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# HOMOTOPY THEORY OF DG CATEGORIES VIA LOCALIZING PAIRS AND DRINFELD'S DG QUOTIENT

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(*communicated by Johannes Huebschmann*)

*Abstract*

Using localizing pairs and Drinfeld's dg quotient we construct a new Quillen model for the homotopy theory of dg categories. We prove that, in contrast with the original model, this new Quillen model carries a natural closed symmetric monoidal structure. As an application, we obtain a simple construction of the internal Hom-objects and a conceptual characterization of Toën's previous work. Making use of this new Quillen model, Lowen has recently developed a derived deformation theory.

## 1. Introduction

A differential graded (=dg) category is a category enriched in the category of complexes of modules over some commutative base ring $k$. Dg categories enhance triangulated categories and are nowadays an important working tool in algebraic geometry, non-commutative geometry, representation theory, mathematical physics, and other areas [**1, 3, 4, 10, 11, 22**]. This new philosophy of enhancing triangulated categories has motivated much foundational work as described in Keller's ICM adress [**8**]. For example, Verdier's triangulated quotient, one of the main tools in triangulated categories, has recently been lifted to the world of dg categories by Drinfeld [**3**, §3.1].

Using Quillen's homotopical algebra formalism, we have constructed in [**20**] the homotopy theory of dg categories with respect to the *Morita dg functors* (i.e., the dg functors $F\colon \mathcal{A} \to \mathcal{B}$ which induce an equivalence $\mathcal{D}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$ between the derived categories). This theory enabled several developments such as the first conceptual characterization [**19**] of Quillen-Waldhausen's $K$-theory [**17, 23**] and the creation by Toën of a derived Morita theory [**22**]. One geometric application of Toën's derived Morita theory, described in [**22**, Theorem 8.9], is the solution of a conjecture due to Orlov [**15**]: all dg functors between derived dg categories of algebraic varieties are induced (using a Fourier-Mukai transform) by some object in the derived dg category of their product.

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The main obstacle in the development of this derived Morita theory was the construction of an internal $\mathsf{Hom}$-functor in the homotopy category $\mathsf{Ho}(\mathsf{dgcat})$ of dg categories. The difficulty is that the Quillen model structure [**20**, Théorème 5.3] (see also [**21**]) on $\mathsf{dgcat}$ is not compatible with the tensor product. Quoting Toën [**22**]:

> The model category $\mathsf{dgcat}$ together with the symmetric monoidal structure $-\otimes-$ is not a symmetric monoidal category, as the tensor product of two cofibrant objects in $\mathsf{dgcat}$ is not cofibrant in general. A direct consequence of this fact is that the internal Hom object between cofibrant-fibrant objects in $\mathsf{dgcat}$ can not be invariant by quasi-equivalences, and thus does not provide internal Hom's for the homotopy category $\mathsf{Ho}(\mathsf{dgcat})$. This is the main difficulty in computing the mapping spaces in $\mathsf{dgcat}$, as the naive approach simply does not work.

In this article we solve this problem, using localizing pairs and Drinfeld's explicit dg quotient construction. We start by constructing in Theorem 4.18 a new Quillen model $\mathsf{Lp}$ for the Morita homotopy category of dg categories. Its objects are the localization pairs, i.e., the inclusions $\mathcal{A}_0 \subset \mathcal{A}_1$ of full dg subcategories, and a morphism $\mathcal{A} \to \mathcal{A}'$ is a weak equivalence if and only if it induces a Morita dg functor $\mathcal{A}_1/\mathcal{A}_0 \to \mathcal{A}'_1/\mathcal{A}'_0$ in Drinfeld's dg quotient. We show in Proposition 5.4 that $\mathsf{Lp}$ carries a natural closed symmetric monoidal structure $(\mathsf{Lp}, -\otimes-, \mathsf{Hom}(-,-))$. Our first main theorem asserts that, in contrast with the case of $\mathsf{dgcat}$, the internal $\mathsf{Hom}$-functor of $\mathsf{Lp}$ admits a right derived functor:

**Theorem 1.1** (see Theorem 6.4)**.** *The internal* $\mathsf{Hom}$*-functor*

$$\mathsf{Hom}(-,-)\colon \mathsf{Lp}^{op} \times \mathsf{Lp} \to \mathsf{Lp},$$

*admits a right derived functor*

$$\mathcal{R}\mathsf{Hom}(-,-)\colon \mathsf{Ho}(\mathsf{Lp}^{op} \times \mathsf{Lp}) \to \mathsf{Ho}(\mathsf{Lp}).$$

We then relate the new Quillen model $\mathsf{Lp}$ with the Morita homotopy theory of dg categories via the Quillen equivalence (see Proposition 7.1)

$$\begin{array}{c}\mathsf{Lp}\\ L\uparrow\ \downarrow Ev_1\\ \mathsf{dgcat},\end{array}$$

where $Ev_1$ associates to a localization pair $(\mathcal{A}_0 \subset \mathcal{A}_1)$ the dg category $\mathcal{A}_1$ and $L$ associates to a dg category $\mathcal{A}$ the localization pair $(\emptyset \subset \mathcal{A})$. Finally, we prove our second main theorem, which asserts that our derived functor $\mathcal{R}\mathsf{Hom}(-,-)$ agrees with Toën's adhoc construction $\mathsf{rep}_{dg}(-,-)$ (see Remark 5.1):

**Theorem 1.2** (see Theorem 7.2)**.** *The derived functors* $-\overset{\mathbb{L}}{\otimes}-$ *and* $\mathcal{R}\mathsf{Hom}(-,-)$ *in*

$\mathsf{Ho}(\mathsf{Lp})$ *agree, under the equivalence*

$$\begin{array}{c} \mathsf{Ho}(\mathsf{Lp}) \\ L \uparrow \downarrow \mathcal{R}Ev_1 \\ \mathsf{Ho}(\mathsf{dgcat}), \end{array}$$

*with the functors* $-\overset{\mathbb{L}}{\otimes}-$ *and* $\mathsf{rep}_{dg}(-,-)$ *in* $\mathsf{Ho}(\mathsf{dgcat})$ *constructed previously by Toën.*

In conclusion, the new Quillen model $\mathsf{Lp}$ provides a simple way to construct the internal $\mathsf{Hom}$-objects in $\mathsf{Ho}(\mathsf{dgcat})$. In contrast with Toën's approach, requiring an involved dg category of "right quasi-representable" bimodules (see Remark 5.1), when using the model $\mathsf{Lp}$ it is enough to derive its natural internal $\mathsf{Hom}$-functor (see Definition 5.2) which only makes use of dg categories of dg functors. We remind the reader that the construction of the internal $\mathsf{Hom}$-objects in $\mathsf{Hmo}$ was the main difficulty in the development of Toën's derived Morita theory [**22**, §7].

The new Quillen model $\mathsf{Lp}$ provides also a conceptual characterization of Toën's adhoc construction as a total derived functor. Intuitively, when we pass from dg categories to localization pairs, we gain an "extra degree of freedom" which allows us to perform derived constructions.

It is expected that, due to its "flexibility", the Quillen model $\mathsf{Lp}$ will be used in different contexts. For example, making use of it Lowen has recently developed a derived deformation theory [**12**] analogous to the deformation theory for abelian categories [**13**]

In the appendix we state a weaker form of the Bousfield-Friedlander localization theorem, which is of independent interest (see Theorem A.4).

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## 2. Background on dg categories

In what follows, $k$ will denote a field. The tensor product $\otimes$ will denote the tensor product over $k$. Let $\mathsf{Ch}(k)$ denote the category of $\mathbb{Z}$-graded complexes of $k$-modules. We consider co-homological notation, i.e., the differential increases the degree. If $M$ is a complex, we will denote by $M[1]$ its shift by one. Recall from [**5**, Theorem 2.3.11] that $\mathsf{Ch}(k)$ carries a projective model structure, whose weak equivalences are the quasi-isomorphisms and whose fibrations are the degreewise surjective maps. We now recall the notions concerning dg categories which will be used throughout the article. For a survey article on dg categories consult Keller's ICM adress [**8**].

**Definition 2.1.** [**8**, §2.2] A *differential graded (=dg) category* $\mathcal{A}$ consists of the following data:

- a class of objects $\mathrm{obj}(\mathcal{A})$ (usually denoted by $\mathcal{A}$ itself);
- for each ordered pair of objects $(X, Y)$ in $\mathcal{A}$, a complex of $k$-modules $\mathsf{Hom}_{\mathcal{A}}(X, Y)$;
- for each ordered triple of objects $(X, Y, Z)$ in $\mathcal{A}$, a composition morphism in $\mathsf{Ch}(k)$
$$\mathsf{Hom}_{\mathcal{A}}(Y, Z) \otimes \mathsf{Hom}_{\mathcal{A}}(X, Y) \to \mathsf{Hom}_{\mathcal{A}}(X, Z),$$
satisfying the usual associativity condition;
- for any object $X$ in $\mathcal{A}$, a morphism $k \to \mathsf{Hom}_{\mathcal{A}}(X, X)$ in $\mathsf{Ch}(k)$, i.e., a unit cocycle $\mathbf{1}_X$ of degree 0 in the complex $\mathsf{Hom}_{\mathcal{A}}(X, X)$, satisfying the usual unit condition with respect to the above composition.

*Remark 2.2.* If $\mathrm{obj}(\mathcal{A})$ is a set we say that $\mathcal{A}$ is a *small* dg category.

**Definition 2.3.** [**8**, §2.3] A *dg functor* $F\colon \mathcal{A} \to \mathcal{B}$ consists of the following data:

- a map $\mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$;
- for each ordered pair of objects $(X, Y)$ in $\mathcal{A}$, a morphism in $\mathsf{Ch}(k)$
$$F(X, Y)\colon \mathsf{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \mathsf{Hom}_{\mathcal{B}}(FX, FY),$$
satisfying the usual unit and associativity conditions.

*Notation 2.4.* We denote by $\mathsf{dgcat}$ the category of small dg categories.

Let $\mathcal{A}$ be a small dg category.

- The *opposite dg category* $\mathcal{A}^{op}$ of $\mathcal{A}$ has the same objects as $\mathcal{A}$ and complexes of morphisms
$$\mathsf{Hom}_{\mathcal{A}^{op}}(X, Y) := \mathsf{Hom}_{\mathcal{A}}(Y, X),$$
where the composition of $f \in \mathsf{Hom}_{\mathcal{A}^{op}}(X, Y)^p$ with $g \in \mathsf{Hom}_{\mathcal{A}^{op}}(Y, Z)^q$ is given by $(-1)^{pq} fg$.
- The *category* $\mathsf{Z}^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms given by
$$\mathsf{Hom}_{\mathsf{Z}^0(\mathcal{A})}(X, Y) := \mathsf{Z}^0(\mathsf{Hom}(X, Y)),$$
where $\mathsf{Z}^0$ is the kernel of $d\colon \mathsf{Hom}_{\mathcal{A}}(X, Y)^0 \to \mathsf{Hom}_{\mathcal{A}}(X, Y)^1$.
- The *homotopy category* $\mathsf{H}^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms given by
$$\mathsf{Hom}_{\mathsf{H}^0(\mathcal{A})}(X, Y) := \mathsf{H}^0(\mathsf{Hom}(X, Y)),$$
where $\mathsf{H}^0$ is the 0-th cohomology of the complex $\mathsf{Hom}_{\mathcal{A}}(X, Y)$.

*Notation 2.5.* We say that a morphism $f\colon X \to Y$ in $\mathsf{Z}^0(\mathcal{A})$ is a *homotopy equivalence* if it becomes invertible in $\mathsf{H}^0(\mathcal{A})$. Two objects $X$ and $Y$ in $\mathcal{A}$ are *homotopy equivalent* if there exists a homotopy equivalence between the two.

An object $X$ in a dg category $\mathcal{A}$ is called *contractible* if the dg algebra $\mathsf{Hom}_{\mathcal{A}}(X,X)$ is acyclic. Notice that $X$ is contractible if and only if the unit cocycle $\mathbf{1}_X$ is a coboundary in the complex $\mathsf{Hom}_{\mathcal{A}}(X,X)$.

**Definition 2.6.** [**8**, §3.1] A *right dg $\mathcal{A}$-module $M$* consists of the following data:

- for each object $X$ in $\mathcal{A}$, a complex of $k$-modules $M(X)$;
- morphisms of complexes

$$M(Y)\otimes \mathsf{Hom}_{\mathcal{A}}(X,Y)\longrightarrow M(X) \quad X,Y\in\mathcal{A},$$

compatible with compositions and units.

*Remark 2.7.* A *left dg $\mathcal{A}$-module* is by definition a right dg $\mathcal{A}^{op}$-module.

*Notation 2.8.* [**8**, §3.1] We denote by $\mathcal{C}_{dg}(\mathcal{A})$ the dg category of right dg $\mathcal{A}$-modules and by

$$\begin{array}{rcl}\widehat{?}\colon \mathcal{A} & \to & \mathcal{C}_{dg}(\mathcal{A})\\ X & \mapsto & \widehat{X} := \mathsf{Hom}_{\mathcal{A}}(?,X)\end{array}$$

the Yoneda dg functor, which associates to $X$ its *representable dg $\mathcal{A}$-module* $\widehat{X}$.

**Definition 2.9.** A dg category $\mathcal{A}$ is *stable under cones* if the following condition holds:

- for each morphism $f\colon A\to B$ of $\mathsf{Z}^0(\mathcal{A})$, there is an object $\mathrm{cone}(f)$ of $\mathcal{A}$ and an isomorphism of right dg $\mathcal{A}$-modules

$$\widehat{\mathrm{cone}(f)} \xrightarrow{\sim} \mathrm{cone}(\widehat{f}),$$

where $\mathrm{cone}(\widehat{f})$ denotes the cone of $\widehat{f}$ in $\mathcal{C}_{dg}(\mathcal{A})$.

*Remark 2.10.* Notice that a morphism $f\colon X\to Y$ in $\mathsf{Z}^0(\mathcal{A})$ is a homotopy equivalence if and only if $\mathrm{cone}(\widehat{f})$ is contractible in $\mathcal{C}_{dg}(\mathcal{A})$.

Recall from [**1**] or [**3**, §2.4] the construction of the functorial *pre-triangulated envelope* $\mathsf{pre\text{-}tr}(\mathcal{A})$ of a dg category $\mathcal{A}$. The idea of the construction is to formally add to $\mathcal{A}$ all cones, cones of morphisms between cones, $\ldots$. The objects of $\mathsf{pre\text{-}tr}(\mathcal{A})$ are formal expressions $C=(\bigoplus_{i=0}^n C_i[r_i],q)$, where $C_i\in\mathcal{A}$, $r_i\in\mathbb{Z}$, $n\geqslant 0$, $q=(q_{ij})$, $q_{ij}\in \mathsf{Hom}^1_{\mathcal{A}}(C_j,C_i)[r_i-r_j]$, $q_{ij}=0$ for $i\geqslant j$, and $dq+q^2=0$. The complex of morphisms $\mathsf{Hom}_{\mathsf{pre\text{-}tr}(\mathcal{A})}(C,C')$ is the space of matrices $f=(f_{ij})$, $f_{ij}\in\mathsf{Hom}(C_j,C'_i)[r'_i-r_j]$, and the composition map is given by matrix multiplication. As is shown in [**1**], one has a canonical fully-faithful dg functor $\mathsf{pre\text{-}tr}(\mathcal{A})\to\mathcal{C}_{dg}(\mathcal{A})$. In what follows we identify the dg category $\mathsf{pre\text{-}tr}(\mathcal{A})$ with its image in $\mathcal{C}_{dg}(\mathcal{A})$.

### 2.1. Quasi-equivalences

It is proven in [**21**, Théorème 2.1] and in [**18**, Proposition 1.13] that the category $\mathsf{dgcat}$ admits a cofibrantly generated Quillen model structure whose weak equivalences and fibrations are as follows:

**Definition 2.11.** A dg functor $F\colon \mathcal{A} \to \mathcal{B}$ is a *quasi-equivalence* if:

- the morphism of complexes

$$F(X,Y)\colon \mathsf{Hom}_{\mathcal{A}}(X,Y) \to \mathsf{Hom}_{\mathcal{B}}(FX,FY)$$

is a quasi-isomorphism for all objects $X, Y \in \mathcal{A}$ and

- the induced functor $\mathsf{H}^0(F)\colon \mathsf{H}^0(\mathcal{A}) \to \mathsf{H}^0(\mathcal{B})$ is essentially surjective.

**Definition 2.12.** A dg functor $F\colon \mathcal{A} \to \mathcal{B}$ is a *fibration* if:

(F1) for all objects $X, Y \in \mathcal{A}$, the induced morphism of complexes

$$F(X,Y)\colon \mathsf{Hom}_{\mathcal{A}}(X,Y) \to \mathsf{Hom}_{\mathcal{B}}(FX,FY)$$

is a degreewise surjection and

(F2) given an object $X$ in $\mathcal{A}$ and a homotopy equivalence $v\colon F(X) \to Y$ in $\mathcal{B}$ (see Notation 2.5), there exists a homotopy equivalence $u\colon X \to X'$ in $\mathcal{A}$, such that $F(X') = Y$ and $F(u) = v$.

*Remark 2.13.* Since the terminal object in $\mathsf{dgcat}$ is the zero category 0 (one object and trivial dg algebra of endomorphisms), Definition 2.12 implies that every object in $\mathsf{dgcat}$ is fibrant.

The cofibrations are defined by left lifting property with respect to the class of trivial fibrations. An explicit set $I = \{Q, S(n)\}$ of generating cofibrations is given as follows (see [**21**, §2]):

**Definition 2.14** (Generating cofibrations)**.** (i) Let $\underline{k}$ be the dg category with one object 3, such that $\mathsf{Hom}_{\underline{k}}(3,3) := k$ (in degree zero) and $Q\colon \emptyset \to \underline{k}$ the unique dg functor (where the empty dg category $\emptyset$ is the initial object in $\mathsf{dgcat}$).

(ii) For $n \in \mathbb{Z}$, let $S^n$ be the complex $k[n]$ (with $k$ concentrated in degree $n$) and let $D^n$ be the mapping cone on the identity of $S^{n-1}$. We denote by $\mathcal{C}(n)$ the dg category with two objects 4 and 5 such that

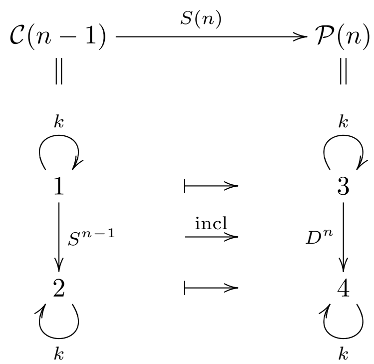
$$\mathsf{Hom}_{\mathcal{C}(n)}(4,4) = \mathsf{Hom}_{\mathcal{C}(n)}(5,5) = k, \quad \mathsf{Hom}_{\mathcal{C}(n)}(5,4) = 0, \quad \mathsf{Hom}_{\mathcal{C}(n)}(4,5) = S^n,$$

and with composition given by multiplication. We denote by $\mathcal{P}(n)$ the dg category with two objects 6 and 7 such that

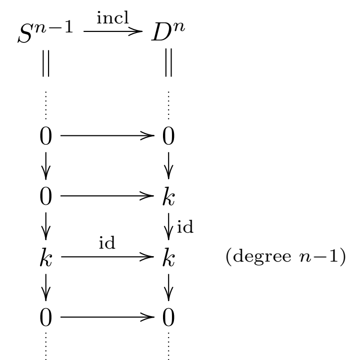
$$\mathsf{Hom}_{\mathcal{P}(n)}(6,6) = \mathsf{Hom}_{\mathcal{P}(n)}(7,7) = k, \quad \mathsf{Hom}_{\mathcal{P}(n)}(7,6) = 0, \quad \mathsf{Hom}_{\mathcal{P}(n)}(6,7) = D^n,$$

and with composition given by multiplication. Finally, let $S(n)\colon \mathcal{C}(n-1) \to \mathcal{P}(n)$ be the dg functor that sends 4 to 6, 5 to 7 and $S^{n-1}$ to $D^n$ by the identity on $k$ in degree

$n-1$:




where




Recall from [**22**, Proposition 2.3] that if $\mathcal{A}$ is a cofibrant dg category then for all objects $X, Y \in \mathcal{A}$, the complex $\mathsf{Hom}_{\mathcal{A}}(X,Y)$ is cofibrant in the projective model structure on $\mathsf{Ch}(k)$.

## 2.2. Morita dg functors

We have also constructed in [**20**, Théorème 5.3] a cofibrantly generated *Morita model structure* on $\mathsf{dgcat}$ whose weak equivalences are the *Morita dg functors*, i.e., the dg functors $F\colon \mathcal{A} \to \mathcal{B}$ which induce an equivalence $\mathcal{D}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$ between derived categories. Notice that a quasi-equivalence is a particular case of a Morita dg functor.

*Notation 2.15.* We will denote by $\mathsf{Hmo}$ the Morita homotopy category of dg categories obtained.

The Morita model structure has the same cofibrations and fewer fibrations than the model structure of subsection 2.1. In particular, its set of generating cofibrations is the one of Definition 2.14. Its set

$$J = \{F(n), R(n), I_n(k_0, \dots, k_n), Lh_n(k_0, \dots, k_n), C\}$$

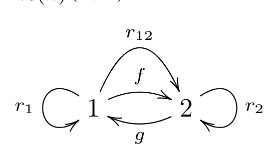
of generating trivial cofibrations is given as follows (see [**18**, §2]):

**Definition 2.16** (Generating trivial cofibrations)**.** (i) Let $\mathcal{B}$ be the dg category with two objects 8 and 9 such that

$$\mathsf{Hom}_{\mathcal{B}}(8,8) = \mathsf{Hom}_{\mathcal{B}}(9,9) = k, \quad \mathsf{Hom}_{\mathcal{B}}(8,9) = \mathsf{Hom}_{\mathcal{B}}(9,8) = 0,$$

and with composition given by multiplication. For $n \in \mathbb{Z}$, we denote by $R(n)\colon \mathcal{B} \to \mathcal{P}(n)$ the dg functor that sends 8 to 6 and 9 to 7.

(ii) For $n \in \mathbb{Z}$, let $\mathcal{K}(n)$ be the dg category with two objects 1 and 2, and complexes of morphisms generated by the morphisms $f \in \mathsf{Hom}_{\mathcal{K}(n)}(1,2)^n$, $g \in \mathsf{Hom}_{\mathcal{K}(n)}(2,1)^{-n}$, $r_1 \in \mathsf{Hom}_{\mathcal{K}(n)}(2,2)^{-1}$, $r_2 \in \mathsf{Hom}_{\mathcal{K}(n)}(2,2)^{-1}$ and $r_{12} \in \mathsf{Hom}_{\mathcal{K}(n)}(1,2)^{n-1}$




subject to the relations $d(f) = d(g) = 0, d(r_1) = g \circ f - \mathbf{1}_1, d(r_2) = f \circ g - \mathbf{1}_2$ and

$d(r_{12}) = f \circ r_1 - r_2 \circ f$. We denote by $F(n) \colon \underline{k} \to \mathcal{K}(n)$ the dg functor that sends 3 to 1. When $n = 0$, $\mathcal{K}(0)$ is denoted by $\mathcal{K}$ and $F(0)$ by $F$.

(iii) For $n > 0$ and $k_0, \ldots, k_n \in \mathbb{Z}$, let $\mathcal{M}_n(k_0, \ldots, k_n)$ be the dg category with $n+1$ objects $0, \ldots, n$ and complexes of morphisms generated by the morphisms $q_{i,j}, 0 \leqslant j < i \leqslant n$, (from $j$ to $i$ of degree $k_i - k_j + 1$) subject to the relations

$$d(Q) + Q^2 = 0,$$

where $Q$ is the lower triangular matrix with coefficients $q_{i,j}$. We denote by $\mathrm{cone}_n(k_0, \ldots, k_n)$ the full dg subcategory of right dg $\mathcal{M}_n(k_0, \ldots, k_n)$-modules whose objects are $\hat{l}, 0 \leqslant l \leqslant n$ (see Notation 2.8) and the *iterated cone* $X_n$, i.e., the graded module

$$X = \bigoplus_{l=0}^{n} \hat{l}[k_l]$$

with differential $d_X + \widehat{Q}$. Let $L \colon \underline{k} \to \mathrm{cone}_n(k_0, \ldots, k_n)$ be the dg functor that sends 3 to $X_n$. Consider the following pushout

$$\begin{array}{ccc} \underline{k} & \xrightarrow{\;L\;} & \mathrm{cone}_n(k_0, \ldots, k_n) \\ {\scriptstyle F}\downarrow & \lrcorner & \downarrow \\ \mathcal{K} & \longrightarrow & \mathrm{cone}_n(k_0, \ldots, k_n) \coprod_{\underline{k}} \mathcal{K}. \end{array}$$

We define $\mathrm{coneh}_n(k_0, \ldots, k_n)$ as the full dg subcategory of the pushout, whose objects are the images of the objects $\hat{l}, 0 \leqslant l \leqslant n$, in $\mathrm{cone}_n(k_0, \ldots, k_n)$ and of the object 2 in $\mathcal{K}$. Finally, we denote by $I_n(k_0, \ldots, k_n)$ the dg functor from $\mathcal{M}_n(k_0, \ldots, k_n)$ to $\mathrm{coneh}_n(k_0, \ldots, k_n)$.

(iv) For $n > 0$ and $k_0, \ldots, k_n$, let $\mathrm{idem}_n(k_0, \ldots, k_n)$ be the dg category obtained from $\mathcal{M}_n(k_0, \ldots, k_n)$ by adding new generators and new relations: we add morphism $e_{i,j}, 0 \leqslant i, j \leqslant n$ (from $j$ to $i$ of degree $k_i - k_j$) subject to the relations

$$d(E) = 0 \quad \text{and} \quad E^2 = E,$$

where $E$ is the matrix with coefficients $e_{i,j}$. We denote by $\mathrm{fact}_n(k_0, \ldots, k_n)$ the full dg subcategory of right dg $\mathrm{idem}_n(k_0, \ldots, k_n)$-modules whose objects are $\hat{l}, 0 \leqslant l \leqslant n$, and the direct factor associated to the idempotent $\widehat{E}$ if the iterated cone $X_n$. Let $L_n(k_0, \ldots, k_n)$ be the natural dg functor from $\mathrm{idem}_n(k_0, \ldots, k_n)$ to $\mathrm{fact}_n(k_0, \ldots, k_n)$. We consider $L_n(k_0, \ldots, k_n)$ as an object in the category of morphisms in $\mathsf{dgcat}$, endowed with the canonical model structure (weak equivalences are quasi-equivalences in each component). Finally, we choose a cofibrant replacement

$$Lh_n(k_0, \ldots, k_n) \colon \mathrm{idemh}_n(k_0, \ldots, k_n) \longrightarrow \mathrm{facth}_n(k_0, \ldots, k_n)$$

of that object.

(v) Let $\mathcal{B}_0$ be the dg category with one object 10, and complexes of morphisms generated by the morphism $h \in \mathsf{Hom}_{\mathcal{B}_0}(10, 10)^{-1}$ subject to the relation $d(h) = \mathbf{1}_{10}$. We denote by $C : \emptyset \to \mathcal{B}_0$ the unique dg functor.

Finally, recall that the Morita fibrant dg categories admit the following simple characterization.

**Proposition 2.17.** *[**18**, Proposition 2.34] The Morita fibrant dg categories are the non-empty dg categories whose essential image of the embedding* $\widehat{\ }\colon \mathsf{H}^0(\mathcal{A}) \hookrightarrow \mathcal{D}(\mathcal{A})$ *is stable under (co)suspensions, cones and direct factors.*

## 3. Homotopy of dg functors

In this chapter we consider always the model structure on $\mathsf{dgcat}$ whose weak equivalences are the quasi-equivalences (Definition 2.11). Let $\mathcal{B}$ be a small dg category.

**Definition 3.1.** Let $P(\mathcal{B})$ be the dg category defined as follows: its objects are the degree zero morphisms in $\mathcal{B}$

$$X \xrightarrow{f} Y$$

which become invertible in $\mathsf{H}^0(\mathcal{B})$. The complexes of morphisms are defined (as $\mathbb{Z}$-graded $k$-vector spaces)

$$\mathsf{Hom}_{P(\mathcal{B})}(X \overset{f}{\to} Y, W \overset{g}{\to} Z) := \mathsf{Hom}_{\mathcal{B}}(X,W) \oplus \mathsf{Hom}_{\mathcal{B}}(Y,Z) \oplus \mathsf{Hom}_{\mathcal{B}}(X,Z)[1].$$

Notice that a homogenous element of degree $r$ of the above $\mathbb{Z}$-graded $k$-vector space, can be represented by a matrix

$$\begin{bmatrix} m_1 & 0 \\ h & m_2 \end{bmatrix},$$

where $m_1 \in \mathsf{Hom}_{\mathcal{B}}(X,W)^r, m_2 \in \mathsf{Hom}_{\mathcal{B}}(Y,Z)^r$ and $h \in \mathsf{Hom}_{\mathcal{B}}(X,Z)^{r-1}$. With this notation, the differential is defined as

$$d(\begin{bmatrix} m_1 & 0 \\ h & m_2 \end{bmatrix}) \quad := \quad \begin{bmatrix} d(m_1) & 0 \\ d(h) + g \circ m_1 - (-1)^r(m_2 \circ f) & d(m_2) \end{bmatrix}.$$

Composition in $P(\mathcal{B})$ corresponds to matrix multiplication and the units to the identity matrices.

*Remark 3.2.* Notice that the complex of morphisms

$$\mathsf{Hom}_{P(\mathcal{B})}(X \overset{f}{\to} Y, W \overset{g}{\to} Z)$$

corresponds to the homotopy pull-back, in $\mathsf{Ch}(k)$, of the diagram

$$\begin{array}{ccc} & & \mathsf{Hom}_{\mathcal{B}}(Y,Z) \\ & & \downarrow{\scriptstyle f^*} \\ \mathsf{Hom}_{\mathcal{B}}(X,W) & \xrightarrow[g_*]{} & \mathsf{Hom}_{\mathcal{B}}(X,Z). \end{array}$$

Moreover we have an inclusion dg functor

$$i\colon \mathcal{B} \longrightarrow P(\mathcal{B})$$

which maps an object $X \in \mathcal{B}$ to $(X = X)$ and a projection dg functor

$$p_0 \times p_1 \colon P(\mathcal{B}) \longrightarrow \mathcal{B} \times \mathcal{B}$$

which maps an object $(X \overset{f}{\to} Y)$ to $(X, Y)$. In conclusion, we have the following commutative diagram in $\mathsf{dgcat}$

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Delta} & \mathcal{B} \times \mathcal{B} \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle p_0 \times p_1} & \\ & P(\mathcal{B}) & \end{array} .$$

**Proposition 3.3.** *The dg category $P(\mathcal{B})$ is a path object for $\mathcal{B}$ (i.e., $i$ is a weak equivalence and $p_0 \times p_1$ a fibration).*

*Proof.* We prove first that the dg functor $i$ is a quasi-equivalence (Definition 2.11). Notice that the dg functor $i$ induces a quasi-isomorphism in $\mathsf{Ch}(k)$

$$\mathsf{Hom}_{\mathcal{B}}(X, Y) \xrightarrow{\sim} \mathsf{Hom}_{P(\mathcal{B})}(i(X), i(Y)),$$

for all objects $X, Y \in \mathcal{B}$. We now show that the functor $\mathsf{H}^0(i)$ is also essentially surjective. In fact, let $X \overset{f}{\to} Y$ be an object of $P(\mathcal{B})$. Consider the following morphism in $P(\mathcal{B})$ from $i(X)$ to $X \overset{f}{\to} Y$,

$$\begin{array}{ccc} X & \xrightarrow{Id} & X \\ \Big\| & {\scriptstyle h=0} & \Big\downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

Notice that this morphism in $P(\mathcal{B})$ becomes an isomorphism in $\mathsf{H}^0(P(\mathcal{B}))$ since $f$ becomes invertible in $\mathsf{H}^0(\mathcal{B})$. This shows that the dg functor $i$ is a quasi-equivalence. We now prove that the dg functor $p_0 \times p_1$ is a fibration (Definition 2.12). Notice first, that by definition of $P(\mathcal{B})$, the dg functor $p_0 \times p_1$ induces a degreewise surjective morphism in $\mathsf{Ch}(k)$

$$\mathsf{Hom}_{P(\mathcal{B})}(X \overset{f}{\to} Y, W \overset{g}{\to} Z) \xrightarrow{p_0 \times p_1} \mathsf{Hom}_{\mathcal{B}}(X, W) \times \mathsf{Hom}_{\mathcal{B}}(Y, Z),$$

for all objects $X \overset{f}{\to} Y$ and $W \overset{g}{\to} Z$ in $P(\mathcal{B})$. This shows condition (F1).

We now show that contractions lift along the dg functor $P(\mathcal{B}) \overset{p_0 \times p_1}{\longrightarrow} \mathcal{B} \times \mathcal{B}$. Let $X \overset{f}{\to} Y$ be an object of $P(\mathcal{B})$. Notice that a contraction of $X \overset{f}{\to} Y$ in $P(\mathcal{B})$ corresponds exactly to the following morphisms in $\mathcal{B}$: $c_X \in \mathsf{Hom}_{\mathcal{B}}^{-1}(X, X)$, $c_Y \in \mathsf{Hom}_{\mathcal{B}}^{-1}(Y, Y)$ and $h \in \mathsf{Hom}_{\mathcal{B}}^{-2}(X, Y)$ subject to the relations $d(c_X) = \mathbf{1}_X$, $d(c_Y) = \mathbf{1}_Y$ and $d(h) = c_Y \circ f + f \circ c_X$. Suppose that we have a contraction $(c_1, c_2)$ of $(X, Y)$ in $\mathcal{B} \times \mathcal{B}$. Notice that we can lift it by considering $c_X = c_1$, $c_Y = c_2$ and $h = c_2 \circ f \circ c_1$. Moreover, since we have the following equivalence of dg categories

$$\mathsf{pre\text{-}tr}(P(\mathcal{B})) \xrightarrow{\sim} P(\mathsf{pre\text{-}tr}(\mathcal{B})),$$

contractions lift also along the dg functor $\mathsf{pre\text{-}tr}(p_0 \times p_1)$. This allows us to prove condition (F2) as follows: let $X \overset{f}{\to} Y$ be an object in $P(\mathcal{B})$ and $M$ a degree zero morphism in $\mathcal{B} \times \mathcal{B}$ from $(p_0 \times p_1)(f)$ to $(W, Z)$ which becomes invertible in

$\mathsf{H}^0(\mathcal{B}\times\mathcal{B})$. Since $p_0 \times p_1$ satisfies condition (F1), there exists an object $W \xrightarrow{g} Z$ in $P(\mathcal{B})$ and a morphism $\underline{M}$ in $\mathsf{Hom}_{P(\mathcal{B})}(X \xrightarrow{f} Y, W \xrightarrow{g} Z)$ such that $(p_0 \times p_1)(\underline{M}) = M$. Notice that the cone of $\widehat{\underline{M}}$ is sent to the cone of $\widehat{M}$ by the dg functor $\mathsf{pre\text{-}tr}(p_0 \times p_1)$. Finally, since contractions lift along $\mathsf{pre\text{-}tr}(p_0 \times p_1)$ and the cone of $\widehat{M}$ is contractible, Remark 2.10 allows us to conclude that $\underline{M}$ is a homotopy equivalence. This finishes the proof. □

Let $\mathcal{A}$ be a cofibrant dg category and $F, G\colon \mathcal{A} \to \mathcal{B}$ dg functors. Since every dg category is fibrant the dg functors $F$ and $G$ are homotopic (see [**6**, Definition 7.3.2]) if and only if there exists a dg functor $H\colon \mathcal{A} \to P(\mathcal{B})$ that makes the following diagram commute

$$\begin{array}{ccc} & & \mathcal{B} \\ & \overset{F}{\nearrow} & \uparrow{\scriptstyle P_0} \\ \mathcal{A} & \xrightarrow{H} & P(\mathcal{B}) \\ & \underset{G}{\searrow} & \downarrow{\scriptstyle P_1} \\ & & \mathcal{B}. \end{array}$$

*Remark 3.4.* Note that a dg functor $H$ as above corresponds exactly to:

- a morphism $\eta A\colon F(A) \to G(A)$ of $Z^0(\mathcal{B})$ which becomes invertible in $\mathsf{H}^0(\mathcal{B})$ for all $A \in \mathcal{A}$ (but which will not be functorial in $A$, in general) and
- a morphism of graded $k$-vector spaces, homogeneous of degree $-1$

$$h = h(A,B)\colon \mathsf{Hom}_{\mathcal{A}}(A,B) \to \mathsf{Hom}_{\mathcal{B}}(F(A), G(B)),$$

  for all $A, B \in \mathcal{A}$ such that we have

$$(\eta B)(F(f)) - (G(f)(\eta A) = d(h(f)) + h(d(f))$$

  and

$$h(fg) = h(f)(F(g)) + (-1)^n (G(f))h(g)$$

  for all composable morphisms $f, g$ of $\mathcal{A}$, where $f$ is of degree $n$.

It is shown in [**9**, §3.3] that if we have a dg functor $H$ as above and the dg category $\mathcal{B}$ is stable under cones (Definition 2.9), we can construct a sequence of dg functors

$$F \to I \to G[1],$$

where $I(A)$ is a contractible object (Notation 2.5) of $\mathcal{B}$, for all $A \in \mathcal{B}$.

## 4. The new Quillen model

In this chapter we construct the new Quillen model $\mathsf{Lp}$ (see Theorem 4.18) and characterize its fibrant objects (see Proposition 4.23).

**Definition 4.1.** A *localization pair* $\mathcal{A}$ is given by a small dg category $\mathcal{A}_1$ and a full dg subcategory $\mathcal{A}_0 \subset \mathcal{A}_1$. A *morphism* $F\colon \mathcal{A} \to \mathcal{B}$ of localization pairs is given by a commutative square

$$\begin{array}{ccc} \mathcal{A}_0 & \hookrightarrow & \mathcal{A}_1 \\ {\scriptstyle F_0}\downarrow & & \downarrow{\scriptstyle F_1} \\ \mathcal{B}_0 & \hookrightarrow & \mathcal{B}_1 \end{array}$$

of dg functors.

*Notation 4.2.* We denote by $\mathsf{Lp}$ the category of localization pairs.

*Notation 4.3.* Let $\mathcal{A}$ be a localization pair. We denote by $\mathcal{A}_1/\mathcal{A}_0$ its *Drinfeld's dg quotient* of $\mathcal{A}$, see [**3**, §3.1]. Recall that it consists of the dg category obtained from $\mathcal{A}_1$ by introducing a new morphism $\epsilon_U$ of degree $-1$ for every object $U$ in $\mathcal{A}_0$ and by imposing the relation $d(h_U) = \mathbf{1}_U$. The dg category $\mathcal{A}_1/\mathcal{A}_0$ has the same objects as $\mathcal{A}_1$ and for objects $X, Y \in \mathcal{A}_1$, we have an isomorphism of graded $k$-vector spaces (but not an isomorphism of complexes)

$$\bigoplus_{n=0}^{\infty} \mathsf{Hom}^n_{\mathcal{A}_1/\mathcal{A}_0}(X,Y) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{A}_1/\mathcal{A}_0}(X,Y),$$

where $\mathsf{Hom}^n_{\mathcal{A}_1/\mathcal{A}_0}(X,Y)$ is the direct sum of tensor products

$$\underbrace{\mathsf{Hom}_{\mathcal{A}_1}(X,U_1) \otimes k[1] \otimes \mathsf{Hom}_{\mathcal{A}_1}(U_1,U_2) \otimes \cdots \otimes k[1] \otimes \mathsf{Hom}_{\mathcal{A}_1}(U_n,Y)}_{n \text{ factors } k[1]},$$

with $U_i \in \mathcal{A}_0$ for $1 \leqslant i \leqslant n$. If we denote by $\epsilon$ the canonical generator of $k[1]$, the differential of an element

$$\underbrace{f_1 \otimes \epsilon \otimes f_2 \otimes \cdots \otimes \epsilon \otimes f_{n+1}}_{n \text{ factors } \epsilon}$$

is equal to

$$d(f_1) \otimes \epsilon \otimes f_2 \otimes \cdots \otimes \epsilon \otimes f_{n+1} + \underbrace{(-1)^{|f_1|} f_1 \otimes \mathbf{1}_{U_1} \otimes \cdots \otimes \epsilon \otimes f_{n+1}}_{(n-1) \text{ factors } \epsilon} + \cdots \quad .$$

This implies that, for every $j \geqslant 0$, the sum

$$\bigoplus_{n \geqslant 0}^{j} \mathsf{Hom}^n_{\mathcal{A}_1/\mathcal{A}_0}(X,Y) \hookrightarrow \mathsf{Hom}_{\mathcal{A}_1/\mathcal{A}_0}(X,Y)$$

is a subcomplex and so we obtain an exhaustive filtration of $\mathsf{Hom}_{\mathcal{A}_1/\mathcal{A}_0}(X,Y)$.

### 4.1. Morita model structure

Let $L$ be the category with two objects 0 and 1 and with a unique non identity morphism $0 \to 1$.

*Remark 4.4.* An immediate application of [**6**, Theorem 11.6.1] implies that the category $\mathsf{dgcat}^L$, i.e., the category of morphisms in $\mathsf{dgcat}$, admits a structure of cofibrantly generated model category whose weak equivalences $W$ are the levelwise Morita dg functors.

Recall from Definitions 2.14 and 2.16, the sets of generating (trivial) cofibrations for the Morita model structure. Following [**6**, Theorem 11.6.1] we denote by $\mathbf{F}_I^L$, resp. by $\mathbf{F}_J^L$, the generating cofibrations, resp. generating trivial cofibrations in $\mathsf{dgcat}^L$. The functor $\mathbf{F}_?^i$, $i = 0, 1$, from $\mathsf{dgcat}$ to $\mathsf{dgcat}^L$ is by definition the left adjoint of the evaluation functor $Ev_i$, $i = 0, 1$, from $\mathsf{dgcat}^L$ to $\mathsf{dgcat}$. More precisely, if $\mathcal{A}$ is a dg category then $\mathbf{F}_{\mathcal{A}}^0 = (\mathcal{A} = \mathcal{A})$ and $\mathbf{F}_{\mathcal{A}}^1 = (\emptyset \to \mathcal{A})$, where $\emptyset$ denotes the empty dg category. Notice that we have $\mathbf{F}_I^L = \mathbf{F}_I^0 \cup \mathbf{F}_I^1$ and $\mathbf{F}_J^L = \mathbf{F}_J^0 \cup \mathbf{F}_J^1$.

The inclusion functor $U\colon \mathsf{Lp} \to \mathsf{dgcat}^L$ admits a left adjoint $S$ which sends an object $G\colon \mathcal{B}_0 \to \mathcal{B}_1$ to the localization pair formed by $\mathcal{B}_1$ and its full dg subcategory $\mathsf{Im}G$, whose objects are those in the image of $G$.

**Proposition 4.5.** *The category* $\mathsf{Lp}$ *admits a structure of cofibrantly generated Quillen model category whose weak equivalences* $W$ *are the levelwise Morita dg functors and with generating cofibrations* $\mathbf{F}_I^L$ *and generating trivial cofibrations* $\mathbf{F}_J^L$.

*Proof.* The proof will consist on verifying the conditions of Theorem A.1 using the adjunction $(S, U)$. We show first that $\mathsf{Lp}$ is complete and cocomplete. Let $\{X_i\}_{i \in I}$ be a diagram in $\mathsf{Lp}$. Notice that the object $S(\operatorname*{colim}_{i \in I} U(X_i))$ in $\mathsf{dgcat}^L$ has the universal property which characterizes the colimit of the diagram $\{X_i\}_{i \in I}$ in $\mathsf{Lp}$. This shows us that the category $\mathsf{Lp}$ is cocomplete. It is also complete, since it is stable under products and equalizers as a full subcategory of $\mathsf{dgcat}^L$. We now prove that conditions (1) and (2) of Theorem A.1 are satisfied:

(1) Since $S(\mathbf{F}_I^L) = \mathbf{F}_I^L$ and $S(\mathbf{F}_J^L) = \mathbf{F}_J^L$ condition (1) is verified.

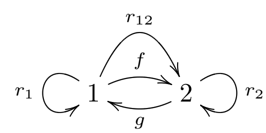
(2) Since the functor $U$ commutes with filtered colimits, it is enough to prove the following: Let $Y \xrightarrow{G} Z$ be an element of the set $\mathbf{F}_J^L$, $X$ an object in $\mathsf{Lp}$ and $Y \to X$ a morphism in $\mathsf{Lp}$. Consider the following pushout in $\mathsf{Lp}$:

$$\begin{array}{ccc} Y & \longrightarrow & X \\ {\scriptstyle G}\downarrow & \lrcorner & \downarrow{\scriptstyle G_*} \\ Z & \longrightarrow & Z \coprod_Y X \end{array}$$

We prove that $U(G_*)$ is a weak equivalence in $\mathsf{dgcat}^L$. We consider two situations:

- if $G$ belongs to the set $\mathbf{F}_J^0 \subset \mathbf{F}_J^L$, then $U(G_*)$ is a weak equivalence simply because the class $J$-cell consists of Morita dg functors, see [**18**, Lemma 2.33].
- if $G$ belongs to the set $\mathbf{F}_J^1 \subset \mathbf{F}_J^L$, then $Ev_1(U(G_*))$ is a Morita dg functor. In particular it induces a quasi-isomorphism in the $\mathsf{Hom}$-spaces. Since the 0-component of $G_*$ is the identity map on objects, the functor $Ev_0(U(G_*))$ is also a Morita dg functor. This implies that $U(G_*)$ is a weak equivalence in $\mathsf{Lp}$ and so condition (2) is proven. □

We will now slightly modify the previous Quillen model structure on $\mathsf{Lp}$. This will furnish us convenient fibrant objects, see Lemma 4.11. Recall from Definition 2.16(ii) the construction of the dg category $\mathcal{K}$:




**Proposition 4.6.** *[**18**, Proposition 1.7] Let $\mathcal{B}$ be a dg category. There is a natural bijection*

$$F\colon \mathcal{K} \longrightarrow \mathcal{B} \mapsto \left( s = F(f), h = \begin{bmatrix} \widehat{F(r_2)} & \widehat{F(r_{12})} \\ \widehat{F(g)} & \widehat{F(r_1)} \end{bmatrix} \right)$$

*between the set of dg functors from $\mathcal{K}$ to $\mathcal{B}$ and the set of couples $(s,h)$, where $s$ belongs to $\mathsf{Z}^0(\mathcal{B})$ and $h$ is a contraction of cone$(\widehat{s})$ in $\mathcal{C}_{dg}(\mathcal{B})$.*

*Remark 4.7.* Let $\mathcal{B}$ be a dg category and $f\colon X \to Y$ a morphism in $\mathsf{Z}^0(\mathcal{B})$. Recall from Remark 2.10 that $f$ is a homotopy equivalence (Notation 2.5) if and only if cone$(\widehat{f})$ is contractible in $\mathcal{C}_{dg}(\mathcal{B})$. Therefore, if $f$ is a homotopy equivalence, choosing a contraction of cone$(\widehat{f})$ provides us with a dg functor from $\mathcal{K}$ to $\mathcal{B}$.

Now, let $\sigma$ be the morphism of localization pairs:

$$\begin{array}{ccc} (\mathrm{End}_{\mathcal{K}}(1) & \hookrightarrow & \mathcal{K}) \\ {\scriptstyle inc}\downarrow & & \| \\ (\mathcal{K} & = & \mathcal{K}), \end{array}$$

where $\mathrm{End}_{\mathcal{K}}(1)$ is the dg algebra of endomorphisms of the object 1 in $\mathcal{K}$ and *inc* is the inclusion dg functor. Since the objects 1 and 2 are homotopy equivalent in $\mathcal{K}$, the morphism $\sigma$ is a levelwise Morita dg functor. We write $\tilde{\mathbf{F}}_I^L$, resp. $\tilde{\mathbf{F}}_J^L$, for the union of $\{\sigma\}$ with $\mathbf{F}_I^L$, resp. with $\mathbf{F}_J^L$.

**Proposition 4.8.** *The category $\mathsf{Lp}$ admits a structure of cofibrantly generated Quillen model category whose weak equivalences $W$ are the levelwise Morita dg functors and with generating cofibrations $\tilde{\mathbf{F}}_I^L$ and generating trivial cofibrations $\tilde{\mathbf{F}}_J^L$.*

*Proof.* The proof will consist on verifying the conditions (1) – (6) of the recognition Theorem A.2. Clearly the class $W$ has the two out of three property and is closed under retracts. This shows condition (1). Notice that the localization pair $(\mathrm{End}_{\mathcal{K}}(1) \subset \mathcal{K})$ is small in $\mathsf{Lp}$, i.e., the functor $\mathsf{Hom}_{\mathsf{Lp}}((\mathrm{End}_{\mathcal{K}}(1) \subset \mathcal{K}), -)$ commutes with filtered colimits. This implies that the domains of $\tilde{\mathbf{F}}_I^L$, resp. of $\tilde{\mathbf{F}}_J^L$, are small relative to $\tilde{\mathbf{F}}_I^L$-cell, resp. to $\tilde{\mathbf{F}}_J^L$-cell, and so conditions (2) and (3) are also satisfied. We have

$$\mathbf{F}_I^L\text{-inj} = \mathbf{F}_J^L\text{-inj} \cap W$$

and so by construction

$$\tilde{\mathbf{F}}_I^L\text{-inj} = \tilde{\mathbf{F}}_J^L\text{-inj} \cap W.$$

This shows conditions (5) and (6). We now prove that $\tilde{\mathbf{F}}_J^L$-cell $\subset W$. Since $\mathbf{F}_J^L$-cell $\subset W$ and the class $W$ is stable under transfinite compositions (see [**6**, Definition 10.2.2]) it is enough to prove that pushouts with respect to $\sigma$ belong to $W$. Let $\mathcal{A}$ be a localization pair and

$$T\colon (\mathrm{End}_{\mathcal{K}}(1) \subset \mathcal{K}) \to (\mathcal{A}_0 \subset \mathcal{A}_1)$$

a morphism in $\mathsf{Lp}$. Consider the following pushout in $\mathsf{Lp}$:

$$\begin{array}{ccc} (\mathrm{End}_{\mathcal{K}}(1) \subset \mathcal{K}) & \xrightarrow{T} & (\mathcal{A}_0 \subset \mathcal{A}_1) \\ {\scriptstyle\sigma}\downarrow & \lrcorner & \downarrow{\scriptstyle R} \\ (\mathcal{K} = \mathcal{K}) & \longrightarrow & (\mathcal{U}_0 \subset \mathcal{U}_1). \end{array}$$

Notice that by Proposition 4.6 and Remark 4.7, the morphism $T$ corresponds to specifying a homotopy equivalence $f$ in $\mathcal{A}_1$ from an object $X$, belonging to $\mathcal{A}_0$, to an object $Y$ and a contraction of $\mathrm{cone}(\widehat{f})$ in $\mathcal{C}_{dg}(\mathcal{A}_1)$. Observe that $\mathcal{U}_1 = \mathcal{A}_1$ and that $\mathcal{U}_0$ identifies with the full dg subcategory of $\mathcal{U}_1$ whose objects consist of $Y$ and those of $\mathcal{A}_0$. Since $X$ and $Y$ are homotopy equivalent, the dg functor $R_0\colon \mathcal{A}_0 \hookrightarrow \mathcal{U}_0$ is a quasi-equivalence (Definition 2.11). This proves condition (4) and so the proof is finished. □

From now on and until the end of this subsection, by Quillen model structure on $\mathsf{Lp}$ we mean that of Proposition 4.8.

*Remark 4.9.* In this new Quillen model structure on $\mathsf{Lp}$ we have more cofibrations and fewer fibrations than the Quillen model structure of Proposition 4.5 since the weak equivalences are the same.

We now give an example showing that our Quillen model structure on $\mathsf{Lp}$ is *not* right proper [**6**, Definition 13.1.1]. This was the main reason for considering a weaker form of Bousfield localization theorem, see Theorem A.4.

*Example 4.10.* We start by showing that the Morita model structure on $\mathsf{dgcat}$ is not right proper: let $\mathcal{A}$ be any Morita fibrant dg category whose derived category $\mathcal{D}(\mathcal{A})$ is not trivial. In particular the dg functor $P\colon \mathcal{A} \to 0$, where 0 denotes the terminal object in $\mathsf{dgcat}$ is a fibration. Let $\underline{k}$ be the dg category of Definition 2.14(i). Consider the following diagram:

$$\begin{array}{ccc} & & \mathcal{A} \\ & & \downarrow{\scriptstyle i_0 \circ P} \\ \underline{k} & \xrightarrow[j_{\underline{k}}]{} & 0 \coprod \underline{k}. \end{array}$$

Notice that $j_{\underline{k}}$ is a Morita dg functor and that the dg functor $i_0 \circ P$ is a fibration, since it has the R.L.P. with respect to the set $J$ of generating trivial cofibrations (see subsection 2.2). This implies that in the fiber product

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathcal{A} \\ \downarrow & \ulcorner & \downarrow{\scriptstyle i_0 \circ P} \\ \underline{k} & \xrightarrow[j_{\underline{k}}]{} & 0 \coprod \underline{k}, \end{array}$$

the dg functor $\emptyset \to \mathcal{A}$ is not a Morita dg functor and so the Morita model structure on $\mathsf{dgcat}$ is not right proper.

As a consequence, the Quillen model structure on $\mathsf{Lp}$ is also *not* right proper: by applying the functor $\mathbf{F}^0_?$ from $\mathsf{dgcat}$ to $\mathsf{Lp}$, we obtain the following fiber product:

$$\begin{array}{ccc} \emptyset = \mathbf{F}^0_{\emptyset} & \longrightarrow & \mathbf{F}^0_{\mathcal{A}} \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \mathbf{F}^0_{i_0 \circ P}} \\ \mathbf{F}^0_{\underline{k}} & \xrightarrow[\mathbf{F}^0_{j_{\underline{k}}}]{} & \mathbf{F}^0_{0 \coprod \underline{k}}. \end{array}$$

The morphism $\mathbf{F}^0_{j_{\underline{k}}}$ is a weak equivalence in $\mathsf{Lp}$. Notice that the morphism $\mathbf{F}^0_{i_0 \circ P}$ belongs to $\sigma$-inj $\cap\, \mathbf{F}^L_J$-inj and so it is a fibration in $\mathsf{Lp}$. Since the morphism $\emptyset \to \mathbf{F}^0_{\mathcal{A}}$ is not a levelwise Morita dg functor we conclude that the Quillen model structure on $\mathsf{Lp}$ is not right proper.

**Lemma 4.11.** *A localization pair $(\mathcal{A}_0 \subset \mathcal{A}_1)$ is fibrant in* $\mathsf{Lp}$ *if and only if $\mathcal{A}_0$ and $\mathcal{A}_1$ are Morita fibrant dg categories (Proposition 2.17) and $\mathcal{A}_0$ is stable under homotopy equivalences (Notation 2.5) in $\mathcal{A}_1$.*

*Proof.* A localization pair $(\mathcal{A}_0 \subset \mathcal{A}_1)$ is fibrant in $\mathsf{Lp}$ if and only if for every morphism $F$ in $\tilde{\mathbf{F}}^L_J$, the following extension problem in $\mathsf{Lp}$ is solvable:

$$\begin{array}{ccc} X & \longrightarrow & (\mathcal{A}_0 \subset \mathcal{A}_1) \\ {\scriptstyle F}\downarrow & \nearrow & \\ Y & & . \end{array}$$

If $F$ belongs to $\mathbf{F}^L_J$ this means that $\mathcal{A}_0$ and $\mathcal{A}_1$ are Morita fibrant dg categories. If $F = \sigma$, Remark 4.7 shows us that this corresponds exactly to the statement that $\mathcal{A}_0$ is stable under homotopy equivalences in $\mathcal{A}_1$. $\square$

**Lemma 4.12.** *If the localization pair $\mathcal{A}$ is cofibrant in* $\mathsf{Lp}$ *then $\mathcal{A}_1$ is cofibrant in* $\mathsf{dgcat}$.

*Proof.* We need to construct a lift to the following problem:

$$\begin{array}{ccc} & & \mathcal{C} \\ & & \sim \downarrow {\scriptstyle P} \\ \mathcal{A}_1 & \longrightarrow & \mathcal{B}, \end{array}$$

where $P$ is a trivial fibration in $\mathsf{dgcat}$ and $\mathcal{A}_1 \to \mathcal{B}$ is a dg functor. Consider the following diagram in $\mathsf{Lp}$:

$$\begin{array}{ccc} & & \mathbf{F}^0_{\mathcal{C}} \\ & \nearrow & \sim \downarrow {\scriptstyle \mathbf{F}^0_P} \\ \mathcal{A} & \longrightarrow & \mathbf{F}^0_{\mathcal{B}}. \end{array}$$

where $\mathcal{A} \to \mathbf{F}^0_{\mathcal{B}}$ is the natural morphism of localization pairs. Notice that $\mathbf{F}^0_P$ belongs to $\sigma$-inj $\cap\, \mathbf{F}_I^L -$ inj and so it is a trivial fibration in $\mathsf{Lp}$. Since $\mathcal{A}$ is cofibrant in $\mathsf{Lp}$ we have a lifting $\mathcal{A} \to \mathbf{F}_{\mathcal{C}}^0$, whose restriction to the 1-component, gives us the desired lift $\mathcal{A}_1 \to \mathcal{C}$. This proves the lemma. □

## 4.2. $Q$-model structure

**Definition 4.13.** Let $Q\colon \mathsf{Lp} \to \mathsf{Lp}$ be the functor that sends a localization pair $\mathcal{A} = (\mathcal{A}_0 \subset \mathcal{A}_1)$ to the localization pair

$$\overline{\mathcal{A}_0} \hookrightarrow \mathcal{A}_1/\mathcal{A}_0,$$

where $\overline{\mathcal{A}_0}$ is the full dg subcategory of $\mathcal{A}_1/\mathcal{A}_0$ (Notation 4.3) whose objects are those of $\mathcal{A}_0$.

*Remark 4.14.* Notice that we have a natural morphism

$$\eta_{\mathcal{A}}\colon (\mathcal{A}_0 \subset \mathcal{A}_1) \to (\overline{\mathcal{A}_0} \subset \mathcal{A}_1/\mathcal{A}_0) \quad \mathcal{A} \in \mathsf{Lp}.$$

**Definition 4.15.** A morphism of localization pairs $F\colon \mathcal{A} \to \mathcal{B}$ is a *$Q$-weak equivalence* if the induced morphism $Q(F)$ is a weak equivalence in the Quillen model structure of Proposition 4.8.

*Remark 4.16.* Observe, that since the objects of $\overline{\mathcal{A}_0}$ and $\overline{\mathcal{B}_0}$ are all contractible (Notation 2.5), the dg functor $\overline{\mathcal{A}_0} \to \overline{\mathcal{B}_0}$ is a Morita dg functor and so the morphism $F$ is a $Q$-weak equivalence if and only if the induced dg functor $\mathcal{A}_1/\mathcal{A}_0 \to \mathcal{B}_1/\mathcal{B}_0$ is a Morita dg functor.

**Definition 4.17.** A morphism in $\mathsf{Lp}$ is a *cofibration* if it is one for the Quillen model structure of Proposition 4.8 and it is a *$Q$-fibration* if it has the right lifting property with respect to all cofibrations of $\mathsf{Lp}$ which are $Q$-weak equivalences.

**Theorem 4.18.** *The category* $\mathsf{Lp}$ *admits a Quillen model category structure, whose weak equivalences are the $Q$-weak equivalences, whose cofibrations are those of* $\mathsf{Lp}$ *and whose fibrations are the $Q$-fibrations.*

*Notation 4.19.* We denote by $\mathsf{Ho}(\mathsf{Lp})$ the homotopy category obtained.

*Proof.* The proof will consist on verifying the conditions (A0)-(A3) of Theorem A.4.

(A0) Let $\mathcal{A}$ be a localization pair such that the morphism

$$\eta_{\mathcal{A}}\colon \mathcal{A} \xrightarrow{\sim} Q(\mathcal{A})$$

is a weak equivalence in $\mathsf{Lp}$ and $F\colon \mathcal{W} \to Q(\mathcal{A})$ a fibration in $\mathsf{Lp}$. We now show that the morphism

$$\eta_{\mathcal{A}}^*\colon \mathcal{A} \underset{Q(\mathcal{A})}{\times} \mathcal{W} \xrightarrow{\sim} \mathcal{W}$$

is a weak equivalence in $\mathsf{Lp}$.

We start by observing that each component of the morphism $\eta_{\mathcal{A}}$ induces the identity map on objects. Since fiber products in $\mathsf{Lp}$ are calculated levelwise, we conclude that each component of the morphism $\eta_{\mathcal{A}}^*$ induces the identity map on objects. Let $X$ and $Y$ be arbitrary objects of $\mathcal{W}_1$. We have the following fiber product in $\mathsf{Ch}(k)$:

$$\begin{array}{ccc}
\mathsf{Hom}_{\mathcal{W}_1 \underset{\mathcal{A}_1/\mathcal{A}_0}{\times} \mathcal{A}_1}(X,Y) & \longrightarrow & \mathsf{Hom}_{\mathcal{A}_1}(F_1X, F_1Y) \\
{\scriptstyle \eta^*(F_1X,F_1Y)} \downarrow & \ulcorner & \sim \downarrow {\scriptstyle \eta(F_1X,F_1Y)} \\
\mathsf{Hom}_{\mathcal{W}_1}(X,Y) & \underset{F_1(X,Y)}{\twoheadrightarrow} & \mathsf{Hom}_{\mathcal{A}_1/\mathcal{A}_0}(F_1X, F_1Y).
\end{array}$$

Since $F$ is a fibration in $\mathsf{Lp}$, $F_1(X,Y)$ is a fibration in the projective model structure on $\mathsf{Ch}(k)$ and since this model structure is right proper, $\eta^*(F_1X, F_1Y)$ is a quasi-isomorphism. The same argument can be done for objects $X$ and $Y$ in $\mathcal{W}_0$ instead of $\mathcal{W}_1$. This proves condition (A0).

(A1) Since $k$ is a field, every complex in $\mathsf{Ch}(k)$ is $k$-flat and so by [**3**, Theorem 3.4] the functor $Q$ preserves weak equivalences.

(A2) The morphisms of localization pairs:

$$Q(\mathcal{A}) \underset{Q(\eta_{\mathcal{A}})}{\overset{\eta_{Q(\mathcal{A})}}{\rightrightarrows}} QQ(\mathcal{A})$$

are weak equivalences in $\mathsf{Lp}$. This follows automatically from the following two facts: in both cases we are introducing contractions to objects that are already contractible and the functor $Q$ induces the identity map on objects.

(A3) Let $\mathcal{A}$ be a localization pair and $F\colon \mathcal{Z} \to Q(\mathcal{A})$ a $Q$-fibration in $\mathsf{Lp}$. We now show that the induced morphism

$$\eta_{\mathcal{A}}^*\colon \mathcal{A} \underset{Q(\mathcal{A})}{\times} \mathcal{Z} \longrightarrow \mathcal{Z}$$

is a $Q$-weak equivalence in $\mathsf{Lp}$. We need to prove that $Q(\eta_{\mathcal{A}}^*)$ is a weak equivalence in $\mathsf{Lp}$. The proof is divided in two parts:

**(1)** We start by proving that the induced morphism:

$$Q(\eta_{\mathcal{A}})^*\colon Q(\mathcal{A}) \underset{QQ(\mathcal{A})}{\times} Q(\mathcal{Z}) \longrightarrow Q(\mathcal{Z})$$

is a weak equivalence in $\mathsf{Lp}$. Notice that since $F$ is a $Q$-fibration, it is also a fibration in $\mathsf{Lp}$ and so the dg functors $F_0$ and $F_1$ are fibrations in the Morita model structure. In particular, $F_0$ and $F_1$ have the right lifting property with respect to the dg functors $R(n), n \in \mathbb{Z}$, (see Definition 2.16(i)) and so they are degreewise surjective at the level of $\mathsf{Hom}$-spaces. We now show that the dg functor $F_0\colon \mathcal{Z}_0 \to \overline{\mathcal{A}_0}$ induces a surjective map on objects. Since every object $X$ in $\overline{\mathcal{A}_0}$ is contractible (Notation 2.5) and the dg functor $F_0$ has the right lifting property with respect to the dg functor $C$ (see Definition 2.16(v)), there exists an object $Y$ in $\mathcal{Z}_0$ such that $F_0(Y) = X$. This implies that each component of the morphism

$$Q(F)\colon Q(\mathcal{Z}) \longrightarrow QQ(\mathcal{A})$$

is a degreewise surjective dg functor at the level of $\mathsf{Hom}$-spaces. By condition

(A2), the morphism

$$(Q\eta_{\mathcal{A}})\colon Q(\mathcal{A}) \longrightarrow QQ(\mathcal{A})$$

is a weak equivalence and so an argument analogous to the one of the proof of condition (A0) (we have just proved that $F_1(X,Y)$ is a fibration in the projective model structure on $\mathsf{Ch}(k)$), implies condition **(1)**.

**(2)** We now prove that the induced morphism:

$$Q(\mathcal{Z} \underset{Q(\mathcal{A})}{\times} \mathcal{A}) \longrightarrow Q(\mathcal{Z}) \underset{QQ(\mathcal{A})}{\times} Q(\mathcal{A})$$

is an isomorphism in $\mathsf{Lp}$. By construction, the functor $Q$ induces the identity map on objects and so both components of the above morphism induce also the identity map on objects. We now consider the 1-component of the above morphism. Let $X$ and $Y$ be objects of $\mathcal{Z}_1/\mathcal{Z}_0$. We have the following fiber product in $\mathsf{Ch}(k)$:

$$\begin{array}{ccc}
\mathsf{Hom}_{\mathcal{Z}_1/\mathcal{Z}_0 \underset{(\mathcal{A}_1/\mathcal{A}_0)/\overline{\mathcal{A}_0}}{\times} \mathcal{A}_1/\mathcal{A}_0}(X,Y) & \longrightarrow & \mathsf{Hom}_{\mathcal{A}_1/\mathcal{A}_0}(F_1(X), F_1(Y)) \\
\downarrow & \lrcorner & \downarrow{\scriptstyle Q\eta_{\mathcal{A}}} \\
\mathsf{Hom}_{\mathcal{Z}_1/\mathcal{Z}_0}(X,Y) & \underset{QF_1}{\twoheadrightarrow} & \mathsf{Hom}_{(\mathcal{A}_1/\mathcal{A}_0)/\overline{\mathcal{A}_0}}(F_1(X), F_1(Y)).
\end{array}$$

Notice that the functor $Q(\eta_{\mathcal{A}})$, resp. $QF_1$, sends the contractions in $\mathcal{A}_1/\mathcal{A}_0$, resp. in $\mathcal{Z}_1/\mathcal{Z}_0$, associated with the objects of $\mathcal{A}_0$, resp. of $\mathcal{Z}_0$, to the new contractions in $(\mathcal{A}_1/\mathcal{A}_0)/\overline{\mathcal{A}_0}$ associated with the objects of $\overline{\mathcal{A}_0}$. Recall that we have the following fiber product in $\mathsf{Ch}(k)$:

$$\begin{array}{ccc}
\mathsf{Hom}_{\mathcal{Z}_1 \underset{\mathcal{A}_1/\mathcal{A}_0}{\times} \mathcal{A}_1}(X,Y) & \longrightarrow & \mathsf{Hom}_{\mathcal{A}_1}(F_1X, F_1Y) \\
\downarrow & \lrcorner & \downarrow{\scriptstyle \eta} \\
\mathsf{Hom}_{\mathcal{Z}_1}(X,Y) & \underset{F_1}{\twoheadrightarrow} & \mathsf{Hom}_{\mathcal{A}_1/\mathcal{A}_0}(F_1X, F_1Y).
\end{array}$$

The previous arguments and the above fiber product shows us that the induced morphism

$$\mathsf{Hom}_{(\mathcal{Z}_1 \underset{\mathcal{A}_1/\mathcal{A}_0}{\times} \mathcal{A}_1)/(\mathcal{Z}_0 \underset{\overline{\mathcal{A}_0}}{\times} \mathcal{A}_0)}(X,Y) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{Z}_1/\mathcal{Z}_0 \underset{(\mathcal{A}_1/\mathcal{A}_0)/\overline{\mathcal{A}_0}}{\times} \mathcal{A}_1/\mathcal{A}_0}(X,Y)$$

is an isomorphism in $\mathsf{Ch}(k)$. The same argument applies to the 0-component of the above morphism. This proves condition **(2)**. Now, conditions **(1)** and **(2)** imply that the morphism

$$Q(\eta_{\mathcal{A}}^{*})\colon Q(\mathcal{Z} \underset{Q(\mathcal{A})}{\times} \mathcal{A}) \longrightarrow Q(\mathcal{Z})$$

is a weak equivalence in $\mathsf{Lp}$, and so condition (A3) is proven. □

### 4.3. $Q$-fibrant objects

Let $\mathcal{A}$ be a localization pair.

**Lemma 4.20.** *If $\mathcal{A}$ is fibrant, in the Quillen model structure of Proposition 4.8, and the morphism $\eta_{\mathcal{A}} \colon \mathcal{A} \to Q(\mathcal{A})$ is a weak equivalence in* $\mathsf{Lp}$ *then $\mathcal{A}$ is $Q$-fibrant.*

*Proof.* We need to show that the morphism $\mathcal{A} \stackrel{P}{\to} 0$ is a $Q$-fibration, where 0 denotes the terminal object in $\mathsf{Lp}$. Consider the following diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\eta_{\mathcal{A}}} & Q(\mathcal{A}) \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q(P)} \\ 0 & \xrightarrow[\eta]{} & Q(0). \end{array}$$

Factorize the morphism $Q(P)$ as

$$\begin{array}{ccc} Q(\mathcal{A}) & \xrightarrow{i} & \mathcal{Z} \\ & {\scriptstyle Q(P)}\searrow & \downarrow{\scriptstyle q} \\ & & Q(0), \end{array}$$

where $i$ is a trivial cofibration and $q$ a fibration in $\mathsf{Lp}$. Since $\eta_{\mathcal{Z}} \colon \mathcal{Z} \to Q(\mathcal{Z})$ is a weak equivalence, Lemma A.5 implies that $q$ is a $Q$-fibration. The morphism $0 \to Q(0)$ is a weak equivalence and so by condition (A0) the induced morphism $0 \underset{Q(0)}{\times} \mathcal{Z} \to \mathcal{Z}$ is a weak equivalence. By hypothesis, the morphism $\eta_{\mathcal{A}}$ is a weak equivalence and so the induced morphism

$$\theta \colon \mathcal{A} \to 0 \underset{Q(0)}{\times} \mathcal{Z}$$

is also a weak equivalence. Factorize the morphism $\theta$ as

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{j} & \mathcal{W} \\ & {\scriptstyle \theta}\searrow & \downarrow{\scriptstyle \pi} \\ & & 0 \underset{Q(0)}{\times} \mathcal{Z}, \end{array}$$

where $\pi$ is a trivial fibration of $\mathsf{Lp}$ and $j$ is a trivial cofibration. Then $q_* \circ \pi$ is a $Q$-fibration and so a lifting exists in the diagram:

$$\begin{array}{ccc} \mathcal{A} & = & \mathcal{A} \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle P} \\ \mathcal{W} & \xrightarrow[q_* \circ \pi]{} & 0. \end{array}$$

Thus $P$ is a retract of a $Q$-fibration, and is therefore a $Q$-fibration itself. This proves the lemma. □

**Lemma 4.21.** *If $\mathcal{A}$ is $Q$-fibrant, then $\mathcal{A}$ is fibrant in* $\mathsf{Lp}$ *and the morphism*

$$\eta_{\mathcal{A}} \colon \mathcal{A} \to Q(\mathcal{A})$$

*is a weak equivalence.*

*Proof.* Since the $Q$-model structure on $\mathsf{Lp}$ has fewer fibrations than the Quillen model structure of Proposition 4.8, the localization pair $\mathcal{A}$ is fibrant in $\mathsf{Lp}$. Consider now the following diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\eta_{\mathcal{A}}} & Q(\mathcal{A}) \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q(P)} \\ 0 & \xrightarrow[\eta]{} & Q(0). \end{array}$$

Factorize $Q(P) = q \circ i$ as in the previous lemma. We have the following diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\theta} & 0 \underset{Q(0)}{\times} \mathcal{Z} \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle q_*} & \\ 0 & & . \end{array}$$

Since $p$ and $q_*$ are $Q$-fibrations, $\mathcal{A}$ and $\mathcal{Z}$ are $Q$-fibrant objects in $\mathsf{Lp}$ and $\theta$ is a $Q$-weak equivalence in $\mathsf{Lp}$. By application of [**6**, Lemma 7.7.1 b)] to $\theta$ we conclude that $\theta$ is a weak equivalence. Since $i$ is also a weak equivalence, $\eta_{\mathcal{A}}$ is a weak equivalence and so the proof is finished. $\square$

*Remark 4.22.* By Lemmas 4.20 and 4.21 a localization pair $\mathcal{A}$ is *$Q$-fibrant* if and only if it is fibrant in $\mathsf{Lp}$ and the morphism

$$\eta_{\mathcal{A}} \colon \mathcal{A} \longrightarrow Q(\mathcal{A})$$

is a weak equivalence.

We now describe explicitly the $Q$-fibrant objects in $\mathsf{Lp}$.

**Proposition 4.23.** *A localization pair $(\mathcal{A}_0 \subset \mathcal{A}_1)$ is $Q$-fibrant if and only if $\mathcal{A}_1$ is a Morita fibrant dg category (Proposition 2.17), and $\mathcal{A}_0$ is the full dg subcategory of contractible objects (Notation 2.5) in $\mathcal{A}_1$.*

*Proof.* Suppose first that $(\mathcal{A}_0 \subset \mathcal{A}_1)$ is $Q$-fibrant. Since it is also fibrant in $\mathsf{Lp}$ the dg category $\mathcal{A}_1$ is fibrant in $\mathsf{dgcat}$. The morphism

$$\eta_{\mathcal{A}} \colon (\mathcal{A}_0 \subset \mathcal{A}_1) \longrightarrow (\overline{\mathcal{A}_0} \subset \mathcal{A}_1/\mathcal{A}_0)$$

is a weak equivalence and so all the objects of $\mathcal{A}_0$ are contractible. Moreover since $\mathcal{A}$ is fibrant in $\mathsf{Lp}$, Lemma 4.11 implies that $\mathcal{A}_0$ is stable under homotopy equivalences in $\mathcal{A}_1$ and so $\mathcal{A}_0$ is in fact the full dg subcategory of contractible objects of $\mathcal{A}_1$. Consider now a localization pair $(\mathcal{A}_0 \subset \mathcal{A}_1)$ as in the statement of the proposition. Since $\mathcal{A}_1$ is a Morita fibrant dg category so is $\mathcal{A}_0$. Notice that $(\mathcal{A}_0 \subset \mathcal{A}_1)$ satisfies the extension condition with regard to $\sigma$ and that the morphism

$$\eta \colon (\mathcal{A}_0 \subset \mathcal{A}_1) \longrightarrow (\overline{\mathcal{A}_0} \subset \mathcal{A}_1/\mathcal{A}_0)$$

is a weak equivalence in $\mathsf{Lp}$. This proves the proposition. $\square$

## 5. Closed monoidal structure

In this chapter we construct a natural closed symmetric monoidal structure on our new Quillen model $\mathsf{Lp}$. Recall from [**8**, §2.3] that $\mathsf{dgcat}$ is a closed symmetric monoidal category. For $\mathcal{A}$ and $\mathcal{B}$ small dg categories, we have:

- A *tensor product category* $\mathcal{A} \otimes \mathcal{B}$. Its set of objects is $obj(\mathcal{A}) \times obj(\mathcal{B})$ and its morphisms spaces are given by
$$\mathsf{Hom}_{\mathcal{A}\otimes\mathcal{B}}((X,Y),(X',Y')) := \mathsf{Hom}_{\mathcal{A}}(X,X') \otimes \mathsf{Hom}_{\mathcal{B}}(Y,Y').$$
The composition and the units are induced from those of $\mathcal{A}$ and $\mathcal{B}$.
- A *dg category of dg functors* $\mathsf{Fun}_{dg}(\mathcal{A},\mathcal{B})$. For two dg functors $F, G\colon \mathcal{A} \to \mathcal{B}$, the complex of graded morphisms $\mathsf{Hom}_{\mathsf{Fun}_{dg}(\mathcal{A},\mathcal{B})}(F,G)$ has as its $n$th component the module formed by the families of morphisms
$$\phi_X \in \mathsf{Hom}^n_{\mathcal{B}}(FX, GX)$$
such that $(Gf)(\phi_X) = (\phi_Y)(Ff)$ for all $f \in \mathsf{Hom}_{\mathcal{A}}(X,Y), X, Y \in \mathcal{A}$. The differential is induced by that of $\mathsf{Hom}_{\mathcal{B}}(FX, GX)$.

*Remark 5.1.* As it is shown in [**20**, Remarque 5.11], the tensor product $-\otimes-$ on $\mathsf{dgcat}$ can be naturally derived into a bi-functor
$$-\overset{\mathbb{L}}{\otimes}-\colon \mathsf{Hmo} \times \mathsf{Hmo} \longrightarrow \mathsf{Hmo}.$$
However, as explained in the introduction, the bi-functor $\mathsf{Fun}_{dg}(-,-)$ can not be naturally derived. By an adhoc procedure requiring an involved dg category of "right quasi-representable" bimodules, Toën constructed in [**22**, Theorem 6.1] the internal $\mathsf{Hom}$-functor for the homotopy category $\mathsf{Hmo}$. We will denoted it by $\mathsf{rep}_{dg}(-,-)$ (notice that Toën used the misleading notation $\mathbb{R}\mathsf{Hom}(-,-)$). Roughly, if $\mathcal{A}$ and $\mathcal{B}$ are dg categories, $\mathsf{rep}_{dg}(\mathcal{A},\mathcal{B})$ is the full dg category of dg $\mathcal{A}_c \otimes \mathcal{B}_f$-modules (where $\mathcal{A}_c$ is a cofibrant resolution of $\mathcal{A}$ and $\mathcal{B}_f$ a fibrant resolution of $\mathcal{B}$) whose objects are the $\mathcal{A}_c$-$\mathcal{B}_f$-bimodules $X$ such that, for every object $A$ in $\mathcal{A}_c$, the $\mathcal{B}_f$-module $X(?,A)$ is isomorphic in the derived category $\mathcal{D}(\mathcal{B}_f)$ of $\mathcal{B}_f$ to a representable $\mathcal{B}_f$-module (Notation 2.8).

**Definition 5.2.** The *internal* $\mathsf{Hom}$*-functor* in $\mathsf{Lp}$
$$\mathsf{Hom}(-,-)\colon \mathsf{Lp}^{op} \times \mathsf{Lp} \longrightarrow \mathsf{Lp},$$
associates to the localization pairs $(\mathcal{A}_0 \subset \mathcal{A}_1)$, $(\mathcal{B}_0 \subset \mathcal{B}_1)$ the localization pair:
$$(\mathsf{Fun}_{dg}(\mathcal{A}_1,\mathcal{B}_0) \subset \mathsf{Fun}_{dg}(\mathcal{A}_0,\mathcal{B}_0) \underset{\mathsf{Fun}_{dg}(\mathcal{A}_0,\mathcal{B}_1)}{\times} \mathsf{Fun}_{dg}(\mathcal{A}_1,\mathcal{B}_1)).$$

**Definition 5.3.** The *tensor product* functor in $\mathsf{Lp}$
$$-\otimes-\colon \mathsf{Lp} \times \mathsf{Lp} \longrightarrow \mathsf{Lp}$$
associates to the localization pairs $(\mathcal{A}_0 \subset \mathcal{A}_1)$, $(\mathcal{B}_0 \subset \mathcal{B}_1)$ the localization pair:
$$(\mathcal{A}_0 \otimes \mathcal{B}_1 \cup \mathcal{A}_1 \otimes \mathcal{B}_0 \subset \mathcal{A}_1 \otimes \mathcal{B}_1),$$
where $\mathcal{A}_0 \otimes \mathcal{B}_1 \cup \mathcal{A}_1 \otimes \mathcal{B}_0$ is the full dg subcategory of $\mathcal{A}_1 \otimes \mathcal{B}_1$ consisting of those objects $a \otimes b$ of $\mathcal{A}_1 \otimes \mathcal{B}_1$, such that $a$ belongs to $\mathcal{A}_0$ or $b$ belongs to $\mathcal{B}_0$.

Let $\mathcal{A} = (\mathcal{A}_0 \subset \mathcal{A}_1)$, $\mathcal{B} = (\mathcal{B}_0 \subset \mathcal{B}_1)$ and $\mathcal{C} = (\mathcal{C}_0 \subset \mathcal{C}_1)$ be localization pairs.

**Proposition 5.4.** *The category* $\mathsf{Lp}$ *endowed with the functors* $\mathsf{Hom}(-,-)$ *and* $-\otimes-$ *is a closed symmetric monoidal category. In particular we have a canonical isomorphism in* $\mathsf{Lp}$*:*

$$\mathsf{Hom}_{\mathsf{Lp}}(\mathcal{A}\otimes\mathcal{B},\mathcal{C}) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{Lp}}(\mathcal{A},\mathsf{Hom}(\mathcal{B},\mathcal{C})).$$

*Proof.* Consider the following commutative square in $\mathsf{dgcat}$:

$$\begin{array}{ccc} \mathcal{A}_0 & \longrightarrow & \mathsf{Fun}_{dg}(\mathcal{B}_1,\mathcal{C}_0) \\ \downarrow & & \downarrow \\ \mathcal{A}_1 & \longrightarrow & \mathsf{Fun}_{dg}(\mathcal{B}_0,\mathcal{C}_0) \underset{\mathsf{Fun}_{dg}(\mathcal{B}_0,\mathcal{C}_1)}{\times} \mathsf{Fun}_{dg}(\mathcal{B}_1,\mathcal{C}_1), \end{array}$$

which corresponds exactly to an element of $\mathsf{Hom}_{\mathsf{Lp}}(\mathcal{A},\mathsf{Hom}(\mathcal{B},\mathcal{C}))$. The category $\mathsf{dgcat}$ endowed with the functors $-\otimes-$ and $\mathsf{Fun}_{dg}(-,-)$ is a closed symmetric monoidal category and so by adjunction the above commutative square corresponds to the following commutative square in $\mathsf{dgcat}$:

$$\begin{array}{ccc} \mathcal{A}_0\otimes\mathcal{B}_1 \underset{\mathcal{A}_0\otimes\mathcal{B}_0}{\times} \mathcal{A}_1\otimes\mathcal{B}_0 & \longrightarrow & \mathcal{C}_0 \\ \downarrow & & \downarrow \\ \mathcal{A}_1\otimes\mathcal{B}_1 & \longrightarrow & \mathcal{C}_1. \end{array}$$

This commutative square can be seen simply, as a morphism in $\mathsf{dgcat}^L$ from

$$\mathcal{A}_0\otimes\mathcal{B}_1 \underset{\mathcal{A}_0\otimes\mathcal{B}_0}{\times} \mathcal{A}_1\otimes\mathcal{B}_0 \longrightarrow \mathcal{A}_1\otimes\mathcal{B}_1$$

to the localization pair $(\mathcal{C}_0 \subset \mathcal{C}_1)$. Notice that the dg functor

$$\mathcal{A}_0\otimes\mathcal{B}_1 \underset{\mathcal{A}_0\otimes\mathcal{B}_0}{\times} \mathcal{A}_1\otimes\mathcal{B}_0 \to \mathcal{A}_1\otimes\mathcal{B}_1$$

induces an injective map on objects and that its image consists of those objects $a\otimes b$ of $\mathcal{A}_1\otimes\mathcal{B}_1$, such that $a$ belongs to $\mathcal{A}_0$ or $b$ belongs to $\mathcal{B}_0$. This implies that

$$\mathsf{Im}(\mathcal{A}_0\otimes\mathcal{B}_1 \underset{\mathcal{A}_0\otimes\mathcal{B}_0}{\times} \mathcal{A}_1\otimes\mathcal{B}_0 \to \mathcal{A}_1\otimes\mathcal{B}_1) = \mathcal{A}\otimes\mathcal{B},$$

and by the adjunction $(S,U)$ from subsection 4.1, this last commutative square in $\mathsf{dgcat}$ corresponds exactly to an element of $\mathsf{Hom}_{\mathsf{Lp}}(\mathcal{A}\otimes\mathcal{B},\mathcal{C})$. This finishes the proof. □

*Remark 5.5.* Note that the unit object in $\mathsf{Lp}$ is the localization pair $(\emptyset \subset \underline{k})$, where $\underline{k}$ is the dg category with one object and whose dg algebra of endomorphisms is $k$.

## 6. Derived internal Hom-functor

In this chapter we prove our first main theorem (Theorem 6.4 below). Let $\mathcal{A}$ be a cofibrant dg category and $\lambda$ an infinite cardinal whose size is greater than or equal to the cardinality of the set of isomorphism classes of objects in the category $\mathsf{H}^0(\mathcal{A})$. Let $\mathcal{B}$ be a Morita fibrant dg category (Proposition 2.17). Recall that we denote by $\widehat{-}\colon \mathcal{B} \to \mathcal{C}_{dg}(\mathcal{B})$ the Yoneda dg functor.

**Definition 6.1.** Let $\mathcal{B}_\lambda$ be the full dg subcategory of $\mathcal{C}_{dg}(\mathcal{B})$, whose objects are:

- the right dg $\mathcal{B}$-modules $M$ such that $M \oplus D$ is representable for a contractible right dg $\mathcal{B}$-module $D$ and
- the right dg $\mathcal{B}$-modules of the form $\widehat{B} \oplus C$, where $B$ is an object of $\mathcal{B}$ and the right dg $\mathcal{B}$-module $C$ is a direct factor of $\bigoplus_{i \in S} \mathrm{cone}(\mathbf{1}_{\widehat{B_i}})$, with $B_i$ an object of $\mathcal{B}$ and $S$ a set of cardinality bounded by $\lambda$.

Let $\mathsf{rep}_{dg}(\mathcal{A}, \mathcal{B})$ be the dg category as in Remark 5.1.

*Remark 6.2.* Notice that the Yoneda dg functor $\widehat{-}\colon \mathcal{B} \to \mathcal{B}_\lambda$ (Notation 2.8) is a quasi-equivalence and that the objects of $\mathcal{B}_\lambda$ are cofibrant and quasi-representable as right dg $\mathcal{B}$-modules, see [**22**, Definition 4.1]. This implies that we have a dg functor:

$$\overline{\mathsf{Fun}}_{dg}(\mathcal{A}, \mathcal{B}_\lambda) := \mathsf{Fun}_{dg}(\mathcal{A}, \mathcal{B}_\lambda)/\mathsf{Fun}_{dg}(\mathcal{A}, (\mathcal{B}_\lambda)_{contr}) \xrightarrow{\Phi} \mathsf{rep}_{dg}(\mathcal{A}, \mathcal{B}),$$

where $(\mathcal{B}_\lambda)_{contr}$ denotes the full dg subcategory of contractible objects.

**Theorem 6.3.** *For a cofibrant dg category $\mathcal{A}$, a Morita fibrant dg category $\mathcal{B}$ and an infinite cardinal $\lambda$ as above, the induced dg functor:*

$$\mathsf{Fun}_{dg}(\mathcal{A}, \mathcal{B}_\lambda)/\mathsf{Fun}_{dg}(\mathcal{A}, (\mathcal{B}_\lambda)_{contr}) \xrightarrow{\Phi} \mathsf{rep}_{dg}(\mathcal{A}, \mathcal{B}),$$

*is a quasi-equivalence (Definition 2.11).*

*Proof.* We prove first that $\mathsf{H}^0(\Phi)$ is essentially surjective. We have the following composition of dg functors

$$\mathsf{Fun}_{dg}(\mathcal{A}, \mathcal{B}) \xrightarrow{I} \overline{\mathsf{Fun}}_{dg}(\mathcal{A}, \mathcal{B}_\lambda) \xrightarrow{\Phi} \mathsf{rep}_{dg}(\mathcal{A}, \mathcal{B}).$$

Since $\mathcal{A}$ is a cofibrant dg category, [**22**, Lemma 4.3] and [**22**, sub-Lemma 4.4] imply that $\mathsf{H}^0(\Phi \circ I)$ is essentially surjective and so we conclude that so is $\mathsf{H}^0(\Phi)$.

We now prove also that the functor $\mathsf{H}^0(I)$ is essentially surjective. Let $F\colon \mathcal{A} \to \mathcal{B}_\lambda$ be a dg functor. Since $\mathcal{A}$ is a cofibrant dg category and $h$ is a quasi-equivalence, there exists a dg functor $F'\colon \mathcal{A} \to \mathcal{B}$ such that $F$ and $h \circ F'$ are homotopic in the Quillen model structure on $\mathsf{dgcat}$ whose weak equivalences are the quasi-equivalences. Notice that since $\mathcal{B}$ is a Morita fibrant dg category so is $\mathcal{B}_\lambda$. In particular $\mathcal{B}_\lambda$ is stable under cones up to homotopy (Proposition 2.17). Since a cone can be obtained from a cone up to homotopy, by adding or factoring out contractible modules, we conclude that by definition, $\mathcal{B}_\lambda$ is also stable under cones (Definition 2.9). By Remark 3.4, we have a sequence of dg functors

$$F \longrightarrow I \longrightarrow h \circ F'[1],$$

such that $I$ belongs to $\mathsf{Fun}_{dg}(\mathcal{A}, (\mathcal{B}_\lambda)_{contr})$. This implies that $F$ and $h \circ F'$ become isomorphic in $\mathsf{H}^0(\overline{\mathsf{Fun}}_{dg}(\mathcal{A}, \mathcal{B}_\lambda))$ and so the functor $\mathsf{H}^0(I)$ is essentially surjective.

Let us now prove that the functor $\mathsf{H}^0(\Phi)$ is fully-faithful. Let $F$ be an object of $\mathsf{Fun}_{dg}(\mathcal{A},\mathcal{B}_\lambda)$. Since $\mathsf{H}^0(I)$ is essentially surjective, we can consider $F$ as belonging to $\mathsf{Fun}_{dg}(\mathcal{A},\mathcal{B})$. We will construct a morphism of dg functors

$$F' \xrightarrow{\mu} F,$$

where $\mu$ becomes invertible in $\mathsf{H}^0(\overline{\mathsf{Fun}}_{dg}(\mathcal{A},\mathcal{B}_\lambda))$ and $F'$ belongs to the left-orthogonal of the category $\mathsf{H}^0(\mathsf{Fun}_{dg}(\mathcal{A},(\mathcal{B}_\lambda)_{contr}))$, i.e.,

$$\mathsf{Hom}_{\mathsf{H}^0(\mathsf{Fun}_{dg}(\mathcal{A},(\mathcal{B}_\lambda)))}(F',G)=0,$$

for every $G\in\mathsf{H}^0(\mathsf{Fun}_{dg}(\mathcal{A},(\mathcal{B}_\lambda)_{contr}))$. We denote by $X_F$ the $\mathcal{A}$-$\mathcal{B}$-bimodule naturally associated to $F$. Consider $X_F$ as a left $\mathcal{A}$-module and let $\mathbf{P}X_F$ be the bar resolution of $X_F$. The left $\mathcal{A}$-module $\mathbf{P}X_F$ is also naturally a right $\mathcal{B}$-module and is moreover cofibrant in the projective model structure on the category of $\mathcal{A}$-$\mathcal{B}$-bimodules, see [**22**, Definition 3.1]. Let $A$ be an object of $\mathcal{A}$. Since the dg category $\mathcal{A}$ is cofibrant in $\mathsf{dgcat}$, $(\mathbf{P}X_F)(?,A)$ is cofibrant as a $\mathcal{B}$-module. Observe that we have the following homotopy equivalence

$$(\mathbf{P}X_F)(?,A)\xrightarrow[\sim]{\mu_A} X_F(?,A),$$

since both $\mathcal{B}$-modules are cofibrant. This implies that the $\mathcal{B}$-module $(\mathbf{P}X_F)(?,A)$ is isomorphic to a direct sum $X_F(?,A)\oplus C$, where $C$ is a contractible and cofibrant $\mathcal{B}$-module. The $\mathcal{B}$-module $C$ is isomorphic to a direct factor of a $\mathcal{B}$-module

$$\bigoplus_{i\in S}(\mathrm{cone}(\mathbf{1}_{\widehat{B_i}}))[n_i],$$

where $S$ is a set whose cardinality is bounded by $\lambda$, $B_i$, $i\in I$ is an object of $\mathcal{B}$ and $n_i$, $i\in S$ is an integer. This implies, by definition of $\mathcal{B}_\lambda$, that the $\mathcal{B}$-module

$$X_F(?,A)\oplus C$$

belongs to $\mathcal{B}_\lambda$ and so the $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathbf{P}X_F$ is in fact isomorphic to $X_{F'}$ for a dg functor $F'\colon\mathcal{A}\to\mathcal{B}_\lambda$. Notice that the previous construction is functorial in $A$ and so we have a morphism of dg functors

$$F' \xrightarrow{\mu} F.$$

Since for each $A$ in $\mathcal{A}$, the morphism $\mu_A\colon F'A\to FA$ is a retraction with contractible kernel, the morphism $\mu$ becomes invertible in

$$\mathsf{H}^0(\overline{\mathsf{Fun}}_{dg}(\mathcal{A},\mathcal{B}_\lambda)).$$

Now, let $G$ be an object in $\mathsf{Fun}_{dg}(\mathcal{A},(\mathcal{B}_\lambda)_{contr})$. We remark that

$$\mathsf{Hom}_{\mathsf{H}^0(\mathsf{Fun}_{dg}(\mathcal{A},\mathcal{B}_\lambda))}(F',G)\xrightarrow{\sim}\mathsf{Hom}_{\mathcal{H}(\mathcal{A}^{op}\otimes\mathcal{B})}(\mathbf{P}X_F,X_G),$$

where $\mathcal{H}(\mathcal{A}^{op}\otimes\mathcal{B})$ denotes the homotopy category of $\mathcal{A}$-$\mathcal{B}$-bimodules. Since $\mathbf{P}X_F$ is a cofibrant $\mathcal{A}$-$\mathcal{B}$-bimodule and $X_G(?,A)$ is a contractible $\mathcal{B}$-module, for every object $A$ in $\mathcal{A}$, the right hand side vanishes and $F'$ belongs to the left-orthogonal of $\mathsf{H}^0(\mathsf{Fun}_{dg}(\mathcal{A},(\mathcal{B}_\lambda)_{contr}))$. This implies that the induced functor

$$\mathsf{H}^0(\mathsf{Fun}_{dg}(\mathcal{A},\mathcal{B}_\lambda)/\mathsf{Fun}_{dg}(\mathcal{A},(\mathcal{B}_\lambda)_{contr}))\to\mathsf{H}^0(\mathsf{rep}_{dg}(\mathcal{A},\mathcal{B}))$$

is fully-faithful and so the proof is finished. □

**Theorem 6.4.** *The internal* $\mathsf{Hom}$ *functor*

$$\mathsf{Hom}(-,-)\colon \mathsf{Lp}^{op} \times \mathsf{Lp} \to \mathsf{Lp},$$

*admits a total right derived functor*

$$\mathcal{R}\mathsf{Hom}(-,-)\colon \mathsf{Ho}(\mathsf{Lp}^{op} \times \mathsf{Lp}) \to \mathsf{Ho}(\mathsf{Lp})$$

*as in [**6**, Definition 8.4.7].*

*Proof.* Let $\mathcal{A}$ and $\mathcal{B}$ be localization pairs. We are now going to define $\mathcal{R}\mathsf{Hom}(\mathcal{A},\mathcal{B})$ and the morphism $\epsilon$ as in [**6**, Definition 8.4.7]. We denote by $\mathcal{A}_c \overset{P}{\to} \mathcal{A}$ a functorial cofibrant resolution of $\mathcal{A}$ in $\mathsf{Lp}$ and by $\mathcal{B} \overset{I}{\to} \mathcal{B}_f$ a functorial $Q$-fibrant resolution of $\mathcal{B}$ in $\mathsf{Lp}$. Remember, that by Proposition 4.23, $\mathcal{B}_f$ is of the form

$$\mathcal{B}_f = ((\mathcal{B}_f)_{contr} \subset \mathcal{B}_f),$$

where $\mathcal{B}_f$ is a Morita fibrant dg category. Let $\lambda$ be an infinite cardinal whose size is greater or equal to the cardinality of the set of isomorphism classes in the category $\mathsf{H}^0((\mathcal{A}_c)_1)$. Consider now the following localization pair

$$(\mathcal{B}_f)_\lambda := (((\mathcal{B}_f)_\lambda)_{contr} \subset (\mathcal{B}_f)_\lambda),$$

where $(\mathcal{B}_f)_\lambda$ is as in Definition 6.1. Remark that we have a canonical weak equivalence in $\mathsf{Lp}$

$$\mathcal{B}_f \overset{F}{\longrightarrow} (\mathcal{B}_f)_\lambda.$$

Now define

$$\mathcal{R}\mathsf{Hom}(\mathcal{A},\mathcal{B}) := \mathsf{Hom}(\mathcal{A}_c, (\mathcal{B}_f)_\lambda)$$

and consider for morphism $\epsilon$ the image in $\mathsf{H}^0(\mathsf{Lp})$ of the following $Q$-equivalence in $\mathsf{Lp}$

$$\eta\colon (\mathcal{A},\mathcal{B}) \overset{(P,I)}{\longrightarrow} (\mathcal{A}_c,\mathcal{B}_f) \overset{(Id,F)}{\longrightarrow} (\mathcal{A}_c,(\mathcal{B}_f)_\lambda),$$

under the functor $\mathsf{Hom}(-,-)$.

We now show that the dg category associated with the localization pair $\mathcal{R}\mathsf{Hom}(\mathcal{A},\mathcal{B})$ is canonically Morita equivalent to

$$\mathsf{rep}_{dg}((\mathcal{A}_c)_1/(\mathcal{A}_c)_0, \mathcal{B}_f).$$

Notice that since $\mathcal{A}_c$ is a cofibrant object in $\mathsf{Lp}$, Lemma 4.12 implies that $(\mathcal{A}_c)_1$ is cofibrant in $\mathsf{dgcat}$ and so we have an exact sequence [**8**, Theorem 4.11]

$$(\mathcal{A}_c)_0 \hookrightarrow (\mathcal{A}_c)_1 \to (\mathcal{A}_c)_1/(\mathcal{A}_c)_0$$

in the Morita homotopy category of dg categories $\mathsf{Hmo}$. Since the dg category $(\mathcal{B}_f)$ is Morita fibrant, the application of the functor $\mathsf{rep}_{dg}(-,\mathcal{B}_f)$ to the previous exact sequence induces a new exact sequence in $\mathsf{Hmo}$

$$\mathsf{rep}_{dg}((\mathcal{A}_c)_0,\mathcal{B}_f) \leftarrow \mathsf{rep}_{dg}((\mathcal{A}_c)_1,\mathcal{B}_f) \leftarrow \mathsf{rep}_{dg}((\mathcal{A}_c)_1/(\mathcal{A}_c)_0,\mathcal{B}_f).$$

Remember that:

$$\mathsf{Hom}(\mathcal{A}_c,(\mathcal{B}_f)_\lambda)_1 = \mathsf{Fun}_{dg}((\mathcal{A}_c)_0,((\mathcal{B}_f)_\lambda)_{contr}) \underset{\mathsf{Fun}_{dg}((\mathcal{A}_c)_0,(\mathcal{B}_f)_\lambda)}{\times} \mathsf{Fun}_{dg}((\mathcal{A}_c)_1,(\mathcal{B}_f)_\lambda).$$

Now, since the dg categories $(\mathcal{A}_c)_1$ and $(\mathcal{B}_f)_\lambda$ satisfy the conditions of Theorem 6.3, we have an inclusion of dg categories

$$\mathsf{Hom}(\mathcal{A}_c, (\mathcal{B}_f)_\lambda)_1/\mathsf{Fun}_{dg}((\mathcal{A}_c)_1, ((\mathcal{B}_f)_\lambda)_{contr}) \longrightarrow \mathsf{rep}_{dg}((\mathcal{A}_c)_1, \mathcal{B}_f).$$

Notice that this inclusion induces the following Morita equivalence

$$\mathsf{Hom}(\mathcal{A}_c, (\mathcal{B}_f)_\lambda)_1/\mathsf{Fun}_{dg}((\mathcal{A}_c)_1, ((\mathcal{B}_f)_\lambda)_{contr}) \xrightarrow{\sim} \mathsf{rep}_{dg}((\mathcal{A}_c)_1/(\mathcal{A}_c)_0, \mathcal{B}_f).$$

We now show that the functor $\mathcal{R}\mathsf{Hom}(-,-)$ preserves $Q$-weak equivalences in $\mathsf{Lp}^{op} \times \mathsf{Lp}$. Consider a $Q$-weak equivalence

$$(\mathcal{A}, \mathcal{B}) \to (\tilde{\mathcal{A}}, \tilde{\mathcal{B}}),$$

in $\mathsf{Lp}^{op} \times \mathsf{Lp}$. By construction it will induce a Morita dg functor

$$(\tilde{\mathcal{A}}_c)_1/(\tilde{\mathcal{A}}_c)_0 \xrightarrow{\sim} (\mathcal{A}_c)_1/(\mathcal{A}_c)_0$$

and also a Morita dg functor

$$\mathcal{B}_f \xrightarrow{\sim} \tilde{\mathcal{B}}_f.$$

This implies that the induced dg functor

$$\mathsf{rep}_{dg}((\mathcal{A}_c)_1/(\mathcal{A}_c)_0, \mathcal{B}_f) \xrightarrow{\sim} \mathsf{rep}_{dg}((\tilde{\mathcal{A}}_c)_1/(\tilde{\mathcal{A}}_c)_0, \tilde{\mathcal{B}}_f)$$

is a Morita dg functor. Now observe that we have the following zig-zag of $Q$-weak equivalences in $\mathsf{Lp}$:

$$\begin{array}{c} (\mathsf{Fun}_{dg}((\mathcal{A}_c)_1, ((\mathcal{B}_f)_\lambda)_{contr}) \subset \mathsf{Hom}(\mathcal{A}_c, (\mathcal{B}_f)_\lambda)_1 \\ \downarrow \\ (\overline{\mathsf{Fun}_{dg}((\mathcal{A}_c)_1, ((\mathcal{B}_f)_\lambda)_{contr})} \subset \mathsf{Hom}(\mathcal{A}_c, (\mathcal{B}_f)_\lambda)_1/\mathsf{Fun}_{dg}((\mathcal{A}_c)_1, ((\mathcal{B}_f)_\lambda)_{contr})) \\ \uparrow \\ (\emptyset \subset \mathsf{Hom}(\mathcal{A}_c, (\mathcal{B}_f)_\lambda)_1/\mathsf{Fun}_{dg}((\mathcal{A}_c)_1, ((\mathcal{B}_f)_\lambda)_{contr})) \end{array} .$$

This allows us to conclude that the functor $\mathcal{R}\mathsf{Hom}(-,-)$ preserves $Q$-weak equivalences in $\mathsf{Lp}^{op} \times \mathsf{Lp}$. The proof is finished. □

**Proposition 6.5.** *Let $\mathcal{A}$ be a cofibrant object in $\mathsf{Lp}$. The induced internal tensor product (Definition 5.3) functor*

$$\mathcal{A} \otimes - \colon \mathsf{Lp} \longrightarrow \mathsf{Lp},$$

*preserves $Q$-weak equivalences.*

*Proof.* Let $F \colon \mathcal{B} \to \mathcal{C}$ be a $Q$-weak equivalence in $\mathsf{Lp}$ between cofibrant objects. We prove that the induced morphism in $\mathsf{Lp}$

$$\mathcal{A} \otimes \mathcal{B} \xrightarrow{F_*} \mathcal{A} \otimes \mathcal{C},$$

is a $Q$-weak equivalence. By Lemma 4.12, $\mathcal{A}_1$, $\mathcal{B}_1$ and $\mathcal{C}_1$ are cofibrant dg categories in dgcat and so we have a morphism of exact sequences in Hmo:

$$\begin{array}{ccccc} \mathcal{B}_0 & \hookrightarrow & \mathcal{B}_1 & \longrightarrow & \mathcal{B}_1/\mathcal{B}_0 \\ \downarrow & & \downarrow & & \downarrow \wr \\ \mathcal{C}_0 & \hookrightarrow & \mathcal{C}_1 & \longrightarrow & \mathcal{C}_1/\mathcal{C}_0, \end{array}$$

where the last column is a Morita dg functor. Since $\mathcal{A}_1$ is cofibrant in dgcat, [**3**, Proposition 1.6.3] implies that by applying the functor $\mathcal{A} \otimes -$ to the previous diagram, we obtain the following morphism of exact sequences in Hmo:

$$\begin{array}{ccccc} \mathcal{A}_1 \otimes \mathcal{B}_0 & \longrightarrow & \mathcal{A}_1 \otimes \mathcal{B}_1 & \longrightarrow & \mathcal{A}_1 \otimes (\mathcal{B}_1/\mathcal{B}_0) \\ \downarrow & & \downarrow & & \downarrow \wr \\ \mathcal{A}_1 \otimes \mathcal{C}_0 & \longrightarrow & \mathcal{A}_1 \otimes \mathcal{C}_1 & \longrightarrow & \mathcal{A}_1 \otimes (\mathcal{C}_1/\mathcal{C}_0). \end{array}$$

In conclusion we have the following Morita dg functor:

$$(\mathcal{A}_1 \otimes \mathcal{B}_1)/(\mathcal{A}_1 \otimes \mathcal{B}_0) \xrightarrow{\sim} (\mathcal{A}_1 \otimes \mathcal{C}_1)/(\mathcal{A}_1 \otimes \mathcal{C}_0).$$

Now let $\mathcal{H}$ be the full dg subcategory of $(\mathcal{A}_1 \otimes \mathcal{B}_1)/(\mathcal{A}_1 \otimes \mathcal{B}_0)$, whose objects are $a \otimes b$ with $a$ belonging to $\mathcal{A}_0$, and $\mathcal{P}$ the full dg subcategory of $(\mathcal{A}_1 \otimes \mathcal{C}_1)/(\mathcal{A}_1 \otimes \mathcal{C}_0)$ whose objects are $a \otimes c$ with $a$ belonging to $\mathcal{A}_0$. We have the following diagram:

$$\begin{array}{ccc} \mathcal{H} & \hookrightarrow & (\mathcal{A}_1 \otimes \mathcal{B}_1)/(\mathcal{A}_1 \otimes \mathcal{B}_0) \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{P} & \hookrightarrow & (\mathcal{A}_1 \otimes \mathcal{C}_1)/(\mathcal{A}_1 \otimes \mathcal{C}_0). \end{array}$$

The dg category $\mathcal{A} \otimes \mathcal{B}$, resp. $\mathcal{A} \otimes \mathcal{C}$, is Morita equivalent to $((\mathcal{A}_1 \otimes \mathcal{B}_1)/(\mathcal{A}_1 \otimes \mathcal{B}_0))/\mathcal{H}$, resp. to $((\mathcal{A}_1 \otimes \mathcal{C}_1)/(\mathcal{A}_1 \otimes \mathcal{C}_0))/\mathcal{P}$, and so we have the following commutative square:

$$\begin{array}{ccc} ((\mathcal{A}_1 \otimes \mathcal{B}_1)/(\mathcal{A}_1 \otimes \mathcal{B}_0))/\mathcal{H} & \xleftarrow{\sim} & \mathcal{A} \otimes \mathcal{B} \\ \downarrow \wr & & \downarrow F^* \\ ((\mathcal{A}_1 \otimes \mathcal{C}_1)/(\mathcal{A}_1 \otimes \mathcal{C}_0))/\mathcal{P} & \xleftarrow[\sim]{} & \mathcal{A} \otimes \mathcal{C}. \end{array}$$

Finally, by the two out of three property, $F^*$ is a $Q$-weak equivalence and so the proof is finished. □

*Remark 6.6.* Since the internal tensor product $- \otimes -$ is symmetric, Proposition 6.5 implies that the total left derived functor $- \otimes -$

$$- \overset{\mathbb{L}}{\otimes} - \colon \mathsf{Ho}(\mathsf{Lp}) \times \mathsf{Ho}(\mathsf{Lp}) \to \mathsf{Ho}(\mathsf{Lp})$$

exists, as in [**6**, Definition 8.4.7].

## 7. Agreement

In this chapter we prove our second main theorem (Theorem 7.2 below). We have the following adjunction:

$$\begin{array}{c} \mathsf{Lp} \\ L \uparrow \downarrow Ev_1 \\ \mathsf{dgcat}, \end{array}$$

where $Ev_1$ associates to a localization pair $(\mathcal{A}_0 \subset \mathcal{A}_1)$ the dg category $\mathcal{A}_1$ and $L$ associates to a dg category $\mathcal{A}$ the localization pair $(\emptyset \subset \mathcal{A})$.

**Proposition 7.1.** *If we consider on* $\mathsf{dgcat}$ *the Morita model structure and on* $\mathsf{Lp}$ *the* $Q$*-model structure, the previous adjunction is a Quillen equivalence.*

*Proof.* The functor $L$ sends Morita dg functors to weak equivalences. Since the cofibrations (and so the trivial fibrations) of the Quillen model structures of Proposition 4.8 and Theorem 4.18 are the same the evaluation functor $Ev_1$ preserves trivial fibrations. This shows us that $L$ is a left Quillen functor.

Let $\mathcal{A}$ be a cofibrant object in $\mathsf{dgcat}$ and $(\mathcal{B}_{contr} \subset \mathcal{B})$ a $Q$-fibrant object in $\mathsf{Lp}$. For a dg functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ in $\mathsf{dgcat}$ we need to show that $F$ is a Morita dg functor if and only if the induced morphism of localization pairs $(\emptyset \subset \mathcal{A}) \to (\mathcal{B}_{contr} \subset \mathcal{B})$ is a $Q$-weak equivalence. But, since the dg functor $\mathcal{B} \to \mathcal{B}/\mathcal{B}_{contr}$ is a Morita dg functor this follows automatically. □

**Theorem 7.2.** *The total derived functors* $-\overset{\mathbb{L}}{\otimes}-$ *and* $\mathcal{R}\mathsf{Hom}(-,-)$ *in* $\mathsf{Ho}(\mathsf{Lp})$ *agree, under the equivalence*

$$\begin{array}{c} \mathsf{Ho}(\mathsf{Lp}) \\ L \uparrow \downarrow \mathcal{R}Ev_1 \\ \mathsf{Hmo}, \end{array}$$

*with the functors* $-\overset{\mathbb{L}}{\otimes}-$ *and* $\mathsf{rep}_{dg}(-,-)$ *(see Remark 5.1) in* $\mathsf{Hmo}$.

*Proof.* Let $\mathcal{A}$ and $\mathcal{B}$ be two small dg categories. Then $\mathcal{A}\overset{\mathbb{L}}{\otimes}\mathcal{B}$ identifies with $\mathcal{A}_c \otimes \mathcal{B}$, where $\mathcal{A}_c$ is a cofibrant resolution of $\mathcal{A}$ in $\mathsf{dgcat}$. Since $L(\mathcal{A}_c)$ is cofibrant in $\mathsf{Lp}$, Proposition 6.5, implies the following zig-zags:

$$L(\mathcal{A})\overset{\mathbb{L}}{\otimes} L(\mathcal{B}) \xleftarrow{\sim} L(\mathcal{A}_c)\otimes L(\mathcal{B}) \xrightarrow{\sim} L(\mathcal{A}_c \otimes \mathcal{B}) = L(\mathcal{A}\overset{\mathbb{L}}{\otimes}\mathcal{B}),$$

of weak equivalences in $\mathsf{Lp}$. This proves that the total left derived tensor products on $\mathsf{Ho}(\mathsf{Lp})$ and $\mathsf{Hmo}$ are identified. Now, notice that $\mathsf{rep}_{dg}(\mathcal{A},\mathcal{B})$ identifies with $\mathsf{rep}_{dg}(\mathcal{A}_c,\mathcal{B}_f)$, where $\mathcal{B}_f$ is a fibrant resolution of $\mathcal{B}$ in $\mathsf{dgcat}$. By definition

$$\mathcal{R}\mathsf{Hom}(L(\mathcal{A}),L(\mathcal{B})) = \mathsf{Hom}((L(\mathcal{A})_c,(L(\mathcal{B})_f)_\lambda),$$

where $\lambda$ denotes an infinite cardinal whose size is greater or equal to the cardinality of the set of isomorphism classes of objects in the category $\mathsf{H}^0(\mathcal{A}_c)$. We have the following $Q$-weakly equivalent objects in $\mathsf{Lp}$:

$$
\begin{array}{l}
\mathcal{R}\mathsf{Hom}(L(\mathcal{A}), L(\mathcal{B})) \\
\mathsf{Hom}((L(\mathcal{A})_c, (L(\mathcal{B})_f)_\lambda) \\
\mathsf{Hom}((\emptyset \subset \mathcal{A}_c), ((\mathcal{B}_f)_\lambda)_{contr} \subset (\mathcal{B}_f)_\lambda)) \\
(\mathsf{Fun}_{dg}(\mathcal{A}_c, ((\mathcal{B}_f)_\lambda)_{contr}) \subset \mathsf{Fun}_{dg}(\mathcal{A}_c, (\mathcal{B}_f)_\lambda)) \\
\overline{\mathsf{Fun}_{dg}(\mathcal{A}_c, ((\mathcal{B}_f)_\lambda)_{contr})} \subset \mathsf{Fun}_{dg}(\mathcal{A}_c, (\mathcal{B}_f)_\lambda))/\mathsf{Fun}_{dg}(\mathcal{A}_c, ((\mathcal{B}_f)_\lambda)_{contr}) \\
(\emptyset \subset \mathsf{rep}_{dg}(\mathcal{A}_c, \mathcal{B}_f)) \\
L(\mathsf{rep}_{dg}(\mathcal{A}, \mathcal{B})).
\end{array}
$$

This proves that the total right derived functor $\mathcal{R}\mathsf{Hom}(-,-)$ in $\mathsf{Ho}(\mathsf{Lp})$ corresponds to the internal $\mathsf{Hom}$-functor $\mathsf{rep}_{dg}(-,-)$. □

*Remark 7.3.* Notice that Theorem 7.2 provides us a conceptual characterization of Toën's adhoc construction as a total derived functor. Intuitively, when we pass from dg categories to localization pairs, we gain an "extra degree of freedom" which allows us to perform derived constructions.

Moreover, Theorems 6.4 and 7.2 provide us a simple way to construct the internal $\mathsf{Hom}$-objects in $\mathsf{Hmo}$. In contrast with Toën's approach, requiring an involved dg category of "right quasi-representable" bimodules (see Remark 5.1), when using the model $\mathsf{Lp}$ (see Theorem 4.18) it is enough to derive its natural internal $\mathsf{Hom}$-functor (see Definition 5.2) which only makes use of dg categories of dg functors. We remind the reader that the construction of the internal $\mathsf{Hom}$-objects in $\mathsf{Hmo}$ was the main difficulty in the development of Toën's derived Morita theory [**22**, §7].

## Appendix A. Homotopical algebra tools

In this appendix we recall some classical results concerning the construction of Quillen model structures. Let us start with Kan's lifting theorem.

**Theorem A.1.** *[**6**, Theorem 11.3.2] Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations $I$ and generating trivial cofibrations $J$. Let $\mathcal{N}$ be a complete and cocomplete category, and let*

$$
\begin{array}{c}
\mathcal{N} \\
F \big\uparrow \big\downarrow U \\
\mathcal{M}
\end{array}
$$

*be a pair of adjoint functors. If we let $FI = \{Fu | u \in I\}$ and $FJ = \{Fu | u \in J\}$ and if*

*(1) both of the sets $FI$ and $FJ$ permit the small object argument and*

*(2) $U$ takes relative $FJ$-cell complexes to weak equivalences,*

*then there is a cofibrantly generated model category structure on $\mathcal{N}$, in which $FI$ is a set of generating cofibrations, $FJ$ is a set of generating trivial cofibrations, and the weak equivalences are the maps that $U$ takes into a weak equivalence in $\mathcal{M}$. Furthermore, with respect to this model structure, $(F, U)$ is a Quillen pair.*

We now recall the following recognition theorem.

**Theorem A.2.** *[**5**, Theorem 2.1.19] Let $\mathcal{M}$ be a complete and cocomplete category, $W$ a class of maps in $\mathcal{M}$ and $I$ and $J$ sets of maps in $\mathcal{M}$ such that:*

*1) The class $W$ satisfies the two out of three axiom and is stable under retracts.*

*2) The domains of the elements of $I$ are small relative to $I$-cell.*

*3) The domains of the elements of $J$ are small relative to $J$-cell.*

*4) $J$-cell $\subseteq W \cap I$-cof.*

*5) $I$-inj $\subseteq W \cap J$-inj.*

*6) $W \cap I$-cof $\subseteq J$-cof or $W \cap J$-inj $\subseteq I$-inj.*

*Then there is a cofibrantly generated model category structure on $\mathcal{M}$ in which $W$ is the class of weak equivalences, $I$ is a set of generating cofibrations, and $J$ is a set of generating trivial cofibrations.*

We now state a weaker form of Bousfield-Friedlander localization [**7**, Theorem X-4.1]. See [**2**] for the original article.

**Definition A.3.** Let $\mathcal{M}$ be a Quillen model category, $Q\colon \mathcal{M} \to \mathcal{M}$ a functor and $\eta\colon \mathrm{Id} \to Q$ a natural transformation between the identity functor and $Q$. A morphism $f\colon A \to B$ in $\mathcal{M}$ is:

- a *$Q$-weak equivalence* if $Q(f)$ is a weak equivalence in $\mathcal{M}$.
- a *cofibration* if it is a cofibration in $\mathcal{M}$.
- a *$Q$-fibration* if it has the R.L.P. with respect to all cofibrations which are $Q$-weak equivalences.

An immediate analysis of the proof of [**7**, Theorem X-4.1] allows us to state the following general Theorem A.4: notice that the proof of [**7**, Lemma X-4.4] only uses a weaker form of right properness (this corresponds to the following condition (A0)) and the proof of [**7**, Lemma 4.6] only uses the following condition (A3).

**Theorem A.4.** *Let $\mathcal{M}$ be a Quillen model structure such that:*

*(A0) Given a diagram*

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[\eta_A]{} & Q(A), \end{array}$$

*with $\eta_{\mathcal{A}}$ a weak equivalence and $P$ a fibration in $\mathcal{M}$, the induced map*

$$\eta_{\mathcal{A}_*}\colon A \underset{Q(A)}{\times} B \to B$$

*is a weak equivalence in $\mathcal{M}$.*

*Suppose that $\mathcal{M}$ is endowed with a functor $Q\colon \mathcal{M} \to \mathcal{M}$ and a natural transformation $\eta\colon Id \to Q$ such that the following three conditions hold:*

*(A1) The functor $Q$ preserves weak equivalences.*

*(A2) The maps* $\eta_{Q(A)}, Q(\eta_A)\colon Q(A) \to QQ(A)$ *are weak equivalences in* $\mathcal{M}$*.*

*(A3) Given a diagram*

$$\begin{array}{ccc} & & B \\ & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow[\eta_A]{} & Q(A) \end{array}$$

*with* $p$ *a* $Q$*-fibration the induced map* $\eta_{\mathcal{A}_*}\colon A \underset{Q(A)}{\times} B \to B$ *is a* $Q$*-weak equivalence.*

*Then there is a Quillen model structure on* $\mathcal{M}$ *for which the weak equivalences are the* $Q$*-weak equivalences, the cofibrations those of* $\mathcal{M}$ *and the fibrations the* $Q$*-fibrations.*

The following Lemma corresponds to [**7**, Lemma X-4.4].

**Lemma A.5.** *Suppose that* $F\colon A \to B$ *is a fibration in* $\mathcal{M}$ *and that* $\eta_A$ *and* $\eta_B$ *are weak equivalences of* $\mathcal{M}$*. Then* $F$ *is a* $Q$*-fibration.*

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