\kappa_\# : \operatorname{con}\mathcal{PP}_{n,m} \leftrightarrow S_{m+n} \leftrightarrow \mathcal{V}_{m+n}$. □

Corollary 9.2. *For all $m, n \geqslant 1$, there is the commutative diagram*

$$\begin{array}{ccc} \left|\operatorname{con}\left(X_m^{n+1} \times Y_n^{m+1}\right)\right| & \hookrightarrow & \left|\operatorname{con}\mathcal{PP}_{n,m}\right| \\ |\kappa| \downarrow\approx & & \approx\downarrow |\kappa_\#| \\ \underline{m}|\left(\underline{n}+m\right) & \hookrightarrow & P_{m+n}. \end{array}$$

Note that if $\chi_\# : \mathcal{V}_{m+n} \to \mathcal{V}_{m+n}$ were induced by the composition $\mathcal{V}_m \times \mathcal{V}_n \xrightarrow{\chi\times\mathbf{1}} \mathcal{V}_m \times \mathcal{V}_n \hookrightarrow \mathcal{V}_{m+n}$ in the same manner as $\kappa_\#$, then $\chi_\#$ and χ would differ on $\mathcal{V}_{m+n}$ even with $n = 1$.

Example 9.3. Continuing Example 7.7, the identification $\operatorname{con}\mathcal{PP}_{1,1} \leftrightarrow \mathcal{V}_2$ is given by

$$\begin{array}{ccc} \left[\begin{array}{c} \curlywedge \\ \curlywedge \end{array}\right][\curlyvee\,\curlyvee] & \xrightarrow{\mathcal{T}} & [\curlyvee]\,[\curlywedge] \\ \updownarrow & & \updownarrow \\ 1|\left(1+1\right) & \underset{(1,1)\text{-shuffle}}{\longrightarrow} & 2|1. \end{array}$$

The identification $\operatorname{con}\mathcal{PP}_{1,2} \leftrightarrow \mathcal{V}_3$:

$$\left[\begin{array}{c} \curlywedge \\ \curlywedge \end{array}\right]\left[\begin{array}{cc} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{array}\right] \leftrightarrow 1|2 \in S_2 \text{ and } [\curlyvee\ \curlyvee\ \curlyvee] \leftrightarrow 1 \in S_1$$

so that

$$u_1 = \left[\begin{array}{c} \curlywedge \\ \curlywedge \end{array}\right]\left[\begin{array}{cc} \curlywedge & \mathbf{1} \\ \curlywedge & \mathbf{1} \end{array}\right][\curlyvee\ \curlyvee\ \curlyvee] \leftrightarrow 1|2|\left(1+2\right) = 1|2|3 \in S_2 \times S_1.$$

Similarly,

$$u_3 = \left[\begin{array}{c} \curlywedge \\ \curlywedge \end{array}\right]\left[\begin{array}{cc} \mathbf{1} & \curlywedge \\ \mathbf{1} & \curlywedge \end{array}\right][\curlyvee\ \curlyvee\ \curlyvee] \leftrightarrow 2|1|\left(1+2\right) = 2|1|3 \in S_2 \times S_1$$

and we have

$$\begin{array}{ccccc}
\left[\begin{array}{c}\curlywedge\\ \curlywedge\end{array}\right]\left[\begin{array}{cc}\curlywedge & \mathbf{1}\\ \curlywedge & \mathbf{1}\end{array}\right][\curlyvee\ \curlyvee\ \curlyvee] & \xrightarrow{\mathcal{T}_{\sigma_1}} & \left[\begin{array}{c}\curlywedge\\ \curlywedge\end{array}\right][\curlyvee\curlyvee]\,[\curlywedge\ \mathbf{1}] & \xrightarrow{\mathcal{T}_{\sigma_2}} & [\curlyvee][\curlywedge]\,[\curlywedge\ \mathbf{1}]\\
\updownarrow & & \updownarrow & & \updownarrow\\
1|2|3 & & 1|3|2 & & 3|1|2
\end{array}$$

$$\underbrace{1|3|2 \qquad 3|1|2}_{(2,1)\text{-shuffles of } 1|2|3}$$

$$\begin{array}{ccccc}
\left[\begin{array}{c}\curlywedge\\ \curlywedge\end{array}\right]\left[\begin{array}{cc}\mathbf{1} & \curlywedge\\ \mathbf{1} & \curlywedge\end{array}\right][\curlyvee\ \curlyvee\ \curlyvee] & \xrightarrow{\mathcal{T}_{\sigma_1}} & \left[\begin{array}{c}\curlywedge\\ \curlywedge\end{array}\right][\curlyvee\curlyvee]\,[\mathbf{1}\ \curlywedge] & \xrightarrow{\mathcal{T}_{\sigma_2}} & [\curlyvee][\curlywedge]\,[\mathbf{1}\ \curlywedge]\\
\updownarrow & & \updownarrow & & \updownarrow\\
2|1|3 & & 2|3|1 & & 3|2|1
\end{array}$$

$$\underbrace{2|3|1 \qquad 3|2|1}_{(2,1)\text{-shuffles of } 2|1|3.}$$

The projection $\vartheta\colon \operatorname{con}\mathcal{PP}_{n,m} \to \operatorname{con}\mathcal{PP}_{n,m}/\!\sim$ has the following simple geometrical interpretation: An element of $\operatorname{con}\left(X_m^{n+1} \times Y_n^{m+1}\right)$ is represented by a fraction with multiple copies of the *same* leveled binary tree in the numerator and likewise in the denominator. Two such elements are equivalent if and only if the trees in their numerators or denominators (possibly both) are isomorphic as PRTs. So equivalence in $\operatorname{con}\left(X_m^{n+1} \times Y_n^{m+1}\right)$ amounts to forgetting levels as in Tonks' projection. The poset structure then propagates this equivalence to general elements of $\operatorname{con}\mathcal{PP}_{n,m}$.

Remark 9.4. Our constructions are independent of the various choices involved here. If $\tilde{\Delta}_P^{(k)}$ iterates Δ_P on factors other than the those on the extreme left, let $\widetilde{X}_m^{n+1} \times \widetilde{Y}_n^{m+1}$ be the poset defined in terms of $\tilde{\Delta}_P^{(k)}$ and let $\widetilde{\mathcal{PP}}_{n,m}$ be the poset produced by our construction. Then there is a canonical bijection $\mathcal{PP}_{n,m} \leftrightarrow \widetilde{\mathcal{PP}}_{n,m}$ and the corresponding geometric realizations are canonically homeomorphic. When Δ_P acts on the extreme right, for example, a (combinatorial) isomorphism $|\mathcal{PP}_{n,m}| \cong |\widetilde{\mathcal{PP}}_{n,m}|$ is evident pictorially: The picture of $|\mathcal{PP}_{n,m}|$ uses the standard orientation of the interval P_2, while the picture of $|\widetilde{\mathcal{PP}}_{n,m}|$ uses the opposite orientation, but nevertheless, these pictures are identical.

9.2. Step 1: The (m,n)-subdivision of $\underline{m}\,|\,(\underline{n}+m)$

The first step in our construction of $PP_{n,m}$ performs an "(m,n)-subdivision" of the codimension 1 cell $\underline{m}\,|\,(\underline{n}+m) \subset P_{n+m}$. In Subsection 5.2 we applied the left-iterated diagonal $\Delta_P^{(n)}$ to construct the n-subdivision $P_m^{(n)}$ of P_m. Since the poset X_m^{n+1} is the 0-skeleton of $\Delta^{(n)}(P_m)$, the geometric realization $\left|X_m^{n+1}\right| = P_m^{(n)}$, and dually $\left|Y_n^{m+1}\right| = P_n^{(m)}$. The cellular subdivision $\left|X_m^{n+1} \times Y_n^{m+1}\right| = P_m^{(n)} \times P_n^{(m)}$ of $\underline{m}\,|\,(\underline{n}+m) = P_m \times P_n$ is called the (m,n)-*subdivision* of $\underline{m}\,|\,(\underline{n}+m)$; thus each cell

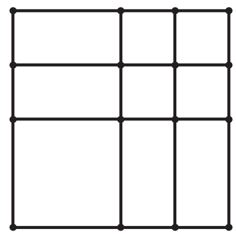


Figure 14: The $(2,2)$-subdivision $P_2^{(2)} \times P_2^{(2)}$.

in this subdivision has a canonical Cartesian product decomposition. The *basic subdivision vertices* of $PP_{n,m}$ are elements of

$$\mathcal{BS}_{n,m} = \left(X_m^{n+1} \times Y_n^{m+1}\right) \setminus \mathcal{V}_{m+n}.$$

Each subdivision cell of $PP_{n,m}$ is a proper subset of some cell of P_{m+n} and is the geometric realization of its poset of vertices.

Example 9.5. The 1-subdivision $P_2^{(1)}$ consists of two 1-cells obtained by subdividing the interval P_2 at its midpoint (see Figure 7). Thus the $(2,1)$-subdivision $P_2^{(1)} \times P_1^{(2)}$ of the edge $12|3 \subset P_3$ contains one basic subdivision vertex represented by the midpoint

$$u_2 = \begin{bmatrix} \curlywedge \\ \curlywedge \end{bmatrix} \begin{bmatrix} \curlywedge & \mathbf{1} \\ \mathbf{1} & \curlywedge \end{bmatrix} [\curlyvee \ \curlyvee \ \curlyvee] \in X_2^2 \times Y_1^3 \setminus \operatorname{con}\left(X_2^2 \times Y_1^3\right),$$

and two 1-cells of $PP_{1,2}$. In fact, $PP_{1,2}$ is exactly the heptagon obtained by subdividing P_3 in this way. The 2-subdivision $P_2^{(2)}$ consists of three 1-cells obtained by subdividing P_2 at its midpoint and again at its three-quarter point. Thus the $(2,2)$-subdivision $P_2^{(2)} \times P_2^{(2)}$ of the square $12|34 \subset P_4$ contains twelve basic subdivision vertices and nine 2-cells of $PP_{2,2}$ as pictured in Figure 14.

The $(3,1)$-subdivision $P_3^{(1)} \times P_1^{(3)}$ of the hexagon $123|4 \subset P_4$ is identified with the 1-subdivision $P_3^{(1)}$ and contains eleven basic subdivision vertices and eight 2-cells of $PP_{3,1}$ as pictured in Figure 15.

The $(3,2)$-subdivision $P_3^{(2)} \times P_2^{(3)}$ of the cylinder $123|45 \subset P_5$, obtained from $P_3^{(2)} \times I$ by subdividing along the horizontal cross-sections $P_3^{(2)} \times \frac{1}{2}$, $P_3^{(2)} \times \frac{3}{4}$, and $P_3^{(2)} \times \frac{7}{8}$, contains 140 basic subdivision vertices and eighty-four 3-cells of $PP_{3,2}$. ($P_3^{(2)}$ is pictured in Figure 11.)

9.3. Step 2: Subdividing cells of $P_{m+n} \setminus \underline{m}|\left(\underline{n}+m\right)$

Recall that elements of $Z_{n,m}$ arise from the non-trivial action of $\mathcal{T}_\sigma$ on $X_m^{n+1} \times Y_n^{m+1}$. When σ ranges over all (m,n)-shuffles (including the identity), we obtain the poset

$$\mathcal{S}_{n,m} = \bigcup_{(m,n)\text{-shuffles } \sigma} \mathcal{T}_\sigma\left(\mathcal{BS}_{n,m}\right)$$

of *subdivision vertices* of $\mathcal{PP}_{n,m}$. Thus as sets, $\mathcal{PP}_{n,m} = \operatorname{con}\mathcal{PP}_{n,m} \sqcup \mathcal{S}_{n,m}$.

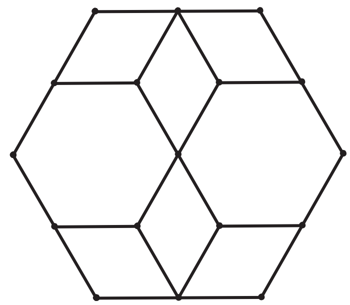


Figure 15: The $(3,1)$-subdivision $P_3^{(1)} \times P_1^{(3)}$.

The second step of our construction detects those cells of $P_{m+n} \setminus \underline{m}|\,(\underline{n}+m)$ that contain subdivision vertices. We refer to such cells as Type I cells; all other cells have Type II. We use the poset structure of subdivision vertices to subdivide Type I cells, and having done so, our construction of $PP_{n,m}$ will be complete.

To begin, let us characterize those Type I cells of minimal dimension that contain *non-basic* subdivision vertices. If e is a cell of some polytope, denote the set of vertices of e by $\mathcal{V}_e$. Given an (m,n)-shuffle σ and a cell $e \subset \underline{m}\,|\,(\underline{n}+m)$, let $\mathcal{T}\,(\sigma,e)$ denote the cell of P_{m+n} of minimal dimension such that $\mathcal{T}_\sigma\,(\mathcal{V}_e) \subseteq \mathcal{V}_{\mathcal{T}(\sigma,e)}$. This defines a map

$$\mathcal{T}\colon\ \{(m,n)\text{-shuffles}\} \times \{\text{partitions of } \underline{m}|\,(\underline{n}+m)\} \to \{\text{partitions of } \underline{m+n}\},$$

which extends the map $(\sigma, \sigma_m \times \sigma_n) \mapsto \sigma \circ (\sigma_m \times \sigma_n)$ in the proof of Proposition 9.1. To define $\mathcal{T}$ at a particular shuffle σ and partition

$$e = A_1|\cdots|A_k|B_1|\cdots|B_l \subseteq \underline{m}|\,(\underline{n}+m)\,,$$

remove all block delimiters of e and think of e as a permutation of $\underline{m+n}$ in which A_i and B_j are contiguous subsequences. Consider the set $\{D_1,\ldots,D_r\}$ of all contiguous subsequences $\sigma(A_i)$ and $\sigma(B_j)$ of $\sigma\,(e)$ that preserve the contiguity of the $B_{j'}$'s and $A_{i'}$'s, respectively, then reinsert block delimiters so that

$$\mathcal{T}(\sigma,e) = C_1|D_{i_1}|\cdots|C_r|D_{i_r}|C_{r+1}.$$

Since each cell of P_{m+n} can be expressed uniquely as a component of the combinatorial join $P_m *_c P_n$, we have

$$\mathcal{T}(\sigma,e) = E *_{(\mathbf{i};\mathbf{j})} F = E_1' \cup (F_1'+m)\,|\,\cdots\,|\,E_s' \cup (F_s'+m), \tag{19}$$

where E_i and F_j are the unions of consecutive blocks $A_{i'}|\cdots|A_{i'+i''}$ and $B_{j'}|\cdots|B_{j'+j''}$ of e, respectively. Thus σ acts on the blocks of e as a (k,l)-shuffle if and only if $C_i = \varnothing$ for all i if and only if $\mathcal{T}(\sigma,e) = A_1|\cdots|A_k *_{(\mathbf{i};\mathbf{j})} B_1|\cdots|B_l$ for some (k,l)-unshuffle $(\mathbf{i};\mathbf{j}) = (i_1<\cdots<i_k\,;\,j_1<\cdots<j_l)$ of $\underline{k+l}$. Clearly, a cell $a \subset P_{n+m}$ contains a non-basic subdivision vertex $\mathcal{T}_\sigma\,(u)$ if and only if $a = \mathcal{T}(\sigma,e)$ for some cell $e \subset \underline{m}\,|\,(\underline{n}+m)$ containing a basic subdivision vertex u on which $\mathcal{T}_\sigma$ acts non-trivially. In fact, a contains at most one non-basic subdivision vertex when $m+n \leqslant 4$.

The following proposition incorporates the property of $\mathcal{T}$ described in Remark 7.5 and will be applied in our subsequent examination of the poset structure of $\mathcal{PP}_{n,m}$.

Proposition 9.6. *If a cell* $e \subset \underline{m} \,|\, (\underline{n}+m)$ *contains a subdivision cell* $a \subset |X_m^{n+1} \times Y_n^{m+1}|$ *and* $\mathcal{T}_\sigma(\mathcal{V}_a) \subset (\mathcal{S}_{n,m} \cap \mathcal{T}(\sigma,e)) \cup \mathcal{V}_{\mathcal{T}(\sigma,e)}$*, then* $|\mathcal{T}_\sigma(\mathcal{V}_a)|$ *is a subdivision cell of* $\mathcal{T}(\sigma,e)$ *(combinatorially) isomorphic to* a*; in particular, if* $a = a_1 \times a_2$*, then*

$$|\mathcal{T}_\sigma(\mathcal{V}_{a_1\times a_2})| = |\mathcal{T}_\sigma(\mathcal{V}_{a_1})| \times |\mathcal{T}_\sigma(\mathcal{V}_{a_2})|.$$

Proof. Since $\mathcal{T}_\sigma(u)$ is defined for all $u \in \mathcal{V}_a$ and $\mathcal{T}_\sigma$ preserves the poset structure of $\mathcal{V}_a$, the cells $a = |\mathcal{V}_a|$ and $|\mathcal{T}_\sigma(\mathcal{V}_a)|$ are combinatorially isomorphic. □

Example 9.7. The action of $\mathcal{T}$ on the four vertices of $12|34$ partitions the 24 vertices of P_4 into four mutually disjoint sets of six vertices each. The vertices $v_1 = 1|2|3|4$ and $v_2 = 1|2|4|3$ of edge $e = 1|2|34$ correspond respectively to

$$\begin{bmatrix}\curlywedge\\ \curlywedge\\ \curlywedge\end{bmatrix}\begin{bmatrix}\curlywedge & \mathbf{1}\\ \curlywedge & \mathbf{1}\\ \curlywedge & \mathbf{1}\end{bmatrix}\begin{bmatrix}\curlyvee & \curlyvee & \curlyvee\\ \mathbf{1} & \mathbf{1} & \mathbf{1}\end{bmatrix}[\curlyvee\ \curlyvee\ \curlyvee] \text{ and } \begin{bmatrix}\curlywedge\\ \curlywedge\\ \curlywedge\end{bmatrix}\begin{bmatrix}\curlywedge & \mathbf{1}\\ \curlywedge & \mathbf{1}\\ \curlywedge & \mathbf{1}\end{bmatrix}\begin{bmatrix}\mathbf{1} & \mathbf{1} & \mathbf{1}\\ \curlyvee & \curlyvee & \curlyvee\end{bmatrix}[\curlyvee\ \curlyvee\ \curlyvee].$$

There are two basic subdivision vertices u_1 and u_2 along e, exactly one of which admits a non-trivial action of $\mathcal{T}$, namely,

$$u_1 = \begin{bmatrix}\curlywedge\\ \curlywedge\\ \curlywedge\end{bmatrix}\begin{bmatrix}\curlywedge & \mathbf{1}\\ \curlywedge & \mathbf{1}\\ \curlywedge & \mathbf{1}\end{bmatrix}\begin{bmatrix}\curlyvee & \curlyvee & \mathbf{1}\\ \mathbf{1} & \mathbf{1} & \curlyvee\end{bmatrix}[\curlyvee\ \curlyvee\ \curlyvee]$$

$$u_1 \overset{\mathcal{T}_{\sigma_1} = \mathcal{T}_{11}^2}{\longrightarrow} \quad z_1 = \begin{bmatrix}\curlywedge\\ \curlywedge\\ \curlywedge\end{bmatrix}\begin{bmatrix}\curlyvee & \mathbf{1}\\ \mathbf{1} & \curlyvee\end{bmatrix}\begin{bmatrix}\curlywedge & \mathbf{1}\\ \curlywedge & \mathbf{1}\end{bmatrix}[\curlyvee\ \curlyvee\ \curlyvee]$$

$$\mathcal{T}_{\sigma_2} \searrow \qquad\qquad \downarrow \mathcal{T}_{11}^3$$

$$z_2 = \begin{bmatrix}\curlywedge\\ \curlywedge\\ \curlywedge\end{bmatrix}\begin{bmatrix}\curlyvee & \mathbf{1}\\ \mathbf{1} & \curlyvee\end{bmatrix}[\curlyvee\curlyvee]\,[\curlywedge\ \mathbf{1}].$$

To physically position z_1 and z_2, first note that $\mathcal{T}_{\sigma_1}(\mathcal{V}_e) = \{\mathcal{T}_{\sigma_1}(v_1) = 1|3|2|4,\ \mathcal{T}_{\sigma_1}(v_2) = 1|4|2|3\}$ and $\mathcal{T}_{\sigma_2}(\mathcal{V}_e) = \{\mathcal{T}_{\sigma_2}(v_1) = 1|3|4|2,\ \mathcal{T}_{\sigma_2}(v_2) = 1|4|3|2\}$. Now $e = A_1|A_2|B_1 = 1|2|34$ and $\sigma_1(e) = 1324$; thus $\sigma_1(B_1)$ is not contiguous in $\sigma_1(e)$ and $\sigma_1(A_2)$ breaks the contiguity of B_1 in $\sigma_1(e)$. Thus $D_1 = \sigma_1(A_1)$ and $\mathcal{T}(\sigma_1, e) = 1|234$. On the other hand, $\sigma_2(e) = 1342$; in this case $\mathcal{T}(\sigma_2, e) = 1|34|2$ since σ_2 acts on the blocks of e as a $(2,1)$-shuffle. Consequently, we represent the vertices z_1 and z_2 as interior points of the faces $1|234$ and $1|34|2$, respectively. To complete the subdivision of $1|234$, use the poset structure to construct new edges from u_1 to z_1 and from z_1 to z_2, and apply Proposition 9.6 to the subdivision cell $a = (v_1, u_1) \subset e$ to construct the edge $|\mathcal{T}_{\sigma_1}(\mathcal{V}_a)|$ from $1|3|2|4$ to z_1. Then $1|234 = d_1 \cup d_2 \cup d_3$ in which $\mathcal{V}_{d_1} = \{u_1, u_2, v_2, \mathcal{T}_{\sigma_1}(v_2), \mathcal{T}_{\sigma_2}(v_2), z_2, z_1\}$, $\mathcal{V}_{d_2} = \{u_1, v_1, \mathcal{T}_{\sigma_1}(v_1), z_1\}$, and $\mathcal{V}_{d_3} = \{\mathcal{T}_{\sigma_1}(v_1), \mathcal{T}_{\sigma_2}(v_1), z_2, z_1\}$ (see Figure 16 and Example 9.8). An algebraic interpretation of these cells appears in the discussion of $KK_{3,3}$ following Theorem 1.

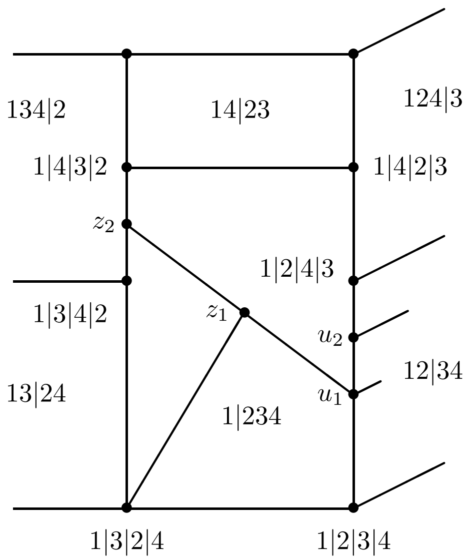


Figure 16: The subdivision of 1|234 in $PP_{3,3}$.

9.4. PP-factorization of proper cells

Recall that an element of $\mathcal{PP}_{n,m}$ is assigned to a unique directed piece-wise linear path from $(m+1,1)$ to $(1,n+1)$ in $\mathbb{N}^2$ with $m+n$ components of unit length (see Figure 13). Let $\Pi_{n,m}$ denote the set of all such paths and consider the map $\pi\colon \mathcal{PP}_{n,m} \to \Pi_{n,m}$. If $u \in \operatorname{con}\left(X_m^{n+1} \times Y_n^{m+1}\right)$, i.e., u is a vertex of $\underline{m}|\,(\underline{n}+m)$, then π restricts to a bijection $\{\mathcal{T}_\sigma(u) \mid (m,n)\text{-shuffles } \sigma\} \leftrightarrow \Pi_{n,m}$, and in view of Proposition 9.1, π assigns each vertex of P_{m+n} to a path in $\Pi_{n,m}$ albeit non-injectively.

Now consider a proper cell $c = C_1|\cdots|C_s \subset |\operatorname{con}\mathcal{PP}_{n,m}| \leftrightarrow P_{m+n}$. Each factor C_t is a permutahedron $P_{m_t+n_t}$ whose vertices are assigned to connected subpaths of paths in $\Pi_{n,m}$. Assign c to a directed piece-wise linear path $\varepsilon_c = \cup\varepsilon_t$ in the following way: Write $c = E_1|\cdots|E_f *_{(\mathbf{i};\mathbf{j})} F_1|\cdots|F_g$ and obtain sequences

$$\gamma = \{m+1 = \gamma_0 > \cdots > \gamma_f = 1\} \quad \text{and} \quad \delta = \{1 = \delta_0 < \cdots < \delta_g = n+1\}, \qquad (20)$$

where $\gamma_{i+1} = \gamma_i - \#E_{f-i}$ and $\delta_{j+1} = \delta_j + \#F_{g-j}$, and assign $C_t = E'_t \cup (F'_t + m)$ to the path

1. $\varepsilon_t\colon (\gamma_{t'-1}, n+1-j) \to (\gamma_{t'}, n+1-j)$, if $C_t = E_{\gamma_{t'}}$ for some t' and maximal j such that $F'_{s_1}, \ldots, F'_{s_j} \neq \varnothing$ and $s_1 < \cdots < s_j < t$;
2. $\varepsilon_t\colon (i, \delta_{t'}) \to (i, \delta_{t'+1})$, if $C_t = F_{\delta_{t'}} + m$ for some t' and maximal i such that $E'_{s_1}, \ldots, E'_{s_i} \neq \varnothing$ and $s_1 < \cdots < s_i < t$;
3. $\varepsilon_t\colon (\gamma_i, \delta_j) \to (\gamma_{i+1}, \delta_{j+1})$, if $C_t = E_{\gamma_i} \cup (F_{\delta_j} + m)$ with $E_{\gamma_i}, F_{\delta_j} \neq \varnothing$ for some i, j.

In particular, a cell $a = A_1 \cdots A_k B_l \cdots B_1 \subset \left|\operatorname{con}\left(X_m^{n+1} \times Y_n^{m+1}\right)\right| \leftrightarrow \underline{m}|\,(\underline{n}+m)$ is assigned to the path

$$\varepsilon_a\colon (m+1,\beta_0) \overset{B_1}{\to} \cdots \overset{B_l}{\to} (m+1,\beta_l) \overset{A_k}{\to} (\alpha_1, n+1) \overset{A_{k-1}}{\to} \cdots \overset{A_1}{\to} (\alpha_k, n+1)\,,$$

where $\alpha = \{m+1 = \alpha_0 > \cdots > \alpha_k = 1\}$ and $\beta = \{1 = \beta_0 < \cdots < \beta_l = n+1\}$ (case (3) does not occur). Thus if $c = \mathcal{T}(\sigma, a)$, the observation in (19) implies that $\gamma \subseteq \alpha$ and $\delta \subseteq \beta$.

Given a subdivision cell $d \subseteq \mathcal{T}(\sigma, a)$, there is a subdivision subcomplex $u \subset a$ such that $d = |\mathcal{T}_\sigma(\mathcal{V}_u)|$. Representing a as a partition $U_1|\cdots|U_s$ of $\underline{m}|\,(\underline{n}+m)$, there is a Cartesian product decomposition $d = D_1 \times \cdots \times D_s$ in which D_t is a subdivision cell of $\mathcal{T}(\sigma, U_t)$. The representation $U_1|\cdots|U_s = E *_{(\mathbf{i};\mathbf{j})} F$ relates the paths associated with the vertices of $\mathcal{T}(\sigma, U_t)$ to the vertices of D_t, and in view of case (3) above, the vertices of D_t are assigned to paths related to those $z \in \mathcal{T}_\sigma(\mathcal{V}_u)$ given by the action of $\mathcal{T}_\sigma$ on the matrix sequences $x_{\gamma_{i+1}} \cdots x_{\gamma_i - 1} y_{\delta_{j+1}-1} \cdots y_{\delta_j}$ associated with the vertices of u as a $(\gamma_i - \gamma_{i+1},\ \delta_{j+1} - \delta_j)$-shuffle. But in every case, there is the Cartesian product decomposition

$$D_t = (e_{y_1^t, x_1^t} \times \cdots \times e_{y_1^t, x_{p_t}^t}) \times \cdots \times (e_{y_{q_t}^t, x_1^t} \times \cdots \times e_{y_{q_t}^t, x_{p_t}^t}),$$

where $e_{y_j^t, x_i^t}$ is some cell of $PP_{y_j^t, x_i^t}$ and $(p_t, q_t) \in \{(\gamma_{t'}, n+1-j), (i, \delta_{t'}), (\gamma_{i+1}, \delta_j)\}$, $p_1 + \cdots + p_t \in \{\gamma_{t'-1}, i, \gamma_i\}$, and $q_1 + \cdots + q_t \in \{n+1-j, \delta_{t'+1}, \delta_{j+1}\}$ (the Cartesian product decomposition of D_t is trivial whenever $m = 1$ or $n = 1$). Therefore every proper cell $e_{n,m} \subset PP_{n,m}$ has a *Cartesian matrix factorization*

$$\begin{aligned} e_{n,m} = [(e_{y_1^1, x_1^1} \times \cdots \times e_{y_1^1, x_{p_k}^1}) \times \cdots \times (e_{y_{q_k}^1, x_1^1} \times \cdots \times e_{y_{q_k}^1, x_{p_k}^1})] \times \cdots \\ \times [(e_{y_1^s, x_1^s} \times \cdots \times e_{y_1^s, x_{p_k}^s}) \times \cdots \times (e_{y_{q_k}^s, x_1^s} \times \cdots \times e_{y_{q_k}^s, x_{p_k}^s})], \end{aligned} \tag{21}$$

where $p_1 = q_s = 1$ and $s \geqslant 2$. The decomposition in (21) is a *PP-factorization* if each factor $e_{y_j^k, x_i^k}$ lies in the family PP.

Indeed, each factor $e_{y_j^k, x_i^k}$ of $e_{n,m}$ has a Cartesian matrix factorization with $x_i^k + y_j^k < m + n$, and we may inductively apply the decomposition in (21) to obtain a decomposition of $e_{n,m}$ as a Cartesian product of polytopes in the family PP. This decomposition involves Cartesian products in two settings: Those within bracketed quantities correspond to tensor products of entries in a bisequence monomial (controlled by $\Delta_P^{(k)}$) and those between bracketed quantities correspond to Υ-products of bisequence monomials. And indeed, this decomposition is encoded by a leveled tree $\Psi(e_{n,m})$ constructed in the same way we constructed $\Psi(\phi(\xi))$ for $\xi \in \mathfrak{G}$. Whereas the levels and the leaves of $\Psi(\phi(\xi))$ are bisequence and Θ-factorizations, the levels and leaves of $\Psi(e_{n,m})$ are Cartesian matrix and PP-factorizations.

Example 9.8. Refer to Example 9.7 and consider the codimension 1 cell $c = C_1|C_2 = 1|234 \subset P_{2+2}$. Write $c = E_1|E_2 * F_1 = 1|2*12$ and obtain $\gamma_0 = 3,\ \gamma_1 = 2,\ \gamma_2 = 1$; and $\delta_0 = 1,\ \delta_1 = 3$. Then $C_1 = E_1' \cup F_1' = E_1 \cup \varnothing$ and $C_2 = E_2' \cup (F_2' + 2) = E_2 \cup (F_1 + 2)$. The path C_1 is assigned to the path component $\varepsilon_1\colon (2,3) \to (1,3)$ and C_2 is assigned to $\varepsilon_2\colon (3,1) \to (2,3)$; in this case there is the action of a $(\gamma_0 - \gamma_1,$ $\delta_1 - \delta_0) = (1,2)$-shuffle on $x_{\gamma_0 - 1} y_{\delta_1 - 1} y_{\delta_0} = x_2 y_2 y_1$, which generates (classes of) vertices of C_2. Let u be the subdivision subcomplex of $1|2|34$ consisting of the two edges (u_1, u_2) and (u_2, v_2) (see Figure 16). Then for $i = 1, 2, 3$, the subdivision cell

$d_i = D_1^i \times D_2^i$, where $D_1^i = C_1 = P_1$ is a vertex and $D_2^i \subset C_2$ has the form $D_2^i = e_{3,2}^i \times e_{3,1}^i$, where $(\dim e_{3,2}^1, \dim e_{3,1}^1) = (2,0)$ and $(\dim e_{3,2}^i, \dim e_{3,1}^i) = (1,1)$ ($e_{3,2}^1 = PP_{2,1}$ is a heptagon and $e_{3,1}^1$ is a vertex of $PP_{2,0}$; $e_{3,2}^i$ is an edge of $PP_{2,1}$ and $e_{3,1}^i = PP_{2,0}$ for $i = 2,3$). Thus up to homeomorphism we have

$$\begin{aligned} d_1 &= [PP_{0,1} \times PP_{0,1} \times PP_{0,1}] \times [PP_{2,1} \times (PP_{1,0} \times PP_{1,0})] \\ d_2 &= [PP_{0,1} \times PP_{0,1} \times PP_{0,1}] \times [(PP_{1,1} \times PP_{1,0}) \times PP_{2,0}] \\ d_3 &= [PP_{0,1} \times PP_{0,1} \times PP_{0,1}] \times [(PP_{1,0} \times PP_{1,1}) \times PP_{2,0}] \,. \end{aligned}$$

9.5. The projection $\vartheta\vartheta \colon PP \to KK$

The final piece of our construction establishes a geometric interpretation of the projection $\vartheta\vartheta \colon PP_{m,n} \to KK_{n+1,m+1}$ induced by the quotient map $\mathcal{PP}_{n,m} \to \mathcal{KK}_{n+1,m+1}$. Let $\mathcal{P}_{n,m} = con\mathcal{PP}_{n,m}$, $P_{n,m} = |\mathcal{P}_{n,m}|$, and $K_{n+1,m+1} = |\mathcal{P}_{n,m}/\sim|$; we obtain $KK_{n+1,m+1}$ as the subdivision of $K_{n+1,m+1}$ that commutes the following diagram:

$$\begin{array}{ccc} PP_{n,m} & \xrightarrow{\approx} & P_{m+n} \\ \vartheta\vartheta \downarrow & & \downarrow \vartheta \\ KK_{n+1,m+1} & \underset{\approx}{\longrightarrow} & K_{n+1,m+1} \end{array}$$

(the horizontal maps are non-cellular homeomorphisms induced by the subdivision process). We identify the cellular chains $C_*(KK)$ with the free matrad $\mathcal{H}_\infty$ and prove that the restriction of the free resolution of prematrads $\rho^{\text{pre}} \colon F^{\text{pre}}(\Theta) \to \mathcal{H}$ to $\mathcal{H}_\infty$ is a free resolution in the category of matrads.

To simplify notation, we suppress the subscripts of $\vartheta_{n,m} \colon P_{n,m} \to K_{n+1,m+1}$ when m and n are clear from context. Since $|\mathcal{P}_{n,m}| = P_{n,m} = P_{m+n}$, a proper face $e \subset P_{n,m}$ is a product of permutahedra

$$e = P_{n_1,m_1} \times \cdots \times P_{n_s,m_s}$$

and projects to a product

$$\tilde{e} = \vartheta(e) = \vartheta(P_{n_1,m_1}) \times \cdots \times \vartheta(P_{n_s,m_s}) = K_{n_1+1,m_1+1} \times \cdots \times K_{n_s+1,m_s+1}.$$

The fact that $\vartheta_{n,m} = Id$ when $1 \leqslant m,n \leqslant 2$ implies $K_{n+1,m+1} = P_{m+n}$; also, $K_{n,2} \cong K_{2,n}$ is the multiplihedron J_n for all n (see [**14**], [**3**], [**10**], [**9**]). The faces $24|13$ and $1|24|3$ of $P_{3,1}$ are degenerate in $K_{4,2}$ since $\vartheta_{3,1}(24|13) = 24|1|3$ and $\vartheta_{3,1}(1|24|3) = 1|2|4|3$; and dually, the faces $24|13$ and $2|13|4$ of $P_{1,3}$ are degenerate in $KK_{2,4}$ since $\vartheta_{1,3}(24|13) = 2|4|13$ and $\vartheta_{1,3}(2|13|4) = 2|1|3|4$. Observe that the product cell

$$K_{m+1} \times K_{n+1} = \vartheta(P_m \times P_n) \subset \vartheta(P_{m+n}) = \vartheta(P_{n,m}) = K_{n+1,m+1}$$

admits the (m,n)-subdivision

$$K_{m+1}^{(n)} \times K_{n+1}^{(m)} = \vartheta\vartheta(P_m^{(n)} \times P_n^{(m)}) \subset \vartheta\vartheta(PP_{n,m}) = KK_{n+1,m+1}.$$

Example 9.9. The $(2,2)$-subdivision $K_3^{(2)} \times K_3^{(2)}$ of the face $\vartheta_{2,2}(12|34) = K_3 \times K_3 \subset K_{3,3}$ produces 9 cells of $KK_{3,3}$ (see Figures 14 and 20); the $(3,1)$-subdivision $K_2^{(3)} \times K_4^{(1)}$ of the face $\vartheta_{1,3}(1|234) = K_2 \times K_4 \subset K_{4,2}$ produces 6 cells of $KK_{4,2}$ (see Figures 3 and 21).

Define

$$\widetilde{\mathcal{BS}}_{n+1,m+1} = (\vartheta_* \times \vartheta_*)\,(\mathcal{BS}_{n,m})\,.$$

Example 9.10. The biassociahedron $KK_{3,3} = PP_{2,2}$ and has 44 vertices, 16 of which lie in $12|34$. Of these 16 vertices, 4 lie in $\mathcal{P}_{2,2}$ and generate the other 19 vertices of $K_{3,3} = P_4$; another 4 lie $\widetilde{\mathcal{BS}}_{3,3}$ and generate the 8 remaining vertices of $KK_{3,3}$ (see Figure 20). By contrast, $KK_{4,2}$ is a non-trivial quotient of $PP_{3,1}$. As in Tonks' projection $\vartheta_0 \colon P_n \to K_{n+1}$, we identify faces of $PP_{3,1}$ indexed by isomorphic graphs (forgetting levels) as pictured in Figure 17. Here an equivalence class of graphs, which labels a face of the target interval, contains the three graphs horizontally to its left.

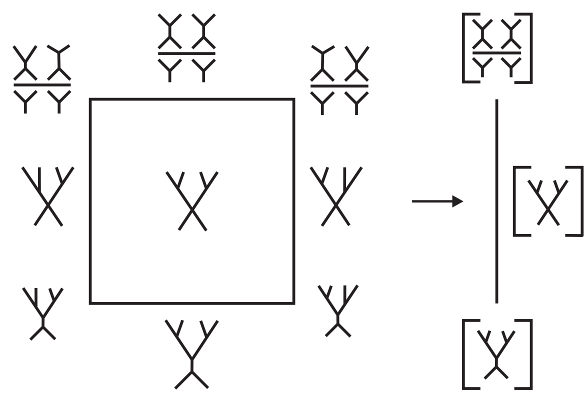


Figure 17: Projection of a degenerate square in $PP_{3,1}$ to $KK_{4,2}$.

The biassociahedra $KK_{1,1}$, $KK_{2,1}$ and $KK_{1,2}$ are isolated vertices and correspond to the free matrad generators $\mathbf{1}$, θ_1^2 and θ_2^1, respectively. The biassociahedra $KK_{n,m}$ with $4 \leqslant m+n \leqslant 6$ are pictured in Figures 18 through 22 below and labelled by partitions and (co)derivation leaf sequences. Note that $KK_{n,m} \cong KK_{m,n}$ for all $m,n \geqslant 1$ and $KK_{2,m}$ is a subdivision of J_m when $m \geqslant 3$.

If $\tilde{e}_{n,m}$ is a proper face of $KK_{n,m}$, the decomposition in (21) induces a product decomposition of the form

$$\begin{aligned}\tilde{e}_{n,m} = {} & [(\tilde{e}_{y_1^1,x_1^1} \times \cdots \times \tilde{e}_{y_1^1,x_{p_k}^1}) \times \cdots \times (\tilde{e}_{y_{q_k}^1,x_1^1} \times \cdots \times \tilde{e}_{y_{q_k}^1,x_{p_k}^1})] \\ & \times \cdots \times [(\tilde{e}_{y_1^s,x_1^s} \times \cdots \times \tilde{e}_{y_1^s,x_{p_k}^s}) \times \cdots \times (\tilde{e}_{y_{q_k}^s,x_1^s} \times \cdots \times \tilde{e}_{y_{q_k}^s,x_{p_k}^s})], \end{aligned} \quad (22)$$

where $p_1 = q_s = 1$ and $s \geqslant 2$. Here "$\times$" within a bracketed quantity corresponds to the tensor product in a bisequence monomial (controlled by certain iterations of Δ_P and the product cell within k^{th} bracket is thought of as a subdivision cell of K_{n_k+1,m_k+1} in the decomposition of $e_{n,m}$ in (21)) and "$\times$" between bracketed quantities corresponds to an Υ-product, and each $\tilde{e}_{y_j^k,x_i^k}$ has the form given by (22) with $x_i^k + y_j^k < m+n$.

We distinguish between two kinds of faces in (22). A Type I face is detected by the diagonal Δ_P and its representation in (22) has $(p_k,q_k) > (1,1)$ for all k; thus Δ_P is only involved in forming the Cartesian products in parentheses. A Type II face $\tilde{e}_{n,m}$

is independent of Δ_P and its representation in (22) satisfies $(p_k, q_k) = (1,1)$ for all k; thus $\tilde{e}_{n,m}$ has the form $KK_{n,i_2} \times K_{i_2-i_1+1} \times \cdots \times K_{i_s-i_{s-1}+1}$, $1 \leqslant i_2 < \cdots < i_s = m$, or $K_{j_0-j_1+1} \times \cdots \times K_{j_{s-2}-_{s-1}+1} \times KK_{j_{s-1},m}$, $1 \leqslant j_{s-1} < \cdots < j_0 = n$. In particular, a codimension 1 face (when $s=2$) has the form $KK_{n,i} \times K_{m+1-i}$ or $K_{n+1-j} \times KK_{j,m}$. Consequently, each cell $\tilde{e}_{n,m} \subset KK_{n,m}$ is associated with a levelled tree $\Psi(\tilde{e}_{n,m})$, whose levels are representations given by (22) and whose leaves are *KK-factorizations.*

The assignment $\iota\colon \theta_p^q \mapsto KK_{q,p}$, which preserves levels, induces a set map

$$\iota\colon \mathfrak{G}_{n,m} \to \{\text{faces of}\, KK_{n,m}\} \tag{23}$$

that sends balanced factorizations to Cartesian matrix factorizations and has the following properties:

(i) The restriction of ι to 0-dimensional module generators of $F_{n,m}(\Theta)$ establishes a bijection with vertices of $KK_{n,m}$ by replacing θ_2^1 with $\curlywedge$ and θ_1^2 with $\curlyvee$ in each entry of $\Psi(\beta)$.

(ii) There is a *location map*

$$\tilde{\iota}\colon \mathfrak{G}_{q+1,p+1} \to \{\text{faces of}\, P_{p+q}\} \tag{24}$$

that commutes the following diagram of set maps

$$\begin{array}{ccc} \mathfrak{G}_{q+1,p+1} & \overset{\tilde{\iota}}{\longrightarrow} & \{\text{ faces of}\, P_{p+q}\} \\ \iota\downarrow & & \downarrow \vartheta_{q,p} \\ \{\text{faces of}\, KK_{q+1,p+1}\} & \overset{\nu}{\longrightarrow} & \{\text{faces of}\, K_{q+1,p+1}\}\,, \end{array}$$

where ν sends a cell of KK to the cell of K of minimal dimension containing it. Indeed, if $\beta = C_s \cdots C_1 \in \mathcal{B}$ is the balanced representative of θ with $C_k \in \mathbf{G}_{\mathbf{x}^k}^{\mathbf{y}^k}$, consider the (co)derivation leaf sequences $((\mathbf{x}^1,\mathbf{y}^1),\ldots,(\mathbf{x}^s,\mathbf{y}^s))$, and let $\mathbf{x}^{i_1},\ldots,\mathbf{x}^{i_k}$ and $\mathbf{y}^{j_1},\ldots,\mathbf{y}^{j_l}$ be the subsequences obtained by removing all $\mathbf{x}^i, \mathbf{y}^j = \mathbf{1}$. Thinking of these subsequences as row and column descent sequences, consider the corresponding faces $A = A_1|\cdots|A_k \subset P_{m-1}$ and $B = B_1|\cdots|B_l \subset P_{n-1}$ and set

$$e_\theta = \chi(A) *_{(\mathbf{i},\mathbf{j})} B \subset P_{m+n-2},$$

where $\chi\colon P_{m-1} \to P_{m-1}$ is the cellular involution defined by $\chi(A_1|\cdots|A_k) = (m-A_k)|\cdots|(m-A_1)$ and $(\mathbf{i};\mathbf{j}) = (i_1 < \cdots < i_k; j_1 < \cdots < j_l)$. Then e_θ is the unique cell of minimal dimension $\geqslant k$ such that $\iota(\theta) \subset \vartheta_{n-1,m-1}(e_\theta)$; in particular, when $s=2$, $\mathbf{x}^1 = \mathbf{x}$ and $\mathbf{y}^2 = \mathbf{y}$, and we recover the special cell $e_\theta = e_{(\mathbf{y},\mathbf{x})}$ defined in (13). Thus, the term "location map" suggests the fact that $\tilde{\iota}$ points out the position of the image cells $\iota(\theta)$ with respect to cells of the permutahedron P_{m+n}. Under $\tilde{\iota}$, the associativity of the Υ-product on $\mathfrak{G}$ is compatible with the associativity of the partitioning procedure in $\underline{p+q}$ by which $a_1|\cdots|a_k$ is obtained from the (ordered) set $a_1\cdots a_k$ by inserting bars: Given $\theta,\zeta \in \mathfrak{G}$, let $\xi \in \mathfrak{G}$ be the component shared by $\partial(\theta)$ and $\partial(\zeta)$ in $\mathcal{H}_\infty$. Then $\vartheta(\tilde{\iota}(\xi)) \subset \vartheta(\tilde{\iota}(\theta)) \cap \vartheta(\tilde{\iota}(\zeta))$ in $K_{q+1,p+1} = \vartheta(P_{p+q})$ (see Example 9.12).

(iii) Let $\dim\theta = k$ and let σ_θ be the set of all 0-dimensional elements of $\mathcal{H}_\infty$ obtained by all possible compositions $\partial_{i_k}\cdots\partial_{i_1}(\theta)$ where ∂_i is a component of $\partial = \sum_i \partial_i$. Then $\iota(\theta)$ is the k-face of $KK_{n,m}$ spanned on the set $\iota(\sigma_\theta)$.

(iv) If $(m,n)\in\{(2,0),(1,1),(0,2)\}$ in item *(ii)* and $k=1$ in item *(iii)*, then $KK_{n+1,m+1}=K_{n+1,m+1}=P_{m+n}$ is an interval and *(ii)* agrees with *(iii)* under the equality $\iota=\tilde{\iota}|_{\mathfrak{G}_{n+1,m+1}}$ for $m+n\leqslant 2$.

Remark 9.11. Since $\tilde{\iota}$ is not surjective, the action of the (pre)matrad axioms on Type II generators forces us to obtain $KK_{n+1,m+1}$ as a quotient of $PP_{n,m}$ modulo combinatorial relations in $PP_{n,m}$ as indicated in Figure 16 above, and thereby extend the equality $K_{n+1}=P_n/\!\sim$ induced by Tonks' projection (see Theorem 9.13 below).

Example 9.12. The action of the map $\theta\to\tilde{\iota}(\theta)$ involving associativity is illustrated by the example in (10):

$$((221),(31))\to 146|2357 \quad\text{and}\quad ((41),(211))\to 12456|37$$

while

$$(((221),(2)),\ ((21),(21)),\ ((2),(211)))\to 146|25|37$$

(on left-hand sides only (co)derivation leaf sequences of underlying matrad module generator are shown). Also, from Example 6.2 we have

$$C_3C_2C_1\xrightarrow{\tilde{\iota}}357|14|26.$$

The properties above imply that ι is a bijection so that $\mathcal{C}_{n,m}$ indexes the faces of $KK_{n,m}$. Define the boundary map in the cellular chain complex $C_*\left(KK_{n,m}\right)$ by

$$\partial(\textstyle\bigvee\!\!\bigwedge_m^n)=\sum_{(\alpha,\beta)\in\mathcal{AB}_{m,n}}(-1)^{\epsilon+\epsilon_\alpha+\epsilon_\beta}e_{\alpha,\beta}, \tag{25}$$

where ϵ is the sign of the cell $e_{(\mathbf{y},\mathbf{x})}\subset P_{n-1,m-1}$ defined by (13). This sign reflects the fact that the sign of a subdivision cell in the boundary inherits (as a component) the sign of the boundary. Therefore, we immediately obtain:

Theorem 9.13. *For each $m,n\geqslant 1$, there is a canonical isomorphism of chain complexes*

$$\iota_*\colon(\mathcal{H}_\infty)_{n,m}\xrightarrow{\approx}C_*(KK_{n,m}) \tag{26}$$

extending the standard isomorphisms

$$\mathcal{A}_\infty(n)=(\mathcal{H}_\infty)_{n,1}\xrightarrow{\approx}C_*(KK_{n,1})=C_*(K_n)$$

and

$$\mathcal{A}_\infty(m)=(\mathcal{H}_\infty)_{1,m}\xrightarrow{\approx}C_*(KK_{1,m})=C_*(K_m).$$

In other words, the cellular chains of the biassociahedra KK realize the free matrad resolution $\mathcal{H}_\infty$ of the bialgebra matrad $\mathcal{H}$. In particular, consider the submodule $\tilde{\mathcal{H}}_\infty\subset\mathcal{H}_\infty$ spanned on the generating set $\tilde{\Theta}$ fixed by summing of all (distinct) elements of $\mathfrak{G}$ in $\mathcal{H}_\infty$ that have the same leaf sequence form. Then (24) induces an isomorphism

$$(\vartheta\circ\tilde{\iota})_*\colon(\tilde{\mathcal{H}}_\infty)_{n,m}\xrightarrow{\approx}C_*(K_{n,m}) \tag{27}$$

by $(\vartheta\circ\tilde{\iota})_*(\tilde{\theta})=\vartheta\left(\tilde{\iota}(\theta_s)\right)$, where $\theta_s\in\mathfrak{G}_{n,m}$ is any summand component of $\tilde{\theta}\in\tilde{\Theta}$, and the following diagram commutes:

$$\begin{array}{ccc} (\tilde{\mathcal{H}}_\infty)_{n,m} & \longrightarrow & (\mathcal{H}_\infty)_{n,m} \\ (\vartheta\circ\tilde{\imath})_*\downarrow\approx & & \approx\downarrow\iota_* \\ C_*(K_{n,m}) & \underset{\nu^\#}{\longrightarrow} & C_*(KK_{n,m}). \end{array}$$

Example 9.14. We have $\tilde{\imath}((111),(21)) = 1|234 \subset P_4$, and apply Example 9.8 for which bijection (26) implies

$$\begin{bmatrix} \theta_2^1 \\ \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \theta_2^3 & \begin{bmatrix} \mathbf{1} \\ \theta_1^2 \end{bmatrix} \theta_1^2 \end{bmatrix} \quad \leftrightarrow \quad d_1$$

$$\begin{bmatrix} \theta_2^1 \\ \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \theta_2^2 \\ \theta_2^1 \end{bmatrix} [\theta_1^2\ \theta_1^2] & \theta_1^3 \end{bmatrix} \quad \leftrightarrow \quad d_2$$

$$\begin{bmatrix} \theta_2^1 \\ \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \theta_1^2 \\ \mathbf{1} \end{bmatrix} \theta_2^2 & \theta_1^3 \end{bmatrix} \quad \leftrightarrow \quad d_3.$$

The edge $(1|3|2|4, z_1)$ in Figure 16 is the intersection $d_2 \cap d_3$ corresponding to

$$A_2A_1 = \begin{bmatrix} \theta_2^1 \\ \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \left(\begin{bmatrix} \theta_1^2 \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} \theta_2^1 \\ \theta_2^1 \end{bmatrix} [\theta_1^2\ \theta_1^2] \right) & \theta_1^3 \end{bmatrix} \in \mathcal{B}_{3,3}. \tag{28}$$

The following proposition applies Proposition 6.14 to reformulate Theorem 9.13 in terms of the $\circledcirc$-operation defined in (15) for $(M,\gamma) = (F^{\mathrm{pre}}(\Theta),\gamma)$.

Proposition 9.15. *Let $(F^{\mathrm{pre}}(\Theta),\gamma)$ be the free prematrad and $(\mathcal{H}_\infty,\partial)$ be the A_∞-matrad. If $\xi = [(\theta \setminus \theta_m^n) \circledcirc (\theta \setminus \theta_m^n)]_{n,m}$ with $mn \geqslant 3$, the components of ξ fit the boundary of $KK_{n,m}$ and $\partial(\xi) = 0$.*

Thus, an A_∞-bialgebra structure on a DGM H is defined by a morphism of matrads $\mathcal{H}_\infty \to U_H$ (compare [**11**]).

In our forthcoming paper [**9**], we construct the theory of relative matrads and use it to define a morphism of A_∞-bialgebras. Using *relative A_∞-matrads,* we prove that over a field, the homology of every biassociative DG bialgebra admits a canonical A_∞-bialgebra structure.

For $KK_{2,2}$:

$$\begin{array}{llll} 1|2 & \leftrightarrow & {}^{11}_{11} & = \gamma(\theta_2^1\theta_2^1\,;\theta_1^2\theta_1^2) \\ 2|1 & \leftrightarrow & {}^{2}_{2} & = \gamma(\theta_1^2\,;\theta_2^1) \end{array}$$

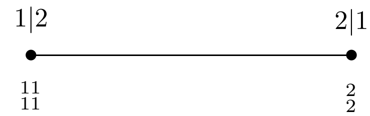


Figure 18: The biassociahedron $KK_{2,2}$ (an interval)

For $KK_{3,2}$:

$$\begin{array}{lcccl}
1|23 & \leftrightarrow & {}^{111}_{11} & = & \gamma(\theta_2^1\theta_2^1\theta_2^1\,;\gamma(\theta_1^2\theta_1^1;\theta_1^2)\theta_1^3+\theta_1^3\gamma(\theta_1^1\theta_1^2;\theta_1^2))\\
13|2 & \leftrightarrow & {}^{12}_{11} & = & \gamma(\theta_2^1\theta_2^2\,;\theta_1^2\theta_1^2)\\
3|12 & \leftrightarrow & {}^{12}_{2} & = & \gamma(\theta_1^1\theta_1^2\,;\theta_2^2)\\
12|3 & \leftrightarrow & {}^{21}_{11} & = & \gamma(\theta_2^2\theta_2^1\,;\theta_1^2\theta_1^2)\\
2|13 & \leftrightarrow & {}^{21}_{2} & = & \gamma(\theta_1^2\theta_1^1\,;\theta_2^2)\\
23|1 & \leftrightarrow & {}^{3}_{2} & = & \gamma(\theta_1^3\,;\theta_2^1)
\end{array}$$

For $KK_{2,3}$:

$$\begin{array}{lcccl}
1|23 & \leftrightarrow & {}^{11}_{21} & = & \gamma(\theta_2^1\theta_2^1\,;\theta_2^2\theta_1^2)\\
13|2 & \leftrightarrow & {}^{2}_{21} & = & \gamma(\theta_2^2\,;\theta_2^1\theta_1^1)\\
3|12 & \leftrightarrow & {}^{3}_{2} & = & \gamma(\theta_1^2\,;\theta_3^1)\\
12|3 & \leftrightarrow & {}^{11}_{111} & = & \gamma(\gamma(\theta_2^1\,;\theta_2^1\theta_1^1)\theta_3^1+\theta_3^1\gamma(\theta_2^1\,;\theta_1^1\theta_2^1)\,;\theta_1^2\theta_1^2\theta_1^2)\\
2|13 & \leftrightarrow & {}^{11}_{12} & = & \gamma(\theta_2^1\theta_2^1\,;\theta_1^2\theta_2^2)\\
23|1 & \leftrightarrow & {}^{2}_{12} & = & \gamma(\theta_2^2\,;\theta_1^1\theta_2^1)
\end{array}$$

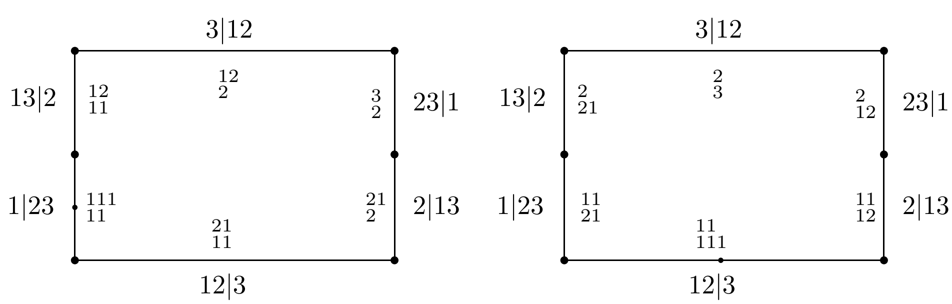


Figure 19: The biassociahedra $KK_{3,2}$ and $KK_{2,3}$ (heptagons)

<u>For $KK_{3,3}$:</u>

$$\begin{array}{llcll}
1|234 & \leftrightarrow & \substack{111\\21} & = & \gamma(\theta_2^1\theta_2^1\theta_2^1\,;\theta_2^3\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)+a+b),\ \text{where}\\
 & & & & \quad a+b=\gamma(\theta_2^2\theta_2^1\,;\theta_1^2\theta_1^2)\theta_1^3+\gamma(\theta_1^2\theta_1^1\,;\theta_2^2)\theta_1^3\\
123|4 & \leftrightarrow & \substack{21\\111} & = & \gamma(c+d+\theta_3^2\gamma(\theta_2^1\,;\theta_1^1\theta_2^1)\,;\theta_1^2\theta_1^2\theta_1^2),\ \text{where}\\
 & & & & \quad c+d=\gamma(\theta_2^1\theta_2^1\,;\theta_2^2\theta_1^2)\theta_3^1+\gamma(\theta_2^2\,;\theta_2^1\theta_1^1)\theta_3^1\\
2|134 & \leftrightarrow & \substack{111\\12} & = & \gamma(\theta_2^1\theta_2^1\theta_2^1\,;\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_2^3+e+f),\ \text{where}\\
 & & & & \quad e+f=\theta_1^3\gamma(\theta_2^1\theta_2^2\,;\theta_1^2\theta_1^2)+\theta_1^3\gamma(\theta_1^1\theta_1^2\,;\theta_2^2)\\
124|3 & \leftrightarrow & \substack{12\\111} & = & \gamma(g+h+\gamma(\theta_2^1\,;\theta_2^1\theta_1^1)\theta_3^2\,;\theta_1^2\theta_1^2\theta_1^2),\ \text{where}\\
 & & & & \quad g+h=\theta_3^1\gamma(\theta_2^1\theta_2^1\,;\theta_1^2\theta_2^2)+\theta_3^1\gamma(\theta_2^2\,;\theta_1^1\theta_2^1)\\
134|2 & \leftrightarrow & \substack{3\\21} & = & \gamma(\theta_2^3\,;\theta_2^1\theta_1^1)\\
234|1 & \leftrightarrow & \substack{3\\12} & = & \gamma(\theta_2^3\,;\theta_1^1\theta_2^1)\\
3|124 & \leftrightarrow & \substack{21\\3} & = & \gamma(\theta_1^2\theta_1^1\,;\theta_3^2)\\
4|123 & \leftrightarrow & \substack{12\\3} & = & \gamma(\theta_1^1\theta_1^2\,;\theta_3^2)\\
13|24 & \leftrightarrow & \substack{21\\21} & = & \gamma(\theta_2^2\theta_2^1\,;\theta_2^2\theta_1^2)\\
24|13 & \leftrightarrow & \substack{12\\12} & = & \gamma(\theta_2^1\theta_2^2\,;\theta_1^2\theta_2^2)\\
14|23 & \leftrightarrow & \substack{12\\21} & = & \gamma(\theta_2^1\theta_2^2\,;\theta_2^2\theta_1^2)\\
23|14 & \leftrightarrow & \substack{21\\12} & = & \gamma(\theta_2^2\theta_2^1\,;\theta_1^2\theta_2^2)\\
34|12 & \leftrightarrow & \substack{3\\3} & = & \gamma(\theta_1^3\,;\theta_3^1)\\
12|34 & \leftrightarrow & \substack{111\\111} & = & \gamma[\theta_3^1\gamma(\theta_2^1\,;\theta_1^1\theta_2^1)\gamma(\theta_2^1\,;\theta_1^1\theta_2^1)+\gamma(\theta_2^1\,;\theta_2^1\theta_1^1)\theta_3^1\gamma(\theta_2^1\,;\theta_1^1\theta_2^1)\\
 & & & & \qquad +\gamma(\theta_2^1\,;\theta_2^1\theta_1^1)\gamma(\theta_2^1\,;\theta_2^1\theta_1^1)\theta_3^1\,;\\
 & & & & \quad \theta_1^3\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)+\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_1^3\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)+\\
 & & & & \qquad +\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_1^3]
\end{array}$$

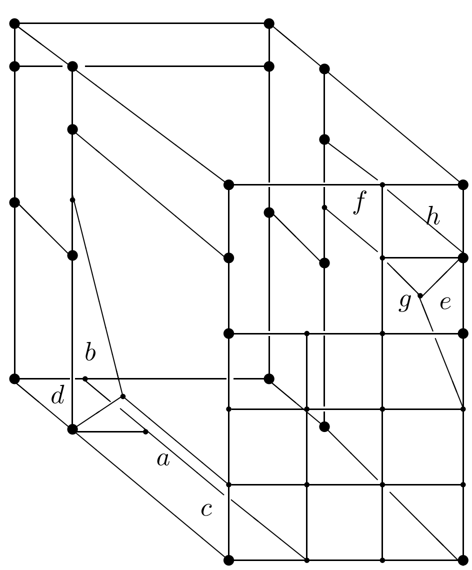


Figure 20: The biassociahedron $KK_{3,3}$ (a subdivision of P_4).

For $KK_{4,2}$:

$$
\begin{array}{llcll}
123|4 & \leftrightarrow & {}^{31}_{11} & = & \gamma(\theta_2^3\theta_2^1\,;\theta_1^2\theta_1^2)\\
2|134 & \leftrightarrow & {}^{211}_{2} & = & \gamma(\theta_1^2\theta_1^1\theta_1^1\,;\theta_2^3)\\
124|3 & \leftrightarrow & {}^{22}_{11} & = & \gamma(\theta_2^2\theta_2^2\,;\theta_1^2\theta_1^2)\\
134|2 & \leftrightarrow & {}^{13}_{11} & = & \gamma(\theta_2^1\theta_2^3\,;\theta_1^2\theta_1^2)\\
234|1 & \leftrightarrow & {}^{4}_{2} & = & \gamma(\theta_1^4\,;\theta_2^1)\\
3|124 & \leftrightarrow & {}^{121}_{2} & = & \gamma(\theta_1^1\theta_1^2\theta_1^1\,;\theta_2^3)\\
4|123 & \leftrightarrow & {}^{112}_{2} & = & \gamma(\theta_1^1\theta_1^1\theta_1^2\,;\theta_2^3)\\
13|24 & \leftrightarrow & {}^{121}_{11} & = & \gamma[\theta_2^1\theta_2^2\theta_2^1\,;\theta_1^3\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)+\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_1^3]\\
14|23 & \leftrightarrow & {}^{112}_{11} & = & \gamma[\theta_2^1\theta_2^1\theta_2^2\,;\theta_1^3\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)+\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_1^3]\\
23|14 & \leftrightarrow & {}^{31}_{2} & = & \gamma(\theta_1^3\theta_1^1\,;\theta_2^2)\\
34|12 & \leftrightarrow & {}^{13}_{2} & = & \gamma(\theta_1^1\theta_1^3\,;\theta_2^2)\\
12|34 & \leftrightarrow & {}^{211}_{11} & = & \gamma[\theta_2^2\theta_2^1\theta_2^1\,;\theta_1^3\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)+\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_1^3]\\
1|234 & \leftrightarrow & {}^{1111}_{11} & = & \gamma(\theta_2^1\theta_2^1\theta_2^1\theta_2^1\,;a+b-c+d+e+f),\ \text{where}\\
 & & & & \quad a=\theta_1^4\gamma(\theta_1^1\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)\,;\theta_1^2)\\
 & & & & \quad b=\gamma(\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_1^1\,;\theta_1^2)\theta_1^4\\
 & & & & \quad c=\gamma(\theta_1^2\theta_1^1\theta_1^1\,;\theta_1^3)\,\gamma(\theta_1^1\theta_1^1\theta_1^2\,;\theta_1^3)\\
 & & & & \quad d=\gamma(\theta_1^3\theta_1^1\,;\theta_1^2)\,\gamma(\theta_1^1\theta_1^2\theta_1^1\,;\theta_1^3)\\
 & & & & \quad e=\gamma(\theta_1^3\theta_1^1\,;\theta_1^2)\,\gamma(\theta_1^1\theta_1^3\,;\theta_1^2)\\
 & & & & \quad f=\gamma(\theta_1^1\theta_1^2\theta_1^1\,;\theta_1^3)\,\gamma(\theta_1^1\theta_1^3\,;\theta_1^2)
\end{array}
$$

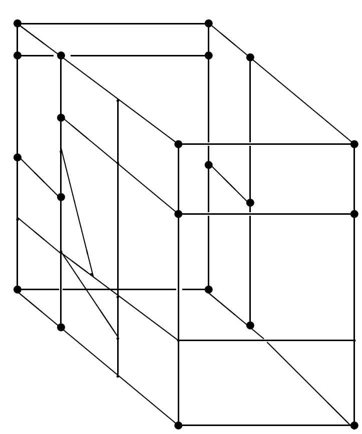


Figure 21: The biassociahedron $KK_{4,2}$ (a subdivision of $J_4 = K_{4,2} = \vartheta_{3,1}(P_4)$).

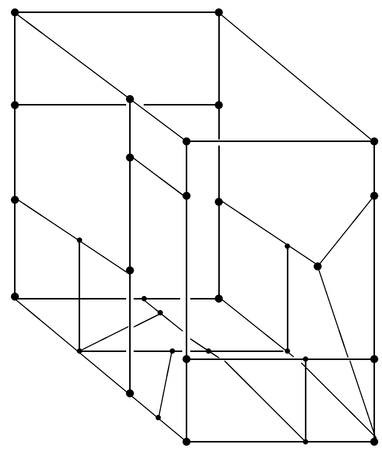


Figure 22: The biassociahedron $KK_{2,4}$ (a subdivision of $J_4 = K_{2,4} = \vartheta_{1,3}(P_4)$).

References

[1] John Frank Adams, *Infinite loop spaces*, Annals of Mathematics Studies, vol. 90, Princeton University Press, Princeton, N.J., 1978.

[2] Murray Gerstenhaber, Anthony Giaquinto, and Samuel D. Schack, *Quantum groups, cohomology, and preferred deformations*, Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. 1, 2 (New York, 1991), World Sci. Publ., River Edge, NJ, 1992, pp. 529–538.

[3] Norio Iwase and Mamoru Mimura, *Higher homotopy associativity*, Algebraic topology (Arcata, CA, 1986), Lecture Notes in Mathematics, Vol. 1370, Springer-Verlag, Berlin, 1989, pp. 193–220.

[4] Jean-Louis Loday and María O. Ronco, *Hopf algebra of the planar binary trees*, Adv. Math. **139** (1998), no. 2, 293–309.

[5] Martin Markl, *A resolution (minimal model) of the PROP for bialgebras*, J. Pure Appl. Algebra **205** (2006), no. 2, 341–374.

[6] Martin Markl, Steve Shnider, and Jim Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002.

[7] J. P. May, *The geometry of iterated loop spaces*, Lectures Notes in Mathematics, Vol. 271, Springer-Verlag, Berlin, 1972.

[8] Teimuraz Pirashvili, *On the PROP corresponding to bialgebras*, Cah. Topol. Géom. Différ. Catég. **43** (2002), no. 3, 221–239.

[9] Samson Saneblidze and Ronald Umble, *Morphisms of A_∞-bialgebra and applications*, in preparation.

[10] ______, *Diagonals on the permutahedra, multiplihedra and associahedra*, Homology Homotopy Appl. **6** (2004), no. 1, 363–411.

[11] ______, *The biderivative and A_∞-bialgebras*, Homology Homotopy Appl. **7** (2005), no. 2, 161–177.

[12] Boris Shoikhet, *The crocs, non-commutative deformations, and (co)associative bialgebras*, preprint, math.QA/0306143.

[13] James Stasheff, *Homotopy associativity of H-spaces. I, II*, Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid. **108** (1963), 293–312.

[14] ______, *H-spaces from a homotopy point of view*, Lecture Notes in Mathematics, Vol. 161, Springer-Verlag, Berlin, 1970.

[15] Andy Tonks, *Relating the associahedron and the permutohedron*, Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995), Contemp. Math., vol. 202, Amer. Math. Soc., Providence, RI, 1997, pp. 33–36.

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