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SUPPORT AND INJECTIVE RESOLUTIONS OF COMPLEXES OVER COMMUTATIVE RINGS

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Abstract

Examples are given to show that the support of a complex of modules over a commutative noetherian ring may not be read off the minimal semi-injective resolution of the complex. The same examples also show that a localization of a semi-injective complex need not be semi-injective.

1. Introduction

Let R be a commutative noetherian ring, and $\operatorname{Spec} R$ the set of prime ideals in R. Recall that the support of a finitely generated R-module M is the set of points $\mathfrak{p}$ in $\operatorname{Spec} R$ such that $M_{\mathfrak{p}} \neq 0$. For arbitrary modules and, more generally, for complexes of modules, different notions of support have been used. From a homological perspective the one introduced by Foxby in [**3**], and recalled in Section 2, has proved to be quite useful. Foxby [**3**, 2.8,2.9] proved that a point $\mathfrak{p}$ is in the support of a complex X with $\mathrm{H}^n(X) = 0$ for $n \ll 0$ if and only if the injective hull of $R/\mathfrak{p}$ appears in the minimal semi-injective resolution of X.

This note gives examples that show that such a result does not extend to arbitrary complexes, contrary to the claims in [**7**, 5.1] and [**2**, 9.2]; see Remark 2.3.

2. Support and injective resolutions

For each point $\mathfrak{p}$ in $\operatorname{Spec} R$, we write $k(\mathfrak{p})$ for the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ of the local ring $R_{\mathfrak{p}}$. The *support* of a complex X of R-modules is the subset

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(X \otimes^{\mathbf{L}}_R k(\mathfrak{p})) \neq 0\}.$$

This notion was introduced by Foxby [**3**, p.157] under the name 'small support', to distinguish it from the 'big support', namely, the set $\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(X)_{\mathfrak{p}} \neq 0\}$. They coincide when the R-module $\mathrm{H}(X)$ is finitely generated—see [**3**, 2.1]—but not in general. Also, $\operatorname{supp}_R X$ and $\operatorname{supp}_R \mathrm{H}(X)$ need not coincide; see [**2**, 9.4].

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A point $\mathfrak{p}$ in $\operatorname{Spec} R$ is *associated* to an R-module M if it is the annihilator of an element in M; see [**9**, §6]. We write $\operatorname{ass}_R M$ for the set of associated primes of M.

Injective modules

In what follows $E_R(M)$ denotes the injective hull of an R-module M; see [**9**, §18]. Using [**9**, 18.4], it is easy to verify that there are equalities

$$\operatorname{supp}_R E_R(R/\mathfrak{p}) = \{\mathfrak{p}\} = \operatorname{ass}_R E_R(R/\mathfrak{p}).$$

Let E be an injective R-module. By the structure theorem for injective R-modules, see [**9**, 18.5], there is an isomorphism

$$E \cong \bigoplus_{\mathfrak{p}\in\operatorname{Spec} R} E(R/\mathfrak{p})^{\mu(\mathfrak{p})},$$

where each $\mu(\mathfrak{p})$, which can be ∞, depends only on E. It follows that one has equalities

$$\operatorname{supp}_R E = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mu(\mathfrak{p}) \neq 0\} = \operatorname{ass}_R E.$$

It is this observation that suggests the possibility of reading the support of a complex from its injective resolutions.

Injective resolutions

We require some basic results concerning injective resolutions; for details see [**1**] and [**6**, Appendix B]. We say that a complex I of R-modules is *homotopically injective* if $\operatorname{Hom}_R(-, I)$ preserves quasi-isomorphisms; it is *semi-injective* if in addition each R-module I^n is injective. For example, a complex I of injective R-modules with $I^n = 0$ for $n \ll 0$ is semi-injective. Each complex X of R-modules admits a *semi-injective resolution*; that is, a quasi-isomorphism $X \to I$, where I is semi-injective. Moreover, one can choose I so that the extension $\operatorname{Ker}(\partial^n) \subseteq I^n$ is essential for each integer n; here ∂ is the differential on I. Such a *minimal* semi-injective resolution of X is unique, up to isomorphism of complexes.

Proposition 2.1. *Let R be a commutative noetherian ring and X a complex of R-modules. If a complex I of injective modules is quasi-isomorphic to X, then*

$$\operatorname{supp}_R X \subseteq \bigcup_{n\in\mathbb{Z}} \operatorname{ass}_R I^n.$$

Equality holds if $I_{\mathfrak{p}}$ is minimal and homotopically injective for each $\mathfrak{p} \in \operatorname{Spec} R$.

Remark 2.2. The additional hypotheses on I hold if R is regular, for then any complex of injectives is semi-injective; see [**5**, 2.4,2.8]. They hold also when I is minimal and $\mathrm{H}^n(X) = 0$ for $n \ll 0$, for then $I^i = 0$ for $i \ll 0$, so I and its localizations are semi-injective. Thus Proposition 2.1 extends Foxby's result mentioned earlier.

Remark 2.3. In [**7**, 5.1] it is claimed that the inclusion in Proposition 2.1 is an equality whenever I is a minimal semi-injective resolution of X. This is, however, not the case; see Proposition 2.7 for counter-examples. The error in the proof of [**7**, 5.1] occurs in the penultimate line, where it is asserted that a certain complex is homotopically injective; what can be salvaged from the argument is Proposition 2.1. The last line of [**2**, 9.2] is also incorrect. Only conditions (2)–(4) in op. cit. are equivalent, and are implied by condition (1).

Proposition 2.1 is implicit in [**4**, 2.1], so we provide only a sketch.

Given an ideal $\mathfrak{a}$ in R, we write $\Gamma_{\mathfrak{a}}(-)$ for the $\mathfrak{a}$-torsion functor on the category of R-modules, and $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$ for its right derived functor; see [**3**] or [**8**].

Proof of Proposition 2.1. By localization, it suffices to prove the following statement: Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. If $\mathfrak{m}$ is in $\operatorname{supp}_R X$, then the complex $\Gamma_{\mathfrak{m}}(I)$ is non-zero; the converse holds if I is minimal semi-injective.

It follows from [**4**, 2.1, 4.1] that the following conditions are equivalent:

(i) $\mathrm{H}(X \otimes^{\mathbf{L}}_R k) \neq 0$;

(ii) $\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k, X)) \neq 0$;

(iii) $\mathrm{H}(\mathbf{R}\Gamma_{\mathfrak{m}}(X)) \neq 0$.

Since the complex I consists of injective modules and is quasi-isomorphic to X, the complexes $\mathbf{R}\Gamma_{\mathfrak{m}}(X)$ and $\Gamma_{\mathfrak{m}}(I)$ are quasi-isomorphic; see [**8**, 3.5.1]. Therefore if $\mathfrak{m}$ is in $\operatorname{supp}_R X$, the complex $\Gamma_{\mathfrak{m}}(I)$ must be non-zero.

Suppose $\mathfrak{m} \notin \operatorname{supp}_R X$ holds, so that $\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k, X)) = 0$. When I is semi-injective there are (quasi-)isomorphisms

$$\mathbf{R}\mathrm{Hom}_R(k, X) \simeq \mathrm{Hom}_R(k, I) \cong \mathrm{Hom}_R(k, \Gamma_{\mathfrak{m}}(I)).$$

When I is also minimal the differential on $\mathrm{Hom}_R(k, I)$ is zero, so $\mathrm{H}(\mathrm{Hom}_R(k, I)) = 0$ implies $\Gamma_{\mathfrak{m}}(I) = 0$. □

Examples

Next we focus on our main task; namely, giving examples that show that the inclusion in Proposition 2.1 can be strict, even when I is a minimal semi-injective complex. Their construction is motivated by an observation of Neeman [**10**, 6.5] and recent work of Iacob and Iyengar [**5**, Section 2]. First, we record an elementary remark about associated primes of products.

Remark 2.4. Let R be a commutative noetherian ring and let $\{M_\lambda\}$ be a family of R-modules. There are inclusions

$$\bigcup_\lambda \operatorname{ass}_R M_\lambda \subseteq \operatorname{ass}_R \big(\prod_\lambda M_\lambda\big) \subseteq \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \subseteq \mathfrak{q} \in \operatorname{ass}_R M_\lambda \text{ for some } \lambda\}.$$

Indeed, the inclusion on the left holds since each M_λ is isomorphic to a submodule of the product. For the one on the right: if a prime $\mathfrak{p}$ is the annihilator of an element (m_λ), then it is contained in the annihilator of each m_λ; pick one that is non-zero.

In the proof of Proposition 2.7 we use the following properties of injective hulls.

Remark 2.5. Let R be a commutative noetherian ring, $\mathfrak{n}$ a prime ideal in R, and E the injective hull of $R/\mathfrak{n}$. The following statements hold:

1. Each r in $R \setminus \mathfrak{n}$ is invertible on E, hence E has a natural $R_{\mathfrak{n}}$-module structure.
2. The $R_{\mathfrak{n}}$-module E is Artinian.
3. As an $R_{\mathfrak{n}}$-module, E has finite length if and only if $\mathfrak{n}$ is a minimal prime.

For (1) see [**9**, 18.4]; for (2), see [**9**, 18.6]; and for (3), see the proof of [**9**, 18.6(iv)].

Construction 2.6. Let R be a commutative noetherian ring of Krull dimension at least one; fix a non-minimal prime ideal $\mathfrak{n}$ in R. Suppose R contains an element x such that $\{r \in R \mid rx = 0\} = (x)$; in particular, $x^2 = 0$.

For example, R could be $\mathbb{Z}[x]/(x^2)$, and $\mathfrak{n} = (p, x)$, where p is a prime number.

In what follows we use properties of injective hulls recalled in Remark 2.5. These can be verified directly in the special case when $R = \mathbb{Z}[x]/(x^2)$.

Let M be the injective hull of $R/\mathfrak{n}$ over R. By the hypothesis on x, the complex of R-modules $\cdots \xrightarrow{x} R \xrightarrow{x} R \to 0 \to \cdots$, with 0 in degree 1, has cohomology only in degree 0. Thus, applying $\operatorname{Hom}_R(-, E)$ to it, one gets a complex of R-modules

$$J = \quad \cdots \longrightarrow 0 \longrightarrow E \xrightarrow{\ x\ } E \xrightarrow{\ x\ } E \xrightarrow{\ x\ } \cdots$$

with 0 in degree -1 and $\mathrm{H}^i(J) = 0$ for $i \neq 0$. Set $M = \mathrm{H}^0(J)$; the inclusion $\iota\colon M \to J$ is then an injective resolution of M over R. It is evidently minimal.

Part (3) of the result below shows that the inclusion in Proposition 2.1 can be strict, while (4) shows that a localization of a semi-injective complex need not be homotopically injective. We write $\Sigma^i X$ for the ith suspension of a complex X.

Proposition 2.7. *Let $X = \prod_{i\in\mathbb{Z}} \Sigma^i M$ and $I = \prod_{i\in\mathbb{Z}} \Sigma^i J$, viewed as complexes of R-modules. The following statements hold.*

1. *The complex I is semi-injective and minimal.*
2. *The natural map $\prod_{i\in\mathbb{Z}} \Sigma^i \iota\colon X \to I$ is a quasi-isomorphism.*
3. $\operatorname{supp}_R X = \{\mathfrak{n}\} \subsetneq \operatorname{ass}_R I^n$*, for each integer n.*
4. *For any prime $\mathfrak{p}$ in $\operatorname{ass}_R I^n$ with $\mathfrak{p} \neq \mathfrak{n}$, the complex of injective $R_{\mathfrak{p}}$-modules $I_{\mathfrak{p}}$ is acyclic but not contractible, and hence not homotopically injective.*

Proof. Recall that $\iota\colon M \to J$ is a quasi-isomorphism.

(1) The complex $\Sigma^i J$ consists of injective R-modules and $(\Sigma^i J)^n = 0$ for $n < -i$, hence $\Sigma^i J$ is semi-injective. Therefore the same holds for I, since a product of semi-injective complexes is semi-injective.

As to the minimality, note that the differential $\partial^n\colon I^n \to I^{n+1}$ is the map

$$\prod_{i\geqslant n} E \xrightarrow{\begin{bmatrix} x \\ 0 \end{bmatrix}} \Big(\prod_{i\geqslant n} E\Big) \oplus E = \prod_{i\geqslant n-1} E\,.$$

Evidently $\operatorname{Ker}(\partial^n)$ is the submodule $\prod_{i\geqslant n} M$ of I^n. It is now straightforward to verify that the extension $\operatorname{Ker}(\partial^n) \subset I^n$ is essential. Thus I is a minimal complex.

(2) holds because a product of quasi-isomorphisms is a quasi-isomorphism.

(3) One has $\operatorname{supp}_R M = \{\mathfrak{n}\}$. Indeed, J is a minimal injective resolution of M over R, so $\operatorname{supp}_R M = \operatorname{ass}_R E = \{\mathfrak{n}\}$. Observe that there is an isomorphism of complexes $X \cong \bigoplus_{i\in\mathbb{Z}} \Sigma^i M$, so $\operatorname{supp}_R X = \{\mathfrak{n}\}$.

Since the R-module I^n is isomorphic to $\prod_{i\geqslant n} E$, Remark 2.4 yields

$$\{\mathfrak{n}\} = \operatorname{ass}_R E \subseteq \operatorname{ass}_R I^n.$$

The claim is that this inclusion is strict; equivalently, that there exist elements in $I^n = \prod_{i\geqslant n} E$ that are not $\mathfrak{n}$-torsion.

Indeed, E is the injective hull of $R/\mathfrak{n}$, so it is a module over the local ring $R_\mathfrak{n}$. Since $\mathfrak{n}$ is not a minimal prime ideal in R, by hypothesis, $R_\mathfrak{n}$ does not have finite length, and hence neither does the $R_\mathfrak{n}$-module E. However E is Artinian, so for each integer $i \geqslant 0$ there must be an element e_i in E such that $\mathfrak{n}^i \cdot e_i \neq 0$. Evidently, the element $(e_{i-n})_{i \geqslant n}$ in I^n is not $\mathfrak{n}$-torsion.

(4) Fix a prime $\mathfrak{p}$ as in the hypothesis. By Remark 2.4, one has $\mathfrak{p} \subset \mathfrak{n}$ so $M_\mathfrak{p} = 0$, since M is $\mathfrak{n}$-torsion, and hence $X_\mathfrak{p} = 0$. As I is quasi-isomorphic to X, the complex $I_\mathfrak{p}$ is quasi-isomorphic to $X_\mathfrak{p}$, and hence an acyclic complex of injective $R_\mathfrak{p}$-modules. It is also minimal since localization preserves minimality. Since the complex $I_\mathfrak{p}$ is non-zero, by the choice of $\mathfrak{p}$, it follows from the minimality that it is not contractible. □

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