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ON TRIVIALITIES OF STIEFEL-WHITNEY CLASSES OF VECTOR BUNDLES OVER ITERATED SUSPENSION SPACES

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Abstract

A space B is described as W-trivial if for every vector bundle over B, all the Stiefel-Whitney classes vanish. We prove that if B is a 9-fold suspension, then B is W-trivial. We also determine all pairs (k, n) of positive integers for which $\Sigma^k FP^n$ is W-trivial, where $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

1. Introduction and results

A space B is called W-trivial if $W(\alpha) = 1$ holds for every vector bundle α over B. Here $W(\alpha)$ denotes the total Stiefel-Whitney class of α. If B is W-trivial, then a kind of Borsuk-Ulam type theorem holds for every vector bundle α over B; precisely, for any integer i with $i > \dim \alpha$, there does not exist a $\mathbb{Z}_2$-map from S^{i-1} to $S(\alpha)$, the sphere bundle of α [**6**, Proposition 2.2]. Thus it would be interesting to ask whether a space is W-trivial or not. As is well-known, the sphere S^n is W-trivial if and only if $n \neq 1, 2, 4, 8$ (see [**5**]). Obviously, the projective space FP^n, where $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, is not W-trivial for any $n > 0$. For the stunted projective space FP^n_m, all (m, n) for which FP^n_m is W-trivial were determined in [**9**]; roughly speaking, FP^n_m is not W-trivial if and only if m is very small compared with n.

As is seen in the case $B = S^n$, it is not true that if B is W-trivial, then its suspension ΣB is also W-trivial. In this paper, we first prove the following theorem.

Theorem 1.1. *For a space B, its 8-fold suspension $\Sigma^8 B$ is W-trivial if either of the following conditions is satisfied:*

1. *B is W-trivial.*
2. *The cup product in $\widetilde{H}^*(B; \mathbb{Z}_2)$ is trivial.*

In general, the cup product in $\widetilde{H}^*(\Sigma B; \mathbb{Z}_2)$ is trivial, so that from the above theorem, we immediately obtain the following result.

Corollary 1.2. *For any space B, its 9-fold suspension $\Sigma^9 B$ is W-trivial.*

As is easily seen by using the suspension theorem, a k-connected complex B with $\dim B \leqslant 2k + 1$ is homotopy equivalent to the suspension of a $(k-1)$-connected complex of dimension $\dim B - 1$. By iterating this, we see that a k-connected complex

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B is homotopy equivalent to the 9-fold suspension of a $(k-9)$-connected complex $(k>9)$ if $\dim B \leqslant 2k-7$. Therefore, from Corollary 1.2, we obtain the following result.

Corollary 1.3. *Let B be a k-connected complex with $k>9$. If $\dim B \leqslant 2k-7$, then B is W-trivial.*

This corollary greatly improves Theorem 1.3 in [**8**]. Since the smallest integer i such that $w_i(\alpha) \neq 0$ is a power of 2 (see [**8**, Lemma 2.1]), the above corollary is actually useful only when $k \geqslant 12$. For example, we see that a 16-dimensional complex is W-trivial if it is 12-connected. It should be also noted that the 16-dimensional stunted projective space $\mathbb{R}P_k^{16}$ is W-trivial for $9<k<16$ while $\mathbb{R}P_9^{16}$ is not W-trivial (see [**8**, Theorem 4.1]).

Next, in this paper, we investigate whether $\Sigma^k FP^n$ is W-trivial or not, where $F=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$. Because of Corollary 1.2, our interests are only in the case when $0<k\leqslant 8$. For $F=\mathbb{R}$, we have the following result.

Theorem 1.4. *For positive integers k and n, the k-fold suspension $\Sigma^k\mathbb{R}P^n$ of $\mathbb{R}P^n$ is not W-trivial if and only if k and n satisfy one of the following conditions:*

(1) $k=1,2,4$ *or* 8 *and* $n \geqslant k$.

(2) $k=3,5$ *or* 7 *and* $n+k=4$ *or* 8.

(3) $k=6$ *and* $n=2$ *or* 3.

This result shows that the condition $k \geqslant 9$ is best possible for $\Sigma^k B$ to be W-trivial in general.

For $F=\mathbb{C}$ and $F=\mathbb{H}$, we have the following results.

Theorem 1.5. *For positive integers k, n with $n>1$, the k-fold suspension $\Sigma^k\mathbb{C}P^n$ of $\mathbb{C}P^n$ is not W-trivial if and only if $k=2$ or 4.*

Theorem 1.6. *For positive integers k, n with $n>1$, the k-fold suspension $\Sigma^k\mathbb{H}P^n$ of $\mathbb{H}P^n$ is not W-trivial if and only if $k=4$.*

It is worth noting that the W-triviality of $\Sigma^k FP^n$ does not depend on n for $F=\mathbb{C}$ or $\mathbb{H}$.

Throughout this paper, all cohomology is assumed to have coefficients $\mathbb{Z}_2$ unless otherwise stated. The total Stiefel-Whitney class of α is denoted by $W(\alpha)$, and the total Chern class by $C(\alpha)$.

The following two lemmas are straightforward to show but they are of fundamental importance for our proofs of theorems.

Lemma 1.7.

(1) *If $\widetilde{KO}(B)=0$, then B is W-trivial.*

(2) *Let $f\colon B \to X$ be a map and suppose that X is W-trivial. If $f^*\colon \widetilde{KO}(X) \to \widetilde{KO}(B)$ is epimorphic, then B is W-trivial.*

Lemma 1.8.

(1) *If $H^{2^r}(B)=0$ for all $r \geqslant 0$, then B is W-trivial.*

(2) *Let $f\colon X \to B$ be a map and suppose that X is W-trivial. If $f^*\colon H^{2^r}(B) \to H^{2^r}(X)$ is monomorphic for all $r \geqslant 0$, then B is W-trivial.*

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We use the Bott periodicity theorem for KO-theory. Let $j\colon S^8 \times B \to \Sigma^8 B$ denote the quotient map and let $p_1\colon S^8 \times B \to S^8$ and $p_2\colon S^8 \times B \to B$ denote the projections. Let α be an arbitrary vector bundle over $\Sigma^8 B$. By the Bott periodicity theorem, we see that $j^*\alpha$ is stably equivalent to $p_1^*(\nu - 8) \otimes p_2^*(\beta - m)$ for some vector bundle β over B. Here ν denotes the Hopf vector bundle over S^8 and $m = \dim \beta$. Then, we have

$$j^*W(\alpha) = W(p_1^*\nu \otimes p_2^*\beta) \cdot W(p_1^*\nu)^{-m} \cdot W(p_2^*\beta)^{-8}. \qquad (*)$$

We compute this and show that $W(\alpha) = 1$. Note that $W(p_1^*\nu) = p_1^*W(\nu) = 1 + s \times 1$, where s denotes the generator of $H^8(S^8)$. Let

$$W(p_1^*\nu) = \prod_{i=1}^{8}(1 + s_i) \quad \text{and} \quad W(p_2^*\beta) = \prod_{j=1}^{m}(1 + t_j)$$

be formal factorizations of $W(p_1^*\nu)$ and $W(p_2^*\beta)$. Then, by an analogous formula to Formula III of Theorem 4.4.3 in [**4**], we have $W(p_1^*\nu \otimes p_2^*\beta) = \prod_{i,j}(1 + s_i + t_j)$. We first calculate the product for i's by using $\prod_{i=1}^{8}(1 + s_i) = 1 + s \times 1$ as follows:

$$\begin{aligned}\prod_{i=1}^{8}((1 + t_j) + s_i) &= \sum_{k=0}^{8}(1 + t_j)^{8-k}\lambda_k(s_1, s_2, \ldots, s_8) \\ &= (1 + t_j)^8 + s_1 s_2 \cdots s_8 \\ &= 1 + t_j^8 + s \times 1,\end{aligned}$$

where λ_k denotes the elementary symmetric polynomial of degree k and we used the fact that $\lambda_k(s_1, s_2, \ldots, s_8) = 0$ for $0 < k < 8$. Therefore we have

$$\begin{aligned}W(p_1^*\nu \otimes p_2^*\beta) &= \prod_{j=1}^{m}((1 + s \times 1) + t_j^8) \\ &= \sum_{k=0}^{m}(1 + s \times 1)^{m-k}\lambda_k(t_1^8, t_2^8, \ldots, t_m^8).\end{aligned}$$

Now, we assume that the cup product in $\widetilde{H}^*(B)$ is trivial. Then, we clearly have $W(\beta)^2 = 1$, so that $W(p_2^*\beta)^8 = p_2^*W(\beta)^8 = 1$. This implies that $\prod_{j=1}^{m}(1 + t_j^8) = 1$, so that $\lambda_k(t_1^8, t_2^8, \ldots, t_m^8) = 0$ for every $k > 0$. Therefore we have

$$W(p_1^*\nu \otimes p_2^*\beta) = (1 + s \times 1)^m.$$

Substituting these results into $(*)$, we obtain

$$j^*W(\alpha) = (1 + s \times 1)^m \cdot (1 + s \times 1)^{-m} \cdot 1^{-1} = 1. \qquad (**)$$

Since $j^*\colon H^*(\Sigma^8 B) \to H^*(S^8 \times B)$ is monomorphic, we conclude that $W(\alpha) = 1$. Thus the proof of Theorem 1.1 under the assumption (2) is completed.

The proof under the assumption (1) is quite similar. Since $W(p_2^*\beta) = 1$ from the assumption that B is W-trivial, we may regard all the t_j's as zeros in our previous calculations. Then we obtain $W(p_1^*\nu \otimes p_2^*\beta) = (1 + s \times 1)^m$ and have the same result as $(**)$. Thus the theorem under the assumption (1) follows.

Here we prepare the following lemma, which will be used to prove Theorems 1.4 and 1.5 in later sections.

Lemma 2.1. *Let d and m be positive integers with $d \leqslant m$.*

(1) *If γ is a vector bundle over S^d with $\dim \gamma = m$ and β is a line bundle over B, then in $H^*(S^d \times B)$ we have*

$$W((p_1^*\gamma - m) \otimes (p_2^*\beta - 1)) = 1 + w_d(\gamma) \times ((1 + w_1(\beta))^{-d} - 1),$$

where $p_1\colon S^d \times B \to S^d$ and $p_2\colon S^d \times B \to B$ are the projections.

(2) *If γ is a complex vector bundle over S^{2d} with $\dim_{\mathbb{C}} \gamma = m$ and β is a complex line bundle over B, then in $H^*(S^{2d} \times B\,; \mathbb{Z})$ we have*

$$C((p_1^*\gamma - m) \otimes_{\mathbb{C}} (p_2^*\beta - 1)) = 1 + c_d(\gamma) \times ((1 + c_1(\beta))^{-d} - 1),$$

where $p_1\colon S^{2d} \times B \to S^{2d}$ and $p_2\colon S^{2d} \times B \to B$ are the projections.

Proof. We prove only (1) since the proof of (2) is quite similar. Let us put $w_d(\gamma) = s$ and $w_1(\beta) = t$. Let $W(p_1^*\gamma) = 1 + s \times 1 = 1^{m-d} \cdot \prod_{i=1}^d (1 + s_i)$ and $W(p_2^*\beta) = 1 + 1 \times t = 1 + t_1$ be formal factorizations. Then, just like before, we can calculate as follows:

$$\begin{aligned} W(p_1^*\gamma \otimes p_2^*\beta) &= (1 + t_1)^{m-d} \cdot \prod_{i=1}^d (1 + s_i + t_1) \\ &= (1 + t_1)^{m-d} \cdot ((1 + t_1)^d + s \times 1) \\ &= (1 + 1 \times t)^m \cdot (1 + s \times (1 + t)^{-d}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} W((p_1^*\gamma - m) \otimes (p_2^*\beta - 1)) &= W(p_1^*\gamma \otimes p_2^*\beta) \cdot W(p_2^*\beta)^{-m} \cdot W(p_1^*\gamma)^{-1} \\ &= (1 + s \times (1 + t)^{-d}) \cdot (1 + s \times 1)^{-1} \\ &= (1 + s \times (1 + t)^{-d}) \cdot (1 - s \times 1) \\ &= 1 + s \times ((1 + t)^{-d} - 1). \end{aligned}$$

Thus the lemma follows. □

3. Proof of Theorem 1.4

In this section, we investigate whether $\Sigma^k \mathbb{R}P^n$ is W-trivial or not. Since $\Sigma^k \mathbb{R}P^n$ is W-trivial for $k > 8$ by Corollary 1.2, our interests are only in the case when $0 < k \leqslant 8$. We divide into three cases: (1) $k = 1, 2, 4$ or 8, (2) $k = 3, 5$ or 7 and (3) $k = 6$.

First we consider the case when $k = 1, 2, 4$ or 8. The result is as follows.

Proposition 3.1. *Let $d = 1, 2, 4$ or 8. Then $\Sigma^d \mathbb{R}P^n$ is not W-trivial if and only if $n \geqslant d$.*

Proof. Recall that for a vector bundle α, the smallest integer i such that $w_i(\alpha) \neq 0$ is a power of 2 (see [**8**, Lemma 2.1]). If $n < d$, then $\Sigma^d \mathbb{R}P^n$ has no cells of dimension a power of 2, so that $\Sigma^d \mathbb{R}P^n$ is W-trivial. Now, let us consider the exact sequence

$$0 \longleftarrow \widetilde{KO}(S^d \vee \mathbb{R}P^n) \overset{i^*}{\longleftarrow} \widetilde{KO}(S^d \times \mathbb{R}P^n) \overset{j^*}{\longleftarrow} \widetilde{KO}(\Sigma^d \mathbb{R}P^n) \longleftarrow 0,$$

where i and j are obvious maps. Let ν denote the Hopf vector bundle over S^d and let ξ denote the canonical line bundle over $\mathbb{R}P^n$. Since $i^*((p_1^*\nu - d) \otimes (p_2^*\xi - 1)) = 0$, there is a vector bundle α over $\Sigma^d\mathbb{R}P^n$ such that $j^*\alpha$ is stably equivalent to $(p_1^*\nu - d) \otimes (p_2^*\xi - 1)$. By Lemma 2.1, in $H^*(S^d \times \mathbb{R}P^n)$ we have

$$\begin{aligned} W(j^*\alpha) &= 1 + s \times ((1+t)^{-d} - 1) \\ &= 1 + s \times (t^d + t^{2d} + t^{3d} + \cdots), \end{aligned}$$

where s and t denote the generator of $H^d(S^d)$ and $H^1(\mathbb{R}P^n)$ respectively. Hence, we see that $j^*W(\alpha) \neq 1$ if $n \geqslant d$. We thus conclude that $W(\alpha) \neq 1$, so that $\Sigma^d\mathbb{R}P^n$ is not W-trivial if $n \geqslant d$. □

Before we consider the second case, we prepare a few lemmas.

Lemma 3.2. *If $\Sigma^k\mathbb{R}P^{2^m-k}$ is W-trivial, then $\Sigma^k\mathbb{R}P^n$ is W-trivial for any integer n with $2^m - k < n < 2^{m+1} - k$.*

Proof. Let $i\colon \Sigma^k\mathbb{R}P^{2^m-k} \to \Sigma^k\mathbb{R}P^n$ be the inclusion map. If $2^m < n + k < 2^{m+1}$, then $i^*\colon H^{2^r}(\Sigma^k\mathbb{R}P^n) \to H^{2^r}(\Sigma^k\mathbb{R}P^{2^m-k})$ is monomorphic for all $r \geqslant 0$ for dimensional reasons. Therefore, the lemma follows from Lemma 1.8. □

Lemma 3.3. *Let α be a vector bundle over a complex B. Let r be an integer with $r \geqslant 2$ and suppose that $w_i(\alpha) = 0$ for $0 < i < 2^r$. Then we have $\mathrm{Sq}^j w_{2^r}(\alpha) = 0$ for $0 < j < 2^{r-1}$.*

Proof. We put $2^{r-1} = m$ and consider the inclusion $i\colon B^{(3m)} \hookrightarrow B$, where $B^{(3m)}$ is the $3m$-skeleton of B. For dimensional reasons, the induced bundle $i^*\alpha$ is stably equivalent to some $3m$-dimensional vector bundle β. Then we clearly have $W(i^*\alpha) = W(\beta)$. We denote by $P(\beta)$ the associated projective bundle of β, and by e the $\mathbb{Z}_2$-Euler class of the line bundle $\beta \to P(\beta)$. The cohomology $H^*(P(\beta))$ is a free $H^*(B^{(3m)})$-module generated by $1, e, e^2, \ldots, e^{3m-1}$, in which we have the relation $e^{3m} = \sum_{i=0}^{3m-1} w_{3m-i}(\beta) \cdot e^i$. Since we have $w_i(\beta) = i^*w_i(\alpha) = 0$ for $0 < i < 2m$ by the assumption, we can write this relation as $e^{3m} = w_{3m} + w_{3m-1} \cdot e + \cdots + w_{2m} \cdot e^m$, where we have abbreviated $w_i(\beta)$ as w_i. We apply the total squaring operation $\mathrm{Sq} = \sum_{i\geqslant 0} \mathrm{Sq}^i$ to this relation. Since $\mathrm{Sq}(e^i) = (\mathrm{Sq}\, e)^i = (e + e^2)^i = e^i(1+e)^i$, we obtain the following equation:

$$e^{3m}(1+e)^{3m} = \mathrm{Sq}\, w_{3m} + \mathrm{Sq}\, w_{3m-1} \cdot e(1+e) + \cdots + \mathrm{Sq}\, w_{2m} \cdot e^m(1+e)^m. \quad (*{*}*)$$

In this equation, we like to compare the coefficients of e^j's. To do this, we must rewrite the left-hand side of $(*{*}*)$ so that all summands have exponents of e less than $3m$. We calculate using the previous relation as follows:

$$\begin{aligned} e^{3m}(1+e)^{3m} &= e^{3m}(1 + e^m + e^{2m} + e^{3m}) \\ &= e^{3m}(1+e^m) + (e^{3m} - w_{2m} \cdot e^m)e^{2m} + w_{2m} \cdot e^{3m} + (e^{3m})^2 \\ &= (w_{3m} + w_{3m-1} \cdot e + \cdots + w_{2m} \cdot e^m)(1 + e^m) \\ &\quad + (w_{3m} + w_{3m-1} \cdot e + \cdots + w_{2m+1} \cdot e^{m-1})e^{2m} \\ &\quad + w_{2m}(w_{3m} + w_{3m-1} \cdot e + \cdots + w_{2m} \cdot e^m) \\ &\quad + (w_{3m} + w_{3m-1} \cdot e + \cdots + w_{2m} \cdot e^m)^2. \end{aligned}$$

With this expression of the left-hand side of $(***)$, we can compare the coefficients of e^j's for $j < 3m$. Comparing the coefficients of e^{2m}, we obtain $\mathrm{Sq}\, w_{2m} = w_{2m} + w_{3m} + w_{2m}^2$. Hence we have $\mathrm{Sq}^j w_{2m} = 0$ for $0 < j < m$ and $\mathrm{Sq}^m w_{2m} = w_{3m}$. We here recall that $w_i = i^* w_i(\alpha)$. Since $i^*\colon H^i(B) \to H^i(B^{(3m)})$ is monomorphic for $i \leqslant 3m$, we conclude that $\mathrm{Sq}^j w_{2m}(\alpha) = 0$ for $0 < j < m$ and $\mathrm{Sq}^m w_{2m}(\alpha) = w_{3m}(\alpha)$. Thus the lemma follows. □

Remark. When $w_i = 0$ for $0 < i < 2^r$, Wu's formula [**10**] turns out to be $\mathrm{Sq}^j w_{2^r} = \binom{2^r-1}{j} w_{2^r+j} = w_{2^r+j}$ $(0 < j < 2^r)$. Lemma 3.3 implies that this is zero for $0 < j < 2^{r-1}$. We also remark that there is a vector bundle over $\Sigma^4\mathbb{H}P^2$ such that $w_8 \neq 0$ and $w_{12} \neq 0$ (see [**8**, Theorem 4.5]). Thus our result is best possible at least for $r = 3$.

Now, we consider the second case: $k = 3, 5$ or 7. The result is as follows.

Proposition 3.4. *Let $k = 3, 5$ or 7. Then $\Sigma^k\mathbb{R}P^n$ is not W-trivial if and only if $n + k = 4$ or 8.*

Proof. We consider the cofibration $\Sigma^k\mathbb{R}P^{n-1} \xrightarrow{i} \Sigma^k\mathbb{R}P^n \xrightarrow{j} S^{n+k}$. First, let $n + k = 8$. Since S^8 is not W-trivial and $j^*\colon H^8(S^8) \to H^8(\Sigma^k\mathbb{R}P^n)$ is monomorphic, it follows from Lemma 1.8 that $\Sigma^k\mathbb{R}P^n$ is not W-trivial. Similarly $\Sigma^k\mathbb{R}P^n$ is not W-trivial when $n + k = 4$. Thus the "if" part of the proposition follows. Next, we suppose $n + k \neq 4, 8$ and show that $\Sigma^k\mathbb{R}P^n$ is W-trivial. Our proof is divided into two cases.

Case 1: $n + k \geqslant 16$.

First we consider the case when $n + k = 2^r$ with $r \geqslant 4$. In this case, we have $\widetilde{KO}(\Sigma^k\mathbb{R}P^{n-1}) = 0$ by [**3**, Theorem 1] since $k = 3, 5, 7$ and $n + k - 1 \equiv 7 \pmod 8$. Hence $j^*\colon \widetilde{KO}(S^{2^r}) \to \widetilde{KO}(\Sigma^k\mathbb{R}P^n)$ is epimorphic. Since S^{2^r} is W-trivial for $r \geqslant 4$, it follows from Lemma 1.7 that $\Sigma^k\mathbb{R}P^n$ is W-trivial, that is, $\Sigma^k\mathbb{R}P^{2^r-k}$ is W-trivial for all $r \geqslant 4$. Hence, by Lemma 3.2, we see that $\Sigma^k\mathbb{R}P^n$ is W-trivial for all $n \geqslant 16 - k$.

Case 2: $k + 1 \leqslant n + k < 16$ $(n + k \neq 4, 8)$.

Let α be a vector bundle over $\Sigma^k\mathbb{R}P^n$ and let r be the smallest integer such that $w_{2^r}(\alpha)$ is (possibly) non-zero. Then we obviously have $r = 2$ or 3 when $k = 3$, and $r = 3$ when $k = 5, 7$. Also, note that $2^r < n + k$ from our assumption $n + k \neq 4, 8$. From Lemma 3.3, we must have $\mathrm{Sq}^1 w_{2^r}(\alpha) = 0$. On the other hand, since k is odd and $2^r < n + k$, $\mathrm{Sq}^1\colon H^{2^r}(\Sigma^k\mathbb{R}P^n) \to H^{2^r+1}(\Sigma^k\mathbb{R}P^n)$ is non-trivial. Therefore, we have $w_{2^r}(\alpha) = 0$. We thus obtain $W(\alpha) = 1$ and conclude that $\Sigma^k\mathbb{R}P^n$ is W-trivial if $n + k < 16$ $(n + k \neq 4, 8)$. This completes the proof of the proposition. □

Finally, we consider the third case: $k = 6$. The result is as follows.

Proposition 3.5. *$\Sigma^6\mathbb{R}P^n$ is not W-trivial if and only if $n = 2$ or 3.*

Proof. The proof is very similar to that of the preceding proposition. Considering the cofibration $S^7 \xrightarrow{i} \Sigma^6\mathbb{R}P^2 \xrightarrow{j} S^8$, we see that $\Sigma^6\mathbb{R}P^2$ is not W-trivial in exactly the same way as before. Let us consider the cofibration $\Sigma^6\mathbb{R}P^2 \xrightarrow{i} \Sigma^6\mathbb{R}P^3 \xrightarrow{j} S^9$. Since $\widetilde{KO}(\Sigma^6\mathbb{R}P^2)$ is a finite group (precisely, $\mathbb{Z}_2$), we see from the exact sequence that $i^*\colon \widetilde{KO}(\Sigma^6\mathbb{R}P^3) \to \widetilde{KO}(\Sigma^6\mathbb{R}P^2)$ is epimorphic. Since $\Sigma^6\mathbb{R}P^2$ is not W-trivial, as shown above, it follows from Lemma 1.7 that $\Sigma^6\mathbb{R}P^3$ is not W-trivial either. Thus the "if" part of the proposition follows. Next, we show that $\Sigma^6\mathbb{R}P^n$ is W-trivial for $n \neq 2, 3$.

Case 1: $n \geqslant 10$.

By [**3**, Theorem 1], we have $\widetilde{KO}(\Sigma^6\mathbb{R}P^{n-1}) = 0$ when $n + 6 = 2^r$ $(r \geqslant 4)$. Hence $j^*\colon \widetilde{KO}(S^{2^r}) \to \widetilde{KO}(\Sigma^6\mathbb{R}P^n)$ is epimorphic, from which we see by Lemma 1.7 that $\Sigma^6\mathbb{R}P^n$ is W-trivial when $n + 6 = 2^r$ $(r \geqslant 4)$. Therefore, it follows from Lemma 3.2 that $\Sigma^6\mathbb{R}P^n$ is W-trivial for all $n \geqslant 10$.

Case 2: $1 \leqslant n < 10$ $(n \neq 2, 3)$.

Obviously $\Sigma^6\mathbb{R}P^n$ is W-trivial when $n = 1$. So we suppose that $n \geqslant 4$. For a vector bundle α over $\Sigma^6\mathbb{R}P^n$, the smallest integer such that $w_{2^r}(\alpha)$ is (possibly) non-zero is 8. Hence, from Lemma 3.3, we have $\mathrm{Sq}^j w_8(\alpha) = 0$ for $0 < j < 4$. Since Sq^1 acts trivially on $H^8(\Sigma^6\mathbb{R}P^n)$, we use Sq^2 in place of Sq^1. Indeed, $\mathrm{Sq}^2\colon H^8(\Sigma^6\mathbb{R}P^n) \to H^{10}(\Sigma^6\mathbb{R}P^n)$ is non-trivial since $n \geqslant 4$. Therefore, we have $w_8(\alpha) = 0$, so that we obtain $W(\alpha) = 1$. Thus $\Sigma^6\mathbb{R}P^n$ is W-trivial when $4 \leqslant n < 10$. This completes the proof of the proposition. □

The proof of Theorem 1.4 is completed by Propositions 3.1, 3.4 and 3.5.

4. Proof of Theorem 1.6

In this section, we investigate whether or not $\Sigma^k FP^n$ is W-trivial for $F = \mathbb{H}$. Because of Corollary 1.2, we have only to consider the case when $0 < k \leqslant 8$. Then, unless $k = 4$ or 8, $\Sigma^k\mathbb{H}P^n$ has no cells of dimension a power of 2, so that we have $H^{2^r}(\Sigma^k\mathbb{H}P^n) = 0$ for all $r \geqslant 0$. Thus, from Lemma 1.8, the possibility for $\Sigma^k\mathbb{H}P^n$ not to be W-trivial is only when $k = 4$ or 8. Therefore, Theorem 1.6 follows if we prove the following proposition.

Proposition 4.1.

(1) $\Sigma^4\mathbb{H}P^n$ *is not W-trivial for all* $n > 1$.

(2) $\Sigma^8\mathbb{H}P^n$ *is W-trivial for all* $n > 1$.

Proof. First, let us consider the cofibration $S^8 \xrightarrow{i} \Sigma^4\mathbb{H}P^n \xrightarrow{j} \Sigma^4(\mathbb{H}P^n/S^4)$. Since $\Sigma^3(\mathbb{H}P^n/S^4)$ has cells only of dimension 3 or 7 modulo 8, we have $\widetilde{KO}(\Sigma^3(\mathbb{H}P^n/S^4)) = 0$ from the Atiyah-Hirzebruch spectral sequence [**2**]. Hence, $i^*\colon \widetilde{KO}(\Sigma^4\mathbb{H}P^n) \to \widetilde{KO}(S^8)$ is epimorphic. Since S^8 is not W-trivial, it follows from Lemma 1.7 that $\Sigma^4\mathbb{H}P^n$ is not W-trivial. This proves (1).

Next we prove (2). Let us consider the cofibration

$$\Sigma^8\mathbb{H}P^{n-1} \xrightarrow{i} \Sigma^8\mathbb{H}P^n \xrightarrow{j} S^{4n+8}.$$

Since $\widetilde{KO}(S^{4n+7}) = 0$, $i^*\colon \widetilde{KO}(\Sigma^8\mathbb{H}P^n) \to \widetilde{KO}(\Sigma^8\mathbb{H}P^{n-1})$ is epimorphic. Hence, we see that if $\Sigma^8\mathbb{H}P^n$ is W-trivial, then $\Sigma^8\mathbb{H}P^{n-1}$ is also W-trivial. Thus, it suffices to prove that $\Sigma^8\mathbb{H}P^{2^m}$ is W-trivial for all $m \geqslant 3$. Now, let α be a vector bundle over $\Sigma^8\mathbb{H}P^{2^m}$. Abusing notation, let i denote the inclusion $\Sigma^8\mathbb{H}P^2 \hookrightarrow \Sigma^8\mathbb{H}P^{2^m}$. From [**7**, Theorem 4.3], $\Sigma^8\mathbb{H}P^2$ is W-trivial. Since $i^*\colon H^{16}(\Sigma^8\mathbb{H}P^{2^m}) \to H^{16}(\Sigma^8\mathbb{H}P^2)$ is monomorphic, we obtain $w_{16}(\alpha) = 0$. Let r be the smallest integer such that $w_{2^r}(\alpha)$ is (possibly) non-zero. Then, we have $r \geqslant 5$ from the above argument. Also note that $r \leqslant m+2$ since $2^r \leqslant 8+4\cdot 2^m$ and $r \geqslant 5$. Now let us consider the operation $\mathrm{Sq}^8\colon H^{2^r}(\Sigma^8\mathbb{H}P^{2^m}) \to H^{2^r+8}(\Sigma^8\mathbb{H}P^{2^m})$. Since $\binom{2^{r-2}-2}{2} \equiv 1 \pmod 2$ and $r \leqslant m+2$, this operation is non-trivial. On the other hand, by Lemma 3.3, we have $\mathrm{Sq}^8 w_{2^r}(\alpha) = 0$ since $r \geqslant 5$. Therefore, we obtain $w_{2^r}(\alpha) = 0$ and conclude that $W(\alpha) = 1$. This completes the proof. □

5. Proof of Theorem 1.5

Finally, in this section, we investigate whether $\Sigma^k\mathbb{C}P^n$ is W-trivial or not. If k is odd, then $\Sigma^k\mathbb{C}P^n$ has no cells of dimension a power of 2. Thus, from Lemma 1.8 and Corollary 1.2, the possibility of $\Sigma^k\mathbb{C}P^n$ being not W-trivial is only when $k = 2, 4, 6$ or 8. For $k = 2$ or 4, we have the following result.

Proposition 5.1. *$\Sigma^2\mathbb{C}P^n$ and $\Sigma^4\mathbb{C}P^n$ are not W-trivial for all $n > 1$.*

Proof. First we consider $\Sigma^2\mathbb{C}P^n$. Analogously to the proof of Proposition 3.1, we consider the exact sequence

$$0 \longleftarrow \widetilde{K}(S^2 \vee \mathbb{C}P^n) \overset{i^*}{\longleftarrow} \widetilde{K}(S^2 \times \mathbb{C}P^n) \overset{j^*}{\longleftarrow} \widetilde{K}(\Sigma^2\mathbb{C}P^n) \longleftarrow 0$$

and the stable class of $(p_1^*\nu - 1) \otimes_{\mathbb{C}} (p_2^*\eta - 1)$. Here, ν is the Hopf vector bundle over S^2 considered as a complex (line) bundle, while η is the canonical complex line bundle over $\mathbb{C}P^n$. Then, we can take a complex vector bundle α over $\Sigma^2\mathbb{C}P^n$ such that $j^*\alpha$ is stably equivalent to $(p_1^*\nu - 1) \otimes_{\mathbb{C}} (p_2^*\eta - 1)$. By Lemma 2.1, in $H^*(S^2 \times \mathbb{C}P^n; \mathbb{Z})$,

$$\begin{aligned} C(j^*\alpha) &= C((p_1^*\nu - 1) \otimes_{\mathbb{C}} (p_2^*\eta - 1)) \\ &= 1 + c_1(\nu) \times ((1 + c_1(\eta))^{-1} - 1) \\ &= 1 + s \times (-t + t^2 - t^3 + \cdots), \end{aligned}$$

where s and t are generators of $H^2(S^2;\mathbb{Z})$ and $H^2(\mathbb{C}P^n;\mathbb{Z})$, respectively. Hence we have $j^*c_2(\alpha) = -s \times t \not\equiv 0 \pmod 2$ for $n \geqslant 1$. Therefore we have $w_4(\alpha) \neq 0$, so that $\Sigma^2\mathbb{C}P^n$ is not W-trivial for $n \geqslant 1$.

Similarly for $\Sigma^4\mathbb{C}P^n$, let us consider the exact sequence

$$0 \longleftarrow \widetilde{K}(S^4 \vee \mathbb{C}P^n) \overset{i^*}{\longleftarrow} \widetilde{K}(S^4 \times \mathbb{C}P^n) \overset{j^*}{\longleftarrow} \widetilde{K}(\Sigma^4\mathbb{C}P^n) \longleftarrow 0.$$

Let ν_2 be a complex vector bundle whose stable class is a generator of $\widetilde{K}(S^4)$. We can take ν_2 as $\dim_{\mathbb{C}}\nu_2 = 2$. From the previous argument, considering S^4 as $\Sigma^2\mathbb{C}P^1$, we see that $c_2(\nu_2) = -s_2$, where s_2 is the generator of $H^4(S^4;\mathbb{Z})$ corresponding to $s \times t$. Now we take a complex vector bundle α over $\Sigma^4\mathbb{C}P^n$ such that $j^*\alpha$ is stably equivalent to $(p_1^*\nu_2 - 2) \otimes_{\mathbb{C}} (p_2^*\eta - 1)$. By Lemma 2.1, in $H^*(S^4 \times \mathbb{C}P^n;\mathbb{Z})$ we have

$$\begin{aligned} C(j^*\alpha) &= C((p_1^*\nu_2 - 2) \otimes_{\mathbb{C}} (p_2^*\eta - 1)) \\ &= 1 + c_2(\nu_2) \times ((1 + c_1(\eta))^{-2} - 1) \\ &= 1 - s_2 \times (-2t + 3t^2 - 4t^3 + \cdots). \end{aligned}$$

Hence we have $j^*c_4(\alpha) = -3\, s_2 \times t^2 \not\equiv 0 \pmod 2$ for $n \geqslant 2$. Therefore we have $w_8(\alpha) \neq 0$, so that $\Sigma^4\mathbb{C}P^n$ is not W-trivial for $n \geqslant 2$. □

Here, before we proceed to consider $\Sigma^k\mathbb{C}P^n$ for $k=6$ or 8, we need to prepare a lemma concerning Steenrod operations. For a non-negative integer m, let $\alpha(m)$ denote the number of ones in the dyadic expansion of m. It is easy to see that if m and ℓ are positive integers such that $\binom{m}{\ell}\equiv 1 \pmod 2$, then $\alpha(m+\ell)\leqslant\alpha(m)$. Also, we clearly have $\alpha(2^{\ell+1}-k)=\alpha(2^{\ell}-k)+1$ for any integer k with $0<k\leqslant 2^{\ell}$, whence we have $\alpha(2^{r-1}-k)>\alpha(2^{j-1}-k)$ for positive integers j and r with $j<r$. Thus, we obtain the following lemma.

Lemma 5.2. *If $k>0$, any Steenrod operation $\varphi\colon H^{2^j}(\Sigma^{2k}\mathbb{C}P^n)\to H^{2^r}(\Sigma^{2k}\mathbb{C}P^n)$ is trivial for $j<r$.*

Now, we are ready to consider $\Sigma^k\mathbb{C}P^n$ for $k=6$ or 8. We have the following result.

Proposition 5.3. *$\Sigma^6\mathbb{C}P^n$ and $\Sigma^8\mathbb{C}P^n$ are W-trivial for all $n>1$.*

Proof. Let α be a vector bundle over $\Sigma^k\mathbb{C}P^n$, where $k=6$ or 8, and let r be the smallest integer such that $w_{2^r}(\alpha)$ is (possibly) non-zero. Clearly we have $r\geqslant 3$ when $k=6$, and $r\geqslant 4$ when $k=8$. Since $\mathrm{Sq}^2 w_8(\alpha)=0$ by Lemma 3.3 and also $\mathrm{Sq}^2\colon H^8(\Sigma^6\mathbb{C}P^n)\to H^{10}(\Sigma^6\mathbb{C}P^n)$ is non-trivial, we have $w_8(\alpha)=0$. So we may suppose that $r\geqslant 4$ also when $k=6$. To prove $w_{2^r}(\alpha)=0$ for $r\geqslant 4$, the above method fails depending on the value of $n+k$. So we use secondary operations. Let $T(\alpha)$ be the Thom space of α and denote the Thom class by U; $U\in H^m(D(\alpha),S(\alpha))=\widetilde{H}^m(T(\alpha))$, where $m=\dim\alpha$. Since $\mathrm{Sq}^\ell U=w_\ell(\alpha)U$ $(\ell>0)$, we have $\mathrm{Sq}^\ell U=0$ for $\ell<2^r$. Since $r\geqslant 4$, secondary operations on U are defined. Indeed, for integers i,j with $0\leqslant i\leqslant j<r$ $(i\neq j-1)$, $\Phi_{i,j}(U)\in H^{m+d(i,j)}(T(\alpha))$ is defined with an indeterminacy $Q^{m+d(i,j)}(T(\alpha);i,j)$, where $d(i,j)=2^i+2^j-1$, and the following formula holds:

$$[\mathrm{Sq}^{2^r}U]=\sum_{\substack{0\leqslant i\leqslant j<r\\ i\neq j-1}} a_{i,j}\Phi_{i,j}(U)\quad\text{modulo}\quad\sum_{\substack{0\leqslant i\leqslant j<r\\ i\neq j-1}} a_{i,j}Q^{m+d(i,j)}(T(\alpha);i,j),$$

where each $a_{i,j}$ is a certain Steenrod operation (see [**1**, Theorem 4.6.1]). Now let us investigate each summand in this decomposition of $\mathrm{Sq}^{2^r}U$. We divide into two cases depending on whether i is zero or not.

Case 1: $i\neq 0$.

In this case, $d(i,j)$ is odd, so that we have $H^{d(i,j)}(\Sigma^k\mathbb{C}P^n)=0$ $(k=6,8)$. Hence, by the Thom isomorphism, we have $H^{m+d(i,j)}(T(\alpha))=0$, so that $\Phi_{i,j}(U)=0$ and $Q^{m+d(i,j)}(T(\alpha);i,j)=0$.

Case 2: $i=0$.

In this case, $d(i,j)=2^j$. Therefore, it follows that $a_{i,j}$ is an operation $H^{m+2^j}(T(\alpha))\to H^{m+2^r}(T(\alpha))$. We claim that the following diagram commutes, where the vertical maps are the Thom isomorphisms.

$$\begin{array}{ccc} H^{m+2^j}(T(\alpha)) & \xrightarrow{a_{i,j}} & H^{m+2^r}(T(\alpha)) \\ \cong\uparrow & & \cong\uparrow \\ H^{2^j}(\Sigma^k\mathbb{C}P^n) & \xrightarrow{a_{i,j}} & H^{2^r}(\Sigma^k\mathbb{C}P^n). \end{array}$$

In fact, for $x\in H^*(\Sigma^k\mathbb{C}P^n)$ and $h\leqslant 2^r-2^j$, we have $\mathrm{Sq}^h(xU)=\mathrm{Sq}^h x\cdot U$ by the

Cartan formula since $\mathrm{Sq}^{\ell}U = w_\ell(\alpha)U = 0$ for $0 < \ell < 2^r$. Thus we have $a_{i,j}(xU) = a_{i,j}x \cdot U$ for $x \in H^{2^j}(\Sigma^k \mathbb{C}P^n)$, so that the diagram commutes. Now, in the above diagram, the lower $a_{i,j}$ is trivial by Lemma 5.2. Therefore, we see that the upper $a_{i,j}$ is also trivial.

Therefore, from the arguments in Cases 1 and 2, we obtain $[\mathrm{Sq}^{2^r}U] = 0$ modulo 0, that is, $\mathrm{Sq}^{2^r}U = 0$. Since $\mathrm{Sq}^{2^r}U = w_{2^r}(\alpha)U$, we conclude that $w_{2^r}(\alpha) = 0$. This completes the proof of the proposition. □

The proof of Theorem 1.5 is completed by Propositions 5.1 and 5.3.

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