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ON THE WU INVARIANTS FOR IMMERSIONS OF A GRAPH INTO THE PLANE

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(*communicated by Alexander Mishchenko*)

Abstract

We give an explicit calculation of the Wu invariants for immersions of a finite graph into the plane and classify all generic immersions of a graph into the plane up to regular homotopy by the Wu invariant. This result is a generalization of the fact that two plane curves are regularly homotopic if and only if they have the same rotation number.

1. Introduction

Throughout this paper we work in the piecewise linear category. In [**6, 7**], Wu defined an isotopy invariant of embeddings and immersions of polyhedra into the Euclidean space in terms of the cohomology of deleted product spaces. In case of embeddings, this invariant classifies all embeddings of a graph into $\mathbb{R}^3$ up to spatial graph-homology (Taniyama [**4**]). But as far as the author knows, little is known about an application of this invariant in case of immersions. Our purpose in this paper is to give an explicit calculation of Wu's invariant of immersions of a graph into $\mathbb{R}^2$ and apply it to a geometric classification.

Let G be a finite, connected and simple graph which has at least one edge. We denote the set of all vertices (resp. edges) of G by $V(G)$ (resp. $E(G)$). If the terminal vertices of an edge e of G are u and v, then we denote $e=(u,v)=(v,u)$. We denote the number of edges incident to a vertex v by $\deg v$. Note that G has a structure of a finite 1-dimensional simplicial complex. We regard G as a topological space by considering its geometric realization, namely G is a compact and connected 1-dimensional polyhedron. In this situation, each of the vertices and the edges of G can be regarded as a subset of G. We call a continuous map $f\colon G\to\mathbb{R}^2$ a *plane immersion* of G if there exists an open covering $\{U_\nu\}$ of G such that $f|_{U_\nu}$ is an embedding for any ν. A plane immersion f of G is said to be *generic* if all of its multipoints are transversal double points away from vertices. We say that two plane immersions f and g of G are *regularly homotopic* if there exists a homotopy $F\colon G\times[0,1]\to\mathbb{R}^2$ from f to g and an open covering $\{U_\nu\}$ of G such that $f_t|_{U_\nu}$ is an embedding for any ν and for any $t\in[0,1]$, where f_t is a continuous map from G to $\mathbb{R}^2$ defined by

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$f_t(x) = F(x,t)$ for any $x \in G$. Note that regular homotopy defines an equivalence relation on plane immersions of a graph.[1]

We give the precise definition of the Wu invariant $\mathcal{R}(f)$ of a plane immersion f of a graph in the next section and also give an explicit calculation of $\mathcal{R}(f)$ in Section 3. It can be calculated as a first cohomology class of a subspace of the symmetric deleted product of the graph, which is called the *symmetric tube* of the graph. Moreover, we have the following classification theorem.

Theorem 1.1. *Let f and g be two generic plane immersions of a graph G. Then the following are equivalent:*

(1) f and g are regularly homotopic.

(2) f and g are transformed into each other by the local moves as illustrated in Figure 1(1), (2), (3) and ambient isotopies.

(3) $\mathcal{R}(f) = \mathcal{R}(g)$.

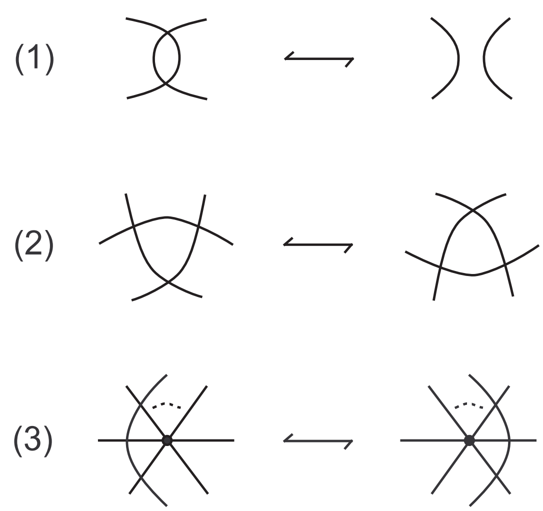

Figure 1.

Let K_m be the *complete graph* on m vertices for a positive integer m, namely $V(K_m) = \{v_1, v_2, \ldots, v_m\}$ and $E(K_m) = \{(v_i, v_j)(1 \leqslant i < j \leqslant m)\}$. A plane immersion f of K_3 is called a *plane curve*. By Theorem 1.1, we have the following corollary.

Corollary 1.2. *Let f and g be two generic plane curves. Then the following are equivalent:*

(1) f and g are regularly homotopic.

(2) f and g are transformed into each other by the local moves as illustrated in Figure 1(1), (2) and ambient isotopies.

[1] This equivalence relation was introduced in [**6**] by the name of *local isotopy*.

(3) $\mathcal{R}(f) = \mathcal{R}(g)$.

We prove Theorem 1.1 in Section 4. As we will see in Example 3.11, $\mathcal{R}(f)$ of a plane curve f coincides with the *rotation number* [**5**] of f. (In this paper we consider the orientation on $\mathbb{R}^2$ with positive rotation numbers in the counterclockwise direction.) Thus Corollary 1.2 coincides with the regular homotopy classification of plane curves by Whitney-Graustein's theorem [**5**] and Kauffman's combinatorial interpretation [**2**]. We remark here that recently Permyakov gives a simple combinatorial interpretation of $\mathcal{R}(f)$ and shows a theorem which corresponds to Theorem 1.1 [**3**].

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2. Wu invariant

In this section we give the definition of the Wu invariant of a plane immersion of a graph G. We refer the reader to [**6**] for the general case. Let X be a topological space. For the embedding $\tilde{d}\colon X \to X \times X$ defined by $\tilde{d}(x) = (x, x)$, we call $\widetilde{X}^* = (X \times X) \setminus \tilde{d}(X)$ the *deleted product* of X. A map $\sigma(x, y) = (y, x)$ gives a free $\mathbb{Z}_2$-action on $\widetilde{X}^*$. We call $X^* = \widetilde{X}^*/\mathbb{Z}_2$ the *symmetric deleted product* of X. We denote the image of $\tilde{d}(X)$ by the natural projection from $X \times X$ to $(X \times X)/\mathbb{Z}_2$ by $d(X)$. Let U be a neighborhood of $\tilde{d}(X)$ in $X \times X$. Then $\widetilde{U}^* = U \setminus \tilde{d}(X)$ is called a *deleted neighborhood of* $\tilde{d}(X)$ *in* $\widetilde{X}^*$. A deleted neighborhood $\widetilde{U}^*$ is said to be *σ-invariant* if $\sigma(U) = U$. Then we call $U^* = \widetilde{U}^*/\mathbb{Z}_2$ a *symmetric deleted neighborhood of* $d(X)$ *in* X^*.

For a graph G, let $\{U_\lambda^*\}$ be the set of all symmetric deleted neighborhoods of $d(G)$ in G^*. Then $\{U_\lambda^*, \prec\}$ forms an oriented set by $U_\lambda^* \prec U_\mu^*$ if $U_\lambda^* \supset U_\mu^*$. For this oriented set, $\{H^1(U_\lambda^*; \mathbb{Z}), i_\lambda^{\mu *}\}$ forms an inductive system of modules, where $H^1(\cdot; \mathbb{Z})$ denotes the integral first cohomology group and

$$i_\lambda^{\mu *} : H^1(U_\lambda^*; \mathbb{Z}) \to H^1(U_\mu^*; \mathbb{Z})$$

is a homomorphism induced by the inclusion. Then we denote the inductive limit $\varinjlim H^1(U_\lambda^*; \mathbb{Z})$ by $R(G)$. We note that we have the following natural homomorphism

$$i_\lambda^* \colon H^1(U_\lambda^*; \mathbb{Z}) \longrightarrow R(G) \tag{2.1}$$

for any symmetric deleted neighborhood U_λ^* of $d(G)$ in G^*.

Let $f\colon G \to \mathbb{R}^2$ be a plane immersion. Namely there exists an open covering $\mathcal{U} = \{U_\nu\}$ of G such that $f|_{U_\nu}$ is an embedding for any ν. Then the set

$$\widetilde{W_{\mathcal{U}}} = \{(x_1, x_2) \in \widetilde{G}^* \mid \text{there exists a } U_\nu \text{ such that } x_1, x_2 \in U_\nu\}$$

forms a σ-invariant deleted neighborhood of $\tilde{d}(G)$ in $\widetilde{G}^*$ and a continuous map $\bar{f}\colon W_{\mathcal{U}} \to (\mathbb{R}^2)^*$ is defined by $\bar{f}[x_1, x_2] = [f(x_1), f(x_2)]$. On the other hand, it is well known that a continuous map $r\colon (\widetilde{\mathbb{R}}^2)^* \to \mathbb{S}^1$ defined by

$$r(y_1, y_2) = (y_1 - y_2)/\|y_1 - y_2\|$$

is a σ-equivariant strong deformation retract and $r\colon (\mathbb{R}^2)^* \to \mathbb{S}^1/\mathbb{Z}_2 \approx \mathbb{S}^1$ is also a strong deformation retract, where $\mathbb{S}^1/\mathbb{Z}_2$ denotes the quotient space of $\mathbb{S}^1$ by identifying the antipodal points. Let Σ be a generator of $H^1(\mathbb{S}^1;\mathbb{Z}) \cong \mathbb{Z}$. Then the image of Σ by the composition

$$H^1(\mathbb{S}^1;\mathbb{Z}) \xrightarrow[\cong]{\bar{r}^*} H^1({\mathbb{R}^2}^*;\mathbb{Z}) \xrightarrow{\bar{f}^*} H^1(W_{\mathcal{U}};\mathbb{Z}) \xrightarrow{i^*_{\mathcal{U}}} R(G)$$

is denoted by $\mathcal{R}(f)$, where $i^*_{\mathcal{U}}$ is the natural homomorphim of (2.1) for $W_{\mathcal{U}}$. We call $\mathcal{R}(f)$ a *Wu invariant*[2] of f. We remark here that the definition above is independent of the choice of $\mathcal{U}$.

Proposition 2.1 ([**6**])**.** *$\mathcal{R}(f)$ is a regular homotopy invariant.*

Proof. Let f and g be two regularly homotopic plane immersions of G. Namely there exists a homotopy $F\colon G \times [0,1] \to \mathbb{R}^2$ from f to g and an open covering $\{U_\nu\}$ of G such that $f_t|_{U_\nu}$ is an embedding for any ν and for any $t \in [0,1]$, where $f_t(x) = F(x,t)$ for $x \in G$. Then we can define a homotopy $F_{\mathcal{U}}\colon W_{\mathcal{U}} \times [0,1] \longrightarrow (\mathbb{R}^2)^*$ from $\bar{f}$ to $\bar{g}$ by $F_{\mathcal{U}}([x_1,x_2],t) = [f_t(x_1), f_t(x_2)]$. Thus we have that $\mathcal{R}(f) = i^*_{\mathcal{U}}\bar{f}^*\bar{r}^*(\Sigma) = i^*_{\mathcal{U}}\bar{g}^*\bar{r}^*(\Sigma) = \mathcal{R}(g)$. This completes the proof. $\square$

3. Symmetric tube of a graph

A precise method to calculate $R(G)$ is provided in [**6**]. Let X and Y be two topological spaces and $M = X \cup (X \times Y \times [0,1]) \cup Y$ the disjoint union. Let us consider a quotient space by identifying $(x,y,0) \in X \times Y \times [0,1]$ with $x \in X$ and $(x,y,1) \in X \times Y \times [0,1]$ with $y \in Y$. We call the quotient space a *join of X and Y* and denote it by $X \circ Y$. We set $[X,Y]^{(0)} = \{[x,y,1/2] \in X \circ Y \mid x \in X,\ y \in Y\}$. For example, the join $v \circ e$ of a vertex v and an edge e is homeomorphic to a 2-simplex, and $[v,e]^{(0)}$ is homeomorphic to a 1-simplex. The following is a special case of what is called a *canonical cellular decomposition* of the product space of X [**1, 6**].

Proposition 3.1. *Let G be a graph. Then $G \times G$ is decomposed into the following cells:*

(1) $\tilde{d}(s)$ for $s \in V(G)$ or $E(G)$.
(2) $s_1 \times s_2$ for $s_i \in V(G)$ or $E(G)$ $(i = 1,2)$ and $s_1 \cap s_2 = \emptyset$.
(3) $\tilde{d}(s) \circ (s_1 \times s_2)$ for $s, s_1, s_2 \in V(G)$ or $E(G)$, $s_1 \cap s_2 = \emptyset$ and $s \cup s_i$ is contained in a vertex or an edge of G $(i = 1,2)$.

In particular, the cellular complex which consists of all cells $[s, s_1 \times s_2]^{(0)}$ for simplices s, s_1 and s_2 of Proposition 3.1(3) is called a *tube* of G and is denoted by $\widetilde{G}^{(0)}$. Clearly $\widetilde{G}^{(0)}$ is σ-invariant in $\widetilde{G}^*$. We call $\widetilde{G}^{(0)}/\mathbb{Z}_2$ a *symmetric tube* of G and denote it by $G^{(0)}$. We denote $[s, s_1 \times s_2]^{(0)}/\mathbb{Z}_2$ by $[s, s_1 * s_2]$, and we note that $[s, s_1 * s_2] = [s, s_2 * s_1]$.

[2]This invariant was introduced in [**6**] by the name of *local isotopy class* and denoted by $\Lambda_f(G)$.

Example 3.2. Let K_3 be the complete graph on three vertices as illustrated in Figure 2. The figure on the right side in Figure 2 illustrates the canonical cellular decomposition of $K_3 \times K_3$ in the sense of Proposition 3.1 as an expanded diagram of the 2-dimensional torus. The dotted thick parts and black thick parts represent the cells of Proposition 3.1(1) and (2), respectively. The gray thick parts represent $\widetilde{K}_3^{(0)}$. Thus we can see that $K_3^{(0)}$ is homeomorphic to the circle.

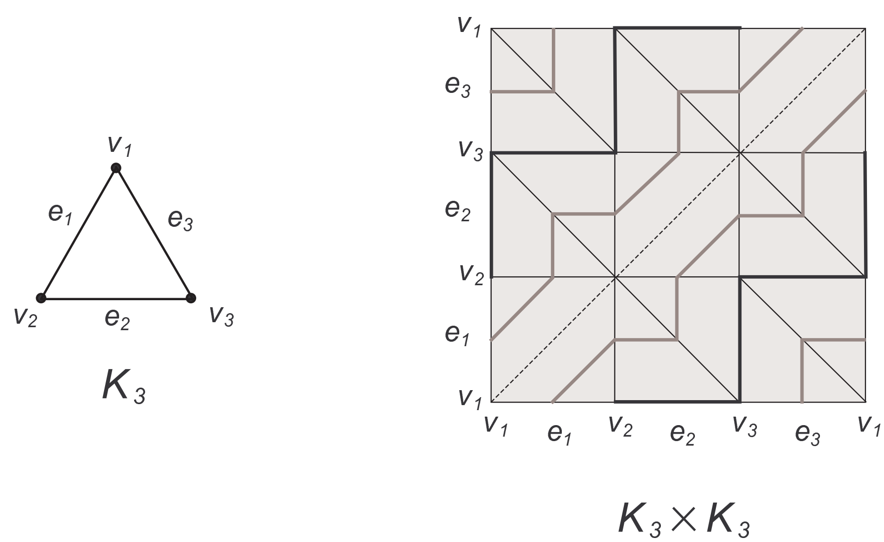

Figure 2.

Let $P(\widetilde{G}^*)$ be the cellular complex which consists of all cells $s_1 \times s_2$ for simplices s_1, s_2 of Proposition 3.1(2). Clearly $P(\widetilde{G}^*)$ is also σ-invariant in $\widetilde{G}^*$. We denote $P(\widetilde{G}^*)/\mathbb{Z}_2$ by $P(G^*)$. We note that $G^* \setminus P(G^*)$ is a symmetric deleted neighborhood of $d(G)$ in G^*. It is known that

$$i^*_{G^* \setminus P(G^*)} \colon H^1(G^* \setminus P(G^*); \mathbb{Z}) \xrightarrow{\cong} R(G),$$

where $i^*_{G^* \setminus P(G^*)}$ is the natural homomorphism of (2.1) for $G^* \setminus P(G^*)$, and there exists a deformation retract $j \colon G^* \setminus P(G^*) \to G^{(0)}$ [**6**]. Therefore we have the following.

Theorem 3.3 ([**6**]). $i^*_{G^{(0)}} = i^*_{G^* \setminus P(G^*)} j^* \colon H^1(G^{(0)}; \mathbb{Z}) \xrightarrow{\cong} R(G)$.

Thus the calculation of $H^1(G^{(0)}; \mathbb{Z})$ provides a precise method to calculate $R(G)$. To calculate $H^1(G^{(0)}; \mathbb{Z})$, we investigate the structure of $G^{(0)}$ directly. We set $V(G) = \{v_1, v_2, \ldots, v_m\}$ and $E(G) = \{e_1, e_2, \ldots, e_n\}$. We choose a fixed orientation on each edge of G. We put

$$\begin{aligned} Z^s_{st} = Z^s_{ts} &= [v_s, v_s * v_t] && \text{for } (v_s, v_t) \in E(G), \\ W^u_{st} = W^u_{ts} &= [v_u, v_s * v_t] && \text{for } (v_u, v_s), (v_u, v_t) \in E(G), v_s \neq v_t, \\ X^i_{st} = X^i_{ts} &= [e_i, v_s * v_t] && \text{for } e_i = (v_s, v_t) \in E(G), \end{aligned}$$

and

$$Y^s_{ti} = [v_s, v_t * e_i] \quad \text{for } (v_s, v_t), e_i \in E(G), (v_s, v_t) \neq e_i \text{ and } v_s \subset e_i.$$

Note that Z^s_{st} and W^u_{st} are 0-dimensional simplices of $G^{(0)}$, and X^i_{st} and Y^s_{ti} are 1-dimensional simplices of $G^{(0)}$. An orientation of X^i_{st} is induced by e_i, and an orientation of Y^s_{ti} is induced by e_i. We can consider Z^s_{st} and W^u_{st} as 0-chains in $C_0(G^{(0)};\mathbb{Z})$ and X^i_{st} and Y^s_{ti} as 1-chains in $C_1(G^{(0)};\mathbb{Z})$. The dual cochain of Z^s_{st}, W^u_{st}, X^i_{st} and Y^s_{ti} are denoted by Z^{st}_s, W^{st}_u, X^{st}_i and Y^{ti}_s, respectively. It is not difficult to see the following.

Proposition 3.4. *For a graph G, a cell of symmetric tube $G^{(0)}$ is one of Z^s_{st}, W^u_{st}, X^i_{st} and Y^s_{ti} as above. Therefore $G^{(0)}$ is also a graph.*

For example, let S_n be a graph as illustrated in Figure 3. By enumerating vertices and edges of $S_n^{(0)}$ and observing the connection between them directly, we have the following.

Lemma 3.5. *The symmetric tube $S_n^{(0)}$ of S_n is a graph as follows:*

(1) $V(S_n^{(0)}) = \{Z^i_{n+1,i}$ *and* $Z^{n+1}_{n+1,i}$ $(i = 1, 2, \ldots, n)$, W^{n+1}_{jk} $(1 \leqslant j < k \leqslant n)\}$.

(2) $E(S_n^{(0)}) = \{X^i_{n+1,i}$ $(i = 1, 2, \ldots, n)$, Y^{n+1}_{jk} *and* Y^{n+1}_{kj} $(1 \leqslant j < k \leqslant n)\}$.

(3) $X^i_{n+1,i} = (Z^{n+1}_{n+1,i}, Z^i_{n+1,i})$ $(i = 1, 2, \ldots, n)$, $Y^{n+1}_{jk} = (Z^{n+1}_{n+1,j}, W^{n+1}_{jk})$ *and* $Y^{n+1}_{kj} = (Z^{n+1}_{n+1,k}, W^{n+1}_{jk})$ $(1 \leqslant j < k \leqslant n)$.

(4) $\deg Z^i_{n+1,i} = 1$, $\deg Z^{n+1}_{n+1,i} = n$ $(i = 1, 2, \ldots, n)$ *and* $\deg W^{n+1}_{jk} = 2$ $(1 \leqslant j < k \leqslant n)$.

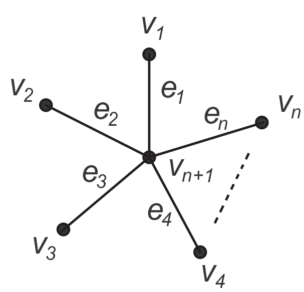

Figure 3.

Example 3.6. Figure 4 illustrates the symmetric tube $S_n^{(0)}$ of S_n for $n = 1, 2$ and 3. We can see that the subgraph H of $S_n^{(0)}$ defined by

$$V(H) = \{Z_{n+1,i}^{n+1}\ (i = 1, 2, \ldots, n),\ W_{jk}^{n+1}\ (1 \leqslant j < k \leqslant n)\},$$
$$E(H) = \{Y_{jk}^{n+1}, Y_{kj}^{n+1}\ (1 \leqslant j < k \leqslant n)\}$$

is homeomorphic to K_n. Precisely speaking, H is isomorphic to the graph which is obtained from K_n by subdividing each edge of K_n once.

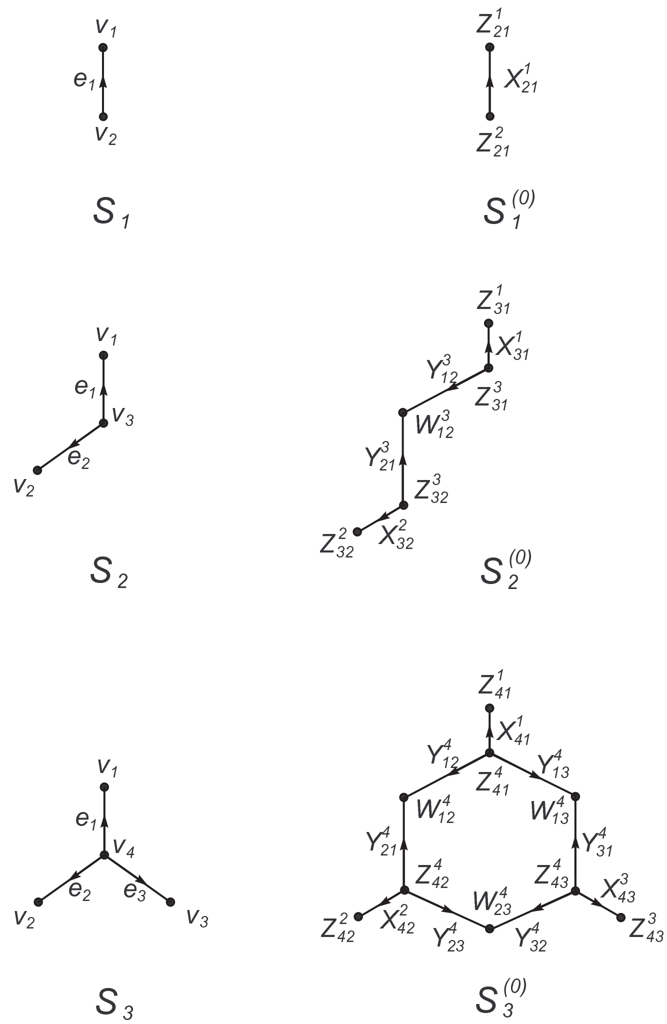

Figure 4.

By Lemma 3.5 and Example 3.6, we can see that the symmetric tube $G^{(0)}$ of a graph

G is obtained from G by "substituting" K_d for each vertex v_s of G as follows, where $d = \deg v_s$. Also see Examples 3.11 and 3.13.

Lemma 3.7. *Let G be a graph. For a vertex v_s of G, let $v_{s_1}, v_{s_2}, \ldots, v_{s_d}$ be all vertices connected to v_s so that $e_{i_l} = (v_s, v_{s_l})$ $(l = 1, 2, \ldots, d,\ 1 \leqslant i_j < i_k \leqslant d$ for $j < k)$. Then $G^{(0)}$ is obtained from G by replacing each v_s with the graph H_s, which is homeomorphic to K_d defined by*

$$\begin{aligned} V(H_s) &= \{Z^s_{ss_l}\ (i = 1, 2, \ldots, d),\ W^s_{s_j s_k}\ (1 \leqslant j < k \leqslant d)\}, \\ E(H_s) &= \{Y^s_{s_j i_k} = (Z^s_{ss_j}, W^s_{s_j s_k}), Y^s_{s_k i_j} = (Z^s_{ss_k}, W^s_{s_j s_k})\ (1 \leqslant j < k \leqslant d)\}, \end{aligned}$$

and by replacing e_{i_l} with $X^{i_l}_{ss_l} = (Z^{s_l}_{ss_l}, Z^s_{ss_l})$ $(l = 1, 2, \ldots, d)$.

By the universal coefficient theorem, it is sufficient to know $H_1(G^{(0)}; \mathbb{Z})$ to calculate $H^1(G^{(0)}; \mathbb{Z})$. So in the following we construct a spanning tree of $G^{(0)}$. First we define a subgraph $T_{S_n^{(0)}}$ of $S_n^{(0)}$ as follows. We set $V(T_{S_n^{(0)}}) = V(S_n^{(0)})$ and

$$\begin{aligned} E(T_{S_n^{(0)}}) = \{&X^i_{n+1,i}\ (i = 1, 2, \ldots, n),\ Y^{n+1}_{nj}\ (j = 1, 2, \ldots, n-1), \\ &Y^{n+1}_{tn}\ (t = 1, 2, \ldots, n-1),\ Y^{n+1}_{kl}\ (1 \leqslant k < l \leqslant n-1)\}. \end{aligned}$$

We note that

$$E(S_n^{(0)}) \setminus E(T_{S_n^{(0)}}) = \{Y^{n+1}_{lk}\ (1 \leqslant k < l \leqslant n-1))\}.$$

Then we easily have the following.

Lemma 3.8. *A subgraph $T_{S_n^{(0)}}$ is a spanning tree of $S_n^{(0)}$.*

Now we construct a spanning tree of $G^{(0)}$ on the outcome of Lemma 3.8. For $v_s \in V(G)$, let $\mathrm{st}(v_s)$ be a subgraph of G consisting of v_s and all edges incident to v_s. Let $v_{s_1}, v_{s_2}, \ldots, v_{s_d}$ be all vertices connected to v_s so that

$$e_{i_l} = (v_s, v_{s_l})\ (l = 1, 2, \ldots, d,\ 1 \leqslant i_j < i_k \leqslant d \text{ for } j < k),$$

where $d = \deg v_s$. We construct a spanning tree $T_{\mathrm{st}(v_s)}$ of $\mathrm{st}(v_s)$ in the same way as $S_n^{(0)}$. Namely, $V(T_{\mathrm{st}(v_s)}) = V(\mathrm{st}(v_s))$ and

$$\begin{aligned} E(T_{\mathrm{st}(v_s)}) = \{&X^{i_l}_{ss_l}\ (l = 1, 2, \ldots, d),\ Y^s_{s_d i_j}\ (j = 1, 2, \ldots, d-1), \\ &Y^s_{s_j i_d}\ (j = 1, 2, \ldots, d-1),\ Y^s_{s_j i_k}\ (1 \leqslant j < k \leqslant d-1)\}. \end{aligned}$$

Let T_G be a spanning tree of G. We define a subgraph $T_{G^{(0)}}$ of $G^{(0)}$ by $V(T_{G^{(0)}}) = V(G^{(0)})$ and

$$\begin{aligned} E(T_{G^{(0)}}) = \{&X^j_{j_1 j_2} \mid e_j = (v_{j_1}, v_{j_2}) \in E(T_G)\} \\ &\cup \bigcup_{v_s \in V(G)} \left(E(T_{\mathrm{st}(v_s)}) \setminus \{X^{i_l}_{ss_l}\ (l = 1, 2, \ldots, d)\}\right). \end{aligned}$$

Then we have the following.

Lemma 3.9. *A subgraph $T_{G^{(0)}}$ is a spanning tree of $G^{(0)}$.*

We note that

$$E(\mathrm{st}(v_s)) \setminus E(T_{\mathrm{st}(v_s)}) = \{Y^s_{s_j i_k} \ (1 \leqslant j < k \leqslant d-1)\}$$

for $v_s \in V(G)$ and this set is empty for $d = 1, 2$. Therefore we have that

$$\begin{aligned} E(G^{(0)}) \setminus E(T_{G^{(0)}}) = \{ & X^j_{j_1 j_2} \mid e_j = (v_{j_1}, v_{j_2}) \in E(G) \setminus E(T_G)\} \\ & \cup \bigcup_{\substack{v_s \in V(G) \\ \deg v_s \geqslant 3}} \{Y^s_{s_j i_k} \ (1 \leqslant j < k \leqslant d-1)\}. \end{aligned}$$

Thus we can determine a structure of $H^1(G^{(0)}; \mathbb{Z})$ completely as follows.

Theorem 3.10.

(1) $H^1(G^{(0)}; \mathbb{Z}) \cong \displaystyle\bigoplus_{\substack{e_j \in E(G) \setminus E(T_G) \\ e_j = (v_{j_1}, v_{j_2})}} \langle X^{j_1 j_2}_j \rangle \oplus \bigoplus_{\substack{v_s \in V(G) \\ \deg v_s \geqslant 3}} \left(\bigoplus_{1 \leqslant j < k \leqslant d-1} \langle Y^{s_j i_k}_s \rangle \right),$

(2) $\operatorname{rank} H^1(G^{(0)}; \mathbb{Z}) = 1 - 2n + \dfrac{1}{2} \displaystyle\sum_{s=1}^{m} (\deg v_s)^2.$

Proof. (1) is clear. We show (2). We have that

$$\begin{aligned} \operatorname{rank} H^1(G^{(0)}; \mathbb{Z}) &= \operatorname{rank} H^1(G; \mathbb{Z}) + \sum_{s=1}^{m} \frac{1}{2} (\deg v_s - 1)(\deg v_s - 2) \\ &= n - m + 1 + \frac{1}{2} \sum_{s=1}^{m} \left\{ (\deg v_s)^2 - 3 \deg v_s + 2 \right\} \\ &= n - m + 1 + \frac{1}{2} \sum_{s=1}^{m} (\deg v_s)^2 - \frac{3}{2} \sum_{s=1}^{m} \deg v_s + m \\ &= n + 1 + \frac{1}{2} \sum_{s=1}^{m} (\deg v_s)^2 - 3n \\ &= 1 - 2n + \frac{1}{2} \sum_{s=1}^{m} (\deg v_s)^2. \end{aligned}$$

This completes the proof. □

Example 3.11. Let K_3 be the complete graph on three vertices as illustrated in the left side of Figure 5. As we saw in Example 3.2, the symmetric tube $K_3^{(0)}$ is a graph as illustrated in Figure 5. For a spanning tree $T_{K_3} = e_1 \cup e_2$ of K_3, by Theorem 3.10 we have that $H^1(K_3^{(0)}; \mathbb{Z}) = \langle X^{12}_1 \rangle \cong \mathbb{Z}$. We note that if $[x, y, 1/2] \in K_3^{(0)}$ rotates once around the one in the direction induced by the orientation of X^1_{12} then the non-ordered pair (x, y) rotates once around K_3; see Figure 6. Here the initial and terminal points of a vector in Figure 6 correspond to x and y, respectively. This shows that $\mathcal{R}(f)$ of a plane curve f coincides with the rotation number of f.

Example 3.12. For S_3 and its symmetric tube $S_3^{(0)}$ as illustrated in Figure 4, by Theorem 3.10 we have that $H^1(S_3^{(0)}; \mathbb{Z}) = \langle Y^{21}_4 \rangle \cong \mathbb{Z}$. We note that if $[x, y, 1/2] \in S_3^{(0)}$

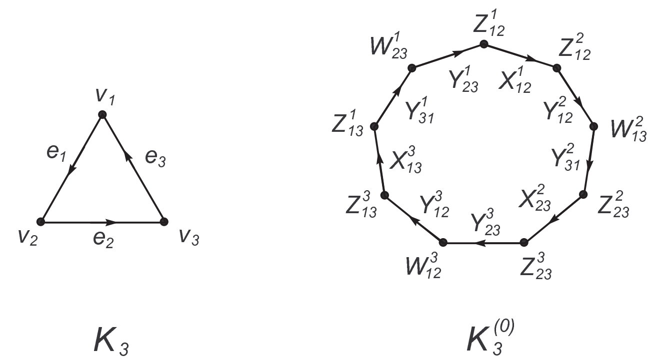

Figure 5.

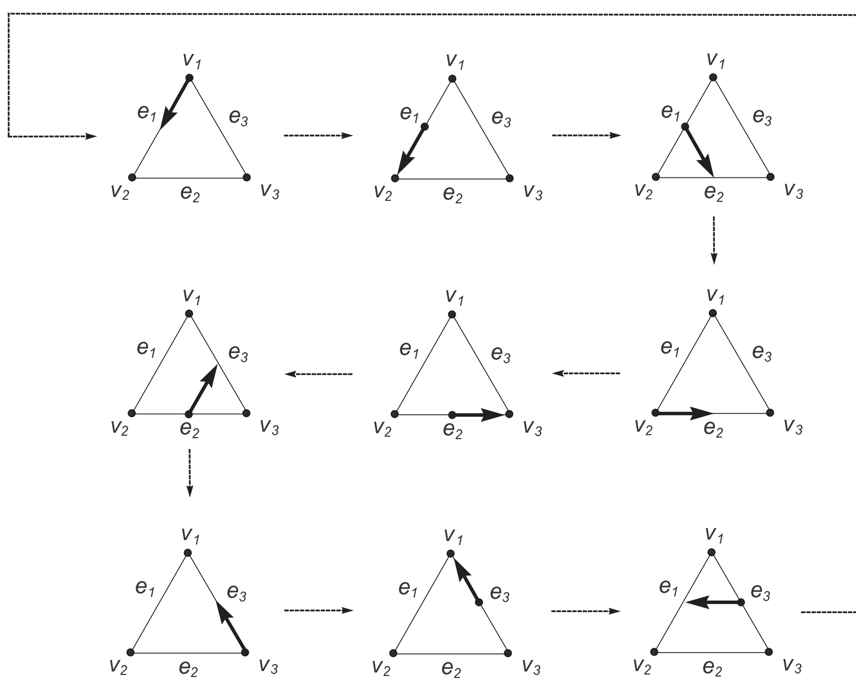

Figure 6.

rotates once around the cycle represented by Y_{21}^4 in the direction induced by the orientation of the one then the non-ordered pair (x, y) rotates once around v_4; see Figure 7. Here the initial and terminal points of a vector in Figure 7 correspond to x and y, respectively.

Let f be a generic plane immersion of S_3. Then there exists a neighbourhood U of v_4 such that $f|_U$ is an embedding. We can see that $\mathcal{R}(f) = 1$ if $f|_U(e_1 \cap U)$, $f|_U(e_2 \cap U)$ and $f|_U(e_3 \cap U)$ are embedded in $\mathbb{R}^2$ as illustrated in Figure 8(1), and $\mathcal{R}(f) = -1$ if $f|_U(e_1 \cap U)$, $f|_U(e_2 \cap U)$ and $f|_U(e_3 \cap U)$ are embedded in $\mathbb{R}^2$ as illustrated in Figure 8(2).

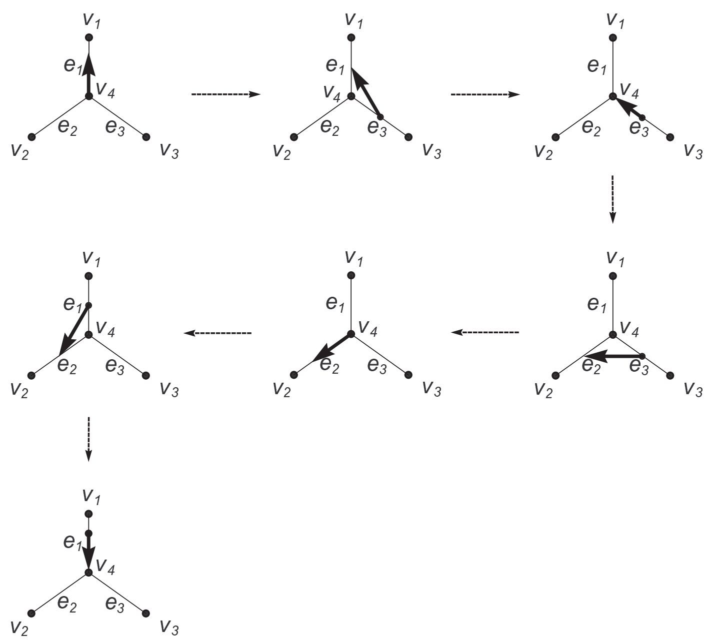

Figure 7.

Example 3.13. Let K_4 be the complete graph on four vertices and f, g and h three generic plane immersions of K_4 as illustrated in Figure 9. Then the symmetric tube of K_4 is a graph as illustrated in Figure 10. For a spanning tree $T_{K_4} = e_1 \cup e_2 \cup e_3$ of K_4, by Theorem 3.10 we have that

$$H^1(K_4^{(0)}; \mathbb{Z}) = \langle X_4^{23}, X_5^{24}, X_6^{34}, Y_1^{31}, Y_2^{31}, Y_3^{22}, Y_4^{23} \rangle \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\text{seven times}}.$$

By calculating on the outcome of Examples 3.11 and 3.12, we have that

$$\mathcal{R}(f) = (-1, 1, -1, 1, 1, 1, 1),$$
$$\mathcal{R}(g) = (1, -1, 1, 1, -1, 1, -1)$$

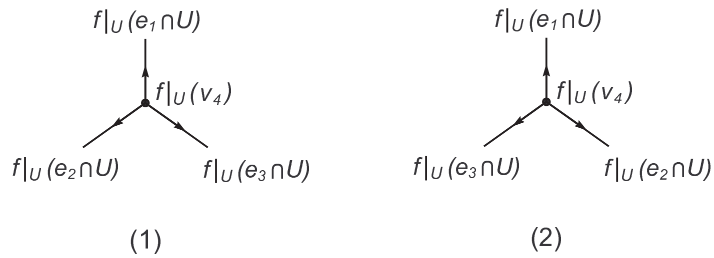

Figure 8.

and

$$\mathcal{R}(h) = (0, 0, 0, 1, 1, -1, 1).$$

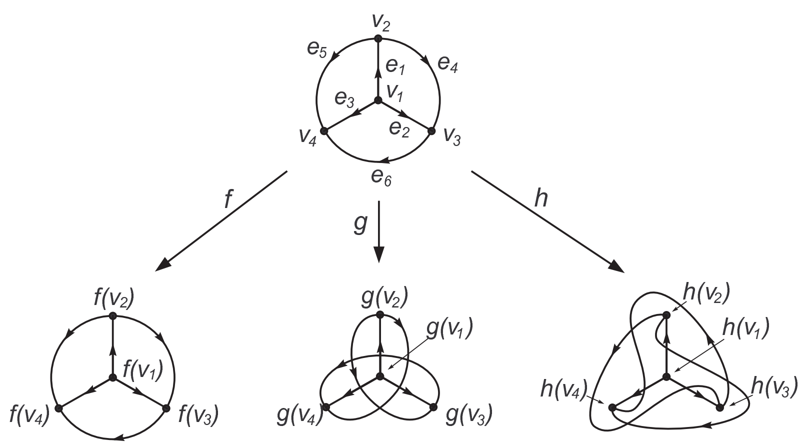

Figure 9.

4. Proof of Theorem 1.1

First we show two lemmas which are needed to prove Theorem 1.1.

Lemma 4.1. *Each of the local moves as illustrated in Figure 11(4) and (5) are represented by a sequence of moves from the list as illustrated in Figure 1(1), (2), (3) and ambient isotopies.*

Proof. See Figures 12 and 13. □

We remark here that the local move as illustrated in Figure 11(4) is none other than the Whitney trick.

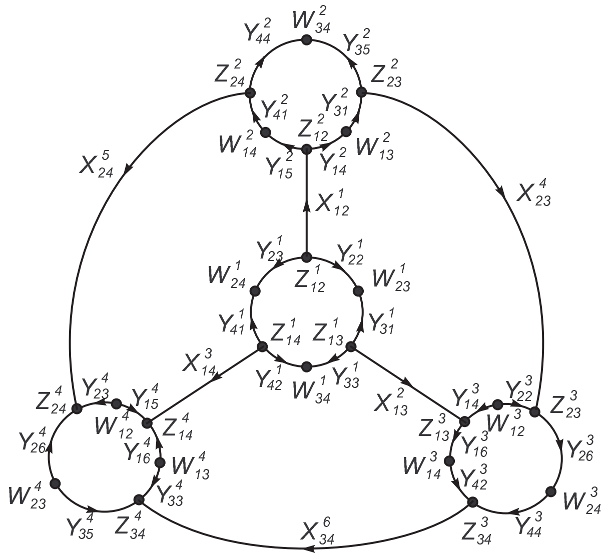

Figure 10.

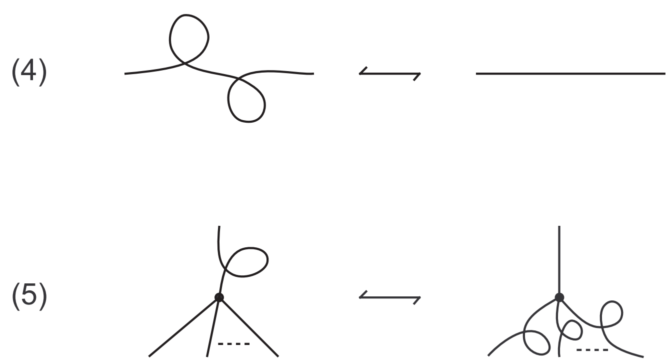

Figure 11.

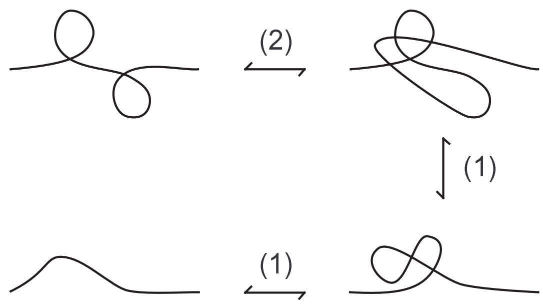

Figure 12.

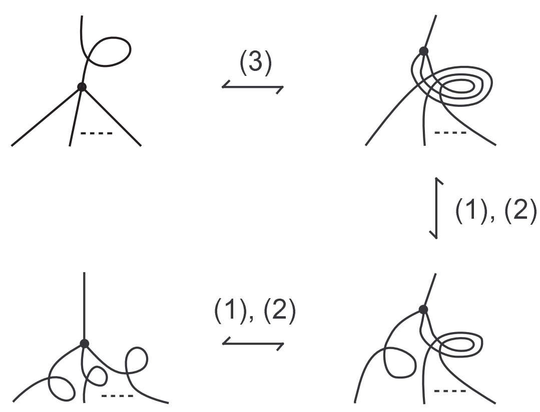

Figure 13.

Lemma 4.2. *Let G be a graph, H a connected subgraph of G and f and g two plane immersions of G. If $\mathcal{R}(f) = \mathcal{R}(g)$, then $\mathcal{R}(f|_H) = \mathcal{R}(g|_H)$.*

Proof. Let $i\colon H \to G$ be the inclusion. Since i is injective, the homomorphism

$$i^*\colon H^1(G^{(0)};\mathbb{Z}) \longrightarrow H^1(H^{(0)};\mathbb{Z})$$

is induced by i. Clearly $i^*(\mathcal{R}(f)) = \mathcal{R}(f|_H)$ and $i^*(\mathcal{R}(g)) = \mathcal{R}(g|_H)$. Therefore by the assumption we have the desired conclusion. □

For a generic plane immersion f and a vertex v_s of a graph G, a cyclic order of the edges of G incident to v_s is determined by considering a neighbourhood U of v_s so that $f|_U$ is an embedding. We call it a *cyclic order of* $f(v_s)$.

Proof of Theorem 1.1. Since (1) ⇒ (3) is shown by Proposition 2.1 and (2)⇒(1) is clear, it is sufficient to show that (3)⇒(2). Assume that $\mathcal{R}(f) = \mathcal{R}(g)$. In the following we show that f and g are transformed into each other by the moves as illustrated in Figure 1(1), (2), (3), Figure 11(4), (5) and ambient isotopies. Then by Lemma 4.1, we have the desired conclusion. Since $\mathcal{R}(f) = \mathcal{R}(g)$, by Lemma 4.2 we have that $\mathcal{R}(f|_{\mathrm{st}(v_s)}) = \mathcal{R}(g|_{\mathrm{st}(v_s)})$ for any vertex v_s of G. Then by Example 3.12 we have that the cyclic order of $f(v_s)$ is equal to the cyclic order of $g(v_s)$ for any vertex v_s of G. Let T_G be a spanning tree of G. By using the moves as illustrated in Figure 1(1), (2), (3), Figure 11(5) and ambient isotopies in case of necessity, we can deform f (resp. g) so that $f|_{T_G}$ (resp. $g|_{T_G}$) is an embedding. Since the cyclic order of $f(v_s)$ is equal to the cyclic order of $g(v_s)$ for any vertex v_s of G, we may assume that $f|_{T_G} = g|_{T_G}$. We set $E(G) \setminus E(T_G) = \{e_{k_1}, e_{k_2}, \ldots, e_{k_\beta}\}$, where β denotes the first Betti number of G. Let p_{k_i} be the unique path on T_G which connects the terminal vertices of e_{k_i}. We denote a cycle $e_{k_i} \cup p_{k_i}$ by γ_{k_i}. Note that the double points of $f|_{\gamma_{k_i}}$ (resp. $g|_{\gamma_{k_i}}$) are only the double points of $f|_{e_{k_i}}$ (resp. $g|_{e_{k_i}}$). Then, by using the moves as illustrated in Figure 1(1), (2), (3), Figure 11(4) and ambient isotopies in case of necessity, we can deform $f|_{\gamma_{k_i}}$ into the generic plane immersion of γ_{k_i} as illustrated in Figure 14(1) or (2) ($i = 1, 2, \ldots, \beta$). Then, by Lemma 4.2 we have that $\mathcal{R}(f|_{\gamma_{k_i}}) = \mathcal{R}(g|_{\gamma_{k_i}})$, namely $f|_{\gamma_{k_i}}$ and $g|_{\gamma_{k_i}}$ have the same rotation number. Thus we may assume that $f|_{\gamma_{k_i}} = g|_{\gamma_{k_i}}$ for $i = 1, 2, \ldots, \beta$. This implies that we can deform f and g identically by the moves as illustrated in Figure 1(1), (2), (3), Figure 11(4), (5) and ambient isotopies. This completes the proof. □

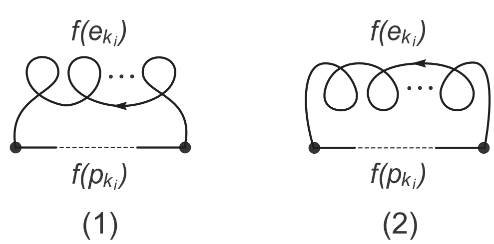

Figure 14.

References

[1] S.T. Hu, Isotopy invariants of topological spaces, *Proc. Roy. Soc. London. Ser. A* **255** (1960), no. 1282, 331–366.

[2] L.H. Kauffman, *Formal knot theory*, Mathematical Notes **30**, Princeton University Press, Princeton, NJ, 1983.

[3] D.A. Permyakov, Classification of immersions of graphs into a plane, *Moscow Univ. Math. Bull.* **63** (2008), no. 5, 208–210.

[4] K. Taniyama, Homology classification of spatial embeddings of a graph, *Topology Appl.* **65** (1995), 205–228.

[5] H. Whitney, On regular closed curves on the plane, *Compositio Math.* **4** (1937), 276–284.

[6] W.T. Wu, *A theory of imbedding, immersion, and isotopy of polytopes in a Euclidean space*, Science Press, Peking, 1965.

[7] W.T. Wu, On the isotopy of a complex in a Euclidean space. I, *Sci. Sinica* **9** (1960), 21–46.

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