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HOMOTOPY NILPOTENCY IN LOCALIZED SU(n)

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Abstract

We determine the homotopy nilpotency of p-localized $\mathrm{SU}(n)$ when p is a quasi-regular prime in the sense of [**9**]. As a consequence, we see that it is not a monotonic decreasing function in p.

1. Introduction

Let G be a compact Lie group and let $-_{(p)}$ stand for the p-localization in the sense of [**2**]. In [**7**], McGibbon asked:

Question 1.1. *For which primes p is $G_{(p)}$ homotopy commutative?*

He answered this question for G simply connected. For example, he showed that $\mathrm{SU}(n)_{(p)}$ is homotopy commutative if and only if $p > 2n$. Later, in [**8**], he studied higher homotopy commutativity of p-local finite loop spaces and, motivated by this work, Saumell [**11**] considered the above question by replacing homotopy commutativity with higher homotopy commutativity in the sense of Williams [**14**]. For example, she showed that if $p > kn$, then $\mathrm{SU}(n)_{(p)}$ is a C_k-space in the sense of Williams [**14**].

One can also approach the problem from the opposite direction:

Question 1.2. *How far from being homotopy commutative is $G_{(p)}$ for a given prime p?*

In [**5**], Kaji and the author approached this question by considering homotopy nilpotency which is defined as follows, where we treat only group-like spaces (see [**15**] for a general definition). Let X be a group-like space, that is, X satisfies all the axioms of groups up to homotopy, and let $\gamma\colon X \times X \to X$ be the commutator map of X. We write the n-iterated commutator map $\gamma \circ (1 \times \gamma) \circ \cdots \circ (1 \times \cdots \times 1 \times \gamma)\colon X^{n+1} \to X$ by γ_n, where X^{n+1} is the direct product of $(n+1)$-copies of X. We say that X is homotopy nilpotent of class n, denoted $\operatorname{nil} X = n$, if $\gamma_n \simeq *$ and $\gamma_{n-1} \not\simeq *$. Namely, $\operatorname{nil} X = n$ means that X is a nilpotent group of class n up to homotopy. Then one can say that $\operatorname{nil} X$ tells how far from being homotopy commutative X is. Note that

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we normalize homotopy nilpotency such that $\operatorname{nil} X = 1$ if and only if X is homotopy commutative. Then, rewriting the above result of McGibbon, we have

$$\operatorname{nil} \mathrm{SU}(n)_{(p)} = 1 \text{ if and only if } p > 2n. \tag{1}$$

In [**5**], Kaji and the author determined $\operatorname{nil} X$ for a p-compact group X when p is a regular prime, that is, X has the homotopy type of the direct product of localized spheres. For example, they showed

$$\operatorname{nil} \mathrm{SU}(n)_{(p)} = \begin{cases} 2 & \text{for } \frac{3}{2}n < p < 2n \\ 3 & \text{for } n \leqslant p \leqslant \frac{3}{2}n \end{cases} \tag{2}$$

when p is odd, and $\operatorname{nil} \mathrm{SU}(2)_{(2)} = 2$.

The aim of this article is to determine $\operatorname{nil} \mathrm{SU}(n)_{(p)}$ when p is a quasi-regular prime in the sense of [**9**], that is, $\mathrm{SU}(n)_{(p)}$ has the homotopy type of the p-localization of the direct product of spheres and sphere bundles over spheres. The result is

Theorem 1.1. *Let p be a prime greater than 5. Then we have:*

1. $\operatorname{nil} \mathrm{SU}(n)_{(p)} = 3$ *if* $p = n+1$ *or* $\frac{n}{2} < p \leqslant \frac{2n+1}{3}$.
2. $\operatorname{nil} \mathrm{SU}(n)_{(p)} = 2$ *if* $\frac{2n+1}{3} < p \leqslant n-2$.

Since the homotopy type of $\mathrm{SU}(n)_{(p)}$ gets easier as p increases, it is natural to expect that $\operatorname{nil} \mathrm{SU}(n)_{(p)}$ is a monotonic decreasing function in p. In fact, (1) and (2) give some evidence for this expectation. However, Theorem 1.1 shows this is false in almost all cases as follows. In [**10**], it is shown that

$$\frac{x}{\log x} < \pi(x) < 1.25506 \frac{x}{\log x}$$

for $x \geqslant 17$, where $\pi(x)$ is the prime counting function, that is, $\pi(x)$ is the number of primes less than or equal to x. This implies that there is a prime p in the range $\frac{2n+1}{3} < p \leqslant n-2$ for $n \geqslant 33$. We can also show that there are such primes for $n = 9$ and $13 \leqslant n \leqslant 32$ by a case-by-case analysis. Thus we obtain

Corollary 1.2. *For $n = 9$ or $n \geqslant 13$, $\operatorname{nil} \mathrm{SU}(n)_{(p)}$ is not a monotonic decreasing function in p.*

In what follows, we will make the conventions: For a map $f\colon X \to Y$, $f_*\colon [A, X] \to [A, Y]$ and $f^*\colon [Y, B] \to [X, B]$ mean the induced maps. If a map $f\colon X \to Y_1 \times Y_2$ satisfies $\pi_1 \circ f \simeq *$, then we say that f falls into Y_2, where π_1 is the first projection. We often assume that the above f is a map from X into Y_2. We denote the adjoint congruence $[X, \Omega Y] \xrightarrow{\cong} [\Sigma X, Y]$ by ad. When X is group-like, we always assume that the homotopy set $[A, X]$ is a group by pointwise multiplication and we denote by 0 unity of this group which is the constant map. We denote the order of an element x of a group by $\operatorname{ord}(x)$.

2. Homotopy groups of B_n

Hereafter, let p denote an odd prime and put $2 \leqslant t \leqslant p$. Each space and map is always assumed to be localized at the prime p.

Let us first recall basic results on the p-primary component of the homotopy groups of spheres.

Theorem 2.1 ([**12**, Chapter XIII])**.**

1. *We have*

$$\pi_{2n-1+k}(S^{2n-1}) \cong \begin{cases} \mathbf{Z}/p & \text{for } k = 2i(p-1)-1, i = 1, \dots, p-1 \\ \mathbf{Z}/p & \text{for } k = 2i(p-1)-2, i = n, \dots, p-1 \\ 0 & \text{otherwise, for } 1 \leqslant k \leqslant 2p(p-1)-3. \end{cases}$$

2. *Let* $\alpha_1(3)$ *be a generator of* $\pi_{2p}(S^3)$ *and define* $\alpha_i(3) \in \pi_{2i(p-1)+2}(S^3)$ *inductively by the Toda bracket* $\{\alpha_{i-1}(3), p, \alpha_1(2i(p-1)+2)\}_1$ *for* $i = 2, \dots, p-1$. *Then* $\pi_{2n+2i(p-1)-2}(S^{2n-1})$ *is generated by* $\alpha_i(2n-1) = \Sigma^{2n-4}\alpha_i(3)$.
3. $\pi_{2i(p-1)+1}(S^3)$ *is generated by* $\alpha_1(3) \circ \alpha_{i-1}(2p)$ *for* $i = 2, \dots, p-1$.
4. $\Sigma^2\colon \pi_{2n+2i(p-1)-3}(S^{2n-1}) \to \pi_{2n+2i(p-1)-1}(S^{2n+1})$ *is the zero map for* $i = n, \dots, p-1$. *In particular,* $\alpha_i(n) \circ \alpha_j(n + 2i(p-1) - 1) = 0$ *for* $i + j < p$ *and* $n \geqslant 5$.

Let B_n be the S^{2n-1}-bundle over $S^{2n+2p-3}$ such that

$$H^*(B_n; \mathbf{Z}/p) = \Lambda(\bar{x}_{2n-1}, \mathcal{P}^1\bar{x}_{2n-1}),$$

where $|\bar{x}_{2n-1}| = 2n-1$. Namely, B_n is induced from the sphere bundle $S^{2n-1} \to \mathrm{O}(2n+1)/\mathrm{O}(2n-1) \to S^{2n}$ by $\frac{1}{2}\alpha_1(2n)$ as in [**9**]. Recall that we have a cell decomposition

$$B_n = S^{2n-1} \cup_{\alpha_1(2n+1)} e^{2n+2p-3} \cup e^{4n+2p-4}.$$

Let A_n denote the $(4n+2p-5)$-skeleton of B_n, that is, $A_n = C_{\alpha_1(2n-1)}$, where C_f stands for the mapping cone of f. In particular, we have

$$A_n = \Sigma^{2n-4}A_2. \tag{3}$$

It follows from a result of McGibbon [**6**] that the cofiber sequence $A_n \to B_n \to S^{4n+2p-4}$ splits after a suspension, that is,

$$\Sigma B_n \simeq \Sigma A_n \vee S^{4n+2p-3}. \tag{4}$$

Mimura and Toda [**9**] showed that SU(n) has the homotopy type of the direct product of odd spheres and B_k's if and only if $p > \frac{n}{2}$. We shall be concerned with SU(n) for $\frac{n}{2} < p < n$, equivalently, SU($p+t-1$) since $2 \leqslant t \leqslant p$. In this case, we have a homotopy equivalence

$$\mathrm{SU}(p+t-1) \simeq B_2 \times \cdots \times B_t \times S^{2t+1} \times \cdots \times S^{2p-1}.$$

We compute the homotopy groups of B_n following Mimura and Toda [**9**] in a slightly larger range than [**9**]. Consider the homotopy exact sequence of the fibration $S^{2n-1} \to B_n \to S^{2n+2p-3}$. Then the connecting homomorphism $\delta\colon \pi_*(S^{2n+2p-3}) \to \pi_{*-1}(S^{2n-1})$ is given by

$$\delta(\Sigma x) = \alpha_1(2n-1) \circ x. \tag{5}$$

Then by Theorem 2.1, we obtain $\pi_*(B_2)$ for $* \leqslant 2p(p-1)$. In particular, each map

$S^m \to B_2$ for $2p+2 \leqslant m \leqslant 2p(p-1)$ lifts to $S^3 \subset B_2$. It also follows from Theorem 2.1 that for $n \geqslant 3$ and $i = 2, \dots, p-1$, we have the short exact sequence

$$0 \to \pi_*(S^{2n-1}) \to \pi_*(B_n) \to \pi_*(S^{2n+2p-3}) \to 0 \tag{6}$$

for $2n+2p-2 \leqslant * \leqslant 2n+2p(p-1)-4$. Then we have only to consider the case that $* = 2n+2i(p-1)-2$ for $i = 2, \dots, p-1$. Let $i_n\colon S^{2n-1} \to A_n$ and $j_n\colon A_n \to B_n$ be the inclusions and let $q_n\colon A_n \to S^{2n+2p-3}$ be the pinch map. Consider the following commutative diagram in which the lower horizontal sequence is the exact sequence (6) and we put $k = 2n+2i(p-1)-2$.

$$\begin{tikzcd}
 & \pi_k(S^{2n-1}) \arrow[r, "i_{n*}"] \arrow[d, equal] & \pi_k(A_n) \arrow[r, "q_{n*}"] \arrow[d, "j_{n*}"] & \pi_k(S^{2n+2p-3}) \arrow[d, equal] & \\
0 \arrow[r] & \pi_k(S^{2n-1}) \arrow[r] & \pi_k(B_n) \arrow[r] & \pi_k(S^{2n+2p-3}) \arrow[r] & 0
\end{tikzcd}$$

Note that a coextension $\underline{\alpha_{i-1}(2n+2p-4)}\colon S^{2n+2i(p-1)-2} \to A_n = C_{\alpha_1(2n-1)}$ satisfies

$$q_{n*}(\underline{\alpha_{i-1}(2n+2p-4)}) = -\alpha_{i-1}(2n+2p-3)$$

and

$$\begin{aligned}
\underline{\alpha_{i-1}(2n+2p-4)} \circ p &= -i_{n*}(\{\alpha_1(2n-1), \alpha_{i-1}(2n+2p-4), p\}_1) \\
&= i_{n*}(\tfrac{1}{i}\{\alpha_{i-1}(2n-1), p, \alpha_1(2n+2p-4)\}_1) \\
&= -i_{n*}(\tfrac{1}{i}\alpha_i(2n-1))
\end{aligned}$$

(see [**12**, p.179]). Then (6) does not split for $* = 2n+2i(p-1)-2$ and hence we have obtained that $\pi_{2n+2i(p-1)-2}(B_n) \cong \mathbf{Z}/p^2$. Moreover, it is generated by the element $j_{n*}(\underline{\alpha_{i-1}(2n+2p-4)})$. In particular, each map $S^m \to B_n$ which is of order p for $2n+2p-2 \leqslant m \leqslant 2n+2p(p-1)-4$ lifts to $S^{2n-1} \subset B_n$. Summarizing, we have calculated

Proposition 2.2. *As for the homotopy groups of B_n, we have:*

1. $\pi_{3+k}(B_2) \cong \begin{cases} \mathbf{Z}/p & \text{for } k = 2i(p-1)-1, i = 2, \dots, p-1 \\ \mathbf{Z}_{(p)} & \text{for } k = 2p-2 \\ 0 & \text{otherwise, for } 1 \leqslant k \leqslant 2p(p-1)-3. \end{cases}$
2. *For $n \geqslant 3$,* $\pi_{2n-1+k}(B_n) \cong \begin{cases} \mathbf{Z}/p & \text{for } k = 2i(p-1)-1, i = 2, \dots, p-1 \\ \mathbf{Z}/p & \text{for } k = 2i(p-1)-2, i = n, \dots, p-1 \\ \mathbf{Z}_{(p)} & \text{for } k = 2p-2 \\ 0 & \text{otherwise, for } 1 \leqslant k \leqslant 2p(p-1)-3. \end{cases}$
3. *For $2p+2 \leqslant m \leqslant 2p(p-1)$, each map $S^m \to B_2$ lifts to $S^3 \subset B_2$.*
4. *For $n \geqslant 3$ and $2n+2p-2 \leqslant m \leqslant 2n+2p(p-1)-4$, each map $S^m \to B_n$ of order p lifts to $S^{2n-1} \subset B_n$.*

By Theorem 2.1 and Proposition 2.2 we can see the homotopy groups of $\mathrm{SU}(p+t-1)$ in a range. It will be useful to list the non-trivial odd homotopy groups of $\mathrm{SU}(p+t-1)$.

Corollary 2.3. *Let $p \geqslant 7$ and $2(p+t)-1 \leqslant k \leqslant 12p-1$. Then $\pi_k(\mathrm{SU}(p+t-1)) = 0$ unless k is odd and not in the following table. Moreover, each element of $\pi_{2k-1}(\mathrm{SU}(p+t-1))$ can be compressed into $S^n \subset \mathrm{SU}(p+t-1)$ for n in the following table.*

k	$6p-3$			
	$8p-5$	$8p-3$		
	$10p-7$	$10p-5$	$10p-3$	
	$12p-9$	$12p-7$	$12p-5$	$12p-3$
n	5	7	9	11

3. Homotopy nilpotency and Samelson products

Let X be a group-like space. For a map $f\colon A \to X$ we write by $-f$ the composition $A \xrightarrow{f} X \xrightarrow{\iota} X$, where $\iota\colon X \to X$ is the homotopy inversion. We will often use the fact that the pinch map $V_1 \times \cdots \times V_k \to V_1 \wedge \cdots \wedge V_k$ induces an injection $[V_1 \wedge \cdots \wedge V_k, X] \to [V_1 \times \cdots \times V_k, X]$ (see [**15**, Lemma 1.3.5]).

Suppose that $X = X_1 \times \cdots \times X_n$ as spaces, not necessarily as group-like spaces. We denote the inclusion $X_k \to X$ and the projection $X \to X_k$ by i_k and p_k respectively for $k = 1, \ldots, n$. Note that we may assume $1_X = (i_1 \circ p_1) \cdots (i_n \circ p_n)$, the pointwise multiplication. Let $\gamma\colon X^2 \to X$ be the commutator map of X and let γ_k be the k-iterated commutator map $\gamma \circ (1 \times \gamma) \circ \cdots \circ (1 \times \cdots \times 1 \times \gamma)\colon X^{k+1} \to X$. By applying a commutator calculus to a certain subgroup of $[X^{k+1}, X]$ together with the above description of 1_X, Kaji and the author [**5**] gave a decomposition of γ_k and obtained

Proposition 3.1. *Let X be a group-like space such that $X = X_1 \times \cdots \times X_n$ as spaces and let $i_m\colon X_k \to X$ be the inclusion for $m = 1, \ldots, n$. Then $\operatorname{nil} X < k$ if and only if $\langle \theta_1, \langle \cdots \langle \theta_k, \theta_{k+1} \rangle \cdots \rangle \rangle = 0$ for each $\theta_1, \ldots, \theta_{k+1} \in \{\pm i_1, \ldots, \pm i_n\}$.*

We produce formulae for Samelson products which will be useful for our purpose.

Proposition 3.2. *Let X be a group-like space and let $\theta_i\colon V_i \to X$ for $i = 1, 2, 3$.*

1. *If $\langle \pm\theta_1, \langle \pm\theta_2, \pm\theta_3 \rangle \rangle = \langle \pm\theta_2, \langle \pm\theta_3, \pm\theta_1 \rangle \rangle = 0$, then $\langle \pm\theta_3, \langle \pm\theta_1, \pm\theta_2 \rangle \rangle = 0$.*
2. *$\langle \theta_1, \theta_2 \rangle = 0$ implies $\langle \theta_1, -\theta_2 \rangle = 0$.*
3. *Let $\theta_3'\colon V_3 \to X$. If $\langle \theta_1, \langle \theta_2, \theta_3 \rangle \rangle = \langle \theta_1, \langle \theta_2, \theta_3' \rangle \rangle = \langle \theta_3, \langle \theta_2, \theta_3' \rangle \rangle = 0$, then we have $\langle \theta_1, \langle \theta_2, \theta_3\theta_3' \rangle \rangle = 0$.*
4. *Suppose that $X = X_1 \times \cdots \times X_n$ as spaces and denote by i_k and p_k the inclusion $X_k \to X$ and the projection $X \to X_k$ respectively for $k = 1, \ldots, n$. Then we have that $\langle \theta_1, i_k \circ p_k \circ \theta_2 \rangle = 0$ for $k = 1, \ldots, n$ implies $\langle \theta_1, \theta_2 \rangle = 0$.*

Proof. *1.* Recall first the Hall-Witt formula of groups. Let G be a group and let $[-,-]$ denote the commutator of G. Then we have the Hall-Witt formula,

$$[y,[z,x^{-1}]]^x[x,[y,z^{-1}]]^z[z,[x,y^{-1}]]^y = 1,$$

for $x, y, z \in G$, where $x^y = yxy^{-1}$.

Let $q_i\colon V_1 \times V_2 \times V_3 \to V_i$ be the i-th projection for $i = 1, 2, 3$. Put $\bar{\theta}_i = \theta_i \circ q_i$ for $i = 1, 2, 3$. For $\sigma \in \Sigma_3$, we define $\sigma\colon V_1 \wedge V_2 \wedge V_3 \to V_{\sigma(1)} \wedge V_{\sigma(2)} \wedge V_{\sigma(3)}$ by $\sigma(v_1, v_2, v_3) = (v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)})$. Then we have

$$[\bar{\theta}_{\sigma(1)}, [\bar{\theta}_{\sigma(2)}, \bar{\theta}_{\sigma(3)}]] = \sigma^{-1} \circ q^*(\langle\theta_1, \langle\theta_2, \theta_3\rangle\rangle),$$

where $[-,-]$ denotes the commutator in the group $[V_1 \times V_2 \times V_3, X]$ and $q\colon X^3 \to X^{(3)}$ is the pinch map, where $X^{(m)}$ denotes the smash product of m-copies of X. Hence, by hypothesis, we have established $[\pm\bar{\theta}_1, [\pm\bar{\theta}_2, \pm\bar{\theta}_3]] = [\pm\bar{\theta}_2, [\pm\bar{\theta}_3, \pm\bar{\theta}_1]] = 0$ and thus it follows from the Hall-Witt formula that we obtain $[\pm\bar{\theta}_3, [\pm\bar{\theta}_1, \pm\bar{\theta}_2]] = 0$. Since σ^{-1} and q^* are monic, we have $\langle\pm\theta_3, \langle\pm\theta_1, \pm\theta_2\rangle\rangle = 0$.

2. This follows from the fact $1_X = (i_1 \circ p_1) \cdots (i_n \circ p_n)$ and the formula

$$[x, yz] = [x, y][x, z]^y$$

for $x, y \in G$.

3. This also follows from the above formula.

4. This follows from the formulae

$$[x, [y, zw]] = [x, [y, z]][x, [z, [y, w]]]^{[y,z]}[x, [y, w]]^{[y,z][z,[y,w]]}$$

for $x, y, z, w \in G$ respectively. □

We denote the inclusions $S^{2i-1} \to \mathrm{SU}(p + t - 1)$, $A_j \to \mathrm{SU}(p + t - 1)$ and $B_j \to \mathrm{SU}(p + t - 1)$ by ϵ_i, λ_j and $\bar{\lambda}_j$ respectively for $2 \leqslant i \leqslant p$ and $2 \leqslant j \leqslant t$. We also denote by π_i the projections $\mathrm{SU}(p + t - 1) \to B_i$ for $2 \leqslant i \leqslant t$ and $\mathrm{SU}(p + t - 1) \to S^{2i-1}$ for $t + 1 \leqslant i \leqslant p$.

Let $W = A_2 \vee \cdots \vee A_t \vee S^{2t+1} \vee \cdots \vee S^{2p-1}$ and let $j = \lambda_2 \vee \cdots \vee \lambda_t \vee \epsilon_{t+1} \vee \cdots \vee \epsilon_p\colon W \to \mathrm{SU}(p + t - 1)$. By (4) there is a homotopy retraction $r\colon \Sigma\mathrm{SU}(p + t - 1) \to \Sigma W$ of Σj and as in [**7**] we can see that there is a self-homotopy equivalence $f\colon \mathrm{SU}(p + t - 1) \to \mathrm{SU}(p + t - 1)$ such that the following square diagram is homotopy commutative.

$$\begin{tikzcd}
\Sigma\mathrm{SU}(p+t-1) \arrow[r, "\Sigma f"] \arrow[d, "r"'] & \Sigma\mathrm{SU}(p+t-1) \arrow[d, "\mathrm{ad}1"] \\
\Sigma W \arrow[r, "\mathrm{ad}j"] & B\mathrm{SU}(p+t-1).
\end{tikzcd}$$

Then for any map $g\colon \Sigma A \to \mathrm{SU}(p + t - 1)$, the Whitehead product $[\pm\mathrm{ad}\bar{\lambda}_i, g] = 0$ if and only if $[\pm\mathrm{ad}\lambda_i, g] = 0$. By adjointness of Whitehead products and Samelson products, we have established

Proposition 3.3. *For any map $f\colon V \to \mathrm{SU}(p + t)$ and each $i = 1, \ldots, t$, the Samelson product $\langle\pm\bar{\lambda}_i, f\rangle = 0$ if and only if $\langle\pm\lambda_i, f\rangle = 0$. In particular, $\langle\pm\bar{\lambda}_k, \pm\bar{\lambda}_l\rangle = 0$ if and only if $\langle\pm\lambda_k, \pm\lambda_l\rangle = 0$.*

4. Computing the Samelson products

Let $\Lambda = \{\epsilon_2, \ldots, \epsilon_p, \lambda_2, \ldots, \lambda_t\}$ and $\bar{\Lambda} = \{\epsilon_2, \ldots, \epsilon_p, \bar{\lambda}_2, \ldots, \bar{\lambda}_t\}$, and let $\pm\Lambda = \{\pm\epsilon_2, \ldots, \pm\epsilon_p, \pm\lambda_2, \ldots, \pm\lambda_t\}$ and $\pm\bar{\Lambda} = \{\pm\epsilon_2, \ldots, \pm\epsilon_p, \pm\bar{\lambda}_2, \ldots, \pm\bar{\lambda}_t\}$. We write the domain of $\theta \in \pm\Lambda$ or $\pm\bar{\Lambda}$ by $X(\theta)$. For example, if $\theta = \lambda_i$, then $X(\theta) = A_i$. For $\theta \in \pm\Lambda$ or $\pm\bar{\Lambda}$, we write $|\theta| = i$ if $\theta = \pm\epsilon_i, \pm\lambda_i$ or $\pm\bar{\lambda}_i$.

By Proposition 3.1, it is sufficient to calculate the iterated Samelson products $\langle\theta_1, \langle\cdots\langle\theta_n, \theta_{n+1}\rangle\cdots\rangle\rangle$ for $\theta_1, \ldots, \theta_{n+1} \in \pm\bar{\Lambda}$ in determining $\operatorname{nil}\mathrm{SU}(p+t-1)$. To do so, we will use the following result of Hamanaka [**3**].

Theorem 4.1 (Hamanaka [**3**])**.** *Let X be a CW-complex with $\dim X \leqslant 2n+2p-4$. Then there is an exact sequence*

$$\widetilde{K}^0(X)_{(p)} \xrightarrow{\Theta} \bigoplus_{i=0}^{p-2} H^{2n+2i}(X, \mathbf{Z}_{(p)}) \to [X, \mathrm{U}(n)]_{(p)} \to \widetilde{K}^1(X)_{(p)} \to \bigoplus_{i=0}^{p-3} H^{2n+2i+1}(X, \mathbf{Z}_{(p)})$$

such that:

1. *$\Theta(x) = \bigoplus_{i=0}^{p-2}(n+i)!ch_{n+i}(x)_{(p)}$ for $x \in \widetilde{K}^0(X)$, where ch_k is the $2k$-dimensional part of the Chern character.*
2. *For $f, g \in [X, \mathrm{U}(n)]_{(p)}$, the commutator $[f, g]$ lies in* **Coker**Θ *and represented by*
$$\bigoplus_{k=0}^{p-2} \sum_{i+j-1=n+k} f^*(x_{2i-1}) \cup g^*(x_{2j-1}),$$
where $x_{2i-1} \in H^{2i-1}(\mathrm{U}(n); \mathbf{Z}_{(p)})$ is the suspension of the universal i-th Chern class $c_i \in H^{2i}(BU(n); \mathbf{Z}_{(p)})$.

As an easy consequence of Theorem 4.1, Hamanaka [**3**] showed:

Proposition 4.2. $\operatorname{ord}(\langle\pm\epsilon_i, \pm\epsilon_j\rangle) = \begin{cases} 0 & \textit{for } i+j \leqslant p+t-1 \\ p & \textit{for } i+j \geqslant p+t. \end{cases}$

Now let us calculate other Samelson products of $\pm\epsilon_i$ and $\pm\lambda_j$ by applying Theorem 4.1. We have that $H^*(B_n; \mathbf{Z}_{(p)}) = \Lambda(x_{2n-1}, x_{2n+2p-3})$ such that the mod p reduction of x_{2n-1} and $x_{2n+2p-3}$ are $\bar{x}_{2n-1}$ and $\mathcal{P}^1\bar{x}_{2n-1}$ respectively. Then $H^*(A_n; \mathbf{Z}_{(p)}) = \mathbf{Z}_{(p)}\langle a_{2n-1}, a_{2n+2p-3}\rangle$ such that $j_n^*(x_i) = a_i$ for $i = 2n-1, 2n+2p-3$, where $R\langle e_1, e_2, \ldots\rangle$ stands for the free R-module with a basis $e_1, e_2, \ldots$ and $j_n\colon A_n \to B_n$ is the inclusion.

Lemma 4.3. *For $n \leqslant p$, $\widetilde{K}(\Sigma A_n)_{(p)} = \mathbf{Z}_{(p)}\langle\xi_n, \eta_n\rangle$ such that*

$$ch(\xi_n) = \Sigma a_{2n-1} + \frac{1}{p!}\Sigma a_{2n+2p-3}, \ ch(\eta_n) = \Sigma a_{2n+2p-3}.$$

Proof. Let γ be the canonical line bundle of $\mathbf{C}P^p$ and let $\epsilon \in \widetilde{K}(\mathbf{C}P^p) = [\mathbf{C}P^p, B\mathrm{U}(\infty)]$ be the composite $\mathbf{C}P^p \xrightarrow{q} S^{2p} \xrightarrow{u} B\mathrm{U}(\infty)$ for the pinch map $q\colon \mathbf{C}P^p \to S^{2p}$ and a generator u of $\pi_{2p}(B\mathrm{U}(\infty))$. Note that $\Sigma\mathbf{C}P^p \simeq A_2 \vee S^5 \vee \cdots \vee S^{2p-1}$. By using (3), we put ξ_n and η_n to be the pullback of $\Sigma^{2n-2}\gamma$ and $\Sigma^{2n-2}\epsilon$ by the inclusion $\Sigma A_n \to \Sigma^{2n-2}\mathbf{C}P^p$. Then Lemma 4.3 follows from an easy calculation of the Chern character of γ and ϵ. $\square$

Proposition 4.4. *1. For $(i,j) \neq (p,t)$,*

$$\mathrm{ord}(\langle \pm\epsilon_i, \pm\lambda_j\rangle) = \mathrm{ord}(\langle \pm\lambda_j, \pm\epsilon_i\rangle) = \begin{cases} 0 & \text{for } i+j \leqslant p+1 \\ p & \text{for } i+j \geqslant p+2. \end{cases}$$

2. For $i+j \leqslant t$, $\mathrm{ord}(\langle \pm\lambda_i, \pm\lambda_j\rangle) = 0$.

3. Let $X(i,j)$ be the $(2i+2j+4p-5)$-skeleton of $A_i \wedge A_j$, that is, $A_i \wedge A_j$ minus the top cell. For $(i,j) \neq (p,p)$,

$$\mathrm{ord}(\langle \pm\lambda_i, \pm\lambda_j\rangle|_{X(i,j)}) = \begin{cases} 0 & \text{for } i+j \leqslant p+1 \\ p & \text{for } i+j \geqslant p+2. \end{cases}$$

Proof. *1.* Note that $\mathrm{U}(n) \simeq \mathrm{SU}(n) \times S^1$ as H-spaces; here we localize at the odd prime p. Then we have $\mathrm{ord}(\langle \epsilon_i, \lambda_j\rangle) = \mathrm{ord}(\langle \epsilon'_i, \lambda'_j\rangle)$, where ϵ'_i and λ'_j are the compositions $S^{2i-1} \xrightarrow{\epsilon_i} \mathrm{SU}(p+t-1) \hookrightarrow \mathrm{U}(p+t-1)$ and $A_i \xrightarrow{\lambda_i} \mathrm{SU}(p+t-1) \hookrightarrow \mathrm{U}(p+t-1)$ respectively. Hence we calculate $\langle \epsilon'_i, \lambda'_j\rangle$. Apply Theorem 4.1 to $X = S^{2i-1} \times A_j$. Then, by Lemma 4.3, the $2(i+j+p-2)$-dimensional part of $\mathbf{Coker}\Theta$ is

$$\mathbf{Z}_{(p)}\langle s_{2i-1} \times a_{2j+2p-3}\rangle / (\tfrac{(i+j+p-2)!}{p!} s_{2i-1} \times a_{2j+2p-3}),$$

where s_{2i-1} is a generator of $H^{2i-1}(S^{2i-1}; \mathbf{Z}_{(p)})$. By definition, $\epsilon'(x_{2i-1}) = s_{2i-1}$ and $\lambda'_j(x_{2j+2p-3}) = a_{2j+2p-3}$. Then, by the above observation, $q^*(\langle \epsilon'_i, \lambda'_j\rangle) \in \mathbf{Coker}\Theta$ is represented by $s_{2i-1} \times a_{2j+2p-3}$. Thus we have calculated $\mathrm{ord}(\langle \epsilon'_i, \lambda'_j\rangle)$.

2. This is quite analogous to *1.*

3. Let $p_i\colon X_1 \times X_2 \to X_i$ be the i-th projection for $i = 1,2$ and let $q\colon X_1 \times X_2 \to X_1 \wedge X_2$ be the pinch map. For $f_i\colon X_i \to \mathrm{U}(n)$, $i = 1,2$, we have

$$[f_1 \circ p_1, f_2 \circ p_2] = q^*(\langle f_1, f_2\rangle) \in [X_1 \times X_2, \mathrm{U}(n)]$$

as in the proof of Proposition 3.2. Since q^* is monic, $\mathrm{ord}([f_1 \circ p_1, f_2 \circ p_2]) = \mathrm{ord}(\langle f_1, f_2\rangle)$. Now if the subcomplex $Y \subset X_1 \times X_2$ satisfies $\dim Y \leqslant 2n+2p-4$, it follows from the construction of the exact sequence in Theorem 4.1 that $[f_1 \circ p_1, f_2 \circ p_2]|_Y$ lies in $\mathbf{Coker}\Theta$ which is represented by

$$\bigoplus_{k=0}^{p-2} \sum_{i+j-1=n+k} g^*(f_1^*(x_{2i-1}) \times f_2^*(x_{2j-1})),$$

where $g\colon Y \to X_1 \times X_2$ is the inclusion (see [**3**] for details). Using this formula, the remaining calculation is analogous to *1.* □

In what follows we will often use the argument below implicitly.

Proposition 4.5. *Let $X \to Y \to Z$ be a cofiber sequence and let W be a space such that $[Z,W] = *$. If a map $f\colon Y \to W$ satisfies $f|_X = 0$, then $f = 0$.*

Proof. Proposition 4.5 follows from the exact sequence $[Z,W] \to [Y,W] \to [X,W]$ induced from the cofiber sequence $X \to Y \to Z$. □

By Theorem 2.1 and Proposition 2.2, the Samelson product $\langle \pm\theta_1, \pm\theta_2\rangle$ for $\theta_1, \theta_2 \in \Lambda$ falls to a single B_i or $S^{2j-1} \subset \mathrm{SU}(p+t-1)$ for $i = 2, \ldots, t$ and $j = t+1, \ldots, p$. We shall consider the lifting problem of the above $\langle \pm\theta_1, \pm\theta_2\rangle$ when it maps to B_i.

Let us first consider $\langle \pm\epsilon_i, \pm\epsilon_j\rangle$. Note that we can assume $i + j \geqslant p + t$ by Proposition 4.2, which implies that $\langle \pm\epsilon_i, \pm\epsilon_j\rangle$ falls to $S^{2(i+j-p)+1}$ for $i + j \leqslant 2p - 1$ and to B_2 for $i = j = p$. Then it is sufficient to look at the case $i = j = p$. By Proposition 4.2, $\mathrm{ord}(\langle \pm\epsilon_p, \pm\epsilon_p\rangle) = p$ and then, by Proposition 2.2, $\langle \pm\epsilon_p, \pm\epsilon_p\rangle$ lifts to $S^3 \subset B_2$. Thus we have obtained

Proposition 4.6. *$\langle \pm\epsilon_i, \pm\epsilon_j\rangle$ falls to $S^{2(i+j-p)+1} \subset \mathrm{SU}(p+t-1)$ if $p + t \leqslant i + j \leqslant 2p - 1$ and lifts to $S^3 \subset B_2$ if $i + j = 2p$.*

Next we consider $\langle \pm\epsilon_i, \pm\lambda_j\rangle$ and $\langle \pm\lambda_j, \pm\epsilon_i\rangle$. In the following calculation, we shall assume the homotopy set $[\Sigma X, Y]$ is a group by the comultiplication of ΣX and the induced map $(\Sigma f)^*\colon [\Sigma X', Y] \to [\Sigma X, Y]$ from $f\colon X \to X'$ as a group homomorphism. Now we have the exact sequence induced from the cofiber sequence $S^{2n+2p-5} \xrightarrow{\alpha_1(2n-2)} S^{2n-2} \to C_{\alpha_1(2n-2)}$ for $n \geqslant 3$:

$$\pi_{2n-1}(S^{2n-1}) \xrightarrow{\alpha_1(2n-1)^*} \pi_{2n+2p-4}(S^{2n-1}) \to [C_{\alpha_1(2n-2)}, S^{2n-1}] \to \pi_{2n-2}(S^{2n-1}).$$

It follows from Theorem 2.1 that $\alpha_1(2n-1)^*$ is epic. Then, for $\pi_{2n-2}(S^{2n-1}) = 0$, we obtain

Proposition 4.7. *For $n \geqslant 3$, $[C_{\alpha_1(2n-2)}, S^{2n-1}] = 0$.*

Corollary 4.8. *For $p + 2 \leqslant i + j \leqslant p + t - 1$, the Samelson products $\langle \pm\lambda_i, \pm\epsilon_j\rangle$ and $\langle \pm\epsilon_j, \pm\lambda_i\rangle$ lift to $S^{2(i+j-p)+1} \subset B_{i+j-p+1}$.*

Proof. We only give a proof for $\langle \epsilon_i, \lambda_j\rangle$ since the others are analogous. It follows from Proposition 2.2 that $\langle \epsilon_i, \lambda_j\rangle$ falls to $B_{i+j-p+1} \subset \mathrm{SU}(p+t-1)$. Since $S^{2i-1} \wedge A_j = C_{\alpha(2i+2j-2)}$, it follows from Proposition 4.7 that $q_*(\langle \epsilon_i, \lambda_j\rangle) = 0$, where $q\colon B_{i+j-p+1} \to S^{2(i+j)-1}$ is the projection. Then $\langle \epsilon_i, \lambda_j\rangle$ lifts to $S^{2(i+j-p)+1}$ and the proof is completed. $\square$

Let us describe the above lift $f\colon A_i \wedge S^{2j-1} \to S^{2(i+j-p)+1}$ of the Samelson product $\langle \lambda_i, \epsilon_j\rangle$. Consider the following commutative diagram in which the row and the column sequences are the exact sequences induced from the cofiber sequence $S^{2n+2p-4} \to C_{\alpha_1(2n+2p-4)} \xrightarrow{q} S^{2n+4p-5}$ and the fiber sequence $S^{2n-1} \to B_n \to S^{2n+2p-3}$ respectively.

$$\begin{tikzcd}
 & {[\Sigma C_{\alpha_1(2n+2p-4)}, S^{2n+2p-3}]} \arrow[d, "\delta"] & \\
\pi_{2n+4p-6}(S^{2n-1}) \arrow[r, "q^*"] & {[C_{\alpha_1(2n+2p-4)}, S^{2n-1}]} \arrow[r] \arrow[d, "i_*"] & \pi_{2n+2p-4}(S^{2n-1}) \\
 & {[C_{\alpha_1(2n+2p-4)}, B_n]} &
\end{tikzcd}$$

Let $\bar{p}\colon C_{\alpha_1(2n+2p-4)} \to S^{2n+2p-4}$ be an extension of the degree p self-map of $S^{2n+2p-4}$.

Then, by (5) and [**12**, Proposition 1.9], we have

$$\delta(\Sigma\bar{p}) = \alpha_1(2n-1)\circ\bar{p} = q^*(\{\alpha_1(2n-1), p, \alpha_1(2n+2p-4)\})b = q^*(\alpha_2(2n-1)).$$

On the other hand, it follows from Theorem 2.1 that

$$\mathbf{Im}q^* = \mathbf{Z}/p\langle q^*(\alpha_2(2n-1))\rangle.$$

Then we have established that if $f\colon C_{\alpha_1(2n+2p-4)} \to S^{2n-1}$ satisfies $f|_{S^{2n+2p-4}} = 0$, then $i_*(f) = 0$. In particular, it follows from Proposition 4.4 that

Proposition 4.9. *For $p+2 \leqslant i+j \leqslant p+t-1$, any lift of $\langle\lambda_i, \epsilon_j\rangle$ to $S^{2(i+j-p)+1} \subset B_{i+j-p+1}$, say f, satisfies $f|_{S^{2i-1}\wedge S^{2j-1}} \neq 0$.*

Next we consider the lifting problem of $\langle\pm\lambda_i, \pm\lambda_j\rangle$. Recall from [**12**, Lemma 3.5] that the cell structure of $C_{\alpha_1(n)} \wedge C_{\alpha_1(n)}$ for $n \geqslant p$ is given by

$$C_{\alpha_1(n)} \wedge C_{\alpha_1(n)} = (C_{\alpha_1(2n)} \vee S^{2n+2p-2}) \cup_{\nu_n} e^{2n+4p-4},$$

where

$$\nu_n = (i_*(\alpha) + (-1)^n \underline{2\alpha_1(2n)}) \vee \alpha_1(2n+2p-2) \tag{7}$$

for the inclusion $i\colon S^{2n} \to C_{\alpha_1(2n)}$ and some $\alpha \in \pi_{2n+4p-5}(S^{2n})$. Since $n \geqslant p$, it follows from the Serre isomorphism $\pi_*(S^{2n}) \cong \Sigma\pi_{*-1}(S^{2n-1}) \oplus \pi_*(S^{4n-1})$ that α is a multiple of $\alpha_2(2n)$.

We shall identify $A_i \wedge A_j$ with $C_{\alpha_1(i+j-1)} \wedge C_{\alpha_1(i+j-1)}$. Consider the following commutative diagram in which the row sequences are the exact sequence induced from the cofiber sequence $A_i \wedge A_j \to S^{2(i+j+2p-3)} \xrightarrow{f} \Sigma X(i,j)$:

$$\begin{tikzcd}
{[\Sigma X(i,j), S^{k-1}]} \arrow[r, "f^*"] \arrow[d, "\Sigma^{2N}"'] & \pi_{k+4p-6}(S^{2k-1}) \arrow[r] \arrow[d, "\Sigma^{2N}"'] & {[A_i \wedge A_j, S^{k-1}]} \arrow[d, "\Sigma^{2N}"'] \\
{[\Sigma^{2N+1} X(i,j), S^{k+2N-1}]} \arrow[r, "(\Sigma^{2N} f)^*"'] & \pi_{k+4p-6+2N}(S^{k+2N-1}) \arrow[r] & {[\Sigma^{2N}(A_i \wedge A_j), S^{k+2N-1}],}
\end{tikzcd}$$

where we put $k = 2(i+j)$. When N is large enough, we have $\Sigma^{2N} f = \Sigma\nu_{i+j+N-1}$. Let $\bar{p}\colon C_{\alpha_1(2(i+j-1))} \to S^{2(i+j-1)}$ be an extension of the degree p self-map of $S^{2(i+j+N-1)}$. Then, by [**12**, p.179], we have

$$\begin{aligned}(\Sigma^{2N} f)^*(\Sigma^{2N}\bar{p}) &= \{p, \alpha_1(2(i+j+N)-1), \alpha_1(2(i+j+N+p-2))\}_1 \\ &= \tfrac{1}{2}\alpha_2(2(i+j+N)-1)\end{aligned}$$

as in the proof of Proposition 2.2. On the other hand, it follows from Theorem 2.1 that $\Sigma^{2N}\colon \pi_{2(i+j+2p-3)}(S^{2(i+j)-1}) \to \pi_{2(i+j+2p-3+N)}(S^{2(i+j+N)-1})$ is an isomorphism. Thus we have obtained

Proposition 4.10. *The inclusion $X(i,j) \to A_i \wedge A_j$ induces an injection $[A_i \wedge A_j, S^{2(i+j)-1}] \to [X(i,j), S^{2(i+j)-1}]$.*

Corollary 4.11. *For $i+j \leqslant p$, $\langle\pm\lambda_i, \pm\lambda_j\rangle = 0$.*

Proof. By Proposition 4.4, it is sufficient to consider the case that $t+1 \leqslant i+j \leqslant p$. In this case, $\langle \pm\lambda_i, \pm\lambda_j \rangle$ falls to $S^{2(i+j)-1} \subset \mathrm{SU}(p+t-1)$ and then the proof is completed by Proposition 4.4 and Proposition 4.10. □

Corollary 4.12. *For $p+1 \leqslant i+j \leqslant 2p-1$, the Samelson products $\langle \pm\lambda_i, \pm\lambda_j \rangle$ can be compressed into $S^{2(i+j-p)+1} \subset \mathrm{SU}(p+t-1)$.*

Proof. We only show the case of $\langle \lambda_i, \lambda_j \rangle$ since other cases are similar. By Proposition 2.1 and Proposition 2.2, $\langle \lambda_i, \lambda_j \rangle$ falls to $B_{i+j-p+1}$. Put $\langle \lambda_i, \lambda_j \rangle|_{X(i,j)} = f \vee g\colon X(i,j) = C_{\alpha_1(2(i+j)-2)} \vee S^{2(i+j+p-2)} \to B_{i+j-p+1}$. By Proposition 4.7, we have $q_*(f) = 0$ for the projection $q\colon B_{i+j-p+1} \to S^{2(i+j)-1}$. By Proposition 4.4, f is of order at most p and then, by Proposition 2.2, $q_*(g) = 0$. Thus, by Proposition 4.10, $q_*(\langle \lambda_i, \lambda_j \rangle) = 0$ and this implies that $\langle \lambda_i, \lambda_j \rangle$ lifts to $S^{2(i+j-p)+1} \subset B_{i+j-p+1}$. □

5. Upper bound for $\mathrm{nil}\,\mathrm{SU}(p+t-1)$

Hereafter, we suppose that $p \geqslant 7$.

The aim of this section is to show:

Theorem 5.1. $\mathrm{nil}\,\mathrm{SU}(p+t-1) \leqslant 3$.

First, here is the proof of Theorem 5.1. By Proposition 3.1 and by Proposition 3.3, it is sufficient to show that

$$\langle \theta_1, \langle \bar\theta_2, \langle \bar\theta_3, \bar\theta_4 \rangle\rangle\rangle = 0 \text{ for } \theta_1 \in \pm\Lambda \text{ and } \bar\theta_2, \bar\theta_3, \bar\theta_4 \in \pm\bar\Lambda.$$

Let $\omega_1 \in \Lambda$ and let $\bar\omega_2, \bar\omega_3, \bar\omega_4 \in \bar\Lambda$. It follows from Proposition 3.2 that if $\langle \pm\langle \pm\bar\omega_3, \pm\bar\omega_4 \rangle, \langle \pm\bar\omega_2, \pm\omega_1 \rangle\rangle = \langle \pm\bar\omega_2, \langle \pm\omega_1, \pm\langle \pm\bar\omega_3, \pm\bar\omega_4 \rangle\rangle\rangle = 0$, then $\langle \pm\omega_1, \langle \pm\langle \pm\bar\omega_3, \pm\bar\omega_4 \rangle, \pm\bar\omega_2 \rangle\rangle = 0$. By Proposition 3.3, this implies $\langle \pm\omega_1, \langle \pm\bar\omega_2, \langle \pm\bar\omega_3, \pm\bar\omega_4 \rangle\rangle\rangle = 0$. On the other hand, by Proposition 3.2, if $\langle \pm\bar\omega_3, \langle \pm\bar\omega_4, \pm\langle \pm\bar\omega_2, \pm\omega_1 \rangle\rangle\rangle = \langle \pm\bar\omega_4, \langle \pm\langle \pm\bar\omega_2, \pm\omega_1 \rangle, \pm\bar\omega_3 \rangle\rangle = 0$, then $\langle \pm\langle \pm\bar\omega_2, \pm\omega_1 \rangle, \langle \pm\bar\omega_3, \pm\bar\omega_4 \rangle\rangle = 0$. By Proposition 3.2, this implies $\langle \pm\langle \pm\bar\omega_3, \pm\bar\omega_4 \rangle, \langle \pm\bar\omega_2, \pm\omega_1 \rangle\rangle = 0$. Thus the proof is completed by the following propositions.

Proposition 5.2. $\langle \theta_1, \langle \theta_2, \langle \theta_3, \theta_4 \rangle\rangle\rangle = 0$ *for $\theta_1, \theta_2 \in \pm\Lambda$ and $\theta_3, \theta_4 \in \pm\bar\Lambda$.*

Proposition 5.3. $\langle \theta_1, \langle \theta_2, \langle \theta_3, \theta_4 \rangle\rangle\rangle = \langle \theta_1, \langle \theta_2, \langle \theta_4, \theta_3 \rangle\rangle\rangle = 0$ *for $\theta_1, \theta_3 \in \pm\Lambda$, $\theta_2, \theta_4 \in \pm\bar\Lambda$ and $|\theta_3| + |\theta_4| \neq 2p$.*

Proposition 5.4. $\langle \pm\lambda_p, \langle \pm\bar\lambda_p, \langle \theta_1, \theta_2 \rangle\rangle\rangle = 0$ *for $\theta_1, \theta_2 \in \pm\bar\Lambda$.*

We will calculate iterated Samelson products in $\pm\bar\Lambda$ from those in $\pm\Lambda$ by using the following lemma.

Lemma 5.5. *Let*

$$X = (\bigvee_{i=1}^{n_1} S_i^{2np-3}) \cup (\bigcup_{i=1}^{n_2} e_i^{2np-3+2(p-1)}) \cup \cdots \cup (\bigcup_{i=1}^{n_k} e_i^{2np-3+2(k-1)(p-1)})$$

and let $f\colon X \to \mathrm{SU}(p+t-1)$. If $n+k \leqslant p$, then f can be compressed into $S^{2n-1} \subset \mathrm{SU}(p+t-1)$ and $\Sigma^{2k} f = 0$.

Proof. If f falls to B_n, it follows from Theorem 2.1 that $q_*(f)=0$ for the projection $q\colon B_n\to S^{2n+2p-3}$ and then f lifts to $S^{2n-1}\subset B_n$. Thus we assume that f is a map from X to S^{2n-1}. Consider the exact sequence induced from the cofiber sequence $\bigvee_{i=1}^{n_1}S_i^{2np-3}\xrightarrow{j}X\xrightarrow{q'}X/(\bigvee_{i=1}^{n_1}S_i^{2np-3})=Y$:

$$[Y,S^{2n-1}]\xrightarrow{(q')^*}[X,S^{2n-1}]\xrightarrow{j^*}\bigoplus_{i=1}^{n_1}\pi_{2np-3}(S_i^{2n-1}).$$

It follows from Theorem 2.1 that $(\Sigma^2 j)^*(\Sigma^2 f)=0$, and then there exists $g\colon\Sigma^2 Y\to S^{2n+1}$ such that $(\Sigma^2 q')^*(g)=\Sigma^2 f$. By induction, we obtain $\Sigma^{2k}f=0$. □

Corollary 5.6. *Let $X=S^{2n-1}$ or $S^{2n-1}\cup e^{2n+2p-3}$ for $n\leqslant 5p-3$ and let $f\colon X\to\mathrm{SU}(p+t-1)$. Then $\langle\theta,f\rangle=\langle f,\theta\rangle=0$ for each $\theta\in\pm\bar{\Lambda}$.*

Proof. By Corollary 2.3, we only have to consider the case $2n-1=6p-3$, $8p-5$, $8p-3$, $10p-7$. Then it follows from Lemma 5.5 that f can be compressed into S^5 or $S^7\subset\mathrm{SU}(p+t-1)$, and that $\Sigma^4 f=0$. By Proposition 4.4, we assume $|\theta|\geqslant p-2$. Since $p\geqslant 7$, $X(\theta)$ is a 6-suspension and then $1_{X(\theta)}\wedge f=f\wedge 1_{X(\theta)}=0$. Thus Proposition 3.3 completes the proof. □

We give candidates for non-zero 2-iterated Samelson products in $\pm\bar{\Lambda}$.

Proposition 5.7. *Let $\theta_1,\theta_2,\theta_3\in\pm\bar{\Lambda}$. If $|\theta_1|+|\theta_2|+|\theta_3|\neq 2p+1,\ 2p+2,\ 2p+3,\ 3p$, then $\langle\theta_1,\langle\theta_2,\theta_3\rangle\rangle=0$.*

Proof. Suppose that $|\theta_1|+|\theta_2|+|\theta_3|\neq 2p+1,2p+2,2p+3,3p$. By Proposition 3.3, it is sufficient to show that $\langle\theta_1,\langle\theta_2,\theta_3\rangle\rangle=0$ for $\theta_1\in\pm\Lambda$ and $\theta_2,\theta_3\in\pm\bar{\Lambda}$.

By Corollary 2.3, $\langle\theta_1,\langle\theta_2,\theta_3\rangle\rangle=0$ if $\theta_1,\theta_2,\theta_3\in\pm\Lambda$. Then by Proposition 3.2 and Proposition 3.3, it is sufficient to show that $\langle\theta_1,\langle\theta_2,\pm\bar{\lambda}_i\rangle\rangle=\langle\theta_1,\langle\pm\bar{\lambda}_i,\theta_2\rangle\rangle=0$ for $\theta_1,\theta_2\in\pm\Lambda$. Since other cases are analogous, we only show $\langle\lambda_i,\langle\lambda_j,\bar{\lambda}_k\rangle\rangle=0$. When $j\geqslant 3$, A_j is a suspension by (3). Then it follows from (4) that $\langle\lambda_j,\bar{\lambda}_k\rangle=\langle\lambda_j,\lambda_k\rangle\vee f\colon A_j\wedge B_k=(A_j\wedge A_k)\vee(A_j\wedge S^{4k+2p-4})\to\mathrm{SU}(p+t-1)$. By Corollary 2.3, we have $\langle\lambda_i,\langle\lambda_j,\lambda_k\rangle\rangle=0$ and, by Corollary 5.6, $\langle\lambda_i,f\rangle=0$. Then we have established $\langle\lambda_i,\langle\lambda_j,\bar{\lambda}_k\rangle\rangle=0$.

When $j=2$, we assume $k=p-1$ or p by Proposition 4.4. It follows from Theorem 2.1 and Proposition 2.2 that $\langle\lambda_2,\bar{\lambda}_{p-1}\rangle$ falls to B_2. By Corollary 4.12 and Theorem 2.1, we have $q_*(\langle\lambda_2,\bar{\lambda}_{p-1}\rangle)=0$ for the projection $q\colon B_2\to S^{2p+1}$. Then $\langle\lambda_2,\bar{\lambda}_{p-1}\rangle$ lifts to $f\colon A_2\wedge B_{p-1}\to S^3$. Hence, by Proposition 4.4, $\langle\lambda_i,f\rangle=0$ if $i\leqslant p-1$ and this shows that $\langle\lambda_i,\langle\lambda_j,\bar{\lambda}_k\rangle\rangle=0$ when $(j,k)=(2,p-1)$. One can analogously show that $\langle\lambda_i,\langle\lambda_j,\bar{\lambda}_k\rangle\rangle=0$ when $(j,k)=(2,p)$. □

Proof of Proposition 5.4. As in the above proof of Theorem 5.1, Proposition 5.2 implies that it is sufficient to prove $\langle\pm\langle\theta_1,\theta_2\rangle,\langle\pm\bar{\lambda}_p,\pm\lambda_p\rangle\rangle=0$.

By Proposition 5.7, we have only to consider the case that $|\theta_1|+|\theta_2|=p+1,p+2,p+3$ or $2p$. When $|\theta_1|+|\theta_2|=p+1$, $\langle\theta_1,\theta_2\rangle$ falls to $B_2\times S^5\times S^7$, $B_2\times B_3\times S^7$ or $B_2\times B_3\times B_4$ by Theorem 2.1 and Proposition 2.2. On the other hand, $\langle\pm\bar{\lambda}_p,\pm\lambda_p\rangle$ falls to $B_2\times S^5$ or $B_2\times B_3$ by Theorem 2.1 and Proposition 2.2. Then, by Proposition 3.2, Proposition 4.4 and Corollary 4.11, we have obtained that $\langle\pm\lambda_p,\langle\pm\bar{\lambda}_p,\langle\theta_1,\theta_2\rangle\rangle\rangle=0$. Other cases are quite analogous. □

Now we proceed with the calculation to show all 3-iterated Samelson products in $\bar{\Lambda}$ vanish. As a first step, we show:

Proposition 5.8. $\langle\theta_1,\langle\theta_2,\langle\theta_3,\theta_4\rangle\rangle\rangle = 0$ *for* $\theta_1,\dots,\theta_4 \in \pm\Lambda$.

Proof. By Proposition 5.7, we assume that $|\theta_2|+|\theta_3|+|\theta_4| = 2p+1$, $2p+2$, $2p+3$ or $3p$. We only show the case that $(\theta_2,\theta_3,\theta_4) = (\lambda_i,\lambda_j,\lambda_k)$ for $i+j+k = 2p+3$ since the other cases are analogous. By Corollary 2.3, there is a homotopy commutative diagram:

$$\begin{tikzcd} A_i\wedge A_j\wedge A_k \arrow[r, "\langle\lambda_i,\langle\lambda_j,\lambda_k\rangle\rangle"] \arrow[d, "q"'] & \mathrm{SU}(p+t-1) \arrow[d, equal] \\ (\bigvee_{i=1}^3 S_i^{8p-3})\cup e^{10p-5} \arrow[r, "f"] & \mathrm{SU}(p+t-1), \end{tikzcd}$$

where q pinches the $(8p-4)$-skeleton of $A_i\wedge A_j\wedge A_k$. It follows from Lemma 5.5 that f can be compressed into $S^7\subset \mathrm{SU}(p+t-1)$ and that $\Sigma^4 f = 0$. Then, by Proposition 4.4, we assume that $i\geqslant p-2$ and this implies that $X(\theta_1)$ is a 6-suspension. Hence we have $1_{A_i}\wedge f = 0$ and this completes the proof. ☐

Corollary 5.9. $\langle\theta_1,\langle\theta_2,\langle\theta_3,\theta_4\rangle\rangle\rangle = \langle\theta_1,\langle\theta_2,\langle\theta_4,\theta_3\rangle\rangle\rangle = 0$ *for* $\theta_1,\theta_3,\theta_3\in\pm\Lambda$ *and* $\theta_4\in\pm\bar{\Lambda}$.

Proof. By Proposition 5.8, we put $\theta_4 = \pm\bar{\lambda}_i$.

We first consider the case that $\theta_3\neq\pm\lambda_2$. Since $X(\theta_3)$ is a suspension, we have the following homotopy commutative diagram by (4).

$$\begin{tikzcd} X(\theta_3)\wedge B_i \arrow[r, "\langle\theta_3,\pm\bar{\lambda}_i\rangle"] \arrow[d, "\simeq"'] & \mathrm{SU}(p+t-1) \arrow[d, equal] \\ (X(\theta_3)\wedge A_i)\vee(X(\theta_3)\wedge S^{4i+2p-4}) \arrow[r, "\langle\theta_3,\pm\lambda_i\rangle\vee f"] & \mathrm{SU}(p+t-1). \end{tikzcd}$$

Then, by using the homotopy equivalence

$$\bigwedge_{j=1}^{3} X(\theta_j)\wedge B_i \simeq (\bigwedge_{j=1}^{3} X(\theta_j)\wedge A_i)\vee(\bigwedge_{j=1}^{3} X(\theta_j)\wedge S^{4i+2p-4}),$$

we have

$$\langle\theta_1,\langle\theta_2,\langle\theta_3,\pm\bar{\lambda}_i\rangle\rangle\rangle = \langle\theta_1,\langle\theta_2,\langle\theta_3,\pm\lambda_i\rangle\rangle\rangle\vee\langle\theta_1,\langle\theta_2,f\rangle\rangle.$$

Thus, by Corollary 5.6 and Proposition 5.8, we have established $\langle\theta_1,\langle\theta_2,\langle\theta_3,\pm\bar{\lambda}_i\rangle\rangle\rangle = 0$. It is analogous to show $\langle\theta_1,\langle\theta_2,\langle\pm\bar{\lambda}_i,\theta_3\rangle\rangle\rangle = 0$.

We next consider the case that $\theta_3 = \pm\lambda_2$. By Corollary 4.11 and Proposition 5.8, we assume that $i = p-1, p$. It follows from Corollary 4.12 that we also assume

$\langle \pm\lambda_2, \pm\bar{\lambda}_i \rangle \colon A_2 \wedge B_i \to S^{2(2+i-p)+1}$. Then, by (4), we have a homotopy commutative diagram

$$\begin{tikzcd} \Sigma^2(A_2 \wedge B_i) \arrow[rr, "\Sigma^2\langle \pm\lambda_2, \pm\bar{\lambda}_i\rangle"] \arrow[d, "\simeq"'] & & S^{2(2+i-p)+3} \arrow[d, equal] \\ \Sigma^2(A_2 \wedge A_i) \vee \Sigma^2(A_2 \wedge S^{4i+2p-4}) \arrow[rr, "\Sigma^2\langle \pm\lambda_2, \pm\lambda_i\rangle \vee g"] & & S^{2(2+i-p)+3}. \end{tikzcd}$$

By Proposition 5.7, we also assume that $|\theta_2| + |\theta_3| + |\lambda_i| = 2p+1$, $2p+2$, $2p+3$ or $3p$ and this implies that $X(\theta_2)$ is a 6-suspension. Then, by applying the homotopy equivalence

$$(X(\theta_2) \wedge A_2 \wedge A_i) \vee (X(\theta_2) \wedge A_2 \wedge S^{2(2+i-p)+1}),$$

we have

$$\langle \theta_2, \langle \pm\lambda_2, \pm\bar{\lambda}_i \rangle\rangle = \langle \theta_2, \langle \pm\lambda_2, \pm\lambda_i \rangle\rangle \vee (\langle \theta_2, \epsilon_{3+i-p}\rangle \circ (1_{\Sigma^{-2}X(\theta_2)} \wedge g)).$$

By Corollary 5.6, we also have $1_{\Sigma^{-2}X(\theta_1)} \wedge g = 0$ and then by Proposition 5.8 we have obtained $\langle \theta_1, \langle \theta_2, \langle \pm\lambda_2, \pm\bar{\lambda}_i \rangle\rangle\rangle = 0$. We can similarly see that $\langle \theta_1, \langle \theta_2, \langle \pm\bar{\lambda}_i, \pm\lambda_2 \rangle\rangle\rangle = 0$ □

Proof of Proposition 5.2. By Proposition 5.8 and Corollary 5.9, we put $\theta_3 = \pm\bar{\lambda}_i$ and $\theta_4 = \pm\bar{\lambda}_j$.

Applying the homotopy extension property of the inclusion $\Sigma A_i \wedge A_j \to \Sigma A_i \wedge B_j$, we replace a homotopy retraction $\Sigma A_i \wedge B_j \to \Sigma A_i \wedge A_j$ with a strict retraction. We also replace a homotopy retraction $\Sigma A_i \wedge B_j \to \Sigma A_i \wedge A_j$ with a strict one.

Let $Y(i,j)$ be the $(4i+4j+4p-7)$-skeleton of $B_i \wedge B_j$, that is, $Y(i,j)$ is $B_i \wedge B_j$ minus the top cell. Since we have strict retractions $\Sigma A_i \wedge B_j \to \Sigma A_i \wedge A_j$ and $\Sigma A_i \wedge B_j \to \Sigma A_i \wedge A_j$, the proof of Corollary 5.9 implies that we can choose contractions of $\langle \theta_1, \langle \theta_2, \langle \pm\bar{\lambda}_i, \pm\lambda_j \rangle\rangle\rangle$ and $\langle \theta_1, \langle \theta_2, \langle \pm\lambda_i, \pm\bar{\lambda}_j \rangle\rangle\rangle$ to coincide on $X(\theta_1) \wedge X(\theta_2) \wedge A_i \wedge A_j$. Then, by gluing the above contractions, we obtain

$$\langle \theta_1, \langle \theta_2, \langle \pm\bar{\lambda}_i, \pm\bar{\lambda}_j \rangle|_{Y(i,j)}\rangle\rangle = 0 \tag{8}$$

for $\theta_1, \theta_2 \in \pm\Lambda$.

Now we consider first the case $\theta_2 \neq \pm\lambda_2$. As in the proof of Corollary 5.9, we have

$$\langle \theta_2, \langle \pm\bar{\lambda}_i, \pm\bar{\lambda}_j \rangle\rangle = \langle \theta_2, \langle \pm\bar{\lambda}_i, \pm\bar{\lambda}_j \rangle|_{Y(i,j)}\rangle \vee f$$

for some map $f \colon X(\theta_2) \wedge S^{4(i+j+p-2)} \to \mathrm{SU}(p+t-1)$, where we use the homotopy equivalence

$$X(\theta_2) \wedge B_i \wedge B_j \simeq (X(\theta_2) \wedge Y(i,j)) \vee (X(\theta_2) \wedge S^{4(i+j+p-2)}).$$

Then for $(\theta_1, \theta_2, \theta_3) \neq (\pm\lambda_p, \pm\bar{\lambda}_p, \pm\bar{\lambda}_p)$, we have $\langle \theta_1, \langle \theta_2, \langle \pm\bar{\lambda}_i, \pm\bar{\lambda}_j \rangle\rangle\rangle = 0$ by Corollary 5.6 and (8).

By Proposition 2.2, $\langle \pm\bar{\lambda}_p, \pm\bar{\lambda}_p \rangle$ falls to $B_2 \times B_3 \subset \mathrm{SU}(p+t-1)$. Then by Proposition 3.2, it is sufficient to show that $\langle \theta_1, \langle \pm\lambda_p, \lambda_i \circ \pi_i \circ \langle \pm\bar{\lambda}_p, \pm\bar{\lambda}_p \rangle\rangle\rangle = 0$ for $i = 2, 3$

for $\theta_1 \in \pm\Lambda$. Analogously to the above case, we have

$$\langle \pm\lambda_p, \lambda_i \circ \pi_i \circ \langle \pm\bar{\lambda}_p, \pm\bar{\lambda}_p\rangle\rangle = \langle \pm\lambda_p, \lambda_i \circ \pi_i \circ \langle \pm\bar{\lambda}_p, \pm\bar{\lambda}_p\rangle|_{Y(p,p)}\rangle \vee f_i$$

for some map $f_i\colon A_p \wedge S^{12p-8} \to \mathrm{SU}(p+t-1)$, where we use the homotopy equivalence

$$A_p \wedge B_p \wedge B_p \simeq (A_p \wedge Y(p,p)) \vee (A_p \wedge S^{12p-8}).$$

By (8), it is sufficient to show $\langle \theta_1, f_i\rangle = 0$ for $i = 2, 3$. By [**13**], we have $\pi_{14p-9}(S^3) = \pi_{16p-11}(S^3) = 0$ and then $\pi_{14p-9}(B_2) = \pi_{16p-11}(B_2) = 0$ by the homotopy exact sequence of the fibration $S^3 \to B_2 \to S^{2p+1}$ and Theorem 2.1. Thus $f_2 = 0$. Similarly, we have $f_3 = 0$.

We next consider the case $\theta_2 = \pm\lambda_2$. By Proposition 5.7, we put $(i,j) = (p-1,p), (p,p-1), (p,p)$. When $(i,j) = (p,p)$, it follows from Proposition 5.7 that $|\theta_2| = 2$ or 3. By Proposition 2.2, $\langle \pm\bar{\lambda}_p, \pm\bar{\lambda}_p\rangle$ falls to $B_2 \times B_3 \subset \mathrm{SU}(p+t-1)$. Then, by Proposition 3.2 and Corollary 4.11, we have $\langle \theta_2, \langle \pm\bar{\lambda}_p, \pm\bar{\lambda}_p\rangle\rangle = 0$. We shall prove the case $(i,j) = (p-1,p)$. The case $(i,j) = (p,p-1)$ is proved quite analogously. By Proposition 2.2, $\langle \pm\bar{\lambda}_{p-1}, \pm\bar{\lambda}_p\rangle$ falls to B_p and then $\langle \pm\lambda_2, \langle \pm\bar{\lambda}_{p-1}, \pm\bar{\lambda}_p\rangle\rangle$ falls to $B_3 \times B_4$. Moreover, by Theorem 2.1 and Corollary 4.12, we can see that $\pi_i \circ \langle \pm\lambda_2, \langle \pm\bar{\lambda}_{p-1}, \pm\bar{\lambda}_p\rangle\rangle$ can be compressed into S^{2i-1} for $i = 3, 4$. Then, by Proposition 3.2, we assume $|\theta_1| \geqslant p-2$ and then $X(\theta_1)$ is a 6-suspension. By an analogous argument as above, there are maps $g_i\colon \Sigma^2(A^2 \wedge S^{12p-12}) \to S^{2i+1}$ such that

$$\Sigma^2\pi_i \circ \langle \pm\lambda_2, \langle \pm\bar{\lambda}_{p-1}, \pm\bar{\lambda}_p\rangle\rangle = \Sigma^2\pi_i \circ \langle \pm\lambda_2, \langle \pm\bar{\lambda}_{p-1}, \pm\bar{\lambda}_p\rangle|_{Y(p-1,p)}\rangle \vee g_i$$

for $i = 3, 4$, where we use the homotopy equivalence

$$\Sigma^2 A_2 \wedge B_{p-1} \wedge B_p \simeq \Sigma^2(A_2 \wedge Y(p-1,p)) \vee \Sigma^2(A^2 \wedge S^{12p-12}).$$

Then, as in the proof of Corollary 5.9, we obtain $\langle \theta_1, \langle \pm\lambda_2, \langle \pm\bar{\lambda}_{p-1}, \pm\bar{\lambda}_p\rangle\rangle\rangle = 0$. □

In order to calculate other Samelson products, we will use

Lemma 5.10. *Let $g\colon V \to W_1 \vee W_2$ and let $f_i\colon W_i \to X$ for $i = 1, 2$. Suppose that $f_i \circ p_i \circ g = 0$ for $i = 1, 2$ and that X is an H-space, where $p_i\colon W_1 \vee W_2 \to W_i$ is the i-th projection. Then $(f_1 \vee f_2) \circ g = 0$.*

Proof. Define $f_1 \cdot f_2\colon W_1 \times W_2 \to X$ by $f_1 \cdot f_2(x,y) = f_1(x)f_2(y)$ for $(x,y) \in W_1 \times W_2$. Then we have a homotopy commutative diagram

$$\begin{tikzcd}
V \arrow[r, "g"] \arrow[d, equal] & W_1 \vee W_2 \arrow[r, "f_1 \vee f_2"] \arrow[d, "j"] & X \arrow[d, equal] \\
V \arrow[r, "j \circ g"] & W_1 \times W_2 \arrow[r, "f_1 \cdot f_2"] & X,
\end{tikzcd}$$

where j is the inclusion. This completes the proof. □

Proof of Proposition 5.3. We first consider the case $\theta_3 \neq \pm\lambda_2$. If $\theta_4 = \pm\bar{\lambda}_i$, we have

$$\langle \theta_3, \theta_4\rangle = \langle \theta_3, \pm\lambda_i\rangle \vee f$$

for some map $f\colon X(\theta_3) \wedge S^{4i+2p-4} \to \mathrm{SU}(p+t-1)$, where we use the homotopy equivalence

$$X(\theta_3) \wedge B_i \simeq (X(\theta_3) \wedge A_i) \vee (X(\theta_3) \wedge S^{4i+2p-4}).$$

We also have an analogous decomposition of $\langle \pm\bar{\lambda}_i, \theta_3 \rangle$. Then by Corollary 5.6 and Proposition 5.8, it is sufficient to show that $\langle \theta_1, \langle \pm\bar{\lambda}_j, \langle \theta_3, \theta_4 \rangle \rangle \rangle = 0$ for $\theta_1, \theta_3, \theta_4 \in \pm\Lambda$. Since $|\theta_3| + |\theta_4| \neq 2p$, we have $\langle \theta_3, \theta_4 \rangle \colon X(\theta_3) \wedge X(\theta_4) \to S^{2(i+j-p)+1}$ by Proposition 4.6, Corollary 4.8 and Corollary 4.12. Since we have

$$\langle \pm\bar{\lambda}_j, \epsilon_{i+j-p+1} \rangle = \langle \pm\lambda_j, \epsilon_{i+j-p+1} \rangle \vee g$$

for some map $g \colon S^{2i+6j-3} \to \mathrm{SU}(p+t-1)$ by applying the homotopy equivalence

$$B_j \wedge S^{2(i+j-p+1)-1} \simeq (A_j \wedge S^{2(i+j-p+1)-1}) \vee S^{2i+6j-3} \to \mathrm{SU}(p+t-1).$$

Then by Corollary 5.6, Proposition 5.8 and Lemma 5.10, we obtain $\langle \theta_1, \langle \pm\bar{\lambda}_j, \langle \theta_3, \theta_4 \rangle \rangle \rangle = 0$.

We next consider the case $\theta_3 = \pm\lambda_2$. By Corollary 4.11, $\theta_4 = \pm\bar{\lambda}_{p-1}$ or $\pm\bar{\lambda}_p$ and then by Proposition 5.8, $\theta_2 = \pm\bar{\lambda}_p$ and $\theta_2 = \bar{\lambda}_{p-1}$ or $\bar{\lambda}_p$ as $\theta_4 = \pm\bar{\lambda}_{p-1}$ or $\pm\bar{\lambda}_p$. Now we consider the case $\theta_4 = \pm\bar{\lambda}_{p-1}$. By Proposition 2.2, $\langle \pm\lambda_2, \pm\bar{\lambda}_{p-1} \rangle$ falls to $B_2 \subset \mathrm{SU}(p+t-1)$. By Theorem 2.1, we have $[A_2 \wedge S^{6p-8}, S^{2p+1}] = 0$ and then the inclusion $A_2 \wedge A_{p-1} \to A_2 \wedge B_{p-1}$ induces an injection $[A_2 \wedge B_{p-1}, S^{2p+1}] \to [A_2 \wedge A_{p-1}, S^{2p+1}]$. On the other hand, it follows from Corollary 4.12 that $q_*(\langle \pm\lambda_2, \pm\lambda_{p-1} \rangle) = 0$ for the projection $q \colon B_2 \to S^{2p+1}$ and then $q_*(\langle \pm\lambda_2, \pm\bar{\lambda}_{p-1} \rangle) = 0$ which is equivalent to that $\langle \pm\lambda_2, \pm\bar{\lambda}_{p-1} \rangle$ can be compressed into $S^3 \subset B_2$. Note that

$$\langle \pm\bar{\lambda}_p, \epsilon_2 \rangle = \langle \pm\lambda_p, \epsilon_2 \rangle \vee f \colon B_p \wedge S^3 \simeq (A_p \wedge S^3) \vee (S^{6p-4} \wedge S^3) \to \mathrm{SU}(p+t-1).$$

By Corollary 5.6, we have $\langle \theta_1, f \rangle = 0$. Then by Lemma 5.10, it is sufficient to show that $\langle \theta_1, \langle \pm\lambda_p, \langle \pm\lambda_2, \pm\bar{\lambda}_{p-1} \rangle \rangle \rangle = 0$ and this is done by Corollary 5.9. The equality $\langle \theta_1, \langle \pm\lambda_p, \langle \pm\bar{\lambda}_{p-1}, \pm\lambda_2 \rangle \rangle \rangle = 0$ can be shown in an analogous way.

Let us consider the case $\theta_4 = \pm\bar{\lambda}_p$. As above, $\langle \pm\lambda_2, \pm\bar{\lambda}_p \rangle$ can be compressed into $S^5 \times S^7$ and then, by Proposition 3.2, it is sufficient to show that $\langle \theta_1, \langle \theta_2, \epsilon_i \circ \pi_i \circ \langle \pm\lambda_2, \pm\bar{\lambda}_p \rangle \rangle \rangle = 0$ for $i = 3, 4$. This is done quite analogously to the above case. We can also see that $\langle \theta_1, \langle \theta_2, \langle \pm\bar{\lambda}_p, \pm\lambda_2 \rangle \rangle \rangle = 0$ as well. □

6. Proof of Theorem 1.1

6.1. $t = 2$

We shall show $\langle \epsilon_{p-1}, \langle \lambda_2, \epsilon_p \rangle \rangle \neq 0$ and then, by Theorem 5.1, the proof of Theorem 1.1 is completed. By Theorem 2.1 and Proposition 2.2, $\langle \lambda_2, \epsilon_p \rangle$ falls to $S^5 \subset \mathrm{SU}(p+1)$. Since $\langle \epsilon_2, \epsilon_p \rangle \neq 0$ by Proposition 4.2, we have $\langle \lambda_2, \epsilon_p \rangle = a\overline{\alpha_1(5)}$ for some integer a such that $a \not\equiv 0$ (p), where $\overline{\alpha_1(5)} \colon C_{\alpha_1(2p+2)} \simeq A_2 \wedge S^{2p-1} \to S^5$ is an extension of $\alpha_1(5)$. Analogously, we have $\langle \epsilon_{p-1}, \epsilon_3 \rangle = b\alpha_1(5)$ for an integer b such that $b \not\equiv 0$ (p). Then, by [**12**, Proposition 1.9],

$$\langle \epsilon_{p-1}, \langle \lambda_2, \epsilon_p \rangle \rangle = ab\alpha_1(5) \circ \Sigma^{2p-3}\overline{\alpha_1(5)} = abq^*(\{\alpha_1(5), \alpha_1(2p+2), \alpha_1(4p-1)\}), \quad (9)$$

where $q \colon S^{2p-3} \wedge A_2 \wedge S^{2p-1} \to S^{6p-3}$ pinches the bottom cell.

Consider the exact sequence induced from the cofiber sequence $S^{2p-3} \wedge A_2 \wedge S^{2p-1} \xrightarrow{q} S^{6p-3} \xrightarrow{\alpha_1(4p)} S^{4p}$:

$$\pi_{4p}(S^5) \xrightarrow{\alpha_1(4p)^*} \pi_{6p-3}(S^5) \xrightarrow{q^*} [S^{2p-3} \wedge A_2 \wedge S^{2p-1}, S^5].$$

By Theorem 2.1, $\alpha_1(4p)^* = 0$ and then q^* is monic. It is known that $\{\alpha_1(5), \alpha_1(2p+2), \alpha_1(4p-1)\} \neq 0$ (see, for example, [**4**, p. 38]) and thus, by (9), we have established $\langle \epsilon_{p-1}, \langle \lambda_2, \epsilon_p \rangle \rangle \neq 0$.

6.2. $3 \leqslant t \leqslant \frac{p-1}{2}$

By Proposition 4.4, possible non-trivial 2-iterated Samelson products in $\pm\bar{\Lambda}$ are:

1. $\langle \pm\epsilon_p, \langle \pm\epsilon_p, \pm\epsilon_p \rangle \rangle$.
2. $\langle \pm\bar{\lambda}_i, \langle \pm\epsilon_j, \pm\epsilon_k \rangle \rangle$, $\langle \pm\epsilon_i, \langle \pm\bar{\lambda}_j, \pm\epsilon_k \rangle \rangle$, $\langle \pm\epsilon_i, \langle \pm\epsilon_j, \pm\bar{\lambda}_k \rangle \rangle$ for $i+j+k = 2p+1$, $2p+2$, $2p+3$.

We shall show these Samelson products are all trivial and then, by Proposition 3.1, the proof is completed.

1. By the Jacobi identity of Samelson products, we have $3\langle \pm\epsilon_p, \langle \pm\epsilon_p, \pm\epsilon_p \rangle \rangle = 0$ and then, for $p > 3$, $\langle \pm\epsilon_p, \langle \pm\epsilon_p, \pm\epsilon_p \rangle \rangle = 0$.
2. By Proposition 3.2, it is sufficient to show $\langle \pm\epsilon_i, \langle \pm\bar{\lambda}_j, \pm\epsilon_k \rangle \rangle = \langle \pm\epsilon_i, \langle \pm\epsilon_j, \pm\bar{\lambda}_k \rangle \rangle = 0$ for $i+j+k = 2p+1$, $2p+2$, $2p+3$. Let us consider $\langle \pm\epsilon_i, \langle \pm\bar{\lambda}_j, \pm\epsilon_k \rangle \rangle$ for $i+j+k = 2p+1$. By (4), we have $\langle \pm\epsilon_i, \langle \pm\bar{\lambda}_j, \pm\epsilon_k \rangle \rangle = \langle \pm\epsilon_i, \langle \pm\lambda_j, \pm\epsilon_k \rangle \rangle \vee \langle \epsilon_i, f \rangle$ for some $f\colon S^{4j+2p-4} \wedge S^{2k-1} \to \mathrm{SU}(p+t-1)$. Then, by Corollary 5.6, it is sufficient to show $\langle \pm\epsilon_i, \langle \pm\lambda_j, \pm\epsilon_k \rangle \rangle = 0$.

Let us consider the case $i+j+k = 2p+1$. By Proposition 4.12, $\langle \pm\lambda_j, \pm\epsilon_k \rangle$ can be compressed into $S^{2(j+k-p)+1} \subset \mathrm{SU}(p+t-1)$ and then we have

$$\langle \pm\epsilon_i, \langle \pm\lambda_j, \pm\epsilon_k \rangle \rangle = \langle \pm\epsilon_i, \epsilon_{j+k-p+1} \rangle \circ (1_{S^{2i-1}} \wedge f),$$

where $f\colon A_j \wedge S^{2k-1} \to S^{2(j+k-p)+1}$. Since $i+j+k-p+1 = p+2 \leqslant p+t-1$, we have $\langle \pm\epsilon_i, \epsilon_{j+k-p+1} \rangle = 0$ and then $\langle \pm\epsilon_i, \langle \pm\lambda_j, \pm\epsilon_k \rangle \rangle = 0$. Analogously, we can see $\langle \pm\epsilon_i, \langle \pm\epsilon_j, \pm\bar{\lambda}_k \rangle \rangle = 0$.

When $i+j+k = 2p+2, 2p+3$, it follows from Corollary 2.3 that $\langle \pm\epsilon_i, \langle \pm\lambda_j, \pm\epsilon_k \rangle \rangle = 0$.

6.3. $\frac{p+1}{2} \leqslant t \leqslant p$

Put $t \neq p$. We shall show $\langle \lambda_{p-t+1}, \langle \lambda_t, \epsilon_p \rangle \rangle \neq 0$ and this completes the proof of Theorem 1.1 by Theorem 5.1. Let X be the $(8p-4)$-skeleton of $A_{p-t+1} \wedge A_t \wedge S^{2p-1}$, that is, $A_{p-t+1} \wedge A_t \wedge S^{2p-1}$ minus the top cell. Then, as in Section 4, the cofiber sequence $S^{2(p-t)+1} \wedge A_t \wedge S^{2p-1} \to X \xrightarrow{q} S^{6p-3}$ splits. We denote a homotopy section of q by s, where q is the restriction of the pinch map $A_{p-t+1} \wedge A_t \wedge S^{2p-1} \to S^{2(2p-t)-1} \wedge A_t \wedge S^{2p-1}$. Then, by Proposition 4.8, we have a homotopy commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "(1_{A_{p-t+1}}\wedge\langle\lambda_t,\epsilon_p\rangle)|_X"] \arrow[d, "\bar{q}"'] & A_{p-t+1}\wedge S^{2t+1} \arrow[r, "\langle\lambda_{p-t+1},\epsilon_{t+1}\rangle"] \arrow[d, "q\wedge 1_{S^{2t+1}}"'] & B_3 \arrow[d, equal] \\
S^{2(2p-t)-1}\wedge S^{2t-1}\wedge S^{2p-1} \arrow[r, "1_{S^{2(2p-t)-1}}\wedge\langle\epsilon_t,\epsilon_p\rangle"] & S^{2(2p-t)-1}\wedge S^{2t+1} \arrow[r, "f"] & B_3.
\end{tikzcd}$$

By Theorem 2.1 and Proposition 4.2, we have $1_{S^{2(2p-t)-1}} \wedge \langle\epsilon_t, \epsilon_p\rangle = a\alpha_1(4p)$ for some integer a such that $a \not\equiv 0\ (p)$.

Let $\underline{\alpha_1(2p+2)}\colon S^{4p} \to A_3$ be a coextension of $\alpha_1(2p+2)$. Then, as in Section 2, we have $f = bi_*(\underline{\alpha_1(2p+2)})$ for some integer b, where $i\colon A_3 \to B_3$ is the inclusion. Suppose that $b = b'p$. Then, by [**12**, Proposition 1.8], we have

$$f = b'i_*(\underline{\alpha_1(2p+2)} \circ p) = -b'i_* \circ j_*(\{\alpha_1(5), \alpha_1(2p+2), p\}) = -\tfrac{b'}{2} i_* \circ j_*(\alpha_2(5)),$$

where $j\colon S^5 \to A_3$ is the inclusion. In particular, f lifts to $S^5 \subset B_3$ and this contradicts Proposition 4.9. Thus we have $b \not\equiv 0\ (p)$.

On the other hand, it follows from [**12**, Proposition 1.8] that

$$\underline{\alpha_1(2p+2)} \circ \alpha_1(2p+2) = -j_*\{\alpha_1(5), \alpha_1(2p+2), \alpha_1(4p-1)\}.$$

It is known that $\{\alpha_1(5), \alpha_1(2p+2), \alpha_1(4p-1)\} \neq 0$ as above and then we have established

$$\begin{aligned} f \circ (1_{S^{2(2p-t)-1}} \wedge \langle\epsilon_t, \epsilon_p\rangle) &= f \circ (1_{S^{2(2p-t)-1}} \wedge \langle\epsilon_t, \epsilon_p\rangle) \circ \bar{q} \circ s \\ &= \langle\lambda_{p-t+1}, \langle\lambda_t, \epsilon_p\rangle\rangle \circ s \\ &\neq 0. \end{aligned}$$

This implies $\langle\lambda_{p-t+1}, \langle\lambda_t, \epsilon_p\rangle\rangle \neq 0$.

When $t = p$, the proof is completed by the homotopy exact sequence induced from the fiber sequence $\mathrm{SU}(2p-2) \to \mathrm{SU}(2p-1) \to S^{4p-3}$.

References

[**1**] R. Bott, A note on the Samelson product in the classical groups, *Comment. Math. Helv.* **34** (1960), 249–256.

[**2**] A.K. Bousfield and D.M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Mathematics **304**, Springer-Verlag, Berlin–New York, 1972.

[**3**] H. Hamanaka, On Samelson products in p-localized unitary groups, *Topology Appl.* **154** (2007), no. 3, 573–583.

[**4**] J.R. Harper, *Secondary Cohomology Operations, Graduate Studies in Mathematics* **49**, American Mathematical Society, Providence, RI, 2002.

[**5**] S. Kaji and D. Kishimoto, Homotopy nilpotency in p-compact groups, preprint.

[**6**] C.A. McGibbon, Some properties of H-spaces of rank 2, *Proc. Amer. Math. Soc.* **81** (1981), no. 1, 121–124.

[**7**] C. McGibbon, Homotopy commutativity in localized groups, *Amer. J. Math.* **106** (1984), 665–687.

[**8**] C.A. McGibbon, Higher forms of homotopy commutativity and finite loop spaces, *Math. Z.* **201** (1989), no. 3, 363–374.

[**9**] M. Mimura and H. Toda, Cohomology operations and homotopy of compact Lie groups I, *Topology* **9** (1970), 317–336.

[**10**] J.B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962), 64–94.

[**11**] L. Saumell, Higher homotopy commutativity in localized groups *Math. Z.* **219** (1995), no. 2, 203–213.

[**12**] H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Ann. of Math. Studies **49**, Princeton Univ. Press, Princeton N.J., 1962.

[**13**] H. Toda, On iterated suspensions I, *J. Math. Kyoto Univ.* **5** (1966), 87–142.

[**14**] F.D. Williams, Higher homotopy-commutativity, Trans. Amer. Math. Soc. **139** (1969), 191–206.

[**15**] A. Zabrodsky, *Hopf Spaces*, North-Holland Mathematics Studies **22**, North-Holland, Amsterdam, 1976.

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