Homology, Homotopy and Applications, vol. 11(1), 2009, pp.349–405

ORIENTED COHOMOLOGY THEORIES OF ALGEBRAIC VARIETIES II (AFTER I. PANIN AND A. SMIRNOV)

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(*communicated by J. F. Jardine*)

Abstract

The concept of oriented cohomology theory is well-known in topology. Examples of these kinds of theories are complex cobordism, complex K-theory, usual singular cohomology, and Morava K-theories. A specific feature of these cohomology theories is the existence of trace operators (or Thom-Gysin operators, or push-forwards) for morphisms of compact complex manifolds. The main aim of the present article is to develop an algebraic version of the concept. Bijective correspondences between orientations, Chern structures, Thom structures and trace structures on a given ring cohomology theory are constructed. The theory is illustrated by singular cohomology, motivic cohomology, algebraic K-theory, the algebraic cobordism of Voevodsky and by other examples.

1. Introduction

The concept of an oriented cohomology theory is well-known in topology [**1**, Part II, p. 37], [**36**, Ch. 1, 4.1.1]. Examples of this kind of theory are complex cobordism, complex K-theory, usual singular cohomology, and Morava K-theories. The significance of these cohomology theories in algebraic topology is well-known due to Adams, Milnor, Novikov, Quillen and many others. A specific feature of these cohomology theories is the existence of trace operators (or Thom-Gysin operators, or push-forwards) for morphisms of compact complex manifolds.

Voevodsky invented machinery producing cohomology theories [**42**]. However, not all interesting cohomology theories are of this kind (see [**9**]). This is why it is reasonable to define what a cohomology theory on algebraic varieties is, what a ring cohomology theory is and what an oriented cohomology theory is. All these were done in [**25**]. An oriented cohomology theory is a pair (A, ω) consisting of a ring cohomology theory A and an orientation ω (the definition is reproduced below in 1.9). The present article concerns a construction of an integration (a trace structure) on an oriented cohomology theory (A, ω) (see Theorem 2.5). This is *the very heart* of the article.

Received June 6, 2008, revised November 30, 2008; published on June 17, 2009.
2000 Mathematics Subject Classification: 14F42, 14F43, 55N22.
Key words and phrases: motivic cohomology, algebraic cobordism, oriented cohomology theories.
This article is available at http://intlpress.com/HHA/v11/n1/a14

In the present article we consider a field k and the category of pairs (X, U) with a smooth variety X over k and its open subset U. Following [**25**], a cohomology theory means a contravariant functor A from this category to the category of abelian groups endowed with a functor transformation $\partial\colon A(U) \to A(X, U)$ and satisfying the localization, Nisnevich excision and homotopy invariance properties (1.1).

For a ring cohomology theory A (see Agreement 1.7), *a trace structure* or equivalently *an integration* on A is a rule assigning to each projective morphism $f\colon Y \to X$ of smooth varieties a grade-preserving two-sided $A(X)$-module operator $\mathrm{tr}_f\colon A(Y) \to A(X)$ satisfying certain natural properties and called *trace operators* or equivalently *push-forwards* (see 2.2). Other terms often used for push-forwards in the literature are Gysin homomorphisms, Thom-Gysin homomorphisms, corestriction homomorphisms, transfers and some others. We prove that all possible trace structures on a ring cohomology theory A are in a natural bijection with all possible orientations ω on A (Theorem 2.5).

To do this, we consider, along with the trace structures on A, three other structures the ring cohomology A can be equipped with: *an orientation on* A, *a Thom structure on* A and *a Chern structure on* A (see [**25**]). *An orientation on* A is a rule ω assigning to each variety X and to each vector bundle E/X a grade-preserving two-sided $A(X)$-module isomorphism

$$\omega_E\colon A(X) \to A(E, E - X)$$

satisfying certain natural properties (1.9) and called the *Thom isomorphism.* A *Thom structure on* A is a rule assigning to each smooth variety X and each line bundle L over X a class $\mathrm{th}(L) \in A^{\mathrm{ev}}(L, L - X)$ satisfying certain natural properties (1.13) and called the *Thom class.* A *Chern structure on* A is a rule assigning to each smooth variety X and each line bundle L over X a class $c(L) \in A^{\mathrm{ev}}(X)$ satisfying certain natural properties (1.12) and called the *first Chern class* (or sometimes called the *Euler class*).

It is proved in the present article that for a given A these structures are in natural bijection with each other. More precisely, we construct the following diagram:

$$\begin{array}{ccc} \textit{Orientations on } A & \xrightarrow{\ \alpha\ } & \textit{Trace structures on } A \\ \delta \uparrow & & \downarrow \beta \\ \textit{Chern structures on } A & \xleftarrow{\ \gamma\ } & \textit{Thom structures on } A \end{array} \tag{1}$$

in which each arrow is a bijection and each round trip coincides with the identity (Theorem 2.5 and [**25**, Thms. 3.35, 3.36]). The constructions of these arrows are described briefly below in this introduction. One of the consequences of the theorem is this: *the existence of at least one of these structures on* A implies *the existence of a trace structure on* A; a trace structure on A is *never defined by* the ring cohomology theory itself (even on usual singular cohomology there are plenty of different trace structures (see an example below in the Introduction).

However, in practice, certain ring cohomology theories are equipped either with a Chern structure or with a Thom structure *on the nose.* Thus they are equipped with distinguished trace structures. For instance, usual singular cohomology with integral coefficients (on complex algebraic varieties) is equipped with the known

Chern structure and algebraic K-theory is equipped with a Chern structure as well ($L \mapsto [\mathbf{1}] - [L^\vee]$). The corresponding trace structures coincide with the well-known ones. The motivic cohomology $H^*(-, \mathbb{Z}(*))$ is equipped with a Chern structure, and the algebraic cobordism theory $\mathbb{MGL}^{*,*}$ (see 2.9.7) is equipped with a natural Thom structure. Thus these two theories are equipped with the corresponding trace structures (see 2.9). The last two examples are *the main motivating examples* for this article.

Using Theorem 2.5 and [**25**, Thms. 3.35, 3.36], we describe rather explicitly all trace structures on A (Theorem 2.15). This description may be considered as one of *the main results of the article.*

To explain this description suppose there is given a ring cohomology theory A which can be equipped with at least one of the above mentioned structures. In this case, the ring $A^{\mathrm{ev}}(\mathbf{P}^\infty)$ is isomorphic to the formal power series $A^{\mathrm{ev}}(pt)[[t]]$ in one variable over the even part $A^{\mathrm{ev}}(pt)$ of the coefficient ring of the theory A (this follows from the projective bundle theorem [**25**, Th. 3.9]). Consider an assignment which takes a trace structure on A to the Chern class $\zeta := c(\mathcal{O}(1))$ of the anti-tautological line bundle over $\mathbf{P}^\infty$ corresponding via $\gamma \circ \beta$ to that trace structure:

$$\textit{Trace structures on } A \xrightarrow{\ c\ } \textit{Local parameters of } A^{\mathrm{ev}}(\mathbf{P}^\infty).$$

Theorem 2.15 states that this assignment is bijective. In particular, this implies that there are plenty of trace structures on a ring cohomology theory A which can be equipped with a Chern structure or with a Thom structure. The injectivity of the assignment c says that two trace structures $f \mapsto \mathrm{tr}_f^{(1)}$ and $f \mapsto \mathrm{tr}_f^{(1)}$ on A coincide if the corresponding two local parameters $\zeta^{(1)}$ and $\zeta^{(2)}$ of $A^{\mathrm{ev}}(\mathbf{P}^\infty)$ coincide.

In the case of singular cohomology $H^*(-, \mathbb{Z})$ on complex algebraic varieties and the standard trace structure, the mentioned local parameter coincides with the generator of the group $H^2(\mathbb{C}\mathbf{P}^\infty, \mathbb{Z})$. In the case of algebraic K-theory and Grothendieck and Quillen's trace structure defined via higher direct images, the local parameter coincides with the class $[\mathbf{1}] - [\mathcal{O}(-1)]$ of the hyperplane in $K_0(\mathbf{P}^\infty)$. In the case of de Rham cohomology and the usual integration of the differential forms, the mentioned local parameter coincides with the class in $H^2(\mathbb{C}\mathbf{P}^n, \mathbb{C})$ of the $(1,1)$-differential form $(dz_0 \wedge d\bar{z}_0 + dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n)/|z|^2$ on the complex projective space $\mathbb{C}\mathbf{P}^n$ (more precisely, this form is the restriction of the local parameter to $\mathbb{C}\mathbf{P}^n$).

For well-known theories (singular cohomology on complex algebraic varieties, étale cohomology theory, motivic cohomology theory) one can use the usual Chern structure in order to get a trace structure (via the assignment $\alpha \circ \delta$) coinciding with the classical ones. In the case of algebraic K-theory, the assignment $L \mapsto [\mathbf{1}] - [L^\vee]$ is a Chern structure which gives, via $\alpha \circ \delta$, the well-known Grothendieck and Quillen trace structure on K-theory defined via the higher direct images of coherent sheaves. If $k = \mathbb{C}$ and the theory is complex cobordism theory restricted to the category of pairs of algebraic varieties and equipped with the Chern structure given by the Conner-Floyd classes [**6**], then the corresponding trace structure coincides with the family of Gysin maps in complex cobordism theory described in [**34**, 1.2 and 1.4].

An example of a *nonstandard Chern structure* on the usual singular cohomology with rational coefficients is given by the assignment $L \mapsto 1 - \exp(-c_1(L))$. This

Chern structure gives a trace structure on $H^*(-,\mathbb{Q})$ which comes via the Chern character from Grothendieck and Quillen's trace structure on algebraic K-theory (see [**27**, Cor. 1.1.10]).

As was already mentioned, to orient a ring theory A is the same as to fix a trace structure on A, a Thom structure on A or a Chern structure on A. An orientation on A is usually denoted ω. The trace structure corresponding to ω via α is usually written $f \mapsto f_\omega$. The Thom structure corresponding to ω via $\beta \circ \alpha$ is often written $L \mapsto \mathrm{th}^\omega(L)$. The Chern structure corresponding to ω via $\gamma \circ \beta \circ \alpha$ is often written $L \mapsto c^\omega(L)$. An oriented cohomology theory is as well an oriented cohomology pretheory in the sense of [**27**] because the mentioned trace structure (the integration) is perfect in the sense of [**27**]. *An orientable ring cohomology theory* is a ring cohomology theory which can be equipped with an orientation.

For an oriented cohomology theory (A,ω), the trace structure on A defines a Borel-Moore theory in the sense of [**19**]. So there is a morphism

$$\varphi_\omega \colon \Omega^{\mathrm{LM}} \to A|_{\mathcal{S}m}$$

of the Borel-Moore theories respecting the push-forwards, where Ω^{LM} is the algebraic cobordism of Levine-Morel [**19**]. More details can be found in 2.9.8.

We now describe briefly the relation of orientable ring cohomology theories to *commutative formal groups* and the relation of the oriented theories to *commutative formal group laws.* The last relation gives one of *the key tools* in order to construct the assignment α. Here are the relations: by the projective bundle theorem one has the Künneth formula $A^{\mathrm{ev}}(\mathbf{P}^\infty \times \mathbf{P}^\infty) = A^{\mathrm{ev}}(\mathbf{P}^\infty) \otimes_{A^{\mathrm{ev}}(pt)} A^{\mathrm{ev}}(\mathbf{P}^\infty)$. The Segre morphism $\mu \colon \mathbf{P}^\infty \times \mathbf{P}^\infty \to \mathbf{P}^\infty$ (corresponding to the line bundle $p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$) induces the pull-back operator

$$\mu^* \colon A^{\mathrm{ev}}(\mathbf{P}^\infty) \to A^{\mathrm{ev}}(\mathbf{P}^\infty \times \mathbf{P}^\infty) = A^{\mathrm{ev}}(\mathbf{P}^\infty) \otimes_{A^{\mathrm{ev}}(pt)} A^{\mathrm{ev}}(\mathbf{P}^\infty)$$

to the completed tensor product, which in turn defines a Hopf-algebra structure on $A^{\mathrm{ev}}(\mathbf{P}^\infty)$. This is *the formal group* F^A associated with the orientable ring cohomology theory A. The assignment $A \mapsto F^A$ gives rise to a functor from the category of all orientable cohomology theories and ring morphisms to the category of commutative formal groups of dimension one. This functor is used in the topological context to identify operations (endomorphisms) of interesting cohomology theories with the endomorphisms of certain formal groups.

An orientation ω on A identifies the ring $A^{\mathrm{ev}}(pt)[[u]]$ with $A^{\mathrm{ev}}(\mathbf{P}^\infty)$ sending the variable u to the local parameter $\xi^\omega := c^\omega(\mathcal{O}(-1))$. In the same way, the orientation ω identifies the ring $A^{\mathrm{ev}}(\mathbf{P}^\infty \times \mathbf{P}^\infty)$ with the ring $A^{\mathrm{ev}}(pt)[[u_1,u_2]]$. Under the last identification, the element $\mu^*(\xi^\omega)$ becomes a formal power series $F^\omega(u_1,u_2)$ in two variables over the ring $A^{\mathrm{ev}}(pt)$. It is *the formal group law* F^ω corresponding to the orientation ω of the theory A. The series $F^\omega(u_1,u_2)$ is a unique series satisfying the following property: for each smooth variety X and each pair of line bundles L_1 and L_2 over X, one has the relation

$$c^\omega(L_1 \otimes L_2) = F^\omega(c^\omega(L_1), c^\omega(L_2))$$

in $A^{\mathrm{ev}}(X)$.

We now sketch the structure of the text. In Section 2, the notion of a trace structure on A is defined (here the requirement 5 concerning a short Gysin sequence is added to the notion of integration introduced in [**30**]). We construct a triangle of correspondences α, ϵ, δ

$$\begin{array}{ccc} \textit{Orientations on } A & \xrightarrow{\ \alpha\ } & \textit{Trace structures on } A \\ {\scriptstyle\delta}\nwarrow & & \swarrow{\scriptstyle\epsilon} \\ & \textit{Chern structures on } A & \end{array}$$

with the same α and δ as above, in which each arrow is a bijection and each round trip coincides with the identity (Theorem 2.5). In [**25**, (1)], a triangle of correspondences

$$\begin{array}{ccc} & \textit{Orientations on } A & \\ {\scriptstyle\delta}\nearrow & & \searrow{\scriptstyle\rho} \\ \textit{Chern structures on } A & \xleftarrow{\ \gamma\ } & \textit{Thom structures on } A \end{array} \tag{2}$$

is constructed with the same γ and δ as above, in which each arrow is a *bijection* and each round trip coincides with the *identity* ([**25**, Thms. 3.5, 3.35, 3.36]). Combining the two triangles, we get the square diagram (1) except for the arrow β. The arrow β is described just below Theorem 2.5. It follows from the description of β that $\rho = \beta \circ \alpha$ and $\epsilon = \gamma \circ \beta$.

The proof of Theorem 2.5 is rather long. It occupies all of Section 2 and is organized as follows. The uniqueness assertions of items 1 and 2 of the theorem are proved in Subsection 2.1. The proof of the uniqueness assertion of item 1 is based on the first variant of the Riemann-Roch type theorem proved in [**27**, Th. 1.1.9]. The proof of the existence assertion is given below in Subsection 2.7 and it requires a construction of a trace structure.

The construction of a trace structure related via α to an orientation is rather long. It occupies several subsections. Namely, in Subsection 2.2 trace operators are constructed for closed imbeddings of smooth varieties (they are called Gysin operators). In Subsection 2.3, trace operators are constructed for the case of projections $X \times \mathbf{P}^n \to X$ (they are called Quillen's operators). In Subsection 2.4, properties of Gysin operators are proved, and in Subsection 2.5, properties of Quillen's operators are proved except for the Key property, which is proved in Subsection 2.6. Trace operators for any projective morphism are constructed in Subsection 2.7. These trace operators form a trace structure on A as it is stated in Theorem 2.12 from the same subsection. The proof of Theorem 2.5 is completed in Subsection 2.8.

The article is completed by Theorem 2.15. In this theorem an orientable ring cohomology theory A is considered and the set of all trace structures is identified with the set of all local parameters of the ring $A^{\mathrm{ev}}(\mathbf{P}^\infty)$ (the even part of the ring $A(\mathbf{P}^\infty)$). Moreover, this identification is given in one direction very explicitly. Namely, it takes a trace structure $f \mapsto \mathrm{tr}_f$ on A to the Chern class $c(\mathcal{O}(1)) \in A^{\mathrm{ev}}(\mathbf{P}^\infty)$ corresponding via Theorem 2.5 to that trace structure.

Since the text is rather long it is reasonable to sketch here our constructions of the assignments α, ϵ and δ (the assignments β and γ are described in Theorems 2.5 and in [**25**, Th. 3.36]). We first describe the arrow ϵ, then sketch the description of δ and finally sketch the description of α.

Suppose we are given a trace structure $f \mapsto \mathrm{tr}_f$ on A. For the zero section $z\colon X \to L$ of a line bundle L over X, set $c(L) = z^A(\mathrm{tr}_z(1)) \in A(X)$ (the pull-back of the push-forward of the element 1). The assignment $L \mapsto c(L)$ is the Chern structure on A corresponding via ϵ to the trace structure (see 2.5).

Suppose we are given a Chern structure $L \mapsto c(L)$ on A. In this case the projective bundle theorem holds (see [**25**, Th. 3.9]) and there is a Chern class theory $E \mapsto c_n(E)$ with values in A. To produce an orientation ω on A, we associate to each vector bundle E/X its Thom class $\mathrm{th}(E) \in A(E, E-X)$. Firstly, for a rank n vector bundle E, we define an element $\bar{\mathrm{th}}(E) := c_n(\mathcal{O}_E(1) \otimes p^*(E)) \in A(\mathbf{P}(\mathbf{1} \oplus E))$. It turns out that the pull-back operator identifies $A(\mathbf{P}(\mathbf{1} \oplus E), \mathbf{P}(\mathbf{1} \oplus E) - \mathbf{P}(\mathbf{1}))$ with a subgroup of $A(\mathbf{P}(\mathbf{1} \oplus E))$ and $\bar{\mathrm{th}}(E)$ belongs to this subgroup. The class $\mathrm{th}(E)$ is defined as the image of the element $\bar{\mathrm{th}}(E)$ under the pull-back isomorphism identifying $A(\mathbf{P}(\mathbf{1} \oplus E), \mathbf{P}(\mathbf{1} \oplus E) - \mathbf{P}(\mathbf{1}))$ with $A(E, E-X)$. The required orientation ω on A is given by the assignment which associates to a vector bundle $p\colon E \to X$ the map

$$\omega_E := (\cup\, \mathrm{th}(E)) \circ p^A \colon A(X) \to A(E, E-X).$$

Details are given in the proof of [**25**, Th. 3.35].

We now describe the construction of the assignment α. This is *the very heart* of the article. So suppose we are given an orientation ω on A. We sketch our construction of the trace structure $f \mapsto \mathrm{tr}^\omega_f$ corresponding via α to ω. First of all, each projective morphism $f\colon Y \to X$ can be presented as a composition of a closed imbedding $i\colon Y \hookrightarrow \mathbf{P}^n \times X$ and the projection $p\colon \mathbf{P}^n \times X \to X$. So it suffices to define trace operators for closed imbeddings and for projections, to set $\mathrm{tr}^\omega_f = \mathrm{tr}^\omega_p \circ \mathrm{tr}^\omega_i$, and to verify that the resulting operator tr^ω_f does not depend on the choice of the decomposition of f. For a closed imbedding $i\colon S \hookrightarrow T$ of smooth varieties, we define the trace operator $\mathrm{tr}^\omega_i\colon A(S) \to A(T)$ as the composition

$$A(S) \xrightarrow{\omega_N} A(N, N-S) \cong A(T, T-S) \to A(T),$$

where $N = N_{T/S}$ is the normal bundle to S in T, ω_N is the Thom isomorphism (given by the orientation ω), and the isomorphism $A(N, N-S) \cong A(T, T-S)$ is the "excision isomorphism" (see [**25**, Th. 2.2]). The homomorphism $A(T, T-S) \to A(T)$ is just the pull-back homomorphism induced by the inclusion of pairs

$$(T, \emptyset) \subset (T, T-S).$$

The trace operators tr^ω_p for the projections $p\colon \mathbf{P}^n \times X \to X$ are defined as follows. We first define the Chern structure on A corresponding to the orientation: given a line bundle L over a smooth X consider the zero section $z\colon X \to L$ and the composition map $A(X) \xrightarrow{\omega_L} A(L, L-X) \to A(L) \xrightarrow{z^A} A(X)$, and set $c^\omega(L)$ to be equal to the evaluation of this composition on the element $1 \in A(X)$. Then we define certain elements $[\mathbf{P}^m]_\omega$ in $A(pt)$ called the classes of projective spaces. By the projective bundle theorem, the ring $A(\mathbf{P}^n \times X)$ as a two-sided $A(X)$-module is a free module with the basis $1, \zeta, \dots, \zeta^n$, where $\zeta = c^\omega(\mathcal{O}(1)) \in A(\mathbf{P}^n)$. So we define the trace operator tr^ω_p to be the unique two-sided $A(X)$-module operator $\mathrm{tr}^\omega_p\colon A(\mathbf{P}^n \times X) \to A(X)$ which takes the element ζ^r to the class $[\mathbf{P}^{n-r}]_\omega$ for $r = 0, 1, \dots, n$. So to finish the definition of the operators tr^ω_p it remains to define the classes $[\mathbf{P}^m]_\omega$.

To do this we need the formal group law corresponding to the orientation ω of

A. This formal group law is the formal power series $F^\omega(u_1, u_2) \in A^{\mathrm{ev}}(pt)[[u_1, u_2]]$ described above. Write down the unique normalized F^ω-invariant differential 1-form ω_F on the formal group (law) in terms of the local parameter u:

$$\omega_F = (P_0 + P_1 u + P_2 u^2 + \cdots)du.$$

Now set

$$[\mathbf{P}^m]_\omega = P_m \in A^{\mathrm{ev}}(pt).$$

This completes the definition of the classes $[\mathbf{P}^m]_\omega$ and thus completes the definition of the operators $\mathrm{tr}_p^\omega \colon A(\mathbf{P}^n \times X) \to A(X)$. Finally, for the morphism $f \colon Y \to X$ and the presentation $f = p \circ i$, set $\mathrm{tr}_f^\omega = \mathrm{tr}_p^\omega \circ \mathrm{tr}_i^\omega$. Theorem 2.12 states that the operator tr_f^ω is well-defined and respects the composition of projective morphisms, and moreover that the assignment $f \mapsto \mathrm{tr}_f^\omega$ is a trace structure required by item 1 of Theorem 2.5.

Another definition (coinciding with the given one) of the classes $[\mathbf{P}^m]_\omega$ uses the complex cobordism theory $\Omega(*)$ and the formal group law F_Ω associated with this theory and its canonical Chern class for line bundles (the Conner-Floyd class) [**6**]. This law was originally introduced by Novikov and Mischenko in [**24**]. It is defined over the ring $\Omega = \Omega(pt)$ and, according to a theorem of Quillen ([**33**, Th. 2]), F_Ω is a universal commutative formal group law of dimension 1. This implies that there exists a unique ring homomorphism $l_\omega \colon \Omega \to A^{\mathrm{ev}}(pt)$ such that the coefficients of F^ω coincide with the l_ω-images of the corresponding coefficients of F_Ω. Now set

$$[\mathbf{P}^m]_\omega = l_\omega([\mathbb{C}P^m]),$$

where $[\mathbb{C}P^m]$ is the class of $\mathbb{C}P^m$ in Ω. This completes the second definition of the classes $[\mathbf{P}^m]_\omega$. The two definitions coincide because by Mischenko's theorem (see [**24**]), the normalized invariant differential form ω_Ω coincides with the form

$$([\mathbb{C}\mathbf{P}^0] + [\mathbb{C}\mathbf{P}^1]u + [\mathbb{C}\mathbf{P}^2]u^2 + \cdots)du$$

and clearly a scalar extension takes the normalized invariant differential form to the normalized invariant differential form.

The normalized invariant form ω of a formal group law $F = F^\omega(u_1, u_2)$ is computed as follows (see [**33**]): $\omega_F = du/F_2(u, 0)$, where F_2 is the derivative of F with respect to the variable u_2. Now let us make a test showing that, for certain examples of cohomology theories and certain choices of the Chern structures, the classes $[\mathbf{P}^m]_\omega$ do coincide with the known ones. Namely, if the cohomology theory is the usual singular cohomology, the de Rham cohomology or the motivic cohomology, and if the Chern structure $L \mapsto c(L)$ is the usual one, then the associated formal group law is additive: $F = u_1 + u_2$. In this case one has $\omega = du$ and the classes are given by $[\mathbf{P}^0]_\omega = 1$ and $[\mathbf{P}^r]_\omega = 0$ for all $r > 0$. This agrees with what is well-known. If the cohomology theory is algebraic K-theory and the Chern structure is given by $L \mapsto [\mathbf{1}] - [L^\vee]$, then the associated formal group law is multiplicative: $F = u_1 + u_2 - u_1 u_2$. In this case one has $\omega = du/(1 - u) = (1 + u + u^2 + \cdots)du$ and the classes are given by $[\mathbf{P}^r]_\omega = 1$ for all $r > 0$. This agrees with Grothendieck's trace structure on the algebraic K-theory given via the higher direct images. In the case of complex K-theory, topologists like to take as the Chern structure the assignment $L \mapsto [L] - [\mathbf{1}]$. In this case the associated formal group law is again multiplicative: $F = u_1 + u_2 + u_1 u_2$, $\omega = du/(1 + u)$ and the classes are given by $[\mathbf{P}^r]_\omega = (-1)^r$ for all $r > 0$. In [**3**] the assignment $L \mapsto [\mathbf{1}] - [L^\vee]$ is chosen as a Chern structure for complex K-theory. This is why the morphism $K^{\mathrm{alg}} \to K^{\mathrm{top}}$ commutes with the push-forwards corresponding to the Chern structures chosen in [**3**].

Finally one should stress that a version of the Poincaré duality isomorphism between a cohomology and a homology theory represented by an oriented T-spectrum is proved in [**32**]. It is shown there that trace operators in cohomology coincide with the expected ones. Namely, for projective varieties $X, Y \in \mathcal{S}m$ and a morphism $f\colon X \to Y$, one has $\mathrm{tr}^\omega_f = (\mathcal{D}^\omega_Y)^{-1} f_* \mathcal{D}^\omega_X$, where D^ω states for a Poincaré duality isomorphism defined by the orientation ω and f_* is the operator on homology induced by f. A trace structure on an oriented homology theory is constructed in [**29**].

1.1. Terminology and notation

Let k be a field. The term "variety" is used in this text to mean a reduced quasi-projective scheme over k. If X is a variety and $U \subset X$ is a Zariski open, then $Z := X - U$ is considered to be a closed subscheme with a unique structure of a reduced scheme, so Z is considered to be a closed subvariety of X. We fix the following notation:

- $\mathcal{A}b$ – the category of abelian groups;
- $\mathcal{S}m$ – the category of smooth varieties;
  $\mathcal{S}m\mathcal{O}p$ - the category of pairs (X, U) with smooth X and open U in X.
  Morphisms are morphisms of pairs.
  We identify the category $\mathcal{S}m$ with a full subcategory of $\mathcal{S}m\mathcal{O}p$ assigning to a variety X the pair $(X, \emptyset)$;
- $pt = \mathrm{Spec}(k)$;
  For a smooth X and an effective divisor $D \subset X$, we write $L(D)$ for a line bundle over X whose sheaf of sections is the sheaf $\mathcal{L}_X(D)$ (see [**12**, Ch. II, §6, 6.13]). This line bundle has a section vanishing exactly on D;
- $\mathbf{P}(V) = \mathrm{Proj}(\mathrm{Sym}^*(V^\vee))$ – the space of lines in a finite-dimensional k-vector space V;
- $L_V = \mathcal{O}_V(-1)$ – the tautological line bundle over $\mathbf{P}(V)$;
- $\mathbf{1}_X$ – the trivial rank-one bundle over X, often we will write $\mathbf{1}$ for $\mathbf{1}_X$;
- For a vector bundle E over X we write $s(E)$ for its section sheaf;
  For a vector bundle E over X we write $E^\vee$ for the vector bundle dual to E;
- $\mathbf{P}(E) := \mathrm{Proj}(\mathrm{Sym}^*(s(E^\vee)))$ – the space of lines in a vector bundle E;
  $L_E = \mathcal{O}_E(-1)$ – the tautological line bundle on $\mathbf{P}(E)$;
  E^0 – the complement to the zero section of E;
  $z\colon X \to E$ – the zero section of a vector bundle E;
- For a contravariant functor A on $\mathcal{S}m$, set
  $$A(\mathbf{P}^\infty) = \varprojlim A(\mathbf{P}(V)), \tag{3}$$
  where the projective system is induced by all the finite-dimensional vector subspaces $V \hookrightarrow k^\infty$.

Similarly, set

$$A(\mathbf{P}^\infty \times \mathbf{P}^\infty) = \varprojlim A(\mathbf{P}(V) \times \mathbf{P}(W)),$$

where the projective system is induced by all the finite-dimensional subspaces $V, W \subset k^\infty$.

1.2. Cohomology theories

In this section we recall briefly the basic notions and certain results from [**25**]. We use the basic definitions, constructions and results from [**25**]. Recall the notion of a cohomology theory from [**25, 30**].

Definition 1.1. A cohomology theory is a contravariant functor $\mathcal{S}m\mathcal{O}p \xrightarrow{A} \mathcal{A}b$ together with a functor morphism $\partial\colon A(U) \to A(X,U)$ satisfying the following properties:

1. Localization: the sequence $A(X) \xrightarrow{j^A} A(U) \xrightarrow{\partial_P} A(X,U) \xrightarrow{i^A} A(X) \xrightarrow{j^A} A(U)$ is exact for each pair $P = (X,U) \in \mathcal{S}m\mathcal{O}p$, where
$$j\colon U \hookrightarrow X \quad \text{and} \quad i\colon (X,\emptyset) \hookrightarrow (X,U)$$
are the natural inclusions;
2. Excision: the operator $A(X,U) \to A(X',U')$ induced by a morphism
$$e\colon (X',U') \to (X,U)$$
is an isomorphism, if the morphism e is étale and for $Z = X - U$, $Z' = X' - U'$ one has $e^{-1}(Z) = Z'$ and $e\colon Z' \to Z$ is an isomorphism;
3. Homotopy invariance: the operator $A(X) \to A(X \times \mathbf{A}^1)$ induced by the projection $X \times \mathbf{A}^1 \to X$ is an isomorphism.

The operator ∂_P is called the boundary operator and is usually written ∂. A morphism of cohomology theories $\varphi\colon (A, \partial^A) \to (B, \partial^B)$ is a functor transformation $\varphi\colon A \to B$ commuting with the boundary morphisms in the sense that for every pair $P = (X,U) \in \mathcal{S}m\mathcal{O}p$ one has $\partial^B_P \circ \varphi_U = \varphi_P \circ \partial^A_P$.

We also write $A_Z(X)$ for $A(X,U)$, where $Z = X - U$, and call the group $A_Z(X)$ the cohomology of X with support on Z. The operator

$$A_Z(X) \xrightarrow{i^A} A(X) \tag{4}$$

is called the support extension operator for the pair (X,U).

Now recall the deformation to the normal cone construction and some of its properties.

1.2.1. Deformation to the normal cone

The deformation to the normal cone is a well-known construction (for example, see [**10**]). Since the construction and its property (6) play an important role in what follows we give here some details.

Let $i\colon Y \hookrightarrow X$ be a closed imbedding of smooth varieties with normal bundle N. There exists a smooth variety X_t together with a smooth morphism $p_t\colon X_t \to \mathbf{A}^1$

and a closed imbedding $i_t\colon Y\times\mathbf{A}^1\hookrightarrow X_t$ such that the map $p_t\circ i_t$ coincides with the projection $Y\times\mathbf{A}^1\to\mathbf{A}^1$, and

- the fiber of p_t over $1\in\mathbf{A}^1$ is canonically isomorphic to X and the base change of i_t by means of the imbedding $1\hookrightarrow\mathbf{A}^1$ coincides with the imbedding $i\colon Y\hookrightarrow X$;
- the fiber of p_t over $0\in\mathbf{A}^1$ is canonically isomorphic to N and the base change of i_t by means of the imbedding $0\hookrightarrow\mathbf{A}^1$ coincides with the zero section $Y\hookrightarrow N$.

Thus we have the diagram

$$(N,N-Y)\xrightarrow{i_0}(X_t,X_t-Y\times\mathbf{A}^1)\xleftarrow{i_1}(X,X-Y). \tag{5}$$

Here and further we identify a variety with its image under the zero section of any vector bundle over this variety.

Let us recall a construction of X_t, p_t and i_t. For that take X'_t to be the blow-up of $X\times\mathbf{A}^1$ with the center $Y\times\{0\}$. Set $X_t=X'_t-\tilde{X}$ where $\tilde{X}$ is the proper preimage of $X\times\{0\}$ under the blow-up map. Let $\sigma\colon X_t\to X\times\mathbf{A}^1$ be the restriction of the blow-up map $\sigma'\colon X'_t\to X\times\mathbf{A}^1$ to X_t and set p_t to be the composition of σ and the projection $X\times\mathbf{A}^1\to\mathbf{A}^1$.

The proper preimage of $Y\times\mathbf{A}^1$ under the blow-up map is mapped isomorphically to $Y\times\mathbf{A}^1$ under the blow-up map. Thus the inverse isomorphism gives the desired imbedding $i_t\colon Y\times\mathbf{A}^1\hookrightarrow X_t$ (observe that $i_t(Y\times\mathbf{A}^1)$ does not cross $\tilde{X}$).

It is not difficult to check that the imbedding i_t satisfies the mentioned two properties. (The preimage of $X\times 0$ under the map σ' consists of two irreducible components: the proper preimage of X and the exceptional divisor $\mathbf{P}(N\oplus 1)$. Their intersection is $\mathbf{P}(N)$ and $i_t(Y\times\mathbb{A}^1)$ crosses $\mathbf{P}(N\oplus 1)$ along $\mathbf{P}(\mathbf{1})=$ the zero section of the normal bundle N.)

We claim that diagram (5) induces isomorphisms on A-cohomology.

Theorem 1.2. *The following diagram consists of isomorphisms:*

$$A_Y(N)\xleftarrow{i_0^A}A_{Y\times\mathbf{A}^1}(X_t)\xrightarrow{i_1^A}A_Y(X). \tag{6}$$

Moreover, for each closed subset $Z\subset Y$ the following diagram

$$A_Z(N)\xleftarrow{i_0^A}A_{Z\times\mathbf{A}^1}(X_t)\xrightarrow{i_1^A}A_Z(X) \tag{7}$$

consists of isomorphisms as well.

This theorem is analogous to the Homotopy Purity Theorem from [**23**, Th. 3.2.3].

Corollary 1.3. *Let $j_0\colon\mathbf{P}(\mathbf{1}\oplus N)\hookrightarrow X'_t$ be the imbedding of the exceptional divisor into X'_t and let $j_1=e_t\circ i_1\colon X\hookrightarrow X'_t$, where $e_t\colon X_t\hookrightarrow X'_t$ is the open inclusion. Then the diagram*

$$A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(\mathbf{1}\oplus N))\xleftarrow{j_0^A}A_{Y\times\mathbf{A}^1}(X'_t)\xrightarrow{j_1^A}A_Y(X) \tag{8}$$

consists of isomorphisms.

Now recall the notion of a ring cohomology theory from [**25, 30**].

Definition 1.4. Let $P = (X, U), Q = (Y, V) \in \mathcal{S}m\mathcal{O}p$. Set

$$P \times Q = (X \times Y, X \times V \cup U \times Y) \in \mathcal{S}m\mathcal{O}p.$$

This product is associative with the obvious associativity isomorphisms. The unit of this product is the variety pt.

This product is commutative with the obvious isomorphisms $P \times Q \cong Q \times P$.

Definition 1.5. One says that a cohomology theory A is a ring cohomology theory if for every $P, Q \in \mathcal{S}m\mathcal{O}p$ there is given a natural bilinear morphism

$$\times\colon A(P) \times A(Q) \to A(P \times Q)$$

called the cross-product which is functorial in both variables and satisfies the following properties:

1. associativity: $(a \times b) \times c = a \times (b \times c) \in A(P \times Q \times R)$ for $a \in A(P)$, $b \in A(Q)$, $c \in A(R)$;
2. there is given an element $1 \in A(pt)$ such that for any pair $P \in \mathcal{S}m\mathcal{O}p$ and any $a \in A(P)$ one has $1 \times a = a = a \times 1 \in A(P)$;
3. partial Leibniz rule: $\partial_{P\times Y}(a \times b) = \partial_P(a) \times b \in A(X \times Y, U \times Y)$ for a pair $P = (X, U) \in \mathcal{S}m\mathcal{O}p$, smooth variety Y and elements $a \in A(U)$, $b \in A(Y)$.

Given cross-products, define cup-products $\cup\colon A_Z(X) \times A_{Z'}(X) \to A_{Z\cap Z'}(X)$ by

$$a \cup b = \Delta^A(a \times b), \tag{9}$$

where $\Delta\colon (X, U \cup V) \hookrightarrow (X \times X, X \times V \cup U \times X)$ is the diagonal. Clearly, cup-products thus defined are bilinear and functorial in both variables. These cup-products are associative as well: $(a \cup b) \cup c = a \cup (b \cup c)$. If p is the projection $X \to pt$, then the element $p^A(1) \in A(X)$ is the unit for the cup-products $\cup\colon A_Z(X) \times A(X) \to A_Z(X)$ and $\cup\colon A(X) \times A_Z(X) \to A_Z(X)$, and a partial Leibniz rule holds:

$$\partial(a \cup b) = \partial(a) \cup b \quad \text{for} \quad a \in A(U), b \in A(X).$$

Given cup-products one can construct cross-products by $a \times b = p_X^A(a) \cup p_Y^A(b)$ for $a \in A(X, U)$ and $b \in A(Y, V)$. Clearly these two constructions are inverse to each other. Thus having products of one kind we have products of the other kind and we can use both products at the same time.

Definition 1.6. A ring morphism $\varphi\colon (A, \partial_A, \times_A, 1_A) \to (B, \partial_B, \times_B, 1_B)$ of ring cohomology theories is a morphism $\varphi\colon (A, \partial_A) \to (B, \partial_B)$ of the underlying cohomology theories which takes the unit 1_A to the unit 1_B and commutes with the $\times$-products:

$$\varphi(1_A) = 1_B \in B(pt)$$

and for every pair $P, Q \in \mathcal{S}m\mathcal{O}p$ and every pairs of elements $a \in A(P)$, $b \in A(Q)$ one has

$$\varphi_{P\times Q}(a \times b) = \varphi(a) \times \varphi(b) \in B(P \times Q).$$

Remark 1.7. The character A is reserved below in the text for a ring cohomology theory in the sense of 1.5, which is the same as [**25**, Def. 2.13]. Moreover, to simplify technicalities we will assume through the text below that:

- all cohomology theories take values in the category of $\mathbb{Z}/2$-graded abelian groups and grade-preserving homomorphisms, the boundary operator ∂ is either grade-preserving or of degree $+1$ and furthermore, all ring cohomology theories are $\mathbb{Z}/2$-graded-commutative ring theories, i.e. for any $a \in A^p(P)$ and $b \in A^q(Q)$ one has the relation $a \times b = (-1)^{pq} b \times a$ in $A^{p+q}(P \times Q)$,
- all Thom isomorphisms in the sense of [**25**, Def. 3.1] are grade-preserving and all Chern and Thom classes [**25**, Def. 3.2, 3.3] are of even degree.
- A ring morphism of ring cohomology theories is a morphism

$$\varphi\colon (A, \partial_A) \to (B, \partial_B)$$

of the underlying cohomology theories which takes the unit 1_A to the unit 1_B, commutes with the $\times$-products and which is grade-preserving.

So for a variety X and a pair $(X, U) \in \mathcal{S}m\mathcal{O}p$ we will write $A^{\mathrm{ev}}(X)$ (respectively $A^{\mathrm{odd}}(X)$) for the subgroup of all even (respectively odd) degree elements of the ring $A(X)$. The subgroup $A^{\mathrm{ev}}(X)$ is a subring and the subgroup $A^{\mathrm{odd}}(X)$ is an $A^{\mathrm{ev}}(X)$-module. We will write $A^{\mathrm{ev}}(X, U)$ (respectively $A^{\mathrm{odd}}(X, U)$) for the subgroup of all even (respectively odd) degree elements of $A(X, U)$. One should remark that under the mentioned agreements the second partial Leibniz rule holds for a ring cohomology theory A:

- suppose that $\partial_{X,U}$ is a graded operator of degree $+1$; then for each $a \in A^p(U)$ and each $b \in A^q(Y)$ the relation $\partial_{Y \times X, Y \times U}(b \times a) = (-1)^q b \times \partial_{X,U}(a)$ holds in $A(Y \times X, Y \times U)$.
- suppose that $\partial_{X,U}$ is a grade-preserving operator; then for each $a \in A^p(U)$ and each $b \in A^q(Y)$ the relation $\partial_{Y \times X, Y \times U}(b \times a) = b \times \partial_{X,U}(a)$ holds in $A(Y \times X, Y \times U)$.

Remark 1.8. Following Agreement 1.7 the reader should replace everywhere through the article [**25**] the concept of “universally central elements” (see [**25**, Def. 2.15]) by the concept of “even degree elements”. For instance, reading [**25**] the reader should replace the ring $A^{\mathrm{uc}}(X)$ of all universally central elements by the ring $A^{\mathrm{ev}}(X)$ of all even degree elements.

Now recall three structures which A can be endowed with: *an orientation, a Chern structure and a Thom structure*. It is proved in [**25**, Thms. 3.5, 3.35, 3.36] that there is a natural one-to-one correspondence between these structures.

Recall that for a vector bundle E over a variety X we identify X with $z(X)$, where $z\colon X \to E$ is the zero section. If X is a smooth variety, then we write $\mathbf{1}_X$ for the trivial rank-one bundle over X. Often we will just write $\mathbf{1}$ for $\mathbf{1}_X$ if it is clear from context what the variety X is.

Definition 1.9. An orientation on the theory A is a rule ω assigning to each smooth variety X, to each closed subset Z of X and to each vector bundle E/X an operator

$$\omega_Z^E\colon A_Z(X) \to A_Z(E),$$

which is a grade-preserving two-sided $A(X)$-module isomorphism and satisfies the following properties:

1. invariance: for each vector bundle isomorphism $\varphi\colon E \to F$ the following diagram commutes:

$$\begin{array}{ccc} A_Z(X) & \xrightarrow{\omega_Z^E} & A_Z(F) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \varphi^A} \\ A_Z(X) & \xrightarrow{\omega_Z^E} & A_Z(E). \end{array}$$

2. base change: for each morphism $f\colon (X', X'-Z') \to (X, X-Z)$ with closed subsets $Z \hookrightarrow X$ and $Z' \hookrightarrow X'$ and for each vector bundle E/X, the following diagram commutes:

$$\begin{array}{ccc} A_Z(X) & \xrightarrow{\omega_Z^E} & A_Z(E) \\ {\scriptstyle f^A}\downarrow & & \downarrow{\scriptstyle g^A} \\ A_{Z'}(X') & \xrightarrow{\omega_{Z'}^{E'}} & A_{Z'}(E') \end{array}$$

   where E' is the pull-back of E to X' and $g\colon E' = E \times_X X' \to E$ is the projection.

3. for each pair of vector bundles $p\colon E \to X$ and $q\colon F \to X$, the following diagram commutes:

$$\begin{array}{ccc} A_Z(X) & \xrightarrow{\omega_Z^E} & A_Z(E) \\ {\scriptstyle \omega_Z^F}\downarrow & & \downarrow{\scriptstyle \omega_Z^{p^*(F)}} \\ A_Z(F) & \xrightarrow{\omega_Z^{q^*(E)}} & A_Z(E \oplus F) \end{array}$$

   and both compositions coincide with the operator $\omega_Z^{E\oplus F}$.

The operators ω_Z^E are called Thom isomorphisms. The theory A is called orientable if there exists an orientation of A. The theory A is called oriented if an orientation ω is chosen and fixed. An oriented cohomology theory is a pair (A, ω) with an orientation ω on A.

It is convenient, say for a construction of an orientation, to give an equivalent definition of orientation using Thom classes rather than Thom isomorphisms. Here is the definition.

Definition 1.10. A Thom classes theory on A is an assignment which associates to each smooth variety X and to each vector bundle $p\colon E \to X$ over X an element $\mathrm{th}(E) \in A_X(E)$ satisfying the following properties:

1. $\mathrm{th}(E)$ is of even degree, that is, $\mathrm{th}(E) \in A_X^{\mathrm{ev}}(E)$;
2. $\varphi^A(\mathrm{th}(F)) = \mathrm{th}(E)$ for each vector bundle isomorphism $\varphi\colon E \to F$;
3. $f^A(\mathrm{th}(E)) = \mathrm{th}(f^*(E))$ for each morphism $f\colon Y \to X$ of smooth varieties;
4. the operator $A(X) \to A_X(E)$, $a \to \mathrm{th}(E) \cup p^A(a)$ is an isomorphism;

5. multiplicativity property: for the projections $q_i\colon E_1 \oplus E_2 \to E_i$ $(i = 1, 2)$ one has
$$q_1^* \mathrm{th}(E_1) \cup q_2^* \mathrm{th}(E_2) = \mathrm{th}(E_1 \oplus E_2) \in A_X(E_1 \oplus E_2). \tag{10}$$

The element $\mathrm{th}(E)$ is called the Thom class of the vector bundle E.

Lemma 1.11. *If ω is an orientation on the theory A, then the assignment*
$$E \mapsto \omega_X^E(1) \in A_X^{\mathrm{ev}}(E)$$
is a Thom classes theory on A. We write $\mathrm{th}_X(E)$ for the element $\omega_X^E(1) \in A_X^{\mathrm{ev}}(E)$.

If an assignment $E/X \mapsto \mathrm{th}(E) \in A_X^{\mathrm{ev}}(E)$ is a Thom classes theory on A, then the family of homomorphisms $\cup \mathrm{th}(E) \circ p^A\colon A_Z(X) \to A_Z(E)$ form an orientation ω on A.

The two mentioned correspondences between orientations and Thom classes theories are inverse to each other.

1.2.2.
To orient a ring theory A one can use a Chern structure on A or a Thom structure on A. Moreover, in [**25**, (1)] a triangle of correspondences is constructed
$$\begin{array}{ccc} & \textit{Orientations on } A & \\ {}^{\delta}\nearrow & & \searrow^{\rho} \\ \textit{Chern structures on } A & \xleftarrow{\ \gamma\ } & \textit{Thom structures on } A \end{array} \tag{11}$$
with the same γ and δ as in the Introduction, in which each arrow is *a bijection* and each round trip coincides with *the identity* ([**25**, Thms. 3.5, 3.35, 3.36]). For an orientation ω on A we often write th^ω for the Thom structure corresponding to ω via the arrow ρ, and c^ω for the Chern structure corresponding to ω via the arrow $\gamma \circ \rho$.

We recall here the definitions of a Chern structure on A and of a Thom structure on A. To do this, it is convenient to fix certain notation. Namely, for any variety X, any vector bundle F over X and any even degree element $\alpha \in A(\mathbf{P}(F))^{\mathrm{ev}}$, we will often write
$$\alpha\colon A(X) \to A(\mathbf{P}(F)) \tag{12}$$
for the operator given by $a \mapsto \alpha \cup p^A(a)$, where $p\colon \mathbf{P}(F) \to X$ is the projection.

Definition 1.12. A Chern structure on A is an assignment $L \mapsto c(L)$ which associates to each smooth X and each line bundle L/X an even degree element $c(L) \in A(X)$ satisfying the following properties:

1. functoriality: $c(L_1) = c(L_2)$ for isomorphic line bundles L_1 and L_2; $f^A(c(L)) = c(f^*(L))$ for each morphism $f\colon Y \to X$;
2. nondegeneracy: the operator $(1, \xi)\colon A(X) \oplus A(X) \to A(X \times \mathbf{P}^1)$ is an isomorphism where $\xi = c(\mathcal{O}(-1))$ and $\mathcal{O}(-1)$ is the tautological line bundle on $\mathbf{P}^1$;
3. vanishing: $c(\mathbf{1}_X) = 0 \in A(X)$ for any smooth variety X.

The element $c(L) \in A^{\mathrm{ev}}(X)$ is called a Chern class of the line bundle L. (It is proved in [**25**, Lem. 3.29] that the elements $c(L)$ are nilpotent.)

Definition 1.13. One says that A is endowed with a Thom structure if for each smooth variety X and each line bundle L/X there is given an even degree element $\mathrm{th}(L) \in A_X(L)$ satisfying the following properties:

1. functoriality: $\varphi^A(\mathrm{th}(L_2)) = \mathrm{th}(L_1)$ for each isomorphism $\varphi\colon L_1 \to L_2$ of line bundles; $f_L^A(\mathrm{th}(L)) = \mathrm{th}(L_Y)$ for each morphism $f\colon Y \to X$ and each line bundle L/X where $L_Y = L \times_X Y$ is the pull-back line bundle over Y and $f_L\colon L_Y \to L$ is the projection to L;
2. nondegeneracy: the cup-product $\cup\,\mathrm{th}(1)\colon A(X) \to A_X(X \times \mathbf{A}^1)$ is an isomorphism (here X is identified with $X \times \{0\}$).

The element $\mathrm{th}(L) \in A_X^{\mathrm{ev}}(L)$ is called *the Thom class* of the line bundle L.

1.2.3.
We use here the notation from Subsection 1.2.1. Let $e_t\colon X_t \hookrightarrow X'_t$ be the open inclusion, let $p\colon \mathbf{P}(\mathbf{1} \oplus N) \to Y$ be the projection and let $s\colon Y \to \mathbf{P}(\mathbf{1} \oplus N)$ be the section of the projection identifying Y with the subvariety $\mathbf{P}(\mathbf{1})$ in $\mathbf{P}(\mathbf{1} \oplus N)$. The following commutative diagram will be repeatedly used below in the text:

$$
\begin{array}{ccccc}
\mathbf{P}(\mathbf{1} \oplus N) & \xrightarrow{j_0} & X'_t & \xleftarrow{j_1} & X \\
{\scriptstyle s}\uparrow & & {\scriptstyle I_t}\uparrow & & \uparrow{\scriptstyle i} \\
Y & \xrightarrow{k_0} & Y \times \mathbf{A}^1 & \xleftarrow{k_1} & Y.
\end{array}
$$

Here $I_t = e_t \circ i_t$, j_0 is the inclusion of the exceptional divisor, $j_1 = e_t \circ i_1$ and k_0 and k_1 are the closed imbeddings given by $y \mapsto (y, 0)$ and $y \mapsto (y, 1)$, respectively. The following result is proved in [**25**, Cor. 3.19].

Corollary 1.14 (Useful lemma)**.** *Let A be an orientable ring cohomology theory, let $V'_t = X'_t - i_t(Y \times \mathbf{A}^1)$ and let $j_t\colon V'_t \hookrightarrow X'_t$ be the open inclusion. Then*

$$\mathrm{Ker}(j_0^A) \cap \mathrm{Ker}(j_t^A) = (0).$$

In other words, the operator $(j_0^A, j_t^A)\colon A(X'_t) \to A(\mathbf{P}(\mathbf{1} \oplus N)) \oplus A(V'_t)$ is a monomorphism.

1.3. Examples of orientations

Here we recall examples of oriented ring cohomology theories from [**25, 30**]. The orientation is described using either a Chern or a Thom structure.

1.3.1.
Let A be algebraic K-theory [**41**]. The rule $L \to [1] - [L^\vee]$ endows A with a Chern structure and thus orients A (property 2 of Definition 1.12 follows from [**34**, §8, Th. 2.1]).

It is interesting to note that the corresponding Chern class c_n of a rank n vector bundle E is exactly the known class

$$\lambda_{-1}(E^\vee) = [1] - [E^\vee] + [\wedge^2 E^\vee] + \cdots + (-1)^n [\wedge^n E^\vee].$$

1.3.2.
Let m be an integer prime to $\mathrm{char}(k)$. Let A be étale cohomology theory

$$A_Z^*(X) = \oplus_{q=-\infty}^{+\infty} H_Z^*(X, \mu_m^{\otimes q})$$

with the obvious $\mathbb{Z}/2$-grading. Consider the short exact sequence of the étale sheaves $0 \to \mu_m \to \mathbb{G} \xrightarrow{\times m} \mathbb{G} \to 0$ and denote by $\partial\colon H^1(X, \mathbb{G}_m) \to H^2(X, \mu_m)$ the boundary map. For a line bundle L over a smooth variety X, let $[L] \in H^1(X, \mathbb{G}_m)$ be its isomorphism class. It is known [**22**] that the rule $L \mapsto \partial([L])$ endows A with a Chern structure. Thus A is oriented.

1.3.3.
Let A be motivic cohomology [**39**]: $A_Z^p(X) = \oplus_{q=0}^{\infty} H_Z^p(X, \mathbb{Z}(q))$ with the obvious $\mathbb{Z}/2$-grading. Recall that $H^2_{\mathcal{M}}(X, \mathbb{Z}(1)) = \mathrm{CH}^1(X)$ for a smooth X [**39**]. For a line bundle L over a smooth variety X, let $D(L) \in \mathrm{CH}^1(X)$ be the associated class divisor. The rule $L \mapsto D(L)$ endows A with a Chern structure in the characteristic zero [**39**, Cor. 4.12.1] (now it is known in any characteristic). Thus A is oriented.

1.3.4.
Let A be K-cohomology [**34**, §7, 5.8]: $A_Z(X) = \oplus_{q=0}^{\infty} \oplus_{p=0}^{\infty} H_Z^p(X, \mathcal{K}_q)$, where $\mathcal{K}$ is the sheaf of K-groups. The summand $H_Z^p(X, \mathcal{K}_q)$ belongs to the even (respectively, the odd) part of $A_Z(X)$ if $p+q$ is even (respectively, odd). Recall that the sheaf $\mathcal{K}_1$ coincides with the sheaf $\mathcal{O}^*$ of invertible functions. For a line bundle L over a smooth variety X, let $[L] \in H^1(X, \mathcal{K}_1) = H^1(X, \mathcal{O}^*)$ be the isomorphism class of L. The rule $L \mapsto [L]$ endows A with a Chern structure [**14**, Th. 8.10] and thus orients A.

1.3.5.
Let $k = \mathbb{R}$ and let $A = A^{\mathrm{ev}} \oplus A^{\mathrm{odd}}$ with

$$A^{\mathrm{ev}}(X, U) = \oplus_0^{\infty} H^p(X(\mathbb{R}), U(\mathbb{R}); \mathbb{Z}/2), \quad \text{and} \quad A^{\mathrm{odd}}(X, U) = 0.$$

Take as a boundary ∂ the usual boundary map for the pair $(X(\mathbb{R}), U(\mathbb{R}))$. Clearly, ∂ is grade-preserving with respect to the grading we choose on A. Now the cup-product makes A a $\mathbb{Z}/2$-graded-commutative ring theory.

For a line bundle L, consider the real line bundle $L(\mathbb{R})$ over the topological space $X(\mathbb{R})$ and set $c(L) = w_1(L(\mathbb{R})) \in H^1(X(\mathbb{R}); \mathbb{Z}/2\mathbb{Z}) \subset A^{\mathrm{ev}}(X)$ (the first Stiefel-Whitney class). Since $\mathbf{P}^n(\mathbb{R}) = \mathbb{R}P^n$ is real projective space, the rule $L \mapsto c(L)$ endows A with a Chern structure and thus orients A.

1.3.6. Semi-topological complex and real K-theories [**9**]
If the ground field k is the field $\mathbb{R}$ of reals, then the semi-topological K-theory of real algebraic varieties $K\mathbb{R}^{\mathrm{semi}}$ defined in [**9**] is an oriented theory as is proved in [**9**]. For a real variety X it interpolates between the algebraic K-theory of X and Atiyah's Real K-theory of the associated Real space of complex points, $X(\mathbb{C})$.

1.3.7. Orienting the algebraic cobordism theory
This example is considered in 2.9.7 below.

1.4. The formal group law F^ω

Let ω be an orientation of A. Thus A is endowed with the Chern structure $L \mapsto c^\omega(L)$ which corresponds to ω (see 1.2.2 or [**25**, Thms. 3.35, 3.36]). Following Novikov, Mischenko [**24**] and Quillen [**33**], we associate a formal group law F^ω with ω. This formal group law is defined over the ring $\bar{A} := A^{\mathrm{ev}}(pt)$ and gives an expression of the first Chern class of $L_1 \otimes L_2$ in terms of the first Chern classes of line bundles L_1, L_2.

Using [**25**, Th. 3.9], identify the formal power series in one variable $\bar{A}[[u]]$ with the ring $A^{\mathrm{ev}}(\mathbf{P}^\infty)$ identifying u with $c^\omega(\mathcal{O}(1)) \in A^{\mathrm{ev}}(\mathbf{P}^\infty)$. The two "projections" $p_i \colon \mathbf{P}^\infty \times \mathbf{P}^\infty \to \mathbf{P}^\infty$ induce two pull-back maps $p_i^A \colon A(\mathbf{P}^\infty) \to A(\mathbf{P}^\infty \times \mathbf{P}^\infty)$. Using [**25**, Th. 3.9] again, identify $A^{\mathrm{ev}}(\mathbf{P}^\infty \times \mathbf{P}^\infty)$ with $\bar{A}[[u_1, u_2]]$ where

$$u_i = p_i^*(u) = c^\omega(p_i^*(\mathcal{O}(1)).$$

Set

$$F^\omega(u_1, u_2) = c^\omega(p_1^*(\mathcal{O}(1) \otimes p_2^*(\mathcal{O}(1))) \in \bar{A}[[u_1, u_2]]. \tag{13}$$

Proposition 1.15. *For any $X \in \mathcal{S}m$ and line bundles L_1/X, L_2/X, one has the following relation in $A(X)$:*

$$c^\omega(L_1 \otimes L_2) = F^\omega(c^\omega(L_1), c^\omega(L_2)).$$

*Here the right-hand side is well-defined since the first Chern classes are of even degree and nilpotent ([**25**, Lem. 3.29]).*

Proposition 1.16. *The formal power series $F^\omega \in \bar{A}[[u_1, u_2]]$ is a commutative formal group law ([**11**]) with the "inverse element"*

$$I^\omega(u) = c^\omega(\mathcal{O}(-1)) \in A^{\mathrm{ev}}(\mathbf{P}^\infty) = \bar{A}[[u]].$$

Definition 1.17. The formal group law F^ω is called the formal group law associated with A endowed with the orientation ω. Its "inverse element" is the series I^ω.

1.4.1. Examples

- If $A = H^*_{\mathcal{M}}(-, \mathbb{Z}(*))$ with first Chern class c^H, then one has the relation $c^H(L_1 \otimes L_2) = c^H(L_1) + c^H(L_2)$;
- If $A = K$-theory with first Chern class defined by $c^K(L) = [1] - [L^\vee]$, then one has the relation $c^K(L_1 \otimes L_2) = c^K(L_1) + c^K(L_2) - c^K(L_1) \cdot c^K(L_2)$;
- Let $k = \mathbb{C}$ and $A = \Omega$ be complex cobordism theory equipped with the Chern structure $L \mapsto \mathrm{cf}(L)$ given by the Conner-Floyd class cf [**6**, pp. 48–52]. The formal group law F_Ω was introduced and used originally in [**24**]. It is the universal commutative formal group law as is proved in [**33**]. The latter means that, for any commutative ring R and any commutative formal group law in one variable F over R, there exists a unique ring homomorphism $l_F \colon \Omega \to R$ such that the coefficients of F coincide with the l_F-images of the corresponding coefficients of F_Ω. Since F_Ω is a universal formal group law its ring of coefficients $\Omega(pt)$ is canonically isomorphic to the Lazard ring L [**33**].

2. Trace structures

The character A is reserved below in the text for a ring cohomology theory in the sense of Agreement 1.7. We use the basic definitions, constructions and results of [**25**] taking into account Remark 1.8.

2.1. Trace structures on a ring cohomology theory

Here we define a notion of a *trace structure* on A. In Theorem 2.5 we claim that an orientation of A gives rise to a unique trace structure on A respecting a certain normalization property. On the other hand, a trace structure defines an orientation and these two constructions are inverse to each other. Let us recall a notion.

Definition 2.1. Let

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{i}} & \tilde{X} \\ {\scriptstyle \tilde{\varphi}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{i} & X \end{array}$$

be a commutative square of varieties with smooth Y, X, $\tilde{X}$, a closed embedding $i\colon Y \hookrightarrow X$ and normal bundles $N := N_{X/Y}$, $\tilde{N} := N_{\tilde{X}/\tilde{Y}}$. The square is called transversal if it is Cartesian in the category of schemes and if the canonical morphism $\tilde{N} \to \tilde{\varphi}^*(N)$ is an isomorphism.

Definition 2.2. Let A be a ring cohomology theory. A trace structure on A is a rule assigning to each projective morphism of smooth varieties $f\colon Y \to X$ a grade-preserving two-sided $A(X)$-module operator

$$\mathrm{tr}_f\colon A(Y) \to A(X)$$

called the trace operator (for f) or the push-forward (for f) and satisfying the following properties:

1. $\mathrm{tr}_{f\circ g} = \mathrm{tr}_f \circ \mathrm{tr}_g$ for any projective morphisms $Z \xrightarrow{g} Y$ and $Y \xrightarrow{f} X$ of smooth varieties;
2. for the transversal square from (2.1) the following diagram commutes:

$$\begin{array}{ccc} A(\tilde{Y}) & \xrightarrow{\mathrm{tr}_{\tilde{i}}} & A(\tilde{X}) \\ {\scriptstyle \tilde{\varphi}^A}\uparrow & & \uparrow{\scriptstyle \varphi^A} \\ A(Y) & \xrightarrow{\mathrm{tr}_i} & A(X); \end{array}$$

3. for any morphism of smooth varieties $f\colon Y \to X$ the following diagram commutes:

$$\begin{array}{ccc} A(\mathbf{P}^n \times Y) & \xleftarrow{(\mathrm{id}\times f)^A} & A(\mathbf{P}^n \times X) \\ {\scriptstyle \mathrm{tr}_{p_Y}}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}_{p_X}} \\ A(Y) & \xleftarrow{f^A} & A(X), \end{array}$$

   where $p_Y\colon \mathbf{P}^n \times Y \to Y$ and $p_X\colon \mathbf{P}^n \times X \to X$ are the natural projections;

4. normalization: for any smooth variety X one has $\mathrm{tr}_{\mathrm{id}_X} = \mathrm{id}_{A(X)}$;
5. localization: for any closed imbedding of smooth varieties $i\colon Y \hookrightarrow X$ and the inclusion $j\colon X - Y \hookrightarrow X$ the sequence $A(Y) \xrightarrow{\mathrm{tr}_i} A(X) \xrightarrow{j^A} A(X - Y)$ is exact. This sequence is often called the Gysin sequence below in the text.

The line bundle $L(D)$ associated with an effective divisor D on a smooth variety X is defined in in 1.1. It has a section vanishing exactly on D.

Remark 2.3. For a ring cohomology theory A, the restriction $A|_{\mathcal{S}m}$ to the category of smooth varieties is a ring cohomology pretheory in the sense of [**27**, Def. 1.1.1]. Furthermore, the notion of "trace structure on A" coincides tautologically with the notion of "integration on A" used in [**27**]. Moreover, Theorem 2.5 and [**25**, Th. 3.9] show that each trace structure on a ring cohomology theory A is a perfect integration on A in the sense of [**27**, Def. 1.1.6]. Thus Theorem 2.5 shows that for an oriented ring cohomology theory A (Definition 1.9) its restriction $A|_{\mathcal{S}m}$ to smooth varieties is an oriented cohomology pretheory in the sense of [**27**, Def. 1.1.7].

The notion of "trace structure on A" coincides tautologically with the notion of "integration on A" used in [**26**]. Here we prefer to use the term "trace structure" for the following reasons. Firstly, the term "trace homomorphism" is used in [**7**] and in other classical books. Secondly, the term "trace homomorphism" is used in the cohomological context as well as in the homological context (see [**7**], [**29**]).

To get the notion of "trace structure on A" we added to the notion of "integration on A" from [**30**] the localization property 5 from Definition 2.2.

Definition 2.4. Let ω be an orientation of A and let $L \mapsto c(L)$ be the Chern structure on A corresponding by [**25**, Th. 3.36] to ω. One says that a trace structure $f \mapsto \mathrm{tr}_f$ on A respects the orientation ω if for each smooth variety X and each smooth divisor $i\colon D \hookrightarrow X$ one has the following relation in $A(X)$:

$$\mathrm{tr}_i(1) = c(L(D)). \tag{14}$$

For a trace structure $f \mapsto \mathrm{tr}_f$ on A respecting the orientation ω we will often write tr^ω_f for tr_f. So the last relation looks now as follows: $\mathrm{tr}^\omega_i(1) = c(L(D))$.

Theorem 2.15 below states that if there exists at least one trace structure on A, then there exist a lot of trace structures. The notation tr^ω_f indicates the fact that the operator depends on the orientation ω rather than on the theory A itself.

Theorem 2.5. *Let A be a ring cohomology theory.*

1. *Let ω be an orientation on A. Then there exists a unique trace structure $f \mapsto \mathrm{tr}_f$ on A respecting this orientation.*
2. *Let $f \mapsto \mathrm{tr}_f$ be a trace structure on A. Then there exists a unique orientation ω on A such that the trace structure respects the ω. Moreover, the assignment $L \mapsto z^A(\mathrm{tr}_z(1)) \in A^{\mathrm{ev}}(X)$ is a Chern structure on A and the required orientation ω is the one corresponding to this Chern structure by [**25**, Th. 3.35].*
3. *The correspondences described in items 1 and 2 of this theorem are inverse to each other.*

Examples illustrating this theorem are given in Subsection 2.9 below. The first item of the theorem describes the arrow α from the Introduction. The second item of the theorem describes the compositions $\delta \circ \gamma \circ \beta$ and $\gamma \circ \beta$ from the Introduction. The arrow β from the Introduction is described rather easily as well. Namely, for a line bundle L over a smooth variety X, let $s\colon X \mapsto \mathbf{P}(1 \oplus L)$ be the section of the projection $p\colon \mathbf{P}(1 \oplus L) \to X$ identifying the variety X with the subvariety $\mathbf{P}(1)$ in the projective bundle $\mathbf{P}(1 \oplus L)$. Let $e\colon L \hookrightarrow \mathbf{P}(1 \oplus L)$ be the open inclusion and let $z\colon X \to L$ be the zero section of L (obviously $s = z \circ e$). Identify the group $A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L))$ with a subgroup of $A(\mathbf{P}(1 \oplus L))$ via the support extension operator from the exact sequence [**25**, (8)]. Now set $\bar{\mathrm{t}}\mathrm{h}(L) = \mathrm{tr}_s(1) \in A^{\mathrm{ev}}_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L))$. Then the assignment $L \mapsto \mathrm{th}(L) = e^A(\bar{\mathrm{t}}\mathrm{h}(L)) \in A^{\mathrm{ev}}_X(L)$ is a Thom structure on A (this is checked in the proof of item 2 below). Moreover, the corresponding Chern structure, by [**25**, Th. 3.5], is the Chern structure from item 2 of Theorem 2.5 (this is also checked in the proof of item 2.).

The proof of Theorem 2.5 is rather long. It occupies all of Section 2 and is organized as follows. In the current subsection the uniqueness assertion of item 1 is proved and item 2 of the theorem is also proved. The proof of the uniqueness assertion of item 1 is based on the first variant of Riemann-Roch type theorem proved in [**27**, Cor. 1.1.11]. The proof of the existence assertion is given below in Subsection 2.7 and it requires a construction of a trace structure.

The construction of a trace structure respecting an orientation is rather long itself and it takes several subsections. Namely, in Subsection 2.2 trace operators are constructed for a closed imbedding of smooth varieties (they are called Gysin operators). In Subsection 2.3 trace operators are constructed for the case of projections $X \times \mathbf{P}^n \to X$ (they are called Quillen's operators). In Subsection 2.4 properties of Gysin operators are proved and in Subsection 2.5 properties of Quillen's operators are proved except for the compatibility with a section of a trivial projective bundle, which is proved in Subsection 2.6. Push-forwards for any projective morphism are constructed in Subsection 2.7. These push-forwards form a trace structure on A as is stated in Theorem 2.12 from the same subsection.

Proof of the uniqueness assertion of item 1. Let $f \mapsto \mathrm{tr}_f$ be a trace structure on A respecting the orientation ω. For each line bundle $p\colon L \to X$ and its zero section $z\colon X \to L$ the line bundle over L associated with the divisor X on L coincides with the line bundle $p^*(L)$. Thus $z^A(\mathrm{tr}_z(1)) = z^A(c(p^*(L))) = c(z^*(p^*(L))) = c(L)$ in $A(X)$. The assignment $L \mapsto z^A(\mathrm{tr}_z(1))$ is the Euler structure associated with the trace structure $f \mapsto \mathrm{tr}_f$ on the pretheory $A|_{\mathcal{S}m}\colon \mathcal{S}m \to Ab$ ([**27**, §1]). The projective bundle theorem [**25**, Th. 3.9] shows that the trace structure $f \mapsto \mathrm{tr}_f$ is a perfect integration on the pretheory $A|_{\mathcal{S}m}$ (in the sense of [**27**, Def. 1.1.6]).

If $f \mapsto \mathrm{tr}^{(1)}_f$ and $f \mapsto \mathrm{tr}^{(2)}_f$ are two trace structures on A respecting the orientation ω, then the two Euler structures $L \mapsto z^A(\mathrm{tr}^{(1)}_z(1))$ and $L \mapsto z^A(\mathrm{tr}^{(2)}_z(1))$ coincide with the Chern structure $L \mapsto c(L)$. By [**27**, Cor. 1.1.11] the two trace structures coincide. The uniqueness assertion is proved. □

Proof of item 2 of Theorem 2.5. We begin with the uniqueness assertion. Assume that the trace structure $f \mapsto \mathrm{tr}_f$ respects two orientations ω and ω' on A. Prove that these two orientations coincide. Let $L \mapsto c(L)$ be the Chern structure on A corresponding to the orientation ω by [**25**, Th. 3.36]. For a line bundle L over a smooth variety X, consider its zero section $z\colon X \to L$ and the projection $p\colon L \to X$. The line bundle $L(z(X))$ associated with the divisor $z(X)$ on the variety L is isomorphic to the line bundle $p^*(L)$. Thus one has a chain of relations

$$z^A(\mathrm{tr}_z(1)) = z^A(c(L(z(X)))) = z^A(c(p^*(L))) = c(z^*(p^*(L))) = c(L).$$

Similarly, one has the relation $z^A(\mathrm{tr}_z(1)) = c'(L)$, where $L \mapsto c'(L)$ is the Chern structure on A corresponding to the orientation ω' by [**25**, Th. 3.36]. Thus $c'(L) = c(L)$ and the relation $\omega = \omega'$ follows.

The first aim is to prove that the assignment $L \mapsto \mathrm{th}(L)$ described just below Theorem 2.5 is a Thom structure. One has to check the functorial behavior of the class $\mathrm{th}(L)$ and its nondegeneracy. To verify the functorial behavior of this class it suffices to check the functorial behavior of the class $\bar{\mathrm{th}}(L)$.

If $\phi\colon L_2 \to L_1$ is a line bundle isomorphism and $\Phi\colon \mathbf{P}(1 \oplus L_2) \to \mathbf{P}(1 \oplus L_1)$ is the induced map of the projective bundles, then the diagram

$$\begin{array}{ccc} A(\mathbf{P}(1 \oplus L_2)) & \xleftarrow{\Phi^A} & A(\mathbf{P}(1 \oplus L_1)) \\ {\scriptstyle \mathrm{tr}_{s_2}}\uparrow & & \uparrow{\scriptstyle \mathrm{tr}_{s_1}} \\ A(X) & \xleftarrow{\mathrm{id}} & A(X) \end{array}$$

commutes because the underlying diagram of varieties is transversal. The support extension operator $A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L_2)) \to A(\mathbf{P}(1 \oplus L_2))$ is injective by the exactness of the short sequence (8) from [**25**]. Thus $\Phi^A(\bar{\mathrm{th}}(L_1)) = \bar{\mathrm{th}}(L_2)$ in $A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L_2))$.

Similarly, for a morphism $f\colon X' \to X$ with smooth X', for $L' = X' \times_X L'$ and for the induced map $\Phi\colon \mathbf{P}(1 \oplus L') \to \mathbf{P}(1 \oplus L)$, one gets the relation $\Phi^A(\bar{\mathrm{th}}(L)) = \bar{\mathrm{th}}(L')$. The functoriality of the class $\mathrm{th}(L)$ is proved.

To prove the nondegeneracy property recall that the Gysin sequence

$$0 \to A(X) \xrightarrow{\mathrm{tr}_s} A(X \times \mathbf{P}^1) \xrightarrow{j^A} A(X \times \mathbf{P}^1 - X \times \{0\}) \to 0$$

is exact in the middle term by 2.2, where $s\colon X \to X \times \mathbf{P}^1$ is defined by $s(x) = (x, 0)$ and $j\colon X \times \mathbf{P}^1 - X \times \{0\} \hookrightarrow X \times \mathbf{P}^1$ is the open inclusion. The operator j^A is clearly surjective and the operator tr_s is injective because $\mathrm{tr}_p \circ \mathrm{tr}_s = \mathrm{tr}_{\mathrm{id}_X} = \mathrm{id}_{A(X)}$, where $p\colon X \times \mathbf{P}^1 \to X$ is the projection. Hence the last five-term sequence is short exact. Now the exactness of the sequence [**25**, (8)] shows that the operator tr_s identifies $A(X)$ with the subgroup $A_{X \times \{0\}}(X \times \mathbf{P}^1)$ of the group $A(X \times \mathbf{P}^1)$.

We claim that the operator tr_s coincides with $(\cup\, \mathrm{tr}_s(1)) \circ p^A$. In fact, for any element $a \in A(X)$, one has $\mathrm{tr}_s(1) \cup p^A(a) = \mathrm{tr}_s(s^A(p^A(a))) = \mathrm{tr}_s(a)$. Since $\bar{\mathrm{th}}(1) = \mathrm{tr}_s(1)$, the last observations show that the operator $\cup\, \bar{\mathrm{th}}(1)\colon A(X) \to A(X \times \mathbf{P}^1)$ identifies $A(X)$ with the subgroup $A_{X \times \{0\}}(X \times \mathbf{P}^1)$ of the group $A(X \times \mathbf{P}^1)$.

Now we are ready to prove the nondegeneracy of property of the Thom class $\mathrm{th}(L)$.

For that consider the commutative diagram

$$\begin{array}{ccc} A_{X\times 0}(X\times\mathbf{A}^1) & \xleftarrow{e^A} & A_{X\times 0}(X\times\mathbf{P}^1) \\ {\scriptstyle \cup\mathrm{th}(1)}\uparrow & & \uparrow{\scriptstyle \cup\bar{\mathrm{th}}(1)} \\ A(X) & \xleftarrow{\mathrm{id}} & A(X). \end{array}$$

The pull-back map e^A is an isomorphism by the excision property. Thus the map $\cup\,\mathrm{th}(1)$ is an isomorphism as well. The nondegeneracy property of the class $\mathrm{th}(L)$ is proved. Thus the assignment $L\mapsto \mathrm{th}(L)$ with $\mathrm{th}(L)=e^A(\bar{\mathrm{th}}(L))\in A_X(L)$ is a Thom structure on A.

Now we show that the assignment $L\mapsto c(L)$ described in item 2 is a Chern structure on A. By Theorem 3.5 from [**25**] it suffices to check the relation $c(L)=[z^A\circ i^A](\mathrm{th}(L))$, where $i^A\colon A_X(L)\to A(L)$ is the support extension operator. To do this, we consider the transversal diagram

$$\begin{array}{ccc} L & \xrightarrow{e} & \mathbf{P}(1\oplus L) \\ {\scriptstyle z_A}\uparrow & & \uparrow{\scriptstyle s_A} \\ X & \xrightarrow{\mathrm{id}} & X \end{array}$$

and the corresponding commutative diagram

$$\begin{array}{ccc} A(L) & \xleftarrow{e^A} & A(\mathbf{P}(1\oplus L)) \\ {\scriptstyle z}\uparrow & & \uparrow{\scriptstyle s} \\ A(X) & \xleftarrow{\mathrm{id}} & A(X). \end{array}$$

Since $e\circ z=s$, one gets the relation

$$z^A(\mathrm{tr}_z(1))=s^A(\mathrm{tr}_s(1)).$$

If $\bar{i}^A\colon A_{\mathbf{P}(1)}(\mathbf{P}(1\oplus L))\to A(\mathbf{P}(1\oplus L))$ is the support extension operator, then the chain of relations

$$\begin{aligned} c(L)&=z^A(\mathrm{tr}_z(1))=s^A(\mathrm{tr}_s(1))\\ &=z^A[e^A(\bar{i}^A(\bar{\mathrm{th}}(L)))]\\ &=z^A[i^A(e^A(\bar{\mathrm{th}}(L)))]\\ &=[z^A\circ i^A](\mathrm{th}(L)) \end{aligned}$$

proves the required relation $c(L)=[z^A\circ i^A](\mathrm{th}(L))$. Thus the assignment $L\mapsto c(L)$ is a Chern structure. Moreover, the Thom structure $L\mapsto \mathrm{th}(L)$ corresponds to the Chern structure in the sense of [**25**, Th. 3.5]. The last chain of relations shows as well that we have the following relation:

$$c(L)=z^A(\mathrm{tr}_z(1))=s^A(\mathrm{tr}_s(1)). \tag{15}$$

Let ω be the orientation on A corresponding to the Chern structure $L\mapsto c(L)$ by Theorem 3.35 from [**25**]. We claim that this orientation is the required one. First of all the Chern structure $L \mapsto c(L)$ corresponds to the orientation ω by Theorem 3.36 from [**25**].

To check that the trace structure $f \mapsto \mathrm{tr}_f$ respects the orientation ω recall that the assignment $L \mapsto z^A(\mathrm{tr}_z(1))$ is the Euler structure on A associated with the trace structure $f \mapsto \mathrm{tr}_f$ [**27**, 1.1.4]. The relations $c(L) = z^A(\mathrm{tr}_z(1))$ show that it coincides with the Chern structure $L \mapsto c(L)$. The projective bundle theorem [**25**, Th. 3.9] states that the trace structure $f \mapsto \mathrm{tr}_f$ is perfect in the sense of [**27**, Def. 1.1.6]. Now the relation (14) holds by [**27**, Theorem 1.1.8] and thus the trace structure $f \mapsto \mathrm{tr}_f$ respects the orientation ω. The proof of item 2 is complete. $\square$

2.2. Trace operators for closed imbedding

Let A be a ring cohomology theory endowed with an orientation ω and with the Chern structure $L \mapsto c(L)$ which corresponds by [**25**, Th. 3.36] to ω. Below we give a construction of trace operators for closed imbeddings. The trace operator to be constructed for a closed imbedding i will be temporarily (until 2.12) denoted by i_{gys}. The deformation to the normal cone construction (see 1.2.1) plays an important role here (the deformation to the normal cone construction is a perfect substitute for the tubular neighborhood in differential topology).

For a closed imbedding $i\colon Y \hookrightarrow X$ of smooth varieties, define an operator

$$i_{\mathrm{th}}\colon A(Y) \to A_Y(X) \tag{16}$$

as the composition $i_{\mathrm{th}}\colon A(Y) \xrightarrow{\mathrm{th}_Y^N} A_Y(N) \xrightarrow{(i_0^A)^{-1}} A_{Y\times\mathbf{A}^1}(X_t) \xrightarrow{i_1^A} A_Y(X)$, where the notation for $N = N_{X/Y}$, i_0^A and i_1^A is taken from (1.2) and th_Y^N is the Thom operator corresponding to the orientation ω by (Definition 1.9). The operator i_{th} is an isomorphism (see 1.11). Define the Gysin operator

$$i_{\mathrm{gys}}\colon A(Y) \to A(X) \tag{17}$$

as the composition $i_{\mathrm{gys}}\colon A(Y) \xrightarrow{i_{\mathrm{th}}} A_Y(X) \xrightarrow{j^A} A(X)$, where j^A is the support extension operator for the pair $(X, X-Y)$ (see 1.1).

It will be checked below that the operator $i_{\mathrm{gys}}\colon A(Y) \to A(X)$ is a *two-sided* $A(X)$-*module homomorphism*. In particular the composite operator

$$i_{\mathrm{gys}} \circ i^A\colon A(X) \to A(X)$$

coincides with the operator given by the cup-product with the element $i_{\mathrm{gys}}(1)$:

$$i_{\mathrm{gys}} \circ i^A = \cup i_{\mathrm{gys}}(1). \tag{18}$$

The following properties of the Gysin operators will be proved below before Theorem 2.12 and are useful when proving this theorem:

1. **Composition property:** One has

$$i_{\mathrm{gys}} \circ j_{\mathrm{gys}} = (i \circ j)_{\mathrm{gys}} \tag{19}$$

for the closed imbedding $Z \overset{j}{\hookrightarrow} Y \overset{i}{\hookrightarrow} X$ of smooth varieties.

2. **Base change for Gysin operator:** The Gysin operators commute with a transversal base change; i.e., for a transversal square from Definition 2.1 the

following diagram commutes:

$$\begin{array}{ccc} A(\tilde{Y}) & \xrightarrow{\tilde{i}_{\mathrm{gys}}} & A(\tilde{X}) \\ \tilde{\varphi}^A \big\uparrow & & \varphi^A \big\uparrow \\ A(Y) & \xrightarrow{i_{\mathrm{gys}}} & A(X), \end{array} \tag{20}$$

3. **Additivity:** Let j_1 and j_2 be the natural imbeddings of smooth varieties Y_1 and Y_2 to $Y = Y_1 \amalg Y_2$. For a closed imbedding $i : Y \hookrightarrow X$ one has
$$i_{\mathrm{gys}} = (i_1)_{\mathrm{gys}} \circ j_1^A + (i_2)_{\mathrm{gys}} \circ j_2^A, \tag{21}$$
where i_r is the composition $i \circ j_r$.
4. **Identity to identity:** For any smooth variety X one has
$$(\mathrm{id}_X)_{\mathrm{gys}} = \mathrm{id}_{A(X)}. \tag{22}$$
5. **Gysin sequence exact:** For any closed imbedding of smooth varieties $i\colon Y \hookrightarrow X$ and the inclusion $k\colon X - Y \hookrightarrow X$, the sequence
$$A(Y) \xrightarrow{i_{\mathrm{gys}}} A(X) \xrightarrow{k^A} A(X - Y) \tag{23}$$
is exact. This sequence is often called the Gysin sequence below in the text.
6. **Smooth divisor case:** For a smooth divisor $i\colon D \hookrightarrow X$ one has
$$i_{\mathrm{gys}}(1) = c(L(D)) \tag{24}$$
in $A(X)$ (see 1.1 for notation). There are more properties of the Gysin operators which are useful themselves and are useful in proving some of the properties listed above. Recall that A is endowed with the Chern structure $L \mapsto c(L)$ corresponding to the orientation ω by [**25**, Th. 3.36] and with the corresponding higher Chern classes (see [**25**, Th. 3.27]).

2.2.1.
Let E be a rank n vector bundle over a smooth Y and let $s\colon Y \to \mathbf{P}(\mathbf{1} \oplus E)$ be the section of the projective bundle $p\colon \mathbf{P}(\mathbf{1} \oplus E) \to Y$ identifying Y with the closed subvariety $\mathbf{P}(\mathbf{1})$ in $\mathbf{P}(\mathbf{1} \oplus E)$. Then one has
$$s_{\mathrm{gys}} = \cup\, \bar{\mathrm{th}}(E) \circ p^A, \tag{25}$$
where $\bar{\mathrm{th}}(E) = c_n(p^*(E) \otimes \mathcal{O}_E(1))$.

2.2.2.
Let $z\colon Y \to E$ be the zero section of a vector bundle E/Y. Then the operator z_{th} coincides with the Thom operator th^E_Y.

2.2.3.
Let F/Y be a rank m vector bundle and let $s\colon Y \to \mathbf{P}(F)$ be a section of the natural projection $p\colon \mathbf{P}(F) \to Y$. Consider the natural inclusion $\mathcal{O}_F(-1) \to p^*(F)$ and let Q be the factor-bundle $p^*(F)/\mathcal{O}_F(-1)$. In $A(\mathbf{P}(F))$ one has
$$s_{\mathrm{gys}}(1) = c_{n-1}(\mathcal{O}_F(1) \otimes p^* s^* Q). \tag{26}$$

2.3. Trace operators for the projection $X \times \mathbf{P}^n \to X$

Let A be a ring cohomology theory endowed with an orientation ω and therefore endowed with the corresponding Chern structure [**25**, Thms. 3.35, 3.36] and with the corresponding higher Chern classes [**25**, Th. 3.27]. Below we give a construction of trace operators for projections. As above, $\bar{A} = A^{\mathrm{ev}}(pt)$ is the subring of all the even degree elements in $A(pt)$.

We give two alternative ways to construct push-forwards for the projections. The first way is based on the fact that the cobordism ring Ω is the coefficient ring of a universal formal group law. The second way uses no complicated facts and is based on residue theory. The push-forward to be constructed for the projection $p\colon X \times \mathbf{P}^n \to X$ will be temporarily (until 2.12) denoted by p_{quil}.

2.3.1. Ω-approach

Consider complex cobordism theory $\Omega(*)$ and the formal group law F_Ω (see Example 1.4.1) associated with this theory and its canonical Chern class for line bundles (the Conner-Floyd class) [**6**]. This law is defined over the ring $\Omega = \Omega(pt)$. According to a theorem of Quillen ([**33**, Th. 2]), F_Ω is a universal commutative formal group law in one variable. This means that for any commutative ring R and any commutative formal group law in one variable F over R, there exists a unique ring homomorphism $l_F\colon \Omega \to R$ such that the coefficients of F coincide with the l_F-images of the corresponding coefficients of F_Ω. For the theory A endowed with the orientation ω, denote by

$$l_\omega\colon \Omega \to \bar{A} \tag{27}$$

the homomorphism l_F, where $F = F^\omega$ is the formal group law associated with the orientation ω on A (see Definition 1.17), and set $[\mathbf{P}^n]_\omega = l_\omega([\mathbb{C}P^n])$, where $[\mathbb{C}P^n]$ is the class of $\mathbb{C}P^n$ in Ω.

For the projection $p\colon X \times \mathbf{P}^n \to X$, define the operator

$$p_{\mathrm{quil}}\colon A(X \times \mathbf{P}^n) \to A(X) \tag{28}$$

as follows. Identify $A(X \times \mathbf{P}^n)$ with $A(X)[t]/(t^{n+1})$ taking t to the element $\zeta = c_1(\mathcal{O}(1))$ (see [**25**, Th. 3.9]), consider the structural morphism $f\colon X \to pt$ and set p_{quil} to be the unique two-sided $A(X)$-module operator which takes the element t^i to the element $f^A([\mathbf{P}^{n-i}]_\omega) \in A(X)$ for $i = 0, 1, \ldots, n$.

By the very construction, the operator $p_{\mathrm{quil}}\colon A(X \times \mathbf{P}^n) \to A(X)$ is a *two-sided* $A(X)$-*module homomorphism.* In particular, the composite operator

$$p_{\mathrm{quil}} \circ p^A\colon A(X) \to A(X)$$

coincides with the operator given by the cup-product with the element $p_{\mathrm{gys}}(1)$:

$$p_{\mathrm{quil}} \circ p^A = \cup p_{\mathrm{quil}}(1) = \cup f^A([\mathbf{P}^{n-i}]_\omega). \tag{29}$$

The following properties of the operator p_{quil} can be proved before Theorem 2.12 and are useful when proving this theorem:

1. **Composition property:** The following diagram commutes:

$$\begin{array}{ccc} A(X \times \mathbf{P}^n \times \mathbf{P}^m) & \xrightarrow{(\tilde{p}_n)_{\text{quil}}} & A(X \times \mathbf{P}^m) \\ {\scriptstyle (\tilde{p}_m)_{\text{quil}}}\big\downarrow & & \big\downarrow{\scriptstyle (p_m)_{\text{quil}}} \\ A(X \times \mathbf{P}^n) & \xrightarrow{(p_n)_{\text{quil}}} & A(X), \end{array}$$

   where $p_n\colon X \times \mathbf{P}^n \to X$, $p_m\colon X \times \mathbf{P}^m \to X$, $\tilde{p}_n\colon X \times \mathbf{P}^n \times \mathbf{P}^m \to X \times \mathbf{P}^m$ and $\tilde{p}_m\colon X \times \mathbf{P}^n \times \mathbf{P}^m \to X \times \mathbf{P}^n$ are the natural projections.
2. **Base change property:** Let $\varphi\colon Y \to X$ be a morphism of smooth varieties and let $\tilde{\varphi}$ be the base change of φ by the natural projection $p_X : X \times \mathbf{P}^n \to X$. Then one has the relation $(p_Y)_{\text{quil}} \circ \tilde{\varphi}^A = \varphi^A \circ (p_X)_{\text{quil}}$, where $p_Y : Y \times \mathbf{P}^n \to Y$ is the natural projection.
3. **Compatibility with linear imbedding:** For a linear imbedding $i\colon \mathbf{P}^n \to \mathbf{P}^m$, the following diagram commutes:

$$\begin{array}{ccc} A(X \times \mathbf{P}^n) & \xrightarrow{i_{\text{gys}}} & A(X \times \mathbf{P}^m) \\ {\scriptstyle (p_n)_{\text{quil}}}\big\downarrow & & \big\downarrow{\scriptstyle (p_m)_{\text{quil}}} \\ A(X) & \xrightarrow{\text{id}} & A(X), \end{array}$$

   where $p_n\colon X \times \mathbf{P}^n \to X$ and $p_m\colon X \times \mathbf{P}^m \to X$ are the natural projections.
4. **Compatibility with Gysin operators:** For a closed imbedding of smooth varieties $i\colon Y \hookrightarrow X$, the following diagram commutes:

$$\begin{array}{ccc} A(Y \times \mathbf{P}^n) & \xrightarrow{(i\times\text{id})_{\text{gys}}} & A(X \times \mathbf{P}^n) \\ {\scriptstyle (p_Y)_{\text{quil}}}\big\downarrow & & \big\downarrow{\scriptstyle (p_X)_{\text{quil}}} \\ A(Y) & \xrightarrow{i_{\text{gys}}} & A(X), \end{array}$$

   where $p_X : X \times \mathbf{P}^n \to X$ and $p_Y : Y \times \mathbf{P}^n \to Y$ are the natural projections.
5. **Compatibility with a section of trivial projective bundles:** For a section $s\colon X \to X \times \mathbf{P}^n$ of the projection $p\colon X \times \mathbf{P}^n \to X$, one has $p_{\text{quil}} \circ s_{\text{gys}} = \text{id}_{A(X)}$.

2.3.2. Approach by means of residues

Here we sketch another way to construct push-forwards for the projections. This way is based on elementary residue theory and uses no complicated facts. Thus it could be applied in a more general situation. Recall that for a commutative ring R and the ring of formal power series $R[[t]]$ in one variable t, the ring of Laurent formal power series is defined as the localization of $R[[t]]$ at the element t: $R((t)) = R[[t]]_t$. Let $\Omega_{R[t]/R}$ be the module of Kähler differentials of $R[t]$ over R. Set $\Omega^{\text{cont}} = \Omega_{R[t]/R} \otimes_{R[t]} R[[t]]$ and $\Omega^{\text{mer}} = \Omega_{R[t]/R} \otimes_{R[t]} R((t))$. Since $\Omega_{R[t]/R}$ is a rank-one free $R[t]$-module generated by the differential form dt, we see that Ω^{cont} (respectively Ω^{mer}) is a rank-one free $R[[t]]$-module (respectively a rank-one free $R((t))$-module) generated by dt. Thus each element of Ω^{mer} can be uniquely written in the form $\Sigma_{n=-N}^{\infty} a_n t^n dt$ for certain elements $a_n \in R$. The residue operator on Ω^{mer} is defined as an operator

$$\mathrm{Res}_{t=0}\colon \Omega^{\mathrm{mer}} \to R \tag{30}$$

which takes a form $\Sigma_{n=-N}^{\infty} a_n t^n dt \in \Omega^{\mathrm{mer}}$ to the element $a_{-1} \in R$. Let $\omega_{\mathrm{inv}} \in \Omega^{\mathrm{cont}}$ be the unique normalized F_A-invariant differential form (see ([**11**])). Define the element

$$[\mathbf{P}^n]_\omega \in \bar{A} \tag{31}$$

as the coefficient at $t^n dt$ in $\omega_{\mathrm{inv}}(t)$ of the form ω_{inv}. For the projection $p\colon \mathbf{P}^n \times X \to X$, we define an operator

$$p_{\mathrm{quil}}\colon A(\mathbf{P}^n \times X) = A(X)[\zeta]/(\zeta^{n+1}) \to A(X) \tag{32}$$

as the unique two-sided $A(X)$-module operator such that $p_{\mathrm{quil}}(\zeta^r) = [\mathbf{P}^{n-r}]_\omega$ for each $r = 0, 1, \ldots, n$. Using the residue operator (30) one can rewrite the operator p_{quil} for the projection $p\colon \mathbf{P}^n \times X \to X$ as follows:

$$p_{\mathrm{quil}}(a) = \mathrm{Res}_{t=0}(\frac{h}{t^{n+1}}\omega_{\mathrm{inv}}), \tag{33}$$

where for an element $a \in A(\mathbf{P}^n \times X)$ we choose $h(t) \in A(X)[[t]]$ to be an arbitrary formal power series with $h(\zeta) = a \in A(\mathbf{P}^n \times X)$ and as above $\zeta = c_1(\mathcal{O}(1)) \in A(\mathbf{P}^n)$. Defined either by (32) or equivalently by (33) the operators p_{quil} are *two-sided* $A(X)$-module operators.

The operators p_{quil} defined by (28) and (33) coincide because, by a theorem of Mischenko [**24**], the unique normalized F_Ω-invariant differential form coincides with the form $\Sigma_{n=1}^{\infty}[\mathbf{P}^n]_\Omega t^n dt$ (it is just the differential form $d\log_\Omega$).

The operators p_{quil} satisfy properties 1–5 listed above. This is proved in Subsections 2.5 and 2.6 below in the text. Moreover, the proofs of properties 1–4 in the case of the residue approach are just the same as the proof of these properties in the case of the Ω-approach (and do not involve cobordism).

However, if we take (33) as the definition of Quillen's operators, then the proof of property 5 is different from the proof of this property in the case of the Ω approach; in fact it is purely algebraic and shorter (see the proof of Proposition 2.10).

Remark 2.6. One concludes this subsection with the following observation: if $E^\vee = \oplus L_i$ is the direct sum of line bundles, then one has (compare with [**33**, Th. 1])

$$q_E = \prod_{i=1}^{d}(t -_F \lambda_i), \tag{34}$$

where $\mathrm{rk}E = d$, $\lambda_i = c_1(L_i)$.

This relation can be proved but we skip this proof because this observation is never used in this text.

2.4. Proofs of properties of the Gysin maps

2.4.1. Identity to identity

The relation (22) is obvious because in this case $Y = X$ and $X_t = Y \times \mathbf{A}^1$ and $N = Y$ and $\mathrm{th}_Y^N = \mathrm{id}\colon A(Y) \to A(X)$.

2.4.2. Gysin sequence exact

This is clear, because the localization sequence

$$A_Y(X) \xrightarrow{j^A} A(X) \xrightarrow{k^A} A(X-Y)$$

for the pair $(X, X-Y)$ is exact and the operator $i_{\mathrm{th}}\colon A(Y) \to A_Y(X)$ is an isomorphism and $i_{\mathrm{gys}} = j^A \circ i_{\mathrm{th}}$.

2.4.3. Two-sided projection formula

In the deformation to the normal cone diagram from Subsection 1.2.1 all morphisms are morphisms over X. Thus i_{gys} is a two-sided $A(X)$-module operator.

2.4.4. Base change for Gysin property

The base change property for the Gysin maps i_{gys} follows from the one for the Thom maps i_{th}. We will prove it now.

Let $N = N_{X/Y}$ and $\tilde{N} = N_{\tilde{X}/\tilde{Y}}$. Since the square from 2.1 is transversal, the canonical map $\tilde{Y} \times_Y N \to \tilde{N}$ is an isomorphism. We will identify $\tilde{Y} \times_Y N$ and $\tilde{N}$ by means of this isomorphism and write $\Phi\colon \tilde{Y} \times_Y N \to N$ for the projection. The functoriality of the Thom class shows that $\Phi^A(\mathrm{th}(N)) = \mathrm{th}(\tilde{N})$. Therefore the diagram

$$\begin{array}{ccc} A(\tilde{Y}) & \xrightarrow{\cup\,\mathrm{th}(\tilde{N})} & A_{\tilde{Y}}(\tilde{N}) \\ {\scriptstyle \tilde{\varphi}^A}\uparrow & & {\scriptstyle \Phi^A}\uparrow \\ A(Y) & \xrightarrow{\cup\,\mathrm{th}(N)} & A_Y(N) \end{array} \tag{35}$$

commutes. Since the square from 2.1 is transversal, the map

$$\phi \times \mathrm{id}\colon \tilde{X} \times \mathbf{A}^1 \to X \times \mathbf{A}^1$$

gives rise to a commutative diagram

$$\begin{array}{ccccccc} \tilde{Y} & \xrightarrow{\tilde{z}} & \tilde{N} & \xrightarrow{\tilde{i}_0} & \tilde{X}_t & \xleftarrow{\tilde{i}_1} & \tilde{X} \\ {\scriptstyle \tilde{\varphi}}\downarrow & & {\scriptstyle \Phi}\downarrow & & {\scriptstyle \varphi_t}\downarrow & & {\scriptstyle \varphi}\downarrow \\ Y & \xrightarrow{z} & N & \xrightarrow{i_0} & X_t & \xleftarrow{i_1} & X \end{array}$$

in which the rows coincide with the deformation to the normal cone diagrams for the pairs $(\tilde{X}, \tilde{Y})$ and (X, Y). The commutativity of the last diagram shows that the diagram consisting of pull-back operators commutes

$$\begin{array}{ccccc} A_{\tilde{Y}}(\tilde{N}) & \xleftarrow{\tilde{i}_0^A} & A_{\tilde{Y}\times\mathbf{A}^1}(\tilde{X}_t) & \xrightarrow{\tilde{i}_1^A} & A_{\tilde{Y}}(\tilde{X}) \\ {\scriptstyle \Phi^A}\uparrow & & {\scriptstyle \varphi_t^A}\uparrow & & {\scriptstyle \varphi^A}\uparrow \\ A_Y(N) & \xleftarrow{i_0^A} & A_{Y\times\mathbf{A}^1}(X_t) & \xrightarrow{i_1^A} & A_Y(X). \end{array}$$

Gluing this diagram with the commutative diagram (35), we get the commutativity of the diagram

$$\begin{array}{ccc} A(\tilde{Y}) & \xrightarrow{\tilde{i}_{\mathrm{th}}} & A_{\tilde{Y}}(\tilde{X}) \\ {\scriptstyle \tilde{\varphi}^A}\uparrow & & {\scriptstyle \varphi^A}\uparrow \\ A(Y) & \xrightarrow{i_{\mathrm{th}}} & A_Y(X). \end{array}$$

The commutativity of the diagram (20) is proved. Property 2 of the Gysin operators follows.

2.4.5. Proof of property 2.2.2

We consider here the zero section $z\colon Y \to E$ of a constant rank n vector bundle E and prove the relation $z_{\mathrm{th}} = \mathrm{th}^E_Y$. It suffices to prove that the composition

$$A_Y(E) \xrightarrow{(i_0^A)^{-1}} A_{Y\times \mathbf{A}^1}(E_t) \xrightarrow{i_1^A} A_Y(E)$$

is the identity map. For that we construct a morphism $q\colon E_t \to E$ such that

- the map q makes E_t into a line bundle over E,
- $q \circ i_1 = \mathrm{id}$,
- $q \circ i_0 = \mathrm{id}$, and
- $q^{-1}(Y) = i_t(Y \times \mathbf{A}^1)$ in E_t,

where $i_t\colon Y \times \mathbf{A}^1 \hookrightarrow E_t$ is the imbedding from the deformation to the normal cone construction (Subsection 1.2.1).

In this case one gets $q^{-1}(Y) = i_t(Y \times \mathbf{A}^1) \subset E_t$. Therefore the relations $i_0^A = (q^A)^{-1} = i_1^A$ hold by the strong homotopy invariance property (see [**25**, 2.2.6]) and we are done. It remains to construct the desired morphism $q\colon E_t \to E$.

Let F/Y be a vector bundle and let F' be the blow-up of F at the zero section. The variety F' coincides with the total space of the line bundle $\mathcal{O}_F(-1)$ over $\mathbf{P}(F)$. Let $q_F\colon F' \to \mathbf{P}(F)$ be the projection of the line bundle to its base $\mathbf{P}(F)$.

If $F = E \oplus 1$ for a vector bundle E over Y, then one has the following commutative diagram:

$$\begin{array}{ccccccc} E' & \longrightarrow & F' & \longleftarrow & F' - E' = E_t & \longleftarrow & \mathbf{P}(1) \times \mathbf{A}^1 = Y \times \mathbf{A}^1 \\ {\scriptstyle q_E}\downarrow & & {\scriptstyle q_F}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle pr}\downarrow \\ \mathbf{P}(E) & \longrightarrow & \mathbf{P}(F) & \longleftarrow & \mathbf{P}(F) - \mathbf{P}(E) = E & \longleftarrow & \mathbf{P}(1) = Y \end{array}$$

in which all the vertical arrows are the projections of the line bundles to their bases.

The projection q has two sections s_0 and s_1. The section s_0 is the zero section and the section s_1 is given by $x \mapsto (x, 1)$.

Observe that the variety $\mathbf{P}(F) - \mathbf{P}(E)$ coincides with E, the variety E_t coincides with the variety $F' - E'$, and the imbedding $i_1\colon E \hookrightarrow E_t$ coincides with the section $s_1\colon E \hookrightarrow F' - E'$. The normal bundle $N = N_{E/Y}$ to Y in E coincides with the bundle E itself and the imbedding $i_0\colon N \hookrightarrow E_t$ coincides with the section $s_0\colon E \hookrightarrow F' - E'$. Finally, the variety $Y \times \mathbf{A}^1$ coincides with $\mathbf{P}(1) \times \mathbf{A}^1$ and the imbedding $Y \times \mathbf{A}^1 \hookrightarrow E_t$ coincides with the imbedding $\mathbf{P}(1) \times \mathbf{A}^1 \hookrightarrow F' - E'$.

Thus the map $q\colon E_t \to E$ makes E_t into a line bundle over E and satisfies the relations $q \circ i_0 = \mathrm{id} = q \circ i_1$ and $q^{-1}(Y) = i_t(Y \times \mathbf{A}^1)$. Thus q is the desired map. The property is proved.

2.4.6. Proof of the relation (25)
Since both sides of the relation (25) are $A(Y)$-linear, it suffices to check the relation $s_{\mathrm{th}}(1) = c_n(\mathcal{O}_F(1) \otimes p^*(E))$. Set $F = \mathbf{1} \oplus E$. Take the zero section z of the vector bundle E and the section s of the projective bundle $\mathbf{P}(F)$ which identifies Y with the closed subvariety $\mathbf{P}(\mathbf{1})$ in $\mathbf{P}(F)$, and consider the commutative diagram

$$\begin{array}{ccc} A(Y) & \xrightarrow{s_{\mathrm{th}}} & A_{\mathbf{P}(1)}(\mathbf{P}(F)) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle e^A} \\ A(Y) & \xrightarrow{z_{\mathrm{th}}} & A_Y(E), \end{array}$$

where $e\colon E \to \mathbf{P}(\mathbf{1} \oplus E) = \mathbf{P}(F)$ is the open imbedding. As was already proved, $z_{\mathrm{th}} = \cup\,\mathrm{th}(E)$ and therefore $z_{\mathrm{th}}(1) = \mathrm{th}(E)$ in $A_Y(E)$. By the very definition of the Thom class, one has $\mathrm{th}(E) = e^A(c_n(\mathcal{O}_F(1) \otimes p^*(E)))$. Since the map e^A is an isomorphism, one gets the desired relation $s_{\mathrm{th}}(1) = c_n(\mathcal{O}_F(1) \otimes p^*(E))$. The property is proved.

2.4.7. Section of a projective bundle (26)
Let F/Y be a rank m vector bundle and let $s\colon Y \to \mathbf{P}(F)$ be a section of the natural projection $p\colon \mathbf{P}(F) \to Y$. Consider the natural inclusion $\mathcal{O}_F(-1) \to p^*(F)$, and let Q_F be the factor-bundle $p^*(F)/\mathcal{O}_F(-1)$ and $E_F = s^*(Q_F)$. We have to prove the relation $s_{\mathrm{gys}}(1) = c_{m-1}(\mathcal{O}_F(1) \otimes p^*(E_F))$ in $A(\mathbf{P}(F))$.

Set $L_F = s^*(\mathcal{O}_F(-1))$. Then one has the obvious exact sequence of vector bundles on X

$$0 \to L_F \to F \to E_F \to 0.$$

If M is a line bundle over X, then replacing F by $F \otimes M$ changes neither $\mathbf{P}(F)$ nor the vector bundle $\mathcal{O}_F(1) \otimes p^*(E_F)$. Thus one may assume that $L_F = \mathbf{1}$. Write in that case E for E_F. Note that the above short exact sequence becomes $0 \to \mathbf{1} \to F \to E \to 0$.

By the splitting principle [**25**, Lem. 3.24] we may assume further that $F = \mathbf{1} \oplus E$. In this case, the relation $s_{\mathrm{gys}}(1) = c_{m-1}(\mathcal{O}_F(1) \otimes p^*(E))$ is just the relation (25) which is proved just above ($n = m - 1$). The relation (26) is proved.

2.4.8. Proof of the relation (24).
For a smooth divisor $i\colon D \hookrightarrow X$ one has to check the relation $i_{\mathrm{gys}}(1) = c(L(D))$ in $A(X)$ (see 1.1 for notation). For that consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{P}(1 \oplus N) & \xrightarrow{j_0} & X'_t & \xleftarrow{j_1} & X \\ {\scriptstyle s}\uparrow & & {\scriptstyle I_t}\uparrow & & \uparrow{\scriptstyle i} \\ D & \xrightarrow{k_0} & D \times \mathbf{A}^1 & \xleftarrow{k_1} & D, \end{array}$$

from 1.2.3. Since both squares in this diagram are transversal the following diagram commutes

$$\begin{array}{ccccc} A(\mathbf{P}(1\oplus N)) & \xleftarrow{j_0^A} & A(X'_t) & \xrightarrow{j_1^A} & A(X) \\ {\scriptstyle s_{\mathrm{gys}}}\uparrow & & {\scriptstyle (I_t)_{\mathrm{gys}}}\uparrow & & \uparrow{\scriptstyle i_{\mathrm{gys}}} \\ A(D) & \xleftarrow{k_0^A} & A(D\times\mathbf{A}^1) & \xrightarrow{k_1^A} & A(D). \end{array}$$

Now consider the line bundle $L_t = L(D\times\mathbf{A}^1)$ over X'_t (see the notation in 1.1). One can show that its restriction $j_1^*(L_t)$ to X is isomorphic to $L(D)$. Therefore one has the relation $j_1^A(c(L_t)) = c(L(D))$ in $A(X)$. These show that it remains to check the relation

$$c(L_t) = (I_t)_{\mathrm{gys}}(1) \tag{36}$$

in $A(X'_t)$. To do this, consider once more the line bundle L_t and note that its restriction $j_0^*(L_t)$ to $\mathbf{P}(1\oplus N)$ is isomorphic to the line bundle $\mathcal{O}_{1\oplus N}(1)\otimes p^*(N)$. Therefore,

$$j_0^A(c(L_t)) = c(\mathcal{O}_{1\oplus N}(1)\otimes p^*(N))$$

in $A(\mathbf{P}(1\oplus N))$. By (25) one has $c(\mathcal{O}_{1\oplus N}(1)\otimes p^*(N)) = s_{\mathrm{gys}}(1)$ in $A(\mathbf{P}(1\oplus N))$. Now the chain of relations in $A(\mathbf{P}(1\oplus N))$

$$j_0^A(c(L_t)) = c(\mathcal{O}_{1\oplus N}(1)\otimes p^*(N)) = s_{\mathrm{gys}}(1) = s_{\mathrm{gys}}(k_0^A(1)) = j_0^A((I_t)_{\mathrm{gys}}(1))$$

proves that $j_0^A(c(L_t)) = j_0^A((I_t)_{\mathrm{gys}}(1))$. Both elements $c(L_t)$ and $(I_t)_{\mathrm{gys}}(1)$ vanish when restricted to the open subset $V_t = X' - D\times\mathbf{A}^1$. In fact, this is clear for the element $c(L_t)$ and this holds for the element $(I_t)_{\mathrm{gys}}(1)$ because the sequence (23) is a complex. So the relation (36) follows from Corollary 1.14.

The proof of the relation $i_{\mathrm{gys}}(1) = c(L(D))$ in $A(X)$ is complete.

2.4.9. Proof of the additivity property (21)

The proof of this property will be given just after three preliminary Lemmas 2.7, 2.8 and 2.9. We begin with the following particular case.

Lemma 2.7. *Let $s = s_1 \amalg s_2\colon S = S_1\amalg S_2 \hookrightarrow T_1\amalg T_2 = T$ be a closed imbedding of smooth varieties. Let $i_m\colon S_1\hookrightarrow S$ and $j_m\colon T_m\hookrightarrow T$ be the open inclusions and let $r_m = j_m\circ s_m\colon S_m\hookrightarrow T$. Then*

$$s_{gys} = (r_1)_{gys}\circ i_1^A + (r_2)_{gys}\circ i_2^A.$$

To prove this lemma consider the isomorphism $(j_1^A, j_2^A)\colon A(T)\to A(T_1)\oplus A(T_2)$. To prove the lemma it suffices to check the following relations:

$$j_1^A\circ s_{\mathrm{gys}} = j_1^A\circ(r_1)_{\mathrm{gys}}\circ i_1^A + j_1^A\circ(r_2)_{\mathrm{gys}}\circ i_2^A \tag{37}$$

$$j_2^A\circ s_{\mathrm{gys}} = j_2^A\circ(r_1)_{\mathrm{gys}}\circ i_1^A + j_2^A\circ(r_2)_{\mathrm{gys}}\circ i_2^A. \tag{38}$$

The transversal square

$$\begin{array}{ccc} T_1 & \xrightarrow{j_1} & T \\ {\scriptstyle s_1}\uparrow & & \uparrow{\scriptstyle s} \\ S_1 & \xrightarrow{i_1} & S \end{array}$$

proves the relation $j_1^A \circ s_{\mathrm{gys}} = (s_1)_{\mathrm{gys}} \circ i_1^A$. The transversal square

$$\begin{array}{ccc} T_1 & \xrightarrow{j_1} & T \\ {\scriptstyle s_1}\uparrow & & \uparrow{\scriptstyle r_1} \\ S_1 & \xrightarrow{\mathrm{id}} & S_1 \end{array}$$

proves the relation $j_1^A \circ (r_1)_{\mathrm{gys}} = (s_1)_{\mathrm{gys}}$. The transversal square

$$\begin{array}{ccc} T_1 & \xrightarrow{j_1} & T \\ \uparrow & & \uparrow{\scriptstyle r_2} \\ \emptyset & \longrightarrow & S_2 \end{array}$$

proves the relation $j_1^A \circ (r_2)_{\mathrm{gys}} = 0$.

Now combining the last three relations one gets the chain of relations

$$j_1^A \circ (r_1)_{\mathrm{gys}} \circ i_1^A + j_1^A \circ (r_2)_{\mathrm{gys}} \circ i_2^A = (s_1)_{\mathrm{gys}} \circ i_1^A = j_1^A \circ s_{\mathrm{gys}},$$

which proves the relation (37). The relation (38) is proved in a similar way. The lemma is proved.

Now suppose we are given varieties V and $U = U_1 \amalg U_2$ and a transversal square of the form

$$\begin{array}{ccc} T_1 \amalg T_2 & \xrightarrow{v} & V \\ {\scriptstyle s=s_1 \amalg s_2}\uparrow & & \uparrow{\scriptstyle t} \\ S_1 \amalg S_2 & \xrightarrow{u=u_1 \amalg u_2} & U_1 \amalg U_2. \end{array}$$

Lemma 2.8. *Let $k_m \colon U_m \hookrightarrow U$ be the open inclusion and let $t_m = t \circ k_m \colon U_m \hookrightarrow V$. Then*

$$v^A \circ t_{gys} = v^A \circ [(t_1)_{gys} \circ k_1^A + (t_2)_{gys} \circ k_2^A].$$

To prove this lemma observe that the following squares are transversal ($m = 1, 2$):

$$\begin{array}{ccc} T_1 \amalg T_2 & \xrightarrow{v} & V \\ {\scriptstyle r_m}\uparrow & & \uparrow{\scriptstyle t_m} \\ S_m & \xrightarrow{u_m} & U_m. \end{array}$$

Thus $v^A \circ (t_m)_{\mathrm{gys}} = (r_m)_{\mathrm{gys}} \circ u_m^A$ for $m = 1, 2$ and one gets the following chain of relations using Lemma 2.7 in the last step:

$$\begin{aligned} v^A \circ [(t_1)_{\mathrm{gys}} \circ k_1^A + (t_2)_{\mathrm{gys}} \circ k_2^A] &= (r_1)_{\mathrm{gys}} \circ u_1^A \circ k_1^A + (r_2)_{\mathrm{gys}} \circ u_2^A \circ k_2^A \\ &= (r_1)_{\mathrm{gys}} \circ i_1^A \circ u^A + (r_2)_{\mathrm{gys}} \circ i_2^A \circ u^A \\ &= s_{\mathrm{gys}} \circ u^A. \end{aligned}$$

Finally, $s_{\mathrm{gys}} \circ u^A = v^A \circ t_{\mathrm{gys}}$ by the transversality of the diagram considered in the lemma. The lemma follows.

Now consider the deformation to the normal cone diagram from Subsection 1.2.1:

$$\begin{array}{ccccc} \mathbf{P}(\mathbf{1}\oplus N) & \xrightarrow{j_0} & X'_t & \xleftarrow{j_1} & X \\ {\scriptstyle s}\uparrow & & {\scriptstyle i_t}\uparrow & & \uparrow{\scriptstyle i} \\ Y & \xrightarrow{k_0} & Y\times\mathbf{A}^1 & \xleftarrow{k_1} & Y. \end{array}$$

Since $Y = Y_1 \amalg Y_2$, one sees that

$$Y\times\mathbf{A}^1 = Y_1\times\mathbf{A}^1 \amalg Y_2\times\mathbf{A}^1 \quad \text{and} \quad \mathbf{P}(\mathbf{1}\oplus N) = \mathbf{P}(\mathbf{1}\oplus N_1)\amalg\mathbf{P}(\mathbf{1}\oplus N_2),$$

where N_i is the normal bundle of X to Y_i. Let $l_m\colon Y_m\times\mathbf{A}^1 \hookrightarrow Y\times\mathbf{A}^1$ be the inclusion for $m=1,2$ and let $i_t^m = i_t\circ l_m$. Since the left-hand side square in the very last diagram is transversal, Lemma 2.8 gives the following relation:

$$j_0^A\circ(i_t)_{\mathrm{gys}} = j_0^A\circ[(i_t^1)_{\mathrm{gys}}\circ l_1^A + (i_t^2)_{\mathrm{gys}}\circ l_2^A]. \tag{39}$$

To complete the proof of the proposition we need the following.

Lemma 2.9. *Let $j_t\colon X'_t - Y\times\mathbf{A}^1 \hookrightarrow X'_t$ be the open inclusion. Then $j_t^A\circ(i_t)_{gys}=0$ and $j_t^A\circ(i_t^m)_{gys}=0$ for $m=1,2$.*

In fact, the first relation follows from the localization property 2.2. The remaining relations for $m=1,2$ follow from the first one because $(i_t^m)_{\mathrm{gys}} = (i_t)_{\mathrm{gys}}\circ(l_m)_{\mathrm{gys}}$.

By this lemma, $j_t^A\circ[(i_t)_{\mathrm{gys}} - (i_t^1)_{\mathrm{gys}}\circ l_1^A - (i_t^2)_{\mathrm{gys}}\circ l_2^A]=0$. By relation (39), one has the relation $j_0^A\circ[(i_t)_{\mathrm{gys}} - (i_t^1)_{\mathrm{gys}}\circ l_1^A - (i_t^2)_{\mathrm{gys}}\circ l_2^A]=0$. Thus

$$[(i_t)_{\mathrm{gys}} - (i_t^1)_{\mathrm{gys}}\circ l_1^A - (i_t^2)_{\mathrm{gys}}\circ l_2^A]=0$$

by Corollary 1.14 and

$$(i_t)_{\mathrm{gys}} = (i_t^1)_{\mathrm{gys}}\circ l_1^A + (i_t^2)_{\mathrm{gys}}\circ l_2^A.$$

Since the right-hand side square from the very last diagram is transversal, one gets the relation

$$j_1^A\circ(i_t)_{\mathrm{gys}} = i_{\mathrm{gys}}\circ k_1^A.$$

Since $k_1 = k_{11}\amalg k_{12}\colon Y_1\amalg Y_2 \hookrightarrow Y_1\times\mathbf{A}^1\amalg Y_2\times\mathbf{A}^1$, the squares

$$\begin{array}{ccc} X'_t & \xleftarrow{j_1} & X \\ {\scriptstyle i_t^{(m)}}\uparrow & & \uparrow{\scriptstyle i_m} \\ Y_m\times\mathbf{A}^1 & \xleftarrow{k_{1m}} & Y_m \end{array}$$

are transversal. Thus one has the relations

$$j_1^A\circ(i_t^m)_{\mathrm{gys}} = (i_m)_{\mathrm{gys}}\circ k_{1m}^A$$

for $m=1,2$.

Recall that $u_m\colon Y_m \hookrightarrow Y$ for $m = 1, 2$ are the inclusions and $i_m = i \circ u_m$. Combining the last relations, one gets a chain of relations

$$\begin{aligned} i_{\mathrm{gys}} \circ k_1^A &= j_1^A \circ (i_t)_{\mathrm{gys}} = j_1^A \circ [(i_t^1)_{\mathrm{gys}} \circ l_1^A + (i_t^2)_{\mathrm{gys}} \circ l_2^A] \\ &= (i_1)_{\mathrm{gys}} \circ k_{11}^A \circ l_1^A + (i_2)_{\mathrm{gys}} \circ k_{12}^A \circ l_2^A \\ &= (i_1)_{\mathrm{gys}} \circ u_1^A \circ k_1^A + (i_2)_{\mathrm{gys}} \circ u_2^A \circ k_1^A, \end{aligned}$$

which proves the following one:

$$i_{\mathrm{gys}} \circ k_1^A = [(i_1)_{\mathrm{gys}} \circ u_1^A + (i_2)_{\mathrm{gys}} \circ u_2^A] \circ k_1^A.$$

Since the operator k_1^A is an isomorphism the additivity relation (21) follows.

2.4.10. Proof of the relation (19)

We begin by proving the following generalization of the relation (24). Namely, let $Y = \cap_{i=1}^n D_i$ be a transversal intersection of n smooth divisors D_r $(r = 1, 2, \ldots, n)$ in X. Writing $i\colon Y \hookrightarrow X$ for the closed imbedding, one has

$$i_{\mathrm{gys}}(1) = c_n(\oplus_{r=1}^n L_r), \tag{40}$$

where $L_r = L(D_r)$ for $(r = 1, 2, \ldots, n)$. To prove this relation consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{P}(1 \oplus N) & \xrightarrow{j_0} & X'_t & \xleftarrow{j_1} & X \\ {\scriptstyle s}\uparrow & & {\scriptstyle I_t}\uparrow & & \uparrow{\scriptstyle i} \\ Y & \xrightarrow{k_0} & Y \times \mathbf{A}^1 & \xleftarrow{k_1} & Y \end{array}$$

from 1.2.3 for the imbedding $i\colon Y \hookrightarrow X$. Observe that the normal bundle $N = N_{X/Y}$ to Y in X is the direct sum of the line bundles $L_r|Y$. Let $(D_r)'_t$ be the proper preimage of the divisor $D_r \times \mathbf{A}^1$ under the blow-up map $\sigma\colon X'_t \to X \times \mathbf{A}^1$. Clearly the subvariety $Y \times \mathbf{A}^1$ in X'_t is a complete intersection of the smooth divisors $(D_r)'_t$. Furthermore, the divisor $(D_r)'_t$ intersects the divisor $\mathbf{P}(\mathbf{1} \oplus N)$ transversally in X'_t and their intersection coincides with the smooth divisor $P_r = \mathbf{P}(\mathbf{1} \oplus E_r)$ in $\mathbf{P}(\mathbf{1} \oplus N)$, where E_r is a direct summand of N which is the direct sum of all $L_s|Y$ except for $L_r|Y$. Clearly the line bundle $L(P_r)$ on $\mathbf{P}(1 \oplus N)$ is isomorphic to the line bundle $p^*(L_r|Y) \otimes \mathcal{O}_{\mathbf{1}\oplus N}(1)$. Thus $c_n(p^*(N) \otimes \mathcal{O}_{\mathbf{1}\oplus N}(1))$ coincides with the cup-product of classes $c_1(p^*(L_r|Y) \otimes \mathcal{O}_{\mathbf{1}\oplus N}(1))$.

Now consider the line bundle $(L_r)_t = L((D_r)'_t)$ over X'_t (see the notation in 1.1). The line bundle $j_0^*((L_r)_t)$ over $\mathbf{P}(\mathbf{1} \oplus N)$ is isomorphic to the line bundle $L(P_r)$ and thus it is isomorphic to the line bundle $\mathcal{O}_{\mathbf{1}\oplus N}(1) \otimes p^*(L_r|Y)$. The line bundle $j_1^*((L_r)_t)$ over X is isomorphic to L_r. Therefore one has the relation $j_0^A(c_1((L_r)_t)) = c_1(\mathcal{O}_{\mathbf{1}\oplus N}(1) \otimes p^*(L_r|Y))$ in $A(\mathbf{P}(1 \oplus N))$ and the relation $j_1^A(c_1((L_r)_t)) = c_1(L_r)$ in $A(X)$. Recall that by (25) $s_{\mathrm{gys}}(1) = c_n(\mathcal{O}_{\mathbf{1}\oplus N}(1) \otimes p^*(N)$ and thus

$$s_{\mathrm{gys}}(1) = \cup_{r=1}^n c_1(\mathcal{O}_{\mathbf{1}\oplus N}(1) \otimes p^*(L_r|Y)).$$

As an intermediate step in the proof of the relation (40) we need to check the relation

$$\cup_{r=1}^{n} c_1((L_r)_t) = I_{\mathrm{gys}}(1) \tag{41}$$

in $A(X'_t)$.

To prove this relation observe that both elements vanish when restricted to the open subset $V_t = X'_t - Y \times \mathbf{A}^1$. In fact, for every integer r, the class $c_1((L_r)_t)$ vanishes on the complement to the divisor $(D_r)'_t$ and thus this class comes from the group $A_{(D_r)'_t}(X'_t)$ via the support extension operator. Therefore the cup-product of the elements $c_1((L_r)_t)$ comes from the group $A_{Y\times\mathbf{A}^1}(X'_t)$ via the support extension operator. Whence this cup-product vanishes on V_t. The element $(I_t)_{\mathrm{gys}}(1)$ comes from the group $A_{Y\times\mathbf{A}^1}(X'_t)$ via the support extension operator by the very definition of the Gysin operator. Therefore both elements $\cup_{r=1}^{n} c_1((L_r)_t)$ and $(I_t)_{\mathrm{gys}}(1)$ vanish on V_t by the localization property 1.1. Now the relation (41) is proved as follows. The chain of relations in $A(\mathbf{P}(1 \oplus N))$

$$\begin{aligned} j_0^A(\cup_{r=1}^{n} c_1((L_r)_t)) &= \cup_{r=1}^{n} c_1(\mathcal{O}_{\mathbf{1}\oplus N}(1) \otimes p^*(L_r|Y)) \\ &= s_{\mathrm{gys}}(1) = s_{\mathrm{gys}}(k_0^A(1)) = j_0^A((I_t)_{\mathrm{gys}}(1)) \end{aligned}$$

proves that $j_0^A(\cup_{r=1}^{n} c_1((L_r)_t)) = j_0^A((I_t)_{\mathrm{gys}}(1))$. Since both elements $\cup_{r=1}^{n} c_1((L_r)_t)$ and $(I_t)_{\mathrm{gys}}(1)$ vanish when restricted to the open subset V_t, the relation (41) follows from Corollary 1.14. The chain of relations in $A(X)$

$$\cup_{r=1}^{n} c_1(L_r) = j_1^A(\cup_{r=1}^{n} c_1((L_r)_t)) = j_1^A((I_t)_{\mathrm{gys}}(1)) = i_{\mathrm{gys}}(k_0^A(1)) = i_{\mathrm{gys}}(1)$$

completes the proof of the relation (40).

The next step in the proof of the composition property is to prove the following relation. Let Y be a smooth variety and let $N \subset M$ be an imbedding of vector bundles on Y such that the quotient $F := M/N$ is a vector bundle as well. Suppose F is of a constant rank, say d. Let $l\colon \mathbf{P}(\mathbf{1} \oplus N) \hookrightarrow \mathbf{P}(\mathbf{1} \oplus M)$ be the obvious closed imbedding and let $p\colon \mathbf{P}(\mathbf{1} \oplus M) \to Y$ be the projection. Then in $A(\mathbf{P}(\mathbf{1} \oplus M))$ one has the relation

$$l_{\mathrm{gys}}(1) = c_d(p^*(F) \otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)). \tag{42}$$

In fact, by the splitting principle [**25**, Lemma 3.24], we may assume that the vector bundle F splits into a direct sum of line bundles $F = \oplus_{r=1}^{m} L_r$. Let F_r be a direct summand of F which is the direct sum of all line bundles L_s except for L_r. Let $M_r \subset M$ be the preimage of F_r under the canonical projection $M \to F$. Let $D_r = \mathbf{P}(\mathbf{1} \oplus M_r)$ be the divisor on $\mathbf{P}(\mathbf{1} \oplus M)$. Clearly the subvariety $\mathbf{P}(\mathbf{1} \oplus N)$ in $\mathbf{P}(\mathbf{1} \oplus M)$ is the transversal intersection of the smooth divisors D_r. Furthermore, the line bundle $L(D_r)$ on $\mathbf{P}(\mathbf{1} \oplus M)$ is isomorphic to the line bundle $p^*(L_r) \otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)$. The relation (40) shows that one has a chain of relations in $A(\mathbf{P}(\mathbf{1} \oplus M))$

$$\begin{aligned} l_{\mathrm{gys}}(1) &= \cup_{r=1}^{d} c_1(L(D_r)) = \cup_{r=1}^{d} c_1(p^*(L_r) \otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)) \\ &= c_d((\oplus_{r=1}^{d} p^*(L_r)) \otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)) = c_d(p^*(F) \otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)). \end{aligned}$$

The relation (42) is proved.

Now we are ready to prove the composition property in one particular case. Let

$s_N\colon Y \to \mathbf{P}(\mathbf{1}\oplus N)$ be the section of the projective bundle $p_N\colon \mathbf{P}(\mathbf{1}\oplus N)\to Y$ identifying Y with the closed subvariety $\mathbf{P}(\mathbf{1})$. Let $s = l\circ s_E\colon Y\to \mathbf{P}(\mathbf{1}\oplus M)$. Clearly s is a section of the projection $p\colon \mathbf{P}(\mathbf{1}\oplus M)\to Y$ identifying Y with the closed subvariety $\mathbf{P}(\mathbf{1})$. Assume that the variety Y is irreducible. We claim that

$$s_{\mathrm{gys}} = l_{\mathrm{gys}}\circ (s_N)_{\mathrm{gys}}. \tag{43}$$

In fact, both sides are two-sided $A(Y)$-module operators. Thus to prove this relation it suffices to prove that $s_{\mathrm{gys}}(1) = l_{\mathrm{gys}}(s_N)_{\mathrm{gys}}(1)$. Since Y is irreducible, all vector bundles on Y are of constant rank. Let $n = rk(N)$. The relations (25) and (42) give rise to a chain of relations in $A(\mathbf{P}(\mathbf{1}\oplus M))$

$$\begin{aligned} l_{\mathrm{gys}}[(s_N)_{\mathrm{gys}}(1)] &= l_{\mathrm{gys}}[c_n(p_N^*(N)\otimes \mathcal{O}_{\mathbf{1}\oplus N}(1))] \\ &= l_{\mathrm{gys}}[l^A(c_n(p^*(N)\otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)))] \\ &= l_{\mathrm{gys}}(1)\cup c_n(p^*(N)\otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)) \\ &= c_d(p^*(F)\otimes \mathcal{O}_{\mathbf{1}\oplus M}(1))\cup c_n(p^*(N)\otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)) \\ &= c_{d+n}(p^*(M)\otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)) = s_{\mathrm{gys}}(1), \end{aligned}$$

which proves the relation (43).

Now we are ready to prove the composition property. For that consider closed imbedding of smooth varieties $j\colon Z\hookrightarrow Y$ and $i\colon Y\hookrightarrow X$. By the additivity property of the Gysin operators (21) one may assume that the variety Z is irreducible. Consider the diagram from 1.2.3 for the closed imbedding $j\circ i\colon Z\hookrightarrow X$:

$$\begin{array}{ccccc} \mathbf{P}(\mathbf{1}\oplus N_{X/Z}) & \xrightarrow{j_0} & X'_t & \xleftarrow{j_1} & X \\ {\scriptstyle s}\uparrow & & {\scriptstyle (I\circ J)_t}\uparrow & & \uparrow{\scriptstyle i\circ j} \\ Z & \xrightarrow{k_0} & Z\times\mathbf{A}^1 & \xleftarrow{k_1} & Z. \end{array}$$

The proper preimage Y'_t of $Y\times\mathbf{A}^1$ under the blow-up map $\sigma\colon X'_t\to X\times\mathbf{A}^1$ and the projective bundle $\mathbf{P}(\mathbf{1}\oplus N_{Y/Z})$ fit in this diagram making a big commutative diagram with transversal squares

$$\begin{array}{ccccc} \mathbf{P}(\mathbf{1}\oplus N_{X/Z}) & \xrightarrow{i_0} & X'_t & \xleftarrow{i_1} & X \\ \uparrow{\scriptstyle l} & & \uparrow{\scriptstyle l_t} & & \uparrow{\scriptstyle i} \\ \mathbf{P}(\mathbf{1}\oplus N_{Y/Z}) & \xrightarrow{i_0^Y} & Y'_t & \xleftarrow{i_1^Y} & Y \\ \uparrow{\scriptstyle s_Y} & & \uparrow{\scriptstyle J_t} & & \uparrow{\scriptstyle j} \\ Z & \xrightarrow{k_0} & Z\times\mathbf{A}^1 & \xleftarrow{k_1} & Z, \end{array}$$

in which the bottom and the middle rows form the deformation to the normal cone diagram for the closed imbedding $j\colon Z\hookrightarrow Y$, and $l\circ s_Y = s$, $l_t\circ J_t = (I\circ J)_t$. Using the transversality of the left-hand side squares in the last diagram, one gets a chain of relations

$$(l_{\mathrm{gys}}\circ (s_Y)_{\mathrm{gys}})\circ k_0^A = l_{\mathrm{gys}}\circ (i_0^Y)^A\circ (J_t)_{\mathrm{gys}} = i_0^A\circ (l_t)_{\mathrm{gys}}\circ (J_t)_{\mathrm{gys}}$$

and

$$s_{\text{gys}} \circ k_0^A = i_0^A \circ ((I \circ J)_t)_{\text{gys}} = i_0^A \circ (l_t \circ J_t)_{\text{gys}}.$$

We already proved the relation

$$l_{\text{gys}} \circ (s_Y)_{\text{gys}} = s_{\text{gys}} \colon A(Z) \to A(\mathbf{P}(\mathbf{1} \oplus N_{X/Z})).$$

Thus one has the relation

$$i_0^A \circ [(l_t \circ J_t)_{\text{gys}} - (l_t)_{\text{gys}} \circ (J_t)_{\text{gys}}] = 0.$$

Observe that for $V_t = X'_t - Z \times \mathbf{A}^1$ and the open inclusion $j_t \colon V_t \hookrightarrow X'_t$ the compositions $j_t^A \circ (l_t \circ J_t)_{\text{gys}}$ and $j_t^A \circ (l_t)_{\text{gys}} \circ (J_t)_{\text{gys}}$ vanish. Now Corollary 1.14 implies the relation

$$(l_t \circ J_t)_{\text{gys}} - (l_t)_{\text{gys}} \circ (J_t)_{\text{gys}} = 0.$$

The transversality of the two right-hand side squares from the last diagram gives rise to the chain of relations

$$\begin{aligned}(i \circ j)_{\text{gys}} \circ k_1^A &= i_1^A \circ (l_t \circ J_t)_{\text{gys}} = i_1^A \circ (l_t)_{\text{gys}} \circ (J_t)_{\text{gys}} \\ &= (i_{\text{gys}} \circ (i_1^Y)^A) \circ (J_t)_{\text{gys}} = i_{\text{gys}} \circ (j_{\text{gys}} \circ k_1^A) \\ &= (i_{\text{gys}} \circ j_{\text{gys}}) \circ k_1^A.\end{aligned}$$

The operator k_1^A is an isomorphism by the homotopy invariance property. The required relation

$$(i \circ j)_{\text{gys}} = i_{\text{gys}} \circ j_{\text{gys}}$$

follows.

2.5. Proofs of properties of Quillen's maps $(p_X)_{\text{quil}}$.

In this subsection we prove the properties of Quillen's operators stated in Subsection 2.3. The projection formula (29) holds by the very definition of the operator $(p_X)_{\text{quil}}$. The base change property 2 is obvious.

To prove property 1 observe that all the maps in the diagram are two-sided $A(X)$-module maps. Thus it suffices to check the relations

$$((p_m)_{\text{quil}} \circ (\tilde{p}_n)_{\text{quil}})(\zeta_1^i \otimes \zeta_2^j) = ((p_n)_{\text{quil}} \circ (\tilde{p}_m)_{\text{quil}})(\zeta_1^i \otimes \zeta_2^j),$$

where $\zeta_1 \in A(X \times \mathbf{P}^n \times \mathbf{P}^m)$ is the pull-back of the element $c_1(\mathcal{O}(1)) \in A(\mathbf{P}^n)$ under the projection $X \times \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^n$ and $\zeta_2 \in A(X \times \mathbf{P}^n \times \mathbf{P}^m)$ is the pull-back of the element $c_1(\mathcal{O}(1)) \in A(\mathbf{P}^m)$ under the projection $X \times \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^m$. Let $f \colon X \to pt$ be the structural morphism. Both sides of the relation coincide with the element $f^A([\mathbf{P}^{n-i}]_\omega \cup [\mathbf{P}^{m-j}]_\omega) \in A(X)$ as follows from the projection formula (29) and the base change property 2. Property 1 is proved.

Now we prove the linear imbedding property 3. It suffices to check the case $n = m - 1$ because of the compatibility of the Gysin homomorphisms with the composition of a closed imbedding (19). Since all of the homomorphisms in the considered diagram are two-sided $A(X)$-module maps, one may assume that $X = pt$. Let

$$\zeta_m = c_1(\mathcal{O}(1)) \in A(\mathbf{P}^m) \quad \text{and} \quad \zeta_n = c_1(\mathcal{O}(1)) \in A(\mathbf{P}^n).$$

The projection formula for the Gysin maps (18) and the relation (24) prove the relations $i_{\rm gys}(\zeta_n^r) = \zeta_m^{r+1}$ in $A(\mathbf{P}^m)$. Now the chain of relations

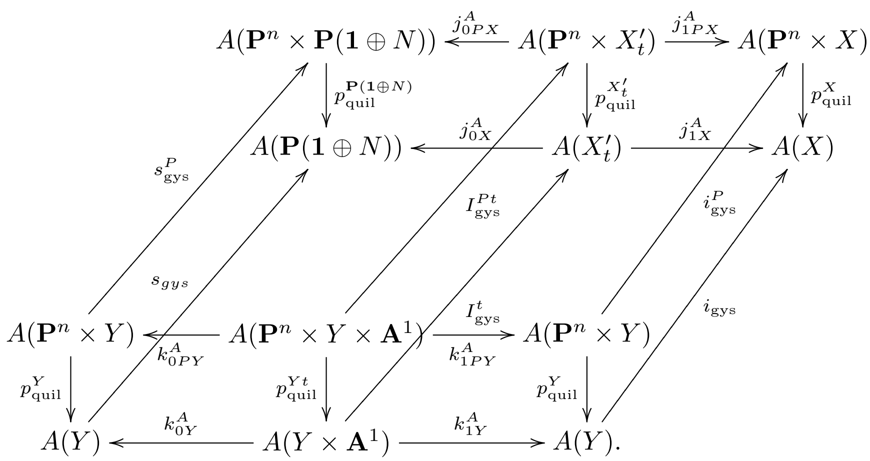
$$(p_m)_{\rm quil}(i_{\rm gys}(\zeta_n^r)) = (p_m)_{\rm quil}(\zeta_m^{r+1}) = [\mathbf{P}^{m-r-1}]_\omega = [\mathbf{P}^{n-r}]_\omega = (p_n)_{\rm quil}(\zeta_n^r)$$

completes the proof of property 3.

Now we prove property 4. Consider the deformation to the normal cone diagram 1.2.3 for the closed imbeddings $i\colon Y \hookrightarrow X$ and $i\colon \mathbf{P}^n \times Y \hookrightarrow \mathbf{P}^n \times X$. Glue them together in the following diagram in which all the vertical arrows are of the form

$$p_{\rm quil}\colon A(\mathbf{P}^n \times S) \to A(S)$$

for various varieties S:



What we need to prove is that the parallelogram on the right-hand side commutes; i.e. we need to prove the relation $i_{\rm gys} \circ p^Y_{\rm quil} = p^X_{\rm quil} \circ i^P_{\rm gys}$. We first prove that the parallelogram on the left-hand side commutes. Then using Corollary 1.14 we check that the parallelogram in the middle commutes. And finally we conclude that the parallelogram on the right-hand side commutes.

We begin with recalling the following notation. For a rank n vector bundle E over a variety S and the projective bundle $p\colon \mathbf{P}(\mathbf{1} \oplus E) \to S$ over S, set

$$\bar{\rm th}(E) = c_n(p^*(E) \otimes \mathcal{O}_E(1)) \in A(\mathbf{P}(\mathbf{1} \oplus E)).$$

To check the commutativity of the left-hand parallelogram, i.e. to check the relation $s_{\rm gys} \circ p^Y_{\rm quil} = p^{\mathbf{P}(\mathbf{1}\oplus N)}_{\rm quil} \circ s^P_{\rm gys}$, recall that $s_{\rm gys} = \cup \bar{\rm th}(N) \circ (p^Y)^A$ with N equal to the normal bundle $N_{X/Y}$ and $s^P_{\rm gys} = \cup \bar{\rm th}(N_{\mathbf{P}^n \times X/\mathbf{P}^n \times Y}) \circ (\mathrm{id}_{\mathbf{P}^n} \times p^Y)^A$. Clearly, $N_{\mathbf{P}^n \times X/\mathbf{P}^n \times Y} = (p^Y)^*(N_{X/Y})$. Let $q\colon \mathbf{P}(\mathbf{1} \oplus N) \to Y$ be the projection. Then for any $a \in A(\mathbf{P}^n \times Y)$ one has a chain of relations (here we write P for $p^{\mathbf{P}(\mathbf{1}\oplus N)}$)

$$\begin{aligned} P_{\rm quil}[(\mathrm{id} \times q)^A(a) \cup (P^A(\bar{\rm th}(N))] &= P_{\rm quil}[((\mathrm{id} \times q)^A(a))] \cup \bar{t}h(N) \\ &= q^A(p^Y_{\rm quil}(a)) \cup \bar{\rm th}(N), \end{aligned}$$

which proves the relation $s_{\rm gys} \circ p^Y_{\rm quil} = P_{\rm quil} \circ s^P_{\rm gys}$.

Recall that $I_t\colon Y \times \mathbf{A}^1 \hookrightarrow X'_t$ and $I^{Pt}\colon \mathbf{P}^n \times Y \times \mathbf{A}^1 \hookrightarrow \mathbf{P}^n \times X'_t$ are the closed imbeddings for the deformation to the normal cone diagram 1.2.3 for the closed imbeddings $Y \hookrightarrow X$ and $\mathbf{P}^n \times Y \hookrightarrow \mathbf{P}^n \times X$ respectively. Now we prove the relation $(I_t)_{\mathrm{gys}} \circ p^{Yt}_{\mathrm{quil}} = p^{X'_t}_{\mathrm{quil}} \circ I^{Pt}_{\mathrm{gys}}$. To do this, we first use the base change property for Gysin and Quillen operators to get a chain of relations

$$\begin{aligned} i_0^A \circ (I_t)_{\mathrm{gys}} \circ p^{Yt}_{\mathrm{quil}} &= s_{\mathrm{gys}} \circ k^A_{0Y} \circ p^{Yt}_{\mathrm{quil}} = s_{\mathrm{gys}} \circ p^{Y}_{\mathrm{quil}} \circ k^A_{0PY} \\ &= P_{\mathrm{quil}} \circ s^P_{\mathrm{gys}} \circ k^A_{0PY} = P_{\mathrm{quil}} \circ j^A_{0PX} \circ I^{Pt}_{\mathrm{gys}} \\ &= i_0^A \circ p^{X'_t}_{\mathrm{quil}} \circ I^{Pt}_{\mathrm{gys}}. \end{aligned}$$

Let $V_t = X'_t - Y \times \mathbf{A}^1_t$ and $j_t \colon V_t \hookrightarrow X'_t$ be the open inclusion. Clearly,

$$j_t^A \circ (I_t)_{\mathrm{gys}} \circ p^{Yt}_{\mathrm{quil}} = 0 \quad \text{and} \quad j_t^A \circ p^{X'_t}_{\mathrm{quil}} \circ I^{Pt}_{\mathrm{gys}} = 0.$$

Now Corollary 1.14 implies the relation $(I_t)_{\mathrm{gys}} \circ p^{Yt}_{\mathrm{quil}} = p^{X'_t}_{\mathrm{quil}} \circ I^{Pt}_{\mathrm{gys}}$.

We are ready to prove the required relation $i_{\mathrm{gys}} \circ p^{Y}_{\mathrm{quil}} = p^{X}_{\mathrm{quil}} \circ i^P_{\mathrm{gys}}$. To do this, consider the closed imbeddings

$$j_{1X} \colon X \hookrightarrow X'_t \quad \text{and} \quad j_{1PX} \colon \mathbf{P}^n \times X \hookrightarrow \mathbf{P}^n \times X'_t$$

from the deformation to the normal cone construction 1.2.3 for the closed imbeddings $Y \hookrightarrow X$ and $\mathbf{P}^n \times Y \hookrightarrow \mathbf{P}^n \times X$, respectively. Using the base change property for Gysin and Quillen operators we get a chain of relations

$$\begin{aligned} i_{\mathrm{gys}} \circ p^{Y}_{\mathrm{quil}} \circ k^A_{1PY} &= i_{\mathrm{gys}} \circ k^A_{1Y} \circ p^{Yt}_{\mathrm{quil}} \\ &= j^A_{1X} \circ (I_t)_{\mathrm{gys}} \circ p^{Yt}_{\mathrm{quil}} = j^A_{1X} \circ p^{X'_t}_{\mathrm{quil}} \circ I^{Pt}_{\mathrm{gys}} \\ &= p^{X}_{\mathrm{quil}} \circ j^A_{1PX} \circ I^{Pt}_{\mathrm{gys}} = p^{X}_{\mathrm{quil}} \circ i^P_{\mathrm{gys}} \circ k^A_{1PY}. \end{aligned}$$

The operator k^A_{1PY} is an isomorphism by the homotopy invariance property. Thus we get the required relation $i_{\mathrm{gys}} \circ p^{Y}_{\mathrm{quil}} = p^{X}_{\mathrm{quil}} \circ i^P_{\mathrm{gys}}$. Property 4 is proved.

The proof of property 5 is postponed to the next subsection.

2.6. Proof of property 5 of Quillen's operator

Let X be a smooth variety, $p \colon X \times \mathbf{P}^n \to X$ be the projection and $s \colon X \to X \times \mathbf{P}^n$ be a section of the projection p. One has to prove the relation

$$p_{\mathrm{quil}} \circ s_{\mathrm{gys}} = \mathrm{id}_{A(X)}. \tag{44}$$

Since both operators s_{gys} and p_{quil} are two-sided $A(X)$-module homomorphisms, it suffices to check the relation in $A(X)$

$$p_{\mathrm{quil}}(s_{\mathrm{gys}}(1)) = 1. \tag{45}$$

Since s is a section of p it has the form (id_X, f) for a morphism $f \colon X \to \mathbf{P}^n$. Consider two diagrams

$$\begin{array}{ccc} \mathbf{P}^n \times \mathbf{P}^n & \xleftarrow{f \times \mathrm{id}} & X \times \mathbf{P}^n \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{P}^n & \xleftarrow{f} & X, \end{array}$$

and

$$\begin{array}{ccc} \mathbf{P}^n \times \mathbf{P}^n & \xleftarrow{f\times \mathrm{id}} & X \times \mathbf{P}^n \\ {\scriptstyle\Delta}\uparrow & & \uparrow{\scriptstyle s} \\ \mathbf{P}^n & \xleftarrow{f} & X, \end{array}$$

where Δ is the diagonal imbedding. The last diagram is Cartesian and transversal. So one has $s_{\mathrm{gys}} \circ f^A = (f \times \mathrm{id})^A \circ \Delta_{\mathrm{gys}}$. In particular, one has the relation

$$s_{\mathrm{gys}}(1) = (f \times \mathrm{id})^A(\Delta_{\mathrm{gys}}(1)). \tag{46}$$

Now this relation, the base change property 3 of trace operators 2.2 and the proposition stated just below give a chain of relations

$$p_{\mathrm{quil}}(s_{\mathrm{gys}}(1)) = p_{\mathrm{quil}}((f \times \mathrm{id})^A(\Delta_{\mathrm{gys}}(1))) = f^A((p_1)_{\mathrm{quil}}(\Delta_{\mathrm{gys}}(1))) = f^A(1) = 1.$$

It remains to prove the following:

Proposition 2.10. *One has the relation* $(p_1)_{\mathrm{quil}}(\Delta_{\mathrm{gys}}(1)) = 1$ *in the group* $A(\mathbf{P}^n)$.

Proof using the residue approach. In this proof we assume that Quillen's operators p_{quil} are defined via (33). Consider a rank n vector bundle Q over $\mathbf{P}^n$ defined via the short exact sequence of vector bundles $0 \to \mathcal{O}(-1) \to 1^{n+1} \to Q \to 0$. To compute $(p_1)_{\mathrm{quil}}(\Delta_{\mathrm{gys}}(1))$ in terms of (33) observe first that

$$\Delta_{\mathrm{gys}}(1) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)) \in A(\mathbf{P}^n \times \mathbf{P}^n). \tag{47}$$

In fact this is just the relation (26) for the cases $Y = \mathbf{P}^n$, $E = 1^{n+1}$, $s = \Delta$, $p = p_1$. To proceed further one needs to find a formal power series $h(t) \in A^{\mathrm{ev}}(\mathbf{P}^n)[[t]]$ such that $h(\zeta) = \Delta_{\mathrm{gys}}(1)$, where as usual $\zeta = c_1(\mathcal{O}(1)) \in A(\mathbf{P}^n)$.

For that consider the two projections

$$q_1 \colon \mathbf{P}^\infty \times \mathbf{P}^n \to \mathbf{P}^\infty \quad \text{and} \quad q_2 \colon \mathbf{P}^\infty \times \mathbf{P}^n \to \mathbf{P}^n,$$

and the projection $p_r \colon \mathbf{P}^n \times \mathbf{P}^n \to \mathbf{P}^n$ for $r = 1, 2$. Set $R = A^{\mathrm{ev}}(\mathbf{P}^n)$ and identify the formal power series in one variable $R[[t]]$ with the ring $A^{\mathrm{ev}}(\mathbf{P}^\infty \times \mathbf{P}^n)$ taking the variable t to the element $c_1(q_1^*(\mathcal{O}(1)))$. This identification induces a ring isomorphism $R[[t]]/(t^{n+1})$ and $A^{\mathrm{ev}}(\mathbf{P}^n \times \mathbf{P}^n)$ which takes the variable t to the element $c_1(p_1^*(\mathcal{O}(1)))$. Now if $i \colon \mathbf{P}^n \hookrightarrow \mathbf{P}^\infty$ is the inclusion then under the mentioned identifications, the ring morphism $(i \times \mathrm{id})^A \colon A^{\mathrm{ev}}(\mathbf{P}^\infty \times \mathbf{P}^n) \to A^{\mathrm{ev}}(\mathbf{P}^n \times \mathbf{P}^n)$ becomes the reduction morphism $R[[t]] \to R[[t]]/(t^{n+1})$. Let F_ω be the formal group law associated with the Chern structure on A and let $\lambda = c_1(\mathcal{O}(1)) \in R$. For brevity write $t -_{F_\omega} \lambda$ for the element $F_\omega(t, I_\omega(\lambda)) \in R[[t]]$. If $R((t))$ is the localization of $R[[t]]$ with respect to the element t, then $R[[t]] \subset R((t))$, the element $t -_{F_\omega} \lambda$ is a unit in $R((t))$ and we claim that

$$c_n(q_1^*(\mathcal{O}(1) \otimes q_2^*(Q))) = t^{n+1}/(t -_{F_\omega} \lambda) \in R[[t]]. \tag{48}$$

This relation shows that the series $h(t) = t^{n+1}/(t -_{F_\omega} \lambda)$ belongs to $R[[t]]$. Furthermore, since $(i \times \mathrm{id})^A(c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q))) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q))$ in $A(\mathbf{P}^\infty \times \mathbf{P}^n)$

the series $h(t)$ satisfies the relation

$$h(\zeta) = \Delta_{\mathrm{gys}}(1) \in A(\mathbf{P}^n \times \mathbf{P}^n). \tag{49}$$

This is the crucial point to finish the proof of the proposition but first we check the relation (48).

For that consider the short exact sequence of vector bundles

$$0 \to \mathcal{O}(-1) \to 1^{n+1} \to Q \to 0$$

over $\mathbf{P}^n$. Then the short exact sequence of vector bundles

$$0 \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(\mathcal{O}(-1)) \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(1^{n+1}) \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q) \to 0$$

on $\mathbf{P}^\infty \times \mathbf{P}^n$ and the Cartan formulae for the total Chern class give rise to the relation in $A(\mathbf{P}^\infty \times \mathbf{P}^n)$

$$c_1(q_1^*(\mathcal{O}(1)) \otimes q_2^*(\mathcal{O}(-1))) \cup c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q)) = c_{n+1}(q_1^*(\mathcal{O}(1)) \otimes q_2^*(1^{n+1})).$$

With the notation fixed above one can rewrite the last relation in terms of the formal group law F_ω and its "inverse element" I_ω (see 1.17) as follows:

$$F_\omega(t, I_\omega(\lambda)) \cup c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q)) = t^{n+1}.$$

Since $F_\omega(t, I_\omega(\lambda)) = t -_{F_\omega} \lambda$, the relation (48) is proved.

Now with the relation (49) in hand and with p_{quil} defined via (33) one computes $(p_1)_{\mathrm{quil}}(\Delta_{\mathrm{gys}}(1)) \in R$ as follows:

$$(p_1)_{\mathrm{quil}}(\Delta_{\mathrm{gys}}(1)) = \mathrm{Res}_{t=0}(\frac{h(t)}{t^{n+1}}\omega_{\mathrm{inv}}) = \mathrm{Res}_{t=0}(\frac{\omega_{\mathrm{inv}}}{t -_{F_A} \lambda}) \in R.$$

So to prove the proposition it remains to prove the following

Lemma 2.11. *Let S be a commutative ring, let $F = F(t_1, t_2) \in S[[t_1, t_2]]$ be a commutative formal group law, and let $\omega \in \Omega_{S[t]/S} \otimes_{S[t]} S[[t]]$ be the unique normalized F-invariant differential form. Let $R = S[u]/(u^{n+1})$ and let $\lambda = \bar{u} \in R$, where $\bar{u}$ is the element u modulo the ideal (u^{n+1}). Then one has the relation in the ring R:*

$$1 = \mathrm{Res}_{t=0}(\frac{\omega_{\mathrm{inv}}}{t -_F \lambda}). \tag{50}$$

The proof of this claim proceeds as follows. If two formal group laws F_1 and F_2 over S are isomorphic and the relation (50) holds for one of them, then it also holds for the other one. It is straightforward to check the relation (50) for the case of the additive formal group law $F = t_1 + t_2$, because in this case the normalized invariant differential form is the form dt and $t -_F \lambda = t - \lambda$. If S contains the field of rational numbers $\mathbb{Q}$, then $F(t_1, t_2)$ is isomorphic to the additive formal group law $t_1 + t_2$. Thus the relation (50) holds in this case for $F(t_1, t_2)$. There exists a universal formal group law F_{un}. It is defined over the Lazar ring L. Since the ring $L_\mathbb{Q} = L \otimes_\mathbb{Z} \mathbb{Q}$ contains $\mathbb{Q}$ the relation (50) holds for the formal group law $F_{\mathrm{un}} \otimes \mathbb{Q}$ over the ring $L_\mathbb{Q}$. Since L is a polynomial ring over $\mathbb{Z}$ it is contained in the ring $L_\mathbb{Q}$. Thus the relation (50) holds for the formal group law F_{un}.

If the relation (50) holds for a formal group law F' over a ring S', then for each ring homomorphism $f\colon S' \to S$ the required relation holds for the scalar extension

$F = F' \otimes_{S'} S$ of the formal group law F' by means of f. Since we already know that the relation holds for the universal formal group law F_{un}, one concludes that it does hold for the given formal group law F over the ring S from the claim. In fact, there exists a ring homomorphism $f\colon L \to S$ such that $F(t_1, t_2) = F_{\text{un}} \otimes_L S$, where S is considered as an L-algebra by means of f.

The claim is proved, which completes the proof of Proposition 2.10. Thus we get a purely algebraic proof of property 5 of Quillen's operators p_{quil}. □

Proof of Proposition 2.10 using the Ω*-approach.* In this proof we assume that Quillen's operators p_{quil} are defined via (28). Let $\Omega(*)$ be complex cobordism theory (on the category of CW-complexes). As above, let $\Omega = \Omega(pt)$ be the coefficient ring of the theory $\Omega(*)$. If E is a complex vector bundle of rank n over a space T, then we let $c_i^{\Omega}(E) \in \Omega^{2i}(T)$ $(i = 1, \dots, n)$ be the Chern classes of E in the sense of Conner-Floyd [**6**]. For a projective morphism $f\colon S \to T$ of smooth algebraic varieties, we let $f_\Omega\colon \Omega(S(\mathbb{C})) \to \Omega(T(\mathbb{C}))$ be the Gysin homomorphism [**34**, 1.2 and 1.4] (if S and T are irreducible of dimensions m and n respectively, then this homomorphism changes the degree by $2(m-n)$). It respects composition: if $g\colon Q \to S$ is a morphism of smooth complex algebraic varieties, then $f_\Omega \circ g_\Omega = (f \circ g)_\Omega$. For the identity map $\text{id}\colon S \to S$, the map id_Ω is the identity. Complex cobordism theory satisfies the projective bundle theorem: $\Omega[t]/(t^{n+1}) = \Omega(\mathbb{C}P^n)$ (here t is identified with $c_1^{\Omega}(\mathcal{O}(1))$).

Let $\bar{A} = A^{\text{ev}}(pt)$ and let $l_\omega\colon \Omega \to \bar{A}$ be the ring homomorphism (27) induced by the orientation ω on A (see 2.3.1). For a smooth variety X we will consider below in this proof the ring $A^{\text{ev}}(X)$ as an Ω-algebra through the homomorphism l_ω.

Let $l_\omega^n\colon \Omega(\mathbb{C}P^n) \to A^{\text{ev}}(\mathbf{P}^n)$ be the only Ω-algebra homomorphism which takes the class $c_1^{\Omega}(\mathcal{O}(1))$ to the class $c_1(\mathcal{O}(1))$. Then the diagram

$$\begin{array}{ccc} \Omega(\mathbb{C}P^n) & \xrightarrow{l_\omega^n} & A^{\text{ev}}(\mathbf{P}^n) \\ {\scriptstyle p_\Omega}\downarrow & & \downarrow{\scriptstyle p_{\text{quil}}} \\ \Omega & \xrightarrow{l_\omega} & \bar{A} \end{array}$$

commutes. In fact, one has a chain of relations (see (28) and [**33**])

$$\begin{aligned} l_\omega(p_\Omega(c_1^{\Omega}(\mathcal{O}(1))^i)) &= l_\omega([\mathbb{C}P^{n-i}]_\Omega) \\ &= [\mathbf{P}^{n-1}]_\omega = p_{\text{quil}}(c_1(\mathcal{O}(1))^i) = p_{\text{quil}}(l_\omega^n(c_1^{\Omega}(\mathcal{O}(1))^i)). \end{aligned}$$

The projective bundle theorem and the projection formulas complete the proof of the commutativity. The projective bundle theorem for complex cobordism shows that the cup-product identifies the Ω-algebras $\Omega(\mathbb{C}P^n) \otimes_\Omega \Omega(\mathbb{C}P^n)$ and $\Omega(\mathbb{C}P^n \times \mathbb{C}P^n)$. The projective bundle theorem for the theory A shows that the cup-product identifies the Ω-algebras $A^{\text{ev}}(\mathbf{P}^n) \otimes_{\bar{A}} A^{\text{ev}}(\mathbf{P}^n)$ and $A^{\text{ev}}(\mathbf{P}^n \times \mathbf{P}^n)$. Now if $p_1\colon \mathbb{C}P^n \times \mathbb{C}P^n \to \mathbb{C}P^n$ is the projection onto the first factor, then the diagram

$$\begin{array}{ccc} \Omega(\mathbb{C}P^n \times \mathbb{C}P^n) & \xrightarrow{l_\omega^{n,n}} & A^{\text{ev}}(\mathbf{P}^n \times \mathbf{P}^n) \\ {\scriptstyle (p_1)_\Omega}\downarrow & & \downarrow{\scriptstyle (p_1)_{\text{quil}}} \\ \Omega(\mathbb{C}P^n) & \xrightarrow{l_\omega^n} & A^{\text{ev}}(\mathbf{P}^n), \end{array}$$

commutes, where $l^{n,n}_\omega = l^n_\omega \otimes l^n_\omega$. This is proved exactly as the commutativity of the previous diagram using the projection formulas in both theories and the projective bundle theorem.

Let $\Delta\colon \mathbb{C}P^n \to \mathbb{C}P^n \times \mathbb{C}P^n$ be the diagonal and let $\Delta\colon \mathbf{P}^n \to \mathbf{P}^n \times \mathbf{P}^n$ be the diagonal as well (the same symbol is used for both imbeddings). We claim that the following relation holds in $A^{\mathrm{ev}}(\mathbf{P}^n \times \mathbf{P}^n)$:

$$l^{n,n}_\omega(\Delta_\Omega(1)) = \Delta_{\mathrm{gys}}(1). \tag{51}$$

Assuming for a moment this relation, one gets a chain of relations

$$1 = l^n_\omega(1) = l^n_\omega((p_1)_\Omega(\Delta_\Omega(1))) = (p_1)_{\mathrm{quil}}(l^{n,n}_\omega(\Delta_\Omega(1))) = (p_1)_{\mathrm{quil}}(\Delta_{\mathrm{gys}}(1))$$

which proves Proposition 2.10. It remains to prove the relation (51).

For that we need the three other relations stated just below. Let Q be a rank n vector bundle over $\mathbf{P}^n$ defined by the short exact sequence

$$0 \to \mathcal{O}(-1) \to 1^{n+1} \to Q \to 0$$

and let $Q_{\mathbb{C}}$ be a rank n algebraic vector bundle over $\mathbb{C}P^n$ defined by the short exact sequence

$$0 \to \mathcal{O}(-1) \to \mathbb{C}^{n+1} \to Q_{\mathbb{C}} \to 0.$$

We claim the following three relations:

$$\Delta_{\mathrm{gys}}(1) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)) \in A^{\mathrm{ev}}(\mathbf{P}^n \times \mathbf{P}^n), \tag{52}$$

$$\Delta_\Omega(1) = c^\Omega_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_{\mathbb{C}})) \in \Omega(\mathbb{C}P^n \times \mathbb{C}P^n), \tag{53}$$

$$l^{n,n}_\omega(c^\Omega_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_{\mathbb{C}}))) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)) \in A^{\mathrm{ev}}(\mathbf{P}^n \times \mathbf{P}^n). \tag{54}$$

Assuming for a minute the last three relations, one gets a chain of relations

$$\Delta_{\mathrm{gys}}(1) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)) = l^{n,n}_\omega(c^\Omega_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_{\mathbb{C}}))) = l^{n,n}_\omega(\Delta_\Omega(1))$$

which proves the relation (51).

It remains to prove relations (52), (53), (54). The relation (52) is a particular case of (26) which is proved in 2.4.7. In fact, take $Y = \mathbf{P}^n$, $E = 1^{n+1}$, $s = \Delta$, $p = p_1$.

Exactly the same argument works for complex cobordism $\Omega(*)$ which proves the relation (53).

To prove the last relation consider the line bundle $\mathcal{O}(1)$ over $\mathbf{P}^\infty$ and consider on $\mathbf{P}^\infty \times \mathbf{P}^n$ the short exact sequence of vector bundles

$$0 \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(\mathcal{O}(-1)) \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(1^{n+1}) \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q) \to 0,$$

where q_1 and q_2 are the projections of $\mathbf{P}^\infty \times \mathbf{P}^n$ onto the first and the second factors, respectively. This exact sequence and the Cartan formula for the total Chern class gives rise to the relation in $A(\mathbf{P}^\infty \times \mathbf{P}^n)$

$$c_1(q_1^*(\mathcal{O}(1)) \otimes q_2^*(\mathcal{O}(-1))) \cup c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q)) = c_{n+1}(q_1^*(\mathcal{O}(1)) \otimes q_2^*(1^{n+1})).$$

Now setting $t_1 = c_1(q_1^*(\mathcal{O}(1)))$ and $t_2 = c_1(q_2^*(\mathcal{O}(1)))$ one can rewrite the last relation in terms of the formal group law F_ω and its "inverse element" I_ω (see 1.17) as follows:

$$F_\omega(t_1, I_\omega(t_2)) \cup c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q)) = t_1^{n+1}. \tag{55}$$

Similarly one gets the following relation in $\Omega(\mathbb{C}P^\infty \times \mathbb{C}P^n)$:

$$F_\Omega(t_1^\Omega, I_\Omega(t_2^\Omega)) \cup c_n^\Omega(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q_\mathbb{C})) = (t_1^\Omega)^{n+1}.$$

Here q_1 and q_2 are the projections of $\mathbb{C}P^\infty \times \mathbb{C}P^n$ onto the first and to the second factors, respectively, and F_Ω and I_Ω are the formal group law and its "inverse element" for complex cobordism 2.3.1. Applying the homomorphism

$$l_\omega^{\infty,n} = l_\omega^\infty \otimes l_\omega^n \colon \Omega(\mathbb{C}P^\infty \times \mathbb{C}P^n) \to A^{\mathrm{ev}}(\mathbf{P}^\infty \times \mathbf{P}^n)$$

to the last relation in $\Omega(\mathbb{C}P^\infty \times \mathbb{C}P^n)$, one gets in the group $A^{\mathrm{ev}}(\mathbf{P}^\infty \times \mathbf{P}^n)$ the relation

$$F_\omega(t_1, I_\omega(t_2)) \cup l_\omega^{\infty,n}(c_n^\Omega(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q_\mathbb{C}))) = t_1^{n+1}. \tag{56}$$

Observe that the element $F_\omega(t_1, I_\omega(t_2)) = t_1 -_{F_\omega} t_2$ is a non-zero-divisor in the ring $A^{\mathrm{ev}}(\mathbf{P}^\infty \times \mathbf{P}^n) = A^{\mathrm{ev}}[t_2][[t_1]]/(t_2^{n+1})$. Thus the relations (55) and (56) prove that

$$l_\omega^{\infty,n}(c_n^\Omega(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q_\mathbb{C}))) = c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q)).$$

Now consider a commutative diagram

$$\begin{array}{ccc} \Omega(\mathbb{C}P^\infty \times \mathbb{C}P^n) & \xrightarrow{l_\omega^{\infty,n}} & A(\mathbf{P}^\infty \times \mathbf{P}^n) \\ {\scriptstyle (i\times \mathrm{id})^\Omega}\downarrow & & \downarrow{\scriptstyle (i\times \mathrm{id})^A} \\ \Omega(\mathbb{C}P^n \times \mathbb{C}P^n) & \xrightarrow{l_\omega^{n,n}} & A(\mathbf{P}^n \times \mathbf{P}^n) \end{array}$$

in which the vertical arrows are the pull-backs induced by the obvious imbedding. Since

$$(i \times \mathrm{id})^\Omega(c_n^\Omega(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q_\mathbb{C}))) = c_n^\Omega(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_\mathbb{C}))$$

and

$$(i \times \mathrm{id})^A(c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q))) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)), \tag{57}$$

the commutativity of the last diagram proves the desired relation (54)

$$l_\omega^{n,n}(c_n^\Omega(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_\mathbb{C}))) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)).$$

Proposition 2.10 is proved.

The proof of property 5 of Quillen's operators p_{quil} is complete. □

2.7. Construction of a trace structure

Let A be a ring cohomology theory endowed with an orientation ω and with the Chern structure $L \mapsto c(L)$ corresponding by [**25**, Th. 3.36] to ω. The main aim of this subsection is to construct a trace structure required by Theorem 2.5. We will use the Gysin operators i_{gys} for closed imbeddings defined by (17) and the Quillen operators p_{quil} for projections defined by (28) or which are equivalently defined by (33).

Let $f \colon Y \to X$ be a projective morphism of smooth varieties. One can present f as a composition of a closed imbedding $i \colon Y \hookrightarrow X \times \mathbf{P}^n$ and the projection

$p\colon X\times\mathbf{P}^n\to X$; i.e. $f=p\circ i$. Define now an operator $\mathrm{tr}_f\colon A(Y)\to A(X)$ by the formula $\mathrm{tr}_f=p_{\mathrm{quil}}\circ i_{\mathrm{gys}}$.

Theorem 2.12. *The operator tr_f does not depend on the particular choice of a decomposition of the morphism f in the form $f=p\circ i$, where $i\colon Y\hookrightarrow X\times\mathbf{P}^n$ is a closed imbedding and $p\colon X\times\mathbf{P}^n\to X$ is the projection.*

Moreover, the assignment $f\mapsto\mathrm{tr}_f$ is a trace structure on A required by Theorem 2.5. Finally, for a closed imbedding $i\colon S\hookrightarrow T$ of smooth varieties, the operator tr_i coincides with the Gysin operator i_{gys}, and for the projection $p\colon T\times\mathbf{P}^n\to T$ the operator tr_p coincides with the Quillen operator p_{quil}.

Proof. For a projective morphism $f\colon Y\to X$ we first check that the map tr_f does not depend on the particular choice of the decomposition of f.

Let $f=p'\circ i'$ be another decomposition of f, where $i'\colon Y\hookrightarrow X\times\mathbf{P}^m$ is a closed imbedding and $p'\colon X\times\mathbf{P}^m\to X$ is the projection. We have to check the relation

$$p_{\mathrm{quil}}\circ i_{\mathrm{gys}}=p'_{\mathrm{quil}}\circ i'_{\mathrm{gys}}. \tag{58}$$

For that consider the commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\;p'\;} & X\times\mathbf{P}^m \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle p_m} \\ Y & \xrightarrow{\;I\;} & X\times\mathbf{P}^n\times\mathbf{P}^m \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_n} \\ X & \xleftarrow{\;p\;} & X\times\mathbf{P}^n, \end{array}$$

where I is the unique imbedding such that $p_m\circ I=i'$ and $p_n\circ I=i$.

Recall that $p_{\mathrm{quil}}\circ p_{n,\mathrm{quil}}=p'_{\mathrm{quil}}\circ p_{m,\mathrm{quil}}$ by property 1 from 2.3.1. Now clearly it suffices to check two relations:

- $p_{m,\mathrm{quil}}\circ I_{\mathrm{gys}}=i'_{\mathrm{gys}}$, and
- $p_{n,\mathrm{quil}}\circ I_{\mathrm{gys}}=i_{\mathrm{gys}}$.

Both of these relations are particular cases of the relation from the following claim.

Lemma 2.13. *Consider a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\;\tilde{j}\;} & T\times\mathbf{P}^k \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\;j\;} & T \end{array}$$

with closed imbeddings j and $\tilde{j}$. One has the following relation: $p_{\mathrm{quil}}\circ\tilde{j}_{\mathrm{gys}}=j_{\mathrm{gys}}$.

Proof of claim. To prove this claim consider the commutative diagram

$$\begin{array}{ccc} Y \times \mathbf{P}^k & \xrightarrow{J} & T \times \mathbf{P}^k \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{j} & T \end{array}$$

with $J = j \times \mathrm{id}$, and consider a section $s\colon Y \to Y \times \mathbf{P}^k$ of the projection q such that $J \circ s = \tilde{j}$.

The relations $\tilde{j}_{\mathrm{gys}} = J_{\mathrm{gys}} \circ s_{\mathrm{gys}}$ (see (19)), $p_{\mathrm{quil}} \circ J_{\mathrm{gys}} = j_{\mathrm{gys}} \circ q_{\mathrm{gys}}$ (see property 4 of 2.3.1) and $q_{\mathrm{quil}} \circ s_{\mathrm{gys}} = \mathrm{id}_{A(Y)}$ give a chain of relations

$$\begin{aligned} p_{\mathrm{quil}} \circ \tilde{j}_{\mathrm{gys}} &= p_{\mathrm{quil}} \circ (J_{\mathrm{gys}} \circ s_{\mathrm{gys}}) = (p_{\mathrm{quil}} \circ J_{\mathrm{gys}}) \circ s_{\mathrm{gys}} \\ &= (j_{\mathrm{gys}} \circ q_{\mathrm{quil}}) \circ s_{\mathrm{gys}} = j_{\mathrm{gys}} \circ (q_{\mathrm{quil}} \circ s_{\mathrm{gys}}) = j_{\mathrm{gys}}. \end{aligned}$$

The claim is proved which completes the proof of the relation (58). Thus the operator tr_f is well-defined. □

The next goal is to prove the relation $\mathrm{tr}_f \circ \mathrm{tr}_g = \mathrm{tr}_{f \circ g}$. For that consider the commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{j} & Y \times \mathbf{P}^m & \xrightarrow{I} & X \times \mathbf{P}^n \times \mathbf{P}^m \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle p_m} & & \downarrow{\scriptstyle q_n} \\ & & Y & \xrightarrow{i} & X \times \mathbf{P}^n \\ & & & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p_n} \\ & & & & X \end{array}$$

with a smooth variety Z, a closed imbedding j, a closed imbedding i with $I = i \times \mathrm{id}$ and the projection $q_n\colon X \times \mathbf{P}^n \times \mathbf{P}^m \to X \times \mathbf{P}^n$.

By the very definition one has relations $\mathrm{tr}_f = p_{n,\mathrm{quil}} \circ i_{\mathrm{gys}}$ and $\mathrm{tr}_g = p_{m,\mathrm{quil}} \circ j_{\mathrm{gys}}$. Thus one has a chain of relations

$$\begin{aligned} \mathrm{tr}_f \circ \mathrm{tr}_g &= p_{n,\mathrm{quil}} \circ (i_{\mathrm{gys}} \circ p_{m,\mathrm{quil}}) \circ j_{\mathrm{gys}} \\ &= p_{n,\mathrm{quil}} \circ (q_{n,\mathrm{quil}} \circ I_{\mathrm{gys}}) \circ j_{\mathrm{gys}} \\ &= (p_{n,quil} \circ q_{n,\mathrm{quil}}) \circ (I \circ j)_{\mathrm{gys}}. \end{aligned}$$

Now consider a closed imbedding $k\colon \mathbf{P}^n \times \mathbf{P}^m \hookrightarrow \mathbf{P}^N$, the projection $p_N\colon X \times \mathbf{P}^N \to X$ and the commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{I \circ j} & X \times \mathbf{P}^n \times \mathbf{P}^m & \xrightarrow{\mathrm{id} \times k} & X \times \mathbf{P}^N. \\ & {\scriptstyle f \circ g}\searrow & \downarrow{\scriptstyle p_n \circ q_n} & \swarrow{\scriptstyle p_N} & \\ & & X & & \end{array}$$

Since

$$\mathrm{tr}_{f \circ g} = p_{N,\mathrm{quil}} \circ ((\mathrm{id} \times k) \circ I \circ j)_{\mathrm{gys}} = p_{N,\mathrm{quil}} \circ (\mathrm{id} \times k)_{\mathrm{gys}} \circ (I \circ j)_{\mathrm{gys}},$$

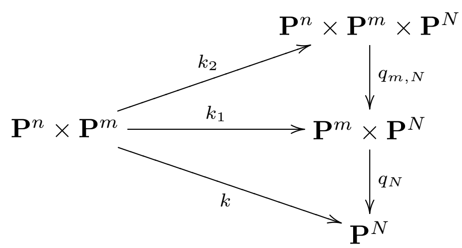
it remains to prove

Lemma 2.14. *The following relation holds:*

$$p_{N,quil} \circ (id \times k)_{gys} = p_{n,quil} \circ q_{n,quil}. \tag{59}$$

Proof of claim. First observe that each of the homomorphisms $p_{N,\mathrm{quil}}$, $(\mathrm{id} \times k)_{\mathrm{gys}}$, $p_{n,\mathrm{quil}}$ and $p_{m,n,\mathrm{quil}}$ is $A(X)$-linear. Thus by the projective bundle theorem it suffices to prove the claim in the case $X = pt$.

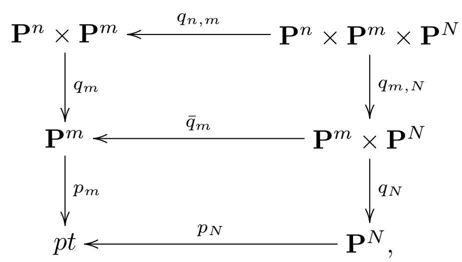
First consider the projections $q_n \colon \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^n$ and $q_m \colon \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^m$ and consider the commutative diagram



with the obvious projections $q_{m,N}$, q_N and with the closed imbeddings $k_1 = (q_m, k)$ and $k_2 = (\mathrm{id}, k)$. Claim 2.13 gives a chain of relations

$$k_{\mathrm{gys}} = q_{N,\mathrm{quil}} \circ k_{1,\mathrm{gys}} = q_{N,\mathrm{quil}} \circ q_{m,N,\mathrm{quil}} \circ k_{2,\mathrm{gys}}. \tag{60}$$

Now consider one more commutative diagram



where $\bar{q}_m$ is the projection. It gives rise to the chain of relations

$$\begin{aligned} p_{N,\mathrm{quil}} \circ q_{N,\mathrm{quil}} \circ q_{m,N,\mathrm{quil}} &= p_{m,\mathrm{quil}} \circ \bar{q}_{m,\mathrm{quil}} \circ q_{m,N,\mathrm{quil}} \\ &= p_{m,\mathrm{quil}} \circ q_{m,\mathrm{quil}} \circ q_{n,m,\mathrm{quil}} \end{aligned} \tag{61}$$

Thus one gets the following relation

$$p_{N,\mathrm{quil}} \circ k_{\mathrm{gys}} = p_{m,\mathrm{quil}} \circ q_{m,\mathrm{quil}} \circ (q_{n,m,\mathrm{quil}} \circ k_{2,\mathrm{gys}}). \tag{62}$$

To complete the proof of the claim, it remains to check the relation

$$\mathrm{id} = q_{n,m,\mathrm{quil}} \circ k_{2,\mathrm{gys}}$$

because $p_{m,\mathrm{quil}} \circ q_{m,\mathrm{quil}} = p_{n,\mathrm{quil}} \circ q_{n,\mathrm{quil}}$ by property 1 from 2.3.1.

To prove the relation $\mathrm{id} = q_{n,m,\mathrm{quil}} \circ k_{2,\mathrm{gys}}$ observe that the map k_2 is a section of the projection $q_{n,m}$. Therefore this relation is just a particular case of property 5 from 2.3.1.

The claim is proved. Thus we checked the relation $\mathrm{tr}_{f \circ g} = \mathrm{tr}_f \circ \mathrm{tr}_g$. The relation $\mathrm{tr}_i = i_{\mathrm{gys}}$ holds for a closed imbedding $i \colon Y \hookrightarrow X$ by the very construction of the map

tr_i. The relation $\mathrm{tr}_p = p_{\mathrm{quil}}$ holds as well by the very construction of the map tr_p for the projection $p\colon X \times \mathbf{P}^n \to X$.

To complete the proof of the theorem, it remains to check properties 2–5 of the trace structure (see Definition 2.2) and to prove the relation $\mathrm{tr}_i(1) = c(L(D))$ for a smooth divisor $i\colon D \hookrightarrow X$. Property 2 coincides with property 2 from Subsection 2.2. Property 3 coincides with property 2 of the maps p_{quil} from 2.3.1. Property 4 is obvious. Property 5 is obvious as well. In fact, the localization sequence

$$A_Y(X) \xrightarrow{k^A} A(X) \xrightarrow{j^A} A(X - Y)$$

is exact (k^A is the support extension operator), $i_{\mathrm{gys}} = k^A \circ i_{th}$ and the operator $i_{th}\colon A(Y) \to A_Y(X)$ defined by (16) is an isomorphism. The relation $\mathrm{tr}_i(1) = c(L(D))$ in $A(X)$ is exactly the relation (24) for the Gysin map i_{gys} which is already proved in Subsection 2.4. □

2.8. Proof of Assertions 1 and 2 of Theorem 2.5

Let A be a ring cohomology theory. Let ω be an orientation of the theory A. A trace structure respecting the orientation ω is constructed in Theorem 2.12. The uniqueness of a trace structure respecting the orientation ω is proved in Subsection 2.1. Assertion 1 of Theorem 2.5 is proved.

Now we prove the third assertion of the theorem. Let ω be the orientation of A from item 1 of the theorem. Let $f \mapsto \mathrm{tr}_f$ be the trace structure on A from the same item 1. Let ω' be the orientation of A corresponding to the trace structure by item 2 of the theorem. The trace structure $f \mapsto \mathrm{tr}_f$ respects the orientation ω and the orientation ω'. Thus $\omega = \omega'$ by item 2 of the theorem.

Now let $f \mapsto \mathrm{tr}_f$ be a trace structure on A, let ω be the orientation of A corresponding to the trace structure by item (2) of the theorem, and let $f \mapsto \mathrm{tr}^{\omega}_f$ be the trace structure respecting the orientation ω as is described in item 1 of the theorem. Both trace structures $f \mapsto \mathrm{tr}_f$ and $f \mapsto \mathrm{tr}^{\omega}_f$ respect the orientation ω. Hence they coincide by item 1 of the theorem.

This completes the proof of Theorem 2.5. □

2.9. Examples of trace structures

Here we consider oriented ring cohomology theories from 1.3. For each such theory A we identify the trace structure on A given by Theorem 2.5 with a well-known trace structure on A. In the very end of the subsection we consider motivic cohomology, semi-topological complex and real K-theories [**9**], and algebraic cobordism.

2.9.1.
Algebraic K-theory can also be made to satisfy Definition 1.1. To do this, we use, for instance, K-groups with support, $K_n(X \text{ on } Z)$ $(n \geqslant 0)$, of [**41**]. So set $A^n(X,U) = K_{-n}(X \text{ on } Z)$, where $Z = X - U$. Further, set $A(X,U) = \oplus_{n=0}^{\infty} A^n(X,U)$. The definition of ∂ and the exactness of the localization sequence are contained in [**41**, Th. 5.1] (except for the surjectivity of the restriction $A^0(X) \to A^0(U)$). If X is quasi-projective, then $K(X \text{ on } X)$ coincides with Quillen's K-groups $K^Q_n(X)$, by [**41**, 3.9, 3.10]. This proves in particular the homotopy invariance of $A^n(X)$ for smooth X. The excision property for A follows from [**41**, 3.19]. It remains now to check the

surjectivity of the restriction $A^0(X) \to A^0(U)$. Clearly, $A^0(X) = K_0^Q(X)$ coincides with the Grothendieck group of vector bundles on X. Since X is smooth, the desired surjectivity follows from [**5**, §8, Prop. 7]. Thus (A, ∂) satisfies Definition 1.1.

Set $A_Z(X) = A_Z^{\mathrm{ev}}(X) \oplus A_Z^{\mathrm{odd}}(X)$, where $A_Z^{\mathrm{ev}}(X) = \oplus_{p=0}^{+\infty} K^{2p}(X \text{ on } Z)$, $A_Z^{\mathrm{odd}}(X) = \oplus_{p=0}^{+\infty} K^{2p+1}(X \text{ on } Z)$. The idea of how to construct pairings

$$A_Z(X) \times A_S(X) \to A_{Z\cap S}(X)$$

based on the tensor product of perfect complexes is described in [**41**, 3.15, (3.15.4)]. The fact that the theory A is indeed a ring cohomology theory is checked in [**38**, Th. 2.1.1].

The rule $L \to c(L) = [1] - [L^\vee]$ endows A with a Chern structure and thus orients A (Property 2 of Definition 1.12 follows from [**34**, §8, Th. 2.1]). Let $f \mapsto \mathrm{tr}_f$ be the corresponding trace structure on A by Theorem 2.5.

The push-forwards $f_*\colon K_*(Y) \to K_*(X)$ from [**41**, 3.16.4] form a trace structure on A because they satisfy conditions 1–5 of Definition 2.2. In fact, f_* is a two-sided $K_*(X)$-module map by [**41**, Prop. 3.17]. Condition 1 is satisfied by [**41**, 3.16.4]. Condition 2 is satisfied by [**41**, 3.18] because, for the transversal square Definition 2.1, the morphisms φ and i are Tor-independent over X. Condition 3 is also satisfied by [**41**, 3.18] because the projection p_X is flat. Clearly condition 4 is satisfied. Finally, condition 5 is satisfied by [**41**, 5.1].

The K_0-groups are just the Grothendieck K-groups of coherent sheaves [**41**, 3.9, 3.10]. The restrictions of the push-forwards f_* to the K_0-groups coincide with the one defined by Grothendieck [**5**] via the higher direct images of coherent sheaves. Thus, for a smooth divisor $i\colon D \hookrightarrow X$, one has (following notation 1.1)

$$i_*(1) = [i_*(\mathcal{O}_D)] = [\mathcal{O}_X] - [\mathcal{O}_X(-D)] = [\mathbf{1}] - [L(D)^\vee] = c(L(D)) \in K_0(X).$$

Now Theorem 2.5 shows that the trace structure given by $f \mapsto f_*$ coincides with the one given by $f \mapsto \mathrm{tr}_f$.

2.9.2.
Let m be an integer prime to $\mathrm{char}(k)$. Let A be the étale cohomology theory $A_Z^*(X) = \oplus_{p=0}^{+\infty} A_Z^p(X)$, where $A_Z^p(X) = \oplus_{q=-\infty}^{+\infty} H_Z^p(X, \mu_m^{\otimes q})$. Set $A_Z^{\mathrm{ev}}(X) = \oplus_{p=0}^{+\infty} A_Z^{2p}(X)$ and $A_Z^{\mathrm{odd}}(X) = \oplus_{p=0}^{+\infty} A_Z^{2p+1}(X)$. The cup-products described in [**22**, Ch. V, §1, 1.17] make A a ring cohomology theory. Consider the short exact sequence of the étale sheaves $0 \to \mu_m \to \mathbb{G} \xrightarrow{\times m} \mathbb{G} \to 0$ and denote by $\partial\colon H^1(X, \mathbb{G}_m) \to H^2(X, \mu_m)$ the boundary map. For a line bundle L over a smooth variety X, let $[L] \in H^1(X, \mathbb{G}_m)$ be its isomorphism class. It is known [**22**] that the rule $L \mapsto c(L) := \partial([L])$ endows A with a Chern structure and thus orients A. Let $f \mapsto \mathrm{tr}_f$ be the trace structure on A given by Theorem 2.5.

Now we will describe a well-known trace structure on A and show that it coincides with $f \mapsto \mathrm{tr}_f$. For that take a projective morphism $f\colon Y \to X$ of smooth varieties. Let D_Y (respectively D_X) be the derived category of bounded above complexes of étale sheaves on Y (respectively on X). Let $Rf_*\colon D_Y \to D_X$ be the total derived functor of the direct image functor f_*. Since f is projective there exists a right adjoint $Rf^!\colon D_X \to D_Y$ to Rf_* constructed in [**8**, §3.1]. Let $\alpha\colon \mathbb{Z} \to Rf_*(\mathbb{Z})$ be the map adjoint to the identity map $\mathbb{Z} \to f^*(\mathbb{Z}) = \mathbb{Z}$. Assuming that Y and X are irreducible, set $d = \dim(X) - \dim(Y)$ and consider the map

$$f_!\colon H^p(Y, \mu_m^{\otimes q}) \to H^{p+2d}(X, \mu_m^{\otimes q+d})$$

defined by the composite

$$\begin{aligned} H^p(Y, \mu_m^{\otimes q}) &= \mathrm{Hom}_{D_Y}(\mathbb{Z}, \mu_m^{\otimes q}[p]) \\ &\cong \mathrm{Hom}_{D_Y}(\mathbb{Z}, Rf^!(\mu_m^{\otimes q+d}[p+2d])) \\ &\to \mathrm{Hom}_{D_X}(Rf_*(\mathbb{Z}), \mu_m^{\otimes q+d}[p+2d]) \\ &\xrightarrow{\alpha^*} \mathrm{Hom}_{D_X}(\mathbb{Z}, \mu_m^{\otimes q+d}[p+2d]) \\ &= H^{p+2d}(X, \mu_m^{\otimes q+d}). \end{aligned}$$

If Y and X are reducible, then define the map $f_!\colon A(Y) \to A(X)$ by components. Using [**8**] one can show that the assignment $f \mapsto f_!$ is a trace structure on X. It is known [**8**] that for a smooth divisor $i\colon D \hookrightarrow X$ one has $i_!(1) = c(L(D))$ in $H^2(X, \mu_m)$. Now Theorem 2.5 shows that the trace structure given by $f \mapsto f_!$ coincides with the one given by $f \mapsto \mathrm{tr}_f$.

2.9.3.
Let A be K-cohomology [**34**, §7, 5.8] $A_Z(X) = \oplus_{q=0}^{\infty} \oplus_{p=0}^{\infty} H_Z^p(X, \mathcal{K}_q)$, where $\mathcal{K}$ is the sheaf of K-groups. To get the group $A_Z^{\mathrm{ev}}(X)$ (resp. $A_Z^{\mathrm{odd}}(X)$) take all of the summands with even (resp. odd) $p+q$; thus $A_Z(X) = A_Z^{\mathrm{ev}}(X) \oplus A_Z^{\mathrm{odd}}(X)$. Following [**14**, Def. 8.1], consider a product structure on A induced by the pairing of sheaves $\mathcal{K}_p \otimes \mathcal{K}_q \to \mathcal{K}_{p+q}$. This product structure makes A a ring cohomology theory by [**14**, Def. 8.1]. For a line bundle L over a smooth variety X let $[L] \in H^1(X, \mathcal{K}_1) = H^1(X, \mathcal{O}^*)$ be the isomorphism class of L. The rule $L \mapsto [L]$ endows A with a Chern structure [**14**, Th. 8.10] and thus orients A. Let $f \mapsto \mathrm{tr}_f$ be the trace structure on A given by Theorem 2.5. For a projective morphism $f\colon Y \to X$ of smooth irreducible varieties with $d = \dim(X) - \dim(Y)$ the trace map $\mathrm{tr}_f\colon A(Y) \to A(X)$ shifts the degree of the K-cohomology as follows:

$$H^p(Y, \mathcal{K}_q) \to H^{p+d}(X, \mathcal{K}_{q+d}).$$

The push-forwards $f_*\colon H^*(Y, \mathcal{K}_*) \to H^*(X, \mathcal{K}_*)$ from [**14**, Thms. 7.18, 7.22] form a trace structure on A because they satisfy conditions 1–5 of Definition 2.2. The groups $H^i(X, \mathcal{K}_i)$ are just the Chow groups $\mathrm{CH}^i(X)$ by a theorem of Quillen [**34**, §7]. Moreover, it is proved in [**14**, §§ 7, 8] that under this identification for a projective morphism $f\colon Y \to X$ of smooth varieties, the push-forward

$$f_*\colon \oplus H^i(Y, \mathcal{K}_i) \to \oplus H^i(X, \mathcal{K}_i)$$

coincides with the classical push-forward $f_*\colon \oplus \mathrm{CH}^i(Y) \to \oplus \mathrm{CH}^i(X)$ (see, for instance, [**10**]). Thus for a smooth divisor $i\colon D \hookrightarrow X$ one has

$$i_*(1) = [L(D)] = c(L(D)) \in H^1(X, \mathcal{K}_1) = \mathrm{CH}^1(X).$$

Now Theorem 2.5 shows that the trace structure given by $f \mapsto f_*$ coincides with the one given by $f \mapsto \mathrm{tr}_f$.

2.9.4.
Let $k = \mathbb{R}$, and let $A = A^{\mathrm{ev}} \oplus A^{\mathrm{odd}}$ with

$$A^{\mathrm{ev}}(X,U) = \oplus_0^\infty H^p(X(\mathbb{R}), U(\mathbb{R}); \mathbb{Z}/2) \quad \text{and} \quad A^{\mathrm{odd}}(X,U) = 0.$$

Take as a boundary ∂ the usual boundary map for the pair $(X(\mathbb{R}), U(\mathbb{R}))$. Clearly ∂ is grade-preserving with respect to the grading we chose on A. Now the cup-product makes A a $\mathbb{Z}/2$-graded-commutative ring theory.

For a line bundle L, consider the real line bundle $L(\mathbb{R})$ over the topological space $X(\mathbb{R})$ and set $c_1(L) = w_1(L(\mathbb{R})) \in H^1(X(\mathbb{R}); \mathbb{Z}/2\mathbb{Z}) \subset A^{\mathrm{ev}}(X)$ (the first Stiefel-Whitney class). Since $\mathbf{P}^n(\mathbb{R}) = \mathbb{R}P^n$ is real projective space, the rule $L \mapsto c_1(L)$ endows A with a Chern structure and thus orients A.

2.9.5.
Let A be motivic cohomology [**39**]: $A^p_Z(X) = \oplus_{q=0}^\infty H^p_Z(X, \mathbb{Z}(q))$. To get a $\mathbb{Z}/2$-grading, set

$$A^{\mathrm{ev}}_Z(X) = \oplus_{-\infty}^{+\infty} A^{2p}_Z(X) \quad \text{and} \quad A^{\mathrm{ev}}_Z(X) = \oplus_{-\infty}^{+\infty} A^{2p+1}_Z(X).$$

Recall that $H^2_{\mathcal{M}}(X, \mathbb{Z}(1)) = \mathrm{CH}^1(X)$ for a smooth X [**39**]. For a line bundle L over a smooth variety X let $D(L) \in \mathrm{CH}^1(X)$ be the associated class divisor. The rule $L \mapsto D(L)$ endows A with a Chern structure. In fact, conditions 1 and 3 of 1.12 are satisfied. To check condition 2 consider the homomorphism from item 2 of 1.12

$$(1,\xi)\colon H^*(X, \mathbb{Z}(q)) \oplus H^{*-2}(X, \mathbb{Z}(q-1)) \to H^*(X \times \mathbf{P}^1, \mathbb{Z}(q)) \tag{63}$$

for the motivic cohomology case. It is known that the map (63) is an isomorphism (see [**39**, Cor. 4.12.1] for the characteristic zero case and [**45**, 4.10] for the general case). Thus A is oriented and equipped with a trace structure $f \mapsto \mathrm{tr}_f$ given by Theorem 2.5.

For a projective morphism $f\colon Y \to X$ of smooth irreducible varieties with $d = \dim(X) - \dim(Y)$, the trace map $\mathrm{tr}_f\colon A(Y) \to A(X)$ shifts the twist and the degree of the motivic cohomology as follows:

$$H^p(Y, \mathbb{Z}(q)) \to H^{p+2d}(X, \mathbb{Z}(q+d)).$$

For a projective space $p\colon \mathbf{P}^n \to pt$ and the class

$$\zeta = c(\mathcal{O}(1)) \in H^2(\mathbf{P}^n, \mathbb{Z}(1)) = \mathrm{Pic}(\mathbf{P}^n),$$

one has $\mathrm{tr}_p(\zeta^i) = 1$ for $i = n$ and $\mathrm{tr}_p(\zeta^i) = 0$ otherwise.

For a closed imbedding $i\colon Y \to X$ there is a Gysin map $i_*\colon A(Y) \to A(X)$ defined in [**39**, 4.9]. Clearly the maps i_* and tr_i should coincide. It would be nice to check this.

2.9.6. Semi-topological complex and real K-theories [**9**]
If the ground field k is the field $\mathbb{R}$ of reals, then the semi-topological K-theory of real algebraic varieties $K\mathbb{R}^{\mathrm{semi}}$ defined in [**9**] is an oriented theory as is proved in [**9**]. Thus it is equipped with the corresponding trace structure. For a real variety X it interpolates between the algebraic K-theory of X and Atiyah's Real K-theory of the associated Real space of complex points, $X(\mathbb{C})$.

2.9.7. Orienting algebraic cobordism theory
In this example the notation in [**28**, §2.1] is used. In particular, T is the motivic pointed space $\mathbf{A}^1/(\mathbf{A}^1-\{0\})$. The symmetric ring T-spectrum $\mathbb{MGL}$ as in [**28**, §2.1] determines a ring cohomology theory. One can check that for any elements $a\in \mathbb{MGL}^{p,r}_Z(X)$ and $b\in\mathbb{MGL}^{q,s}_Z(X)$, one has $a\cup b=(-1)^{pq}b\cup a$. So setting $A^{\mathrm{ev}}_Z(X)=\oplus_{p,r}\mathbb{MGL}^{2p,r}_Z(X)$ and $A^{\mathrm{odd}}_Z(X)=\oplus_{p,r}\mathbb{MGL}^{2p+1,r}_Z(X)$, we get a $\mathbb{Z}/2$-graded-commutative ring theory

$$(X,X-Z)\mapsto A^{\mathrm{ev}}_Z(X)\oplus A^{\mathrm{odd}}_Z(X).$$

For an arbitrary symmetric T-spectrum $E=(E_0,E_1,\ldots)$ there is a canonical element in $E^{2n,n}(E_n)$ represented by the map

$$(*,\ldots,*,E_n,E_n\wedge T,\ldots)\to(E_0,E_1,\ldots,E_n,\ldots)$$

of T-spectra. Taking $E=\mathbb{MGL}$ we get in that way an element $[u_1]\in\mathbb{MGL}^{2,1}(\mathbb{MGL}_1)$ (see [**28**, Ex. 1.4]). By the very definition, $\mathbb{MGL}_1=\mathrm{Th}(\mathcal{T}(1))$ and $\mathcal{T}(1)$ is the tautological line bundle $\mathcal{O}(-1)$ over the space $\mathbf{Gr}(1)=\mathbf{P}^\infty$. Now set

$$\mathrm{th}=[u_1]\in\mathbb{MGL}^{2,1}(\mathbb{MGL}_1)=\mathbb{MGL}^{2,1}_{\mathbf{P}^\infty}(\mathcal{O}(-1)).$$

Consider the fiber $\mathbb{A}^1$ of $\mathcal{T}(1)$ over the point $g_1\in\mathbf{Gr}(1)$ as in [**28**, §2.1] and the Thom space $\mathbb{A}^1/(\mathbb{A}^1-\{0\})$ of that fibre. The restriction of the element th to this Thom space $\mathbb{A}^1/(\mathbb{A}^1-\{0\})=\mathrm{Th}(\mathbb{A}^1)$ coincides with the T-suspension $\sigma\in\mathbb{MGL}^{2,1}(\mathrm{Th}(\mathbb{A}^1))=\mathbb{MGL}^{2,1}_{\{0\}}(\mathbb{A}^1)$ of the unit $1\in\mathbb{MGL}^{0,0}(pt)$. Thus the element th is a Thom element in the sense of [**30**, Def. 3.5.1].

The inclusion $i\colon\mathbf{A}^1\hookrightarrow\mathbf{P}^1$ induces an isomorphism $\mathbb{MGL}^{2,1}_{\{0\}}(\mathbf{P}^1)\xrightarrow{i^*}\mathbb{MGL}^{2,1}_{\{0\}}(\mathbf{A}^1)$. Let $\bar\sigma\in\mathbb{MGL}^{2,1}_{\{0\}}(\mathbf{P}^1)$ be such that $i^*(\bar\sigma)=\sigma\in\mathbb{MGL}^{2,1}_{\{0\}}(\mathbb{A}^1)=\mathbb{MGL}^{2,1}(\mathrm{Th}(\mathbb{A}^1))$. Let $z\colon\mathbf{P}^\infty\to\mathcal{O}(-1)$ be the zero section and let $c=z^{\mathbb{MGL}}(\mathrm{th})\in\mathbb{MGL}^{2,1}(\mathbf{P}^\infty)$. As is proved in [**31**, Prop. 6.5.1], the element c satisfies the relation

$$c|_{\mathbf{P}^1}=-\bar\sigma\in\mathbb{MGL}^{2,1}(\mathbf{P}^1),$$

so c is a Chern element in the sense of [**30**, Def. 3.5.1]. The element c gives rise to a unique Chern structure $L\mapsto c(L)$ on $\mathbb{MGL}$ such that $c(\mathcal{O}(-1))=c\in\mathbb{MGL}^{2,1}(\mathbf{P}^\infty)$ (see [**30**, Th. 3.5.2] for the statement and [**28**, §1.3.7] or [**37**, Th. 3] for the proof). This Chern structure defines by [**25**, Th. 3.35] the orientation ω of the algebraic cobordism theory $\mathbb{MGL}$. Let $f\mapsto\mathrm{tr}^\omega_f$ be the corresponding trace structure by Theorem 2.5. Since the Chern class has bidegree $(2,1)$, the trace operator shifts bidegrees. Namely, if $f\colon Y\to X$ is a projective morphism of smooth irreducible varieties and $d=\dim X-\dim Y$, then one has

$$\mathrm{tr}^\omega_f\colon\mathbb{MGL}^{p,q}(Y)\to\mathbb{MGL}^{p+2d,q+d}(X).$$

For the chosen orientation ω, one has the relation

$$\mathrm{th}=\mathrm{th}^\omega(\mathcal{O}(-1))\in\mathbb{MGL}^{2,1}_{\mathbf{P}^\infty}(\mathcal{O}(-1)).$$

2.9.8. Relation to the Levine-Morel cobordism Ω^{LM}
Let $\Omega^{\mathrm{LM}}:=\oplus_{p=-\infty}^{\infty}\Omega^p$ be the Levine-Morel cobordism functor [**19**]. For an oriented ring cohomology theory (A,ω) the trace structure $f\mapsto\mathrm{tr}^\omega_f$ on A determines a structure of an oriented Borel-Moore theory on $A|_{\mathcal{S}m}$ in the sense of [**19**]. So there is a

unique morphism of oriented Borel-Moore theories

$$\varphi_\omega\colon \Omega^{\mathrm{LM}} \to A|_{\mathcal{S}m}$$

taking an element $[Y \xrightarrow{f} X] \in \Omega^{\mathrm{LM}}(X)$ to $\mathrm{tr}^\omega_f(1) \in A(X)$. This morphism respects the push-forwards. This morphism respects the pull-backs provided that $\mathrm{char}(k) = 0$. In fact, this follows from the transversality lemma [**20**, Prop. 3.3.1] and properties 2 and 3 of a trace structure (Definition 2.2). So, in the case that $\mathrm{char}(k) = 0$, the morphism φ_ω is a ring morphism of the oriented cohomology pretheories (in the sense of [**27**, Definition 1.1.7]). Applying this observation to the algebraic cobordism theory of Voevodsky oriented as in 2.9.7, we get an oriented ring morphism

$$\varphi\colon \Omega^{\mathrm{LM}} \to \oplus_p\ \mathbb{MGL}^{2p,p}|_{\mathcal{S}m}.$$

It is proved in a recent preprint of M. Levine that φ is an isomorphism under the assumption that $\mathrm{char}(k) = 0$.

2.10. All possible trace structures on an orientable theory A

In this section we consider an orientable cohomology theory A and identify the set of all trace structures on it with the set of all local parameters of the ring $A^{\mathrm{ev}}(\mathbf{P}^\infty)$ (Theorem 2.15). One should consider this theorem as *the main result* of the article. To state this theorem recall that an element $\pi \in A^{\mathrm{ev}}(\mathbf{P}^\infty)$ is called *a local parameter* of the ring $A^{\mathrm{ev}}(\mathbf{P}^\infty)$ if it satisfies the two conditions:

1. its restriction to a rational point vanishes, and
2. the ring homomorphism $A^{\mathrm{ev}}(pt)[[t]] \to A^{\mathrm{ev}}(\mathbf{P}^\infty)$ sending the variable t to the element π is an isomorphism.

Theorem 2.15 (All trace structures)**.** *The assignment from item 2 of Theorem 2.5, associating to each trace structure on A the Chern class* $c(\mathcal{O}(1)) \in A^{\mathrm{ev}}(\mathbf{P}^\infty)$*, is a bijection*

$$c\colon \textit{Trace Structures on } A \ \to \textit{local parameters of}\, A^{\mathrm{ev}}(\mathbf{P}^\infty)$$

from the set of all trace structures on A to the set of all local parameters of the ring $A^{\mathrm{ev}}(\mathbf{P}^\infty)$*.*

Proof. Consider the assignment from item 2 of Theorem 2.5 which associates to a trace structure $f \mapsto \mathrm{tr}_f$ on A the Chern structure $L \mapsto c(L) = z^A(\mathrm{tr}_z(1))$. By Theorem 2.5 and [**25**, Th. 3.36], this assignment is a bijection of the set of all trace structures on A with the set of all Chern structures on A. To prove the theorem it remains to check that the assignment

$$c\colon \textit{Chern structures} \to \textit{local parameters of } A^{\mathrm{ev}}(\mathbf{P}^\infty),$$

which takes a Chern structure $L \mapsto c(L)$ to the element $c(\mathcal{O}(1)) \in A^{\mathrm{ev}}(\mathbf{P}^\infty)$, is bijective.

To prove the injectivity of c consider two Chern structures $L \mapsto c^{(1)}(L)$ and $L \mapsto c^{(2)}(L)$ on A and assume that $c^{(1)}(\mathcal{O}(1)) = c^{(2)}(\mathcal{O}(1))$. By [**25**, Claim 3.23], for each variety X and each line bundle L over X, there exists a finite-dimensional vector space V and a diagram of the form

$$X \xleftarrow{p} X' \xrightarrow{f} \mathbf{P}(V)$$

in which X' is a torsor under a vector bundle over X and the morphism f is such that the line bundles $p^*(L)$ and $f^*(\mathcal{O}_V(1))$ are isomorphic. The pull-back operator $p^*\colon A(X) \to A(X')$ is an isomorphism by the strong homotopy invariance of the theory A [**25**, 2.2.6]. Now $c^{(1)}(L) = c^{(2)}(L)$ by the functoriality of the classes $c^{(1)}$ and $c^{(2)}$.

To prove the surjectivity of c choose a Chern structure $L \mapsto c(L)$ on A. Then the element $\pi = c(\mathcal{O}(1))$ is a local parameter of $A^{\mathrm{ev}}(\mathbf{P}^\infty)$ by [**25**, Th. 3.9]. Let $\pi^{\mathrm{new}} \in A^{\mathrm{ev}}(\mathbf{P}^\infty)$ be one more local parameter. Let $g(t) \in A^{\mathrm{ev}}(pt)[[t]]$ be a formal power series such that $g(\pi) = \pi^{\mathrm{new}}$ in $A(\mathbf{P}^\infty)$. Now consider an assignment $L \mapsto c^{\mathrm{new}}(L) := g(c(L))$. Clearly it is a Chern structure on A and $c^{\mathrm{new}}(\mathcal{O}(1)) = g(\pi) = \pi^{\mathrm{new}}$ in $A(\mathbf{P}^\infty)$. The theorem is proved. □

In Remark 2.16 below we describe explicitly the bijection

$$c^{-1}\colon \textit{local parameters of } A^{\mathrm{ev}}(\mathbf{P}^\infty) \to \textit{Trace structures on } A$$

inverse to the bijection c from Theorem 2.15. For that let $\pi^{\mathrm{new}} \in A^{\mathrm{ev}}(\mathbf{P}^\infty)$ be a local parameter, let $g(t)$ be the series from the proof of Theorem 2.15 and let $r(t) = g(t)/t \in A(pt)^{\mathrm{ev}}[[t]]$. Clearly $r(t)$ is a unit in $A(pt)^{\mathrm{ev}}[[t]]$. Let $r^{-1}(t)$ be its multiplicative inverse. For a vector bundle E over a variety X, let $r(E), r^{-1}(E) \in A(X)^{\mathrm{ev}}$ be the invariants from [**27**, Prop. 2.2.3]. Recall that $r(E)r^{-1}(E) = 1$.

Remark 2.16. For a projective morphism of varieties $f\colon Y \to X$, let T_Y and T_X be the tangent bundles to Y and X respectively. Set

$$\mathrm{tr}_f^{\mathrm{new}} = (\cup r(T_X)) \circ \mathrm{tr}_f \circ (\cup r^{-1}(T_Y))\colon A(Y) \to A(X). \tag{64}$$

Then the assignment $f \mapsto \mathrm{tr}_f^{\mathrm{new}}$ is a new trace structure on A and the corresponding Chern structure is given by $c^{\mathrm{new}}(L) = c(L) \cup r(L)$ [**27**, Th. 2.3.2]. Now

$$c^{\mathrm{new}}(\mathcal{O}(1)) = c(\mathcal{O}(1)) \cup r(\mathcal{O}(1)) = \pi \cup r(\pi) = g(\pi) = \pi^{\mathrm{new}}$$

in $A(\mathbf{P}^\infty)$. Thus the assignment c^{-1} takes the local parameter π^{new} to the trace structure $f \mapsto \mathrm{tr}_f^{\mathrm{new}}$ on A.

Acknowledgements

The author thanks A. Suslin for his very stimulating interest in the subject. His remark on the possible existence of trace operators in a general situation was the origin of our work on the subject. The author thanks as well A. Merkurjev, E. Friedlander, F. Morel and S. Yagunov for their stimulating interest in the subject. The joint paper with S.Yagunov [**31**] should be regarded as a first draft attempt in the realization of Suslin's remark.

The author grateful acknowledges the support of the SFB-701 at the University of Bielefeld, the RTN Network HPRN-CT-2002-00287, the grant INTAS-05-1000008-8118, the joint DFG - RFBR-project 09-01-91333-NNIO-a and the Program of the year 2008–09 "Research in the fundamental domains of the modern mathematic" at the Presidium of RAS.

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