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UMKEHR MAPS

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Abstract

In this note, we study umkehr maps in generalized (co)homology theories arising from the Pontrjagin-Thom construction, from integrating along fibers, pushforward homomorphisms, and other similar constructions. We consider the basic properties of these constructions and develop axioms which any umkehr homomorphism must satisfy. We use a version of Brown representability to show that these axioms completely characterize these homomorphisms, and a resulting uniqueness theorem follows. Finally, motivated by constructions in string topology, we extend this axiomatic treatment of umkehr homomorphisms to a fiberwise setting.

1. Introduction

The classical umkehr homomorphism of Hopf [**11**] assigns to a map $f\colon M \to N$ of closed manifolds of the same dimension a "wrong way" homomorphism $f_!\colon H_*(N) \to H_*(M)$ on singular homology. Hopf showed that this map is compatible with intersection pairings. Freudenthal [**6**] showed that $f_!$ corresponds to the homomorphism $f^*\colon H^*(N) \to H^*(M)$ induced by f on cohomology by means of the Poincaré duality isomorphisms for M and N. This identification allows one to give a definition of the umkehr homomorphism for a map between closed manifolds of any dimension.

Variants of the umkehr homomorphism, such as those defined by the Pontrjagin-Thom construction, intersections of chains, integration along fibers, and the Becker-Gottlieb transfer, have played central roles in the development of differential and algebraic topology. Similarly, the "push-forward" constructions in cohomology, Chow groups, and K-theory, have been important techniques in algebraic geometry and index theory. Topological generalizations of umkehr mappings have played important roles in recent developments in topology, such as Madsen and Weiss's proof of the Mumford conjecture and its generalizations [**7, 8, 14**], and the development of string topology [**2, 3**].

Considering these various different, but related constructions, it is natural to ask how they are related? Similarly, one might ask: what properties characterize or classify umkehr homomorphisms?

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The goal of this note is to describe naturally occurring axioms which completely classify umkehr homomorphisms. These axioms come as a result of considering the basic properties of the umkehr homomorphisms mentioned above. We will show that a Brown-type representability theorem classifies these umkehr maps. In more recent applications, such as those in string topology, umkehr homomorphisms were needed in the setting of pullback squares of Serre fibrations,

$$\begin{array}{ccc} E_1 & \xrightarrow{\tilde{f}} & E_2 \\ \downarrow & & \downarrow \\ P & \xrightarrow[f]{} & N, \end{array}$$

where $f\colon P \to N$ is a smooth map of manifolds. That is, one wanted an umkehr homomorphism, $\tilde{f}_!\colon H_*(E_2) \to H_*(E_1)$ (with a dimension shift of $\dim N - \dim P$). This leads us to consider axioms for the existence and uniqueness of umkehr homomorphism in this fiberwise setting, using a fiberwise version of Brown representability, which we prove in the appendix.

2. Preliminaries

We will work in the category $\mathcal{T}$ of compactly generated weak Hausdorff spaces. Products are to be re-topologized using the compactly generated topology. Mapping spaces are to be given the compactly generated, compact open topology. A *weak equivalence* of spaces denotes a (chain of) weak homotopy equivalence(s).

We will assume that the reader is familiar with the standard machinery of algebraic topology including homotopy limits and colimits (the standard reference for the latter is [**1**]).

A spectrum E is a sequence of based spaces E_n, $n \geqslant 0$ together with (structure) maps $\Sigma E_n \to E_{n+1}$, where Σ denotes reduced suspension. A spectrum has homotopy groups $\pi_n(E)$ for $n \in \mathbb{Z}$ defined by the colimit of the system $\{\pi_{n+j}(E_j)\}_{j\geqslant 0}$. A morphism of spectra $E \to E'$ consists of maps $E_n \to E'_n$ which are compatible with the structure maps. A morphism is a weak equivalence if it induces an isomorphism in homotopy groups in each degree. The category of spectra is denoted $\mathcal{S}$.

We say that a spectrum E is an *Ω-spectrum* if the adjoint maps $E_n \to \Omega E_{n+1}$ are weak equivalences. For any spectrum E, there a weak equivalence $E \simeq E^{\mathrm{f}}$ with E^{f} an Ω-spectrum. This weak equivalence is natural.

For an unbased space K, we let $\mathrm{map}(K, E)$ denote the mapping spectrum whose j-th space is given by the space of (unbased) maps $K \to E_j$. The basepoint of this mapping space is taken to be the constant map at the basepoint of E_j. The structure maps in this case are induced by suspending and taking the adjunction. For this to have the "correct" homotopy type, it should be assumed that E is an Ω-spectrum and that K has the homotopy type of a CW complex.

Although it will not be emphasized in the paper, the above discussion fits naturally within the context of a Quillen model category structure on the category of spectra (see, for example, [**17**]).

3. What should an umkehr map do?

Umkehr homomorphisms are known to occur in all cohomology theories. Now, every cohomology theory is representable, so one can view the umkehr homomorphism as arising from a certain map of spectra.

Minimally, an umkehr map should assign to an embedding $f\colon P \subset N$ of closed manifolds a wrong way stable map

$$f^!\colon N^+ \to P^\nu ,$$

where ν is the normal bundle of f. One definition of $f^!$ is given by taking the Pontryagin-Thom construction of the embedding f.

For a variety of reasons, it is also desirable to twist the above by an arbitrary vector bundle ξ over N. In this case, the umkehr map should give a map of Thom spectra

$$f^!_\xi\colon N^\xi \to P^{\nu+\xi}.$$

Classically, such an $f^!_\xi$ is produced by taking the Pontryagin-Thom construction of the composite

$$P \subset N \xrightarrow{\text{zero section}} D(\xi),$$

where $D(\xi)$ is the unit disk bundle of ξ. This directly motivates one to consider umkehr maps as being defined not only for closed manifolds, but more generally for maps of compact manifolds *having a boundary.* The twisting by bundles then becomes a special case, as $D(\xi)$ is a manifold with boundary.

For example, if f fails to be an embedding, then we can always approximate the composite

$$f_j\colon P \subset N = N \times 0 \subset N \times D^j$$

by an embedding when j is sufficiently large, and therefore, assuming umkehr maps have been defined for manifolds with boundary, we obtain an umkehr map for f_j which we simply declare to be the umkehr map for f.

The above suggests the following: Let $\mathcal{M}$ be the category whose objects are compact manifolds P (possibly with boundary) in which a morphism $P \to Q$ is a continuous map (not necessarily preserving the boundary). Let $\mathcal{S}$ be the category of spectra. We will consider *contravariant* functors

$$u\colon \mathcal{M} \to \mathcal{S}$$

satisfying certain axioms.

The first two axioms will be:

- *Vacuum axiom.* If $\emptyset$ is the empty manifold, then $u(\emptyset)$ is contractible.
- *Homotopy invariance axiom.* If $f\colon P \to Q$ is a weak (homotopy) equivalence, then so is $u(f)$.

The vacuum axiom is motivated by the fact that the Pontryagin-Thom collapse of an empty submanifold yields a constant map.

The homotopy invariance axiom is motivated by the following: Let $P \subset N$ be a homotopy equivalence, where P is closed. Then the Pontryagin-Thom collapse $N/\partial N \to P^\nu$ is a stable homotopy equivalence.

The last axiom is that umkehr functors are required to satisfy is *locality*. In its most geometric form, locality will mean that a decomposition of manifolds yields a corresponding decomposition of their Pontryagin-Thom collapses. Suppose, for example, that $P \subset S^n$ is a closed submanifold with normal bundle ν such that P is transverse to the equator $S^{n-1} \subset S^n$. Let $D^n_\pm$ denote the upper and lower hemispheres. Setting $P_\pm = P \cap D^n_\pm$ and $Q = P \cap S^{n-1}$, we obtain a decomposition $P = P_- \cup_Q P_+$. The Pontryagin-Thom collapse of each inclusion $(P_\pm, Q) \subset (D^n_\pm, S^{n-1})$ gives the maps

$$k_\pm \colon (D^n_\pm, S^{n-1}) \to (P^\nu_\pm, Q^\nu),$$

which may be glued to a yield a map

$$S^n = D^n_- \cup_{S^{n-1}} D^n_+ \xrightarrow{k_- \cup k_+} P^\nu_- \cup_{Q^\nu} P^\nu_+ = P^\nu,$$

which is just the Pontryagin-Thom construction of $P \subset S^n$.

In general, it seems that the cleanest way to formulate the locality axiom is in terms of a (right homotopy) Kan extension of u to the category $\mathcal{T}$ of topological spaces. The resulting functor will also be homotopy invariant. The Kan extension $u^\# \colon \mathcal{T} \to \mathcal{S}$ is the contravariant functor given by

$$u^\#(Y) = \operatorname*{holim}_{P \xrightarrow{\sim} Y} u(P),$$

where the homotopy limit is indexed over the category of compact manifolds P equipped with a weak equivalence to Y. Notice that $u^\#$ restricted to $\mathcal{M}$ coincides with u up to natural equivalence.

- *Locality axiom.* The functor $u^\#$ is excisive, i.e., it preserves homotopy cocartesian squares.

The above axioms imply, by a version of Brown's representability theorem (cf. appendix), that the composite $u^\#$ is representable: there is an Ω-spectrum E, unique up to weak equivalence, and a natural weak equivalence

$$u^\#(X) \simeq \operatorname{map}(X, E),$$

where $\operatorname{map}(X, E)$ denotes the spectrum of (unbased) maps from X to E, i.e., the spectrum whose j-th space is the space of unbased maps $X \to E_j$. (In the above, we are implicitly using the fact that every compact manifold has the homotopy type of a finite complex. This will imply that $u^\#$ is determined up to equivalence by its restriction to the category of finite complexes over X.)

Notice that we can recover E by taking of $u(*)$, where $*$ is the one-point manifold. Summarizing,

Theorem 3.1. *An umkehr functor $u \colon \mathcal{M} \to \mathcal{S}$ is characterized up to natural weak equivalence by its value $E := u(*)$ at the one-point manifold.*

Conversely, an Ω-spectrum E gives rise to an umkehr functor u by the rule

$$u(P) := \operatorname{map}(P, E).$$

Examples

(1) *The Pontryagin-Thom construction.* The traditional Pontryagin-Thom construction comes from the umkehr functor corresponding to the spectrum

$E = u(*) = S^0$, the sphere spectrum. That is,

$$u(P) = \mathrm{map}(P, S^0),$$

the Spanier-Whitehead dual of P. The fact that this functor yields the Pontrjagin-Thom construction comes from Atiyah duality, which gives a natural equivalence of spectra,

$$\mathrm{map}(P, S^0) \simeq P^{-\tau_P},$$

where $P^{-\tau_P}$ is the Thom spectrum of the stable normal bundle, that is, the virtual bundle $-\tau_P$, where τ_P is the tangent bundle of P.

Given an embedding $f \colon P \to N$ with normal bundle $\nu(f)$, the map of Spanier-Whitehead duals, $f_! \colon \mathrm{map}(N, S^0) \to \mathrm{map}(P, S^0)$, is equivalent to the Pontryagin-Thom map

$$N^{-\tau_N} \to P^{-\tau_N \oplus \nu(f)} = P^{-\tau_P}.$$

(2) *Integration along the fibers.* Consider a smooth submersion of closed oriented manifolds,

$$P^{n+k} \xrightarrow{p} M^n,$$

where the superscript denotes dimension. Then integration along fibers defines a homomorphism in de Rham cohomology,

$$p^{\int} \colon H^q_{\mathrm{dR}}(P) \to H^{q-k}_{\mathrm{dR}}(M).$$

This can be seen in terms of the umkehr functor defined by setting $u(*) = h\mathbb{R}$, the Eilenberg-Mac Lane spectrum for the real numbers. In other words, $u(N) = \mathrm{map}(N, h\mathbb{R})$. The homomorphism induced by the bundle gives a homomorphism,

$$\mathrm{map}(M, h\mathbb{R}) \xrightarrow{p_!} \mathrm{map}(P, h\mathbb{R}),$$

which can be written as $p_! \colon \mathrm{map}(M, S^0) \wedge h\mathbb{R} \to \mathrm{map}(P, S^0) \wedge h\mathbb{R}$. Using Atiyah duality, as in the previous example, this is equivalent to a map which, by abuse of notation, we also call $p_!$,

$$p_! \colon M^{-\tau_M} \wedge h\mathbb{R} \to P^{-\tau_P} \wedge h\mathbb{R}.$$

When one applies homotopy groups to this map, and the Thom isomorphism, one obtains a homomorphism,

$$p_! \colon H_{q-k}(M; \mathbb{R}) \to H_q(P; \mathbb{R}),$$

which is linearly dual to the integration map $p^{\int}$. That is,

$$\int_\alpha p^{\int}(\omega) = \int_{p_! \alpha} \omega.$$

(3) *Oriented bordism.* For a space X, let $\mathrm{MSO}_p(X)$ denote bordism classes of maps $P \to X$, where P is a closed smooth oriented p-manifold. If

$$f \colon Q \to N$$

is a map of closed smooth oriented manifolds, then we obtain an umkehr homomorphism

$$f^!_* \colon \mathrm{MSO}_p(N) \to \mathrm{MSO}_{p+q-n}(Q)$$

as follows. Let $\gamma \in \mathrm{MSO}_p(N)$. Choose a representative $g\colon P \to N$ of γ in such a way that f and g are mutually transverse. Then the fiber product $P \times_N Q$ is an oriented manifold of dimension $p + q - n$, and the bordism class of the evident map $P \times_N Q \to Q$ defines the umkehr homomorphism. Of course, the spectrum representing the associated umkehr functor is the Thom spectrum, MSO.

4. A generalization

A generalization of the umkehr map arises naturally within the framework of string topology.

The context is this: one has an embedding $P \subset N$ of closed manifolds with normal bundle ν, and also a (not necessarily smooth) fiber bundle $p\colon E \to N$. Let $q\colon E_{|P} \to P$ be the restriction of p to P. Then we have a cartesian square

$$\begin{array}{ccc} E_{|P} & \longrightarrow & E \\ \downarrow & & \downarrow \\ P & \longrightarrow & N. \end{array}$$

The spaces E and $E_{|P}$ may not be smooth, and may even be infinite-dimensional, (for example in string topology the total space E is typically built from path or loop spaces in the manifold N). However one still observes that the codimension of $E_{|P}$ in E is finite, and that one can find a regular neighborhood which is homeomorphic to the pullback of ν along q. Collapsing a complement of this tubular neighborhood to a point, we obtain a based map

$$E^+ \to (E_{|P})^{q^*\nu},$$

where the target is the Thom space of $q^*\nu$. Given this construction, which seems to depend on certain choices, it is not entirely clear that it carries with it any uniqueness properties. We will show in fact that it does.

It will be convenient for us to categorify the above. The idea will be that the above umkehr map can be thought of as arising from a suitable functor on the category of manifolds *over* N. The representing objects in this setting will be fiberwise spectra. For the sake of completeness, we begin with a digression describing those aspects of fiberwise spectra that we will need for our purposes. The reader is referred to [**15**] for a more complete discussion.

Fibered spectra

One may regard a spectrum as a generalization of an abelian group, where the latter objects appear as the Eilenberg-Mac Lane spectra. Analogously, a fibered spectrum on a space X can be thought of as a system of local coefficients on X in which the fibers, which were formerly abelian groups, are now replaced by spectra.

For a space X, let $\mathcal{T}_X$ be the category of spaces over X. Its objects are spaces Y equipped with a fixed choice of map $Y \to X$. A morphism is a map of underlying spaces which is compatible with maps to X. Similarly, let $\mathcal{R}_X$ be the category of *retractive spaces* over X. An object is a space Y equipped with maps $s_Y\colon X \to Y$ and $r_Y\colon Y \to X$ such that $r_Y \circ s_Y$ is the identity map (the structure maps r_Y, s_Y are usually suppressed from the notation). A morphism $f\colon Y \to Z$ is a map of underlying spaces which commutes with their structure maps: $r_Z \circ f = r_Y$ and $f \circ s_Y = s_Z$.

A morphism in either of the above categories is said to be a *weak equivalence* if it is a weak homotopy equivalence of underlying spaces.

One has a forgetful functor $u\colon \mathcal{R}_X \to \mathcal{T}_X$. There is also a functor v in the other direction given by $Y \mapsto Y^+$, where Y^+ is the retractive space $Y \amalg X$. One readily verifies that u is a right adjoint to v.

Given objects $Y, Z \in \mathcal{R}_X$, the hom-set $\hom_{\mathcal{R}_X}(Y, Z)$ may be topologized as a subspace of the function space of all continuous maps $Y \to Z$ of underlying spaces, where the function space is equipped with the compactly generated compact open topology. This gives $\mathcal{R}_X$ the structure of a topological category.

Definition 4.1 (Fiberwise suspension)**.** Given an object $Y \in \mathcal{T}_X$, its *unreduced fiberwise suspension* is defined to be the double mapping cylinder

$$S_X Y := X \times 0 \cup Y \times [0,1] \cup X \times 1.$$

It comes with an evident map $S_X Y \to X$, so it is an object of $\mathcal{T}_X$.

Given an object $Y \in \mathcal{R}_X$, its *reduced fiberwise suspension* is given by

$$\Sigma_X Y = S_X Y \cup_{S_X X} X.$$

Note that Σ_X defines an endo-functor of $\mathcal{R}_X$.

If Y, Z are objects of $\mathcal{R}_X$, then its fiberwise smash product $Y \wedge_X Z$ is the object given by the pushout of the diagram

$$X \leftarrow Y \cup_X Z \to Y \times_X Z.$$

Definition 4.2 (Fibered spectra)**.** A *fibered spectrum* $\mathcal{E}$ over X consists of objects $\mathcal{E}_j \in \mathcal{R}_X$ for $j \in \mathbb{N}$ together with (structure) maps

$$\Sigma_X \mathcal{E}_j \to \mathcal{E}_{j+1},$$

for each $j \geqslant 0$. A *morphism* $\mathcal{E} \to \mathcal{E}'$ is given by maps $\mathcal{E}_j \to \mathcal{E}'_j$ which are compatible with the structure maps.

We say that $\mathcal{E}$ is *fibrant* if the adjoints to the structure maps are weak homotopy equivalences of underlying spaces. Any fibered spectrum $\mathcal{E}$ can be converted into a fibrant one $\mathcal{E}^{\mathrm{f}}$ in which

$$\mathcal{E}^{\mathrm{f}}_j := \operatorname*{hocolim}_n \ \Omega^n_X \mathcal{E}_{j+n},$$

where the homotopy colimit is taken in $\mathcal{R}_X$, and Ω^j_X is the adjoint to n-fold reduced fiberwise suspension. The above is called *fibrant replacement.*

A morphism $\mathcal{E} \to \mathcal{E}'$ is a *weak equivalence* if the associated morphism of fibrant replacements $\mathcal{E}^{\mathrm{f}} \to (\mathcal{E}')^{\mathrm{f}}$ is a *levelwise* weak equivalence: for each j, the map $\mathcal{E}^{\mathrm{f}}_j \to (\mathcal{E}')^{\mathrm{f}}_j$ is required to be weak equivalence of $\mathcal{R}_X$.

Examples

(1) *Fiberwise suspension spectra.* Let $Y \in \mathcal{R}_X$ be an object. Let $\Sigma^\infty_X Y$ be the fibered spectrum over X given by the collection $\Sigma^j_X Y$ of iterated fiberwise suspensions of Y.

(2) *Trivial fibered spectra.* Let C be a spectrum. The collection of spaces $C_j \times X$ as j-varies forms a fibered spectrum over X. The maps

$$\Sigma_X(C_j \times X) \to C_{j+1} \times X$$

use the identification $\Sigma_X(C_j \times X) = (\Sigma C_j) \times X$ together with the structure maps of C.

(3) *Fibered Eilenberg-Mac Lane spectra.* Let $\mathcal{F}$ be a bundle of abelian groups on X. Let $\mathcal{F}_x$ denote the fiber at x. Then we have a fibered spectrum $h\mathcal{F}$ on X, in which $h\mathcal{F}_j$ can be described as follows: the fiber at $x \in X$ is given by $K(\mathcal{F}_x, j)$, the Eilenberg-Mac Lane space based on the abelian group $\mathcal{F}_x$.

(4) *Fiberwise smash product with a spectrum.* Let C be a spectrum and let $E \to X$ be a fibration. Then we obtain a fibered spectrum $E \otimes C$ in whose j-th total space is given by the pushout of the diagram

$$X \longleftarrow E \times * \xrightarrow{\subset} E \times C_j.$$

If E_x is the fiber to $E \to X$ at $x \in X$, then the fiber of $(E \otimes C)_j \to X$ is given by $(E_x)_+ \wedge C$.

(5) *Twisted suspension.* Let $\mathcal{E}$ be a fibered spectrum over X. If ξ is a vector bundle over X, then we can form a new fibered spectrum ${}^{\xi}\mathcal{E}$ called the *twist* of $\mathcal{E}$ by ξ. The j-th total space of ${}^{\xi}\mathcal{E}$ takes the form of a fiberwise smash product

$$S^{\xi} \wedge_X \mathcal{E}_j,$$

where S^{ξ} is the object of $\mathcal{R}_X$ given by the fiberwise one-point compactification of ξ. The notion of twisting extends to the case when ξ is a virtual bundle (we omit the details, but see Example (2) below of the Poincaré duality equivalence (Theorem 4.3)).

Homology

A fibered spectrum $\mathcal{E}$ gives rise to a covariant spectrum-valued functor

$$H_\bullet(-;\mathcal{E})\colon \mathcal{T}_X \to \mathcal{S}$$

called *homology* with $\mathcal{E}$-coefficients.

Consider the following construction: let $Y \in \mathcal{T}_X$ be an object and call the structure map $f\colon Y \to X$. Let $f^*\mathcal{E}$ be the fibered spectrum over Y given by the collection of fiber products $Y \times_X \mathcal{E}_j$. The set of quotient spaces

$$(Y \times_X \mathcal{E}_j)/Y$$

yields a spectrum. However, we must take the derived version of this construction to insure homotopy invariance.

Here are the details. First of all, we need to replace $\mathcal{E}$ with its fibrant replacement $\mathcal{E}^{\mathrm{f}}$. Secondly, we must replace the above quotient, by a homotopy quotient, i.e., the mapping cone. The result of these changes will produce a spectrum with j-th space

$$(Y \times_X \mathcal{E}^{\mathrm{f}}_j) \cup_Y CY.$$

This spectrum is $H_\bullet(Y;\mathcal{E})$.

Examples

- The homology spectrum of the trivial fibered spectrum (Example (2) above) is $C \wedge Y_+$.
- For the fibered Eilenberg-Mac Lane spectrum $h\mathcal{F}$ (Example (3) above), the homology spectrum of Y has homotopy groups isomorphic to the homology Y with coefficients in the bundle of coefficients $\mathcal{F}$ pulled back to Y.

Cohomology

Given a fibered spectrum $\mathcal{E}$, we obtain a contravariant spectrum-valued functor

$$H^\bullet(-;\mathcal{E})\colon \mathcal{T}_X \to \mathcal{S}$$

called *cohomology* with $\mathcal{E}$-coefficients. Roughly, it is given at an object Y by taking the spectrum of sections of $\mathcal{E}$ along $Y \to X$.

More precisely, we consider the spectrum whose j-th space is the hom-space $\hom_{\mathcal{T}_X}(Y, \mathcal{E}_j)$ (or equivalently, the space of sections of $\mathcal{E}_j \to X$ along Y). The structure maps for $\mathcal{E}$ yield structure maps on these hom-spaces, so we obtain a spectrum.

To get a homotopy invariant version of this construction, we need to replace $\mathcal{E}$ by its fibrant replacement, and Y by a functorial cellular approximation (for example, we can replace Y by $|SY|$, the realization of the simplicial total singular complex of X). The result of these manipulations yields a spectrum $H^\bullet(Y;\mathcal{E})$ which is homotopy invariant in Y.

Poincaré duality

Let N be a closed manifold of dimension d with tangent bundle τ_N. Let $-\tau_N$ be the virtual bundle of dimension $-d$ representing the stable normal bundle of P.

We now state the Poincaré duality theorem with coefficients in a fibered spectrum.

Theorem 4.3 (Poincaré duality)**.** *For any fibered spectrum $\mathcal{E}$ over N, there is a weak equivalence of spectra*

$$H_\bullet(N; {}^{-\tau_N}\mathcal{E}) \simeq H^\bullet(N;\mathcal{E}).$$

The equivalence is natural in $\mathcal{E}$.

Although usually stated differently, this result appears in the literature (see [**12**, Thms. A, D], [**13**, §5,8], [**16**, Thm. 4.9], [**19**, Prop. 2.4]).

Definition 4.4. A closed n-manifold N is *$\mathcal{E}$-orientable* if there is a weak equivalence of fibered spectra

$${}^{-\tau_N}\mathcal{E} \simeq \mathcal{E}[-n],$$

where $\mathcal{E}[-n]$ is the n-fold fiberwise desuspension of $\mathcal{E}$.

Corollary 4.5. *Assume N is $\mathcal{E}$-orientable. Then there is a weak equivalence of spectra*

$$H_\bullet(N;\mathcal{E}[-n]) \simeq H^\bullet(N;\mathcal{E}).$$

Examples

(1) *Atiyah and Spanier-Whitehead duality.* Let $\mathcal{E}$ be the trivial suspension spectrum, $\Sigma^\infty_N N$. In other words, the j^{th}-space is given by

$$(\Sigma^\infty_N N)_j = S^j \times N \to N,$$

so the fiber over any point is the sphere S^j. We can describe the twisted spectrum, ${}^{-\tau_N}(\Sigma^\infty_N N)$ in the following way: Suppose we have an embedding in Euclidean space, $N \hookrightarrow \mathbb{R}^L$, with normal bundle $\nu_L \to N$. Then for any $j \geqslant 0$, the $(j+L)^{th}$ space of the twisted spectrum is given by

$$({}^{-\tau_N}\Sigma^\infty_N N)_{j+L} = S^{\nu_L} \wedge_N (\Sigma^\infty_N N)_j = S^{\nu_L} \wedge S^j.$$

Then clearly we have an identification

$$H_\bullet(N; {}^{-\tau_N}\Sigma^\infty_N N) = N^{-\tau_N},$$

where the right side is the Thom spectrum of the virtual bundle $-\tau_N$. On the other hand, the cohomology spectrum, $H^\bullet(N; \Sigma^\infty_N N)$ has as its j^{th}-space the space of sections,

$$\hom_{\mathcal{T}_N}(N, S^j \times N) = \mathrm{map}(N, S^j).$$

In other words, this cohomology spectrum is the mapping spectrum $\mathrm{map}(N, S^0)$, or the Spanier-Whitehead dual of N_+. Thus the Poincaré duality equivalence, Theorem 4.3 in this case, gives the Atiyah duality,

$$N^{-\tau_N} \simeq \mathrm{map}(N, S^0).$$

(2) *The free loop space and string topology.*
Let $LN = \mathrm{map}(S1, N)$ be the free loop space, and let $e\colon LN \to N$ be the fibration that evaluates a loop at the basepoint $0 \in \mathbb{R}/\mathbb{Z} = S1$. The fiber at x_0 of e is the based loop space, $\Omega_{x_0} N$.
Let L^+N denote the disjoint union $LN \amalg N$, and define $e^+\colon L^+N \to N$ to be $e \amalg \mathrm{id}$. Then e^+ defines a retractive space over N (where the section is given by the summand N). We consider the fiberwise suspension spectrum, $\mathcal{E} = \Sigma^\infty_N L^+N$. This fibered spectrum has as its j^{th} space the j-fold fiberwise reduced suspension, $\Sigma^j_N L^+N$, which fibers over N, with fiber $\Sigma^j(\Omega N)_+$. We consider the Poincaré duality equivalence (Theorem 4.3) in the case of this fibered spectrum.
We consider the twisted spectrum ${}^{-\tau_N}(\Sigma^\infty_N L^+N)$. This fibered spectrum can be described in the following way. Suppose, as above, $N \hookrightarrow \mathbb{R}^L$ with normal bundle $\nu_L \to N$. Then for any $j \geqslant 0$, the $(j+L)^{th}$ space of the twisted spectrum is given by

$$({}^{-\tau_N}\Sigma^\infty_N L^+N)_{j+L} = S^{\nu_L} \wedge_N (\Sigma^j_N L^+N).$$

Then clearly the homology spectrum is given by

$$H_\bullet(N; {}^{-\tau_N}\Sigma^\infty_N L^+N) = LN^{-\tau_N}, \tag{1}$$

the Thom spectrum of the virtual bundle $e^*(-\tau_N)$. It was shown in [**3**] that the spectrum $LN^{-\tau_N}$ is a ring spectrum, whose induced product in homology reflects the Chas-Sullivan loop product in *string topology* [**2**] after one applies the

Thom isomorphism. This product can be seen by applying the Poincaré duality equivalence (Theorem 4.3) as follows.
The cohomology spectrum, $H^\bullet(N; \Sigma^\infty_N L^+N)$ has as its j^{th}-space the space of sections, $\hom_{\mathcal{T}_N}(N, \Sigma^j_N L^+N)$. We therefore write $\hom_{\mathcal{T}_N}(N, \Sigma^\infty_N L^+N)$ for this spectrum. The Poincaré duality equivalence in this setting gives an equivalence,

$$LN^{-\tau_N} \simeq \hom_{\mathcal{T}_N}(N, \Sigma^\infty_N L^+N). \tag{2}$$

Now notice that the fiberwise spectrum $\Sigma^\infty_N L^+N$ is a fiberwise ring spectrum, since the fibration $L^+N \to N$ is a fiberwise monoid. (More precisely it is a fiberwise A_∞-monoid. See [**10**].) Thus the spectrum of sections, $\hom_{\mathcal{T}_N}(N, \Sigma^\infty_N L^+N)$ is a ring spectrum, which by Poincare duality is equivalent to the Thom spectrum $LN^{-\tau_N}$. As was seen in [**10**] and [**4**], the ring structure coming from the fiberwise monoid corresponds to the ring spectrum structure on $LN^{-\tau_N}$ described in [**3**], and thus reflects the string topology loop product.

Generalized umkehr functors

Let X be a topological space. Let $\mathcal{M}_X$ be the category whose objects are compact manifolds P (possibly with boundary) equipped with a map $P \to X$; the map will not usually be specified in the notation. A morphism is a map $f\colon P \to Q$ which is compatible with maps to N in the obvious way (again, we do not require that f preserves boundaries). A morphism is a weak equivalence if and only if the underlying map of spaces is a weak homotopy equivalence.

We will consider contravariant functors

$$u\colon \mathcal{M}_X \to \mathcal{S}.$$

Definition 4.6. A functor u will be called a *generalized umkehr functor* if it satisfies three axioms. The first two axioms are:

- **Axiom 1** (*Vacuum*). The value of u at the empty manifold $\emptyset$ is contractible.
- **Axiom 2** (*Homotopy invariance*). u is a homotopy functor, i.e., if a morphism $f\colon P \to Q$ is a weak (homotopy) equivalence, then so is $u(f)$.

Recall that $\mathcal{T}_X$ is the category of spaces over X. As before, we can perform a homotopy Kan extension to u along the full inclusion $\mathcal{M}_X \subset \mathcal{T}_X$ to obtain a contravariant homotopy functor

$$u^\#\colon \mathcal{T}_X \to \mathcal{S}.$$

The final axiom for generalized umkehr functors is:

- **Axiom 3** (*Locality*). The functor $u^\#$ preserves homotopy cocartesian squares. (A square of $\mathcal{T}_X$ is homotopy cocartesian if it is one when considered in $\mathcal{T}$ by means of the forgetful functor.)

Again, we see that these axioms imply that $u^\#$ is representable. (The appropriate fiberwise version of Brown representability will be proved in the appendix.) In this fiberwise setting, representability means there is a fibered spectrum $\mathcal{E}$, unique up to equivalence, and a natural weak equivalence

$$u^\#(X) \simeq H^\bullet(X; \mathcal{E}).$$

In particular, $\mathcal{E}$ and u determine one another. Summarizing,

Theorem 4.7. *A fibered spectrum $\mathcal{E} \to N$ gives rise to an umkehr functor by the rule*

$$u(P) := H^\bullet(P;\mathcal{E}).$$

Conversely, a functor u that satisfies axioms 1–3 determines a fibered spectrum $\mathcal{E} \to N$, unique up to weak equivalence, whose associated cohomology recovers u up to natural equivalence.

The generalized umkehr homomorphism

Let $\mathcal{E} \to X$ be a fibered spectrum, and suppose

$$f\colon P \to Q$$

is a morphism of $\mathcal{M}_X$ such that P and Q are closed manifolds. We then have an induced map on cohomology spectra

$$f^\bullet\colon H^\bullet(Q;\mathcal{E}) \to H^\bullet(P;\mathcal{E}).$$

Using the Poincaré duality equivalence, we can rewrite this up to homotopy as a map

$$f^!\colon H_\bullet(Q; {}^{-\tau_Q}\mathcal{E}) \to H_\bullet(P; {}^{-\tau_P}\mathcal{E}).$$

Assume now that P and Q are $\mathcal{E}$-oriented. Then taking homotopy groups of $f^!$, we get a homomorphism

$$f^!_*\colon H_*(Q;\mathcal{E}) \to H_{*+q-p}(P;\mathcal{E}).$$

This is the generalized umkehr homomorphism.

Umkehr maps in string topology

As seen in Example 2 of the Poincaré duality equivalence, the basic ring structure arising in string topology can be seen via the equivalence (2) of $LM^{-\tau_M}$ and the ring spectrum, $\hom_{\mathcal{T}_M}(M, \Sigma^\infty_M L^+M)$. Here M is a closed manifold. However, in its original form [**2**] and [**3**], the string topology product was created via an umkehr map. We now see how this fits into our framework.

Let $L^\infty M$ be the space of maps from the figure eight $S^1 \vee S^1$ to M. This space is the fiber product, $LM \times_M LM$. That is, we have a pullback square

$$\begin{array}{ccc} L^\infty M & \xrightarrow{\subset} & LM \times LM \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e\times e} \\ M & \xrightarrow[\Delta]{} & M \times M, \end{array}$$

where Δ is the a diagonal map, the vertical maps of the square are the fibrations given by evaluation at the basepoint, and the upper horizontal map arises from the quotient map $S^1 \amalg S^1 \to S^1 \vee S^1$ by taking maps into M.

Let $h\mathbb{Z}$ be the Eilenberg-Mac Lane spectrum on the integers. Consider the product

$$LM \times LM \xrightarrow{e\times e} M \times M$$

as an object in $\mathcal{R}_{M\times M}$. We consider the fiberwise smash product spectrum $\mathcal{E} = (LM \times LM) \otimes h\mathbb{Z}$. (See Example (4) after Definition (4.2) above.)

In particular, $(LM \times LM) \otimes h\mathbb{Z}$ is the fibered spectrum whose j-th space is given by the pushout of the diagram

$$M \times M \longleftarrow LM \times LM \times * \xrightarrow{\subset} LM \times LM \times (h\mathbb{Z})_j .$$

The umkehr homomorphism taken with respect to the fibered spectrum $(LM \times LM) \otimes h\mathbb{Z}$, applied to the diagonal map $\Delta\colon M \to M \times M$, which is viewed as a morphism in $\mathcal{M}_{M\times M}$, is then computed by the induced map in cohomology spectra,

$$\begin{aligned} H^\bullet(M \times M; (LM \times LM) \otimes h\mathbb{Z}) &\xrightarrow{\Delta^\bullet} H^\bullet(M; (LM \times LM) \otimes h\mathbb{Z}) \qquad (3)\\ \hom_{\mathcal{T}_{M\times M}}(M \times M, (LM \times LM) \otimes h\mathbb{Z}) &\xrightarrow{\Delta^*} \hom_{\mathcal{T}_{M\times M}}(M, (LM \times LM) \otimes h\mathbb{Z}) \\ &= \hom_{\mathcal{T}_M}(M, L_\infty M \otimes h\mathbb{Z}), \end{aligned}$$

and then we apply the Poincaré duality equivalence (Theorem 4.3). However, by an argument completely analogous to that used to verify (1) in Example (2) of the Poincaré duality equivalence, we see that

$$H_\bullet(M \times M; {}^{-\tau_{M\times M}}((LM \times LM) \otimes h\mathbb{Z})) = LM^{-\tau_M} \wedge LM^{-\tau_M} \wedge h\mathbb{Z}.$$

If M is oriented in singular homology, then this last spectrum is equivalent to $\Sigma^{-2m}(LM \wedge LM)_+ \wedge h\mathbb{Z}$. Similarly, from the above pullback square we see that

$$H_\bullet(M; {}^{-\tau_{M\times M}}((LM \times LM) \otimes h\mathbb{Z})) = L_\infty M^{-\tau_M} \wedge h\mathbb{Z},$$

where the last spectrum is equivalent to $\Sigma^{-m}(L_\infty M)_+ \wedge h\mathbb{Z}$ assuming M is equipped with an orientation. Thus the umkehr map in this situation gives a map

$$\Sigma^{-2m} LM \wedge LM \wedge h\mathbb{Z} \to \Sigma^{-m} L_\infty M \wedge h\mathbb{Z},$$

or, by taking homotopy groups, this takes the form

$$H_*(LM \times LM) \to H_{*-m}(L^\infty M).$$

The Chas-Sullivan loop product is given by the composite

$$\begin{aligned} H_p(LM) \otimes H_q(LM) &\to H_{p+q}(LM \times LM) \\ &\to H_{p+q-m}(L^\infty M) \to H_{p+q-m}(LM), \end{aligned}$$

where the first homomorphism is the external product, the second is the umkehr homomorphism and the third is given by taking the homology of the map of spaces $L^\infty M \to LM$ arising from pinch map $S^1 \to S^1 \vee S^1$.

Appendix A. Representability

The purpose of this section is to outline a proof of the representability theorem for contravariant homotopy functors $f\colon \mathcal{T}_X \to \mathcal{S}$.

Definition A.1. A (contravariant) homotopy functor f is said to be *strongly excisive* if:

- f preserves homotopy pushouts, and

- f converts coproducts to products up to weak equivalence, i.e., for any collection of objects Y_α, the natural map
$$f(\coprod_\alpha Y_\alpha) \to \prod_\alpha f(Y_\alpha)$$
is a weak equivalence.

Remark A.2. The conditions imply that up to weak equivalence, f is determined by its restriction to the full subcategory of finite complexes over X. This means that the natural map
$$f(Y) \to \operatorname*{holim}_{Z \in C_Y} f(Z)$$
is a weak equivalence, where the homotopy limit is indexed over the category C_Y consisting of spaces over Y which are homeomorphic to a finite complex. (To prove this, note that any object Y is, up to equivalence, a filtered (homotopy) colimit of objects Y_α, where the underlying space of K_α is a finite CW complex. Furthermore, such a filtered homotopy colimit can be expressed as a pushout, whose terms have the form $\amalg_\beta Z_\beta$, where Z_β is a finite complex. The proof is now completed using the strong excisiveness of f.)

Definition A.3. A functor $f\colon \mathcal{T}_X \to \mathcal{S}$ is *cohomological* if:

- f is a homotopy functor.
- The value of f at the initial object $\emptyset$ is contractible.
- f is strongly excisive.

Theorem A.4 (Representability)**.** *For cohomological functors f, there is a fibered spectrum $\mathcal{E}$ and a natural weak equivalence of functors*
$$f(Y) \simeq H^\bullet(Y;\mathcal{E}).$$

Remark A.5.

(1) The fibered spectrum $\mathcal{E}$ is unique up to equivalence. Heuristically, the value of $H^\bullet(-;\mathcal{E})$ at the one-point maps $x \to X$ recovers the fibers $\mathcal{E}_x$ of $\mathcal{E}$. The homotopy colimit in the category of *un*based spaces of $(\mathcal{E}_x)_j$ recovers the j-th total space $\mathcal{E}_j$ up to equivalence.

(2) Our method of proof can be adapted to show that the functor
$$\mathcal{E} \mapsto H^\bullet(-;\mathcal{E})$$
defines an equivalence the homotopy category of fibered spectra over X and the homotopy category of cohomological functors. We will not need this statement.

The main tool in the proof of Theorem A.4 is a natural transformation,
$$f(Y) \to f^\natural(Y),$$
called the *coassembly map*, which is defined for any homotopy functor f. The target functor $f^\natural$ is always strongly excisive.

The idea will then be to show that the coassembly map is a weak equivalence when f satisfies our assumptions, and that $f^\natural$ is representable.

The coassembly map

Let Δ_Y be category whose objects are maps $\Delta^p \to Y$ (for $p \geqslant 0$), where Δ^p is the standard p-simplex, and morphisms are given by inclusions of faces of the simplex. Let $f^\natural$ be the functor defined by

$$f^\natural(Y) = \operatorname*{holim}_{\Delta_Y} f_{|\Delta_Y},$$

where $f_{|\Delta_Y}$ is the restriction of f to Δ_Y. Given any map $\Delta^p \to Y$ we obtain a map $f(Y) \to f(\Delta^p)$. This assignment is compatible with taking faces, so we get a natural map

$$c\colon f(Y) \to f^\natural(Y).$$

This is the coassembly map.

We now verify the properties of $f^\natural$. Note that $f^\natural$ is a homotopy functor since f is and the homotopy limit construction is homotopy invariant. Since

$$\Delta_{\amalg_\alpha Y_\alpha} = \coprod_\alpha \Delta_{Y_\alpha}$$

and the homotopy limit indexed over a coproduct of categories is the product of the corresponding homotopy limits, we see that f maps coproducts to products up to weak equivalence. One can also show that $f^\natural$ preserves homotopy pushouts. One way to see this is to note that there is a natural equivalence between the functor $f_{|\Delta_Y}$ and the constant functor with value $f(*)$, but the homotopy limit of the latter will clearly map homotopy pushouts to homotopy pullbacks. Since homotopy pullback diagrams in spectra are the same as homotopy pushouts, we infer that $f^\natural$ is strongly excisive.

Proof of Theorem A.4. Assuming that f is cohomological, we first show that the coassembly map

$$c\colon f(Y) \to f^\natural(Y)$$

is a weak equivalence. It clearly is a weak equivalence when Y is the initial object. It is also a weak equivalence when Y is a point, since in this case the map $\Delta^p \to *$ is a weak equivalence and f is a homotopy functor. Since f is strongly excisive, c is a weak equivalence when Y is a finite set over X.

A Mayer-Vietoris argument then shows that the coassembly map is a weak equivalence whenever Y is a finite complex over X. Because f is strongly excisive, this is enough to show that c is a weak equivalence in general, since f is determined up to weak equivalence by its restriction to the category of finite complexes over X.

To complete the proof of Theorem A.4, we will show that $f^\natural$ is representable. For $Y \in \mathcal{T}_X$, consider the functor

$$f_j^Y : \Delta_Y \to \mathcal{T}$$

$f_j^Y(\sigma) = f(\Delta^p)_j$, the j-th space of the spectrum $f(\Delta^p)$, which we will consider here as an unbased space. Define

$$\mathcal{E}(Y)_j := \operatorname{hocolim} f_j^Y.$$

If we let j vary, the $\mathcal{E}(Y)_j$ define a fibered spectrum $\mathcal{E}(Y)$.

By considering the constant map $f_j^Y(Y) \to *$ and taking homotopy colimits, we have a map

$$\mathcal{E}(Y)_j \to B\Delta_Y,$$

where $B\Delta Y$ is the classifying space of the category Δ_Y, i.e, the geometric realization of its nerve (recall that the homotopy colimit of the constant functor to a point is $B\Delta_Y$). This map has the following properties:

- It is a quasifibration, i.e., the map from each fiber to its corresponding homotopy fiber is a weak equivalence.
- The space of sections of the associated fibration is weak equivalent to the homotopy limit of f_j^Y. This is an observation of Dwyer [**5**, Prop. 3.12].
- By definition, the collection

  $$\{\operatorname{holim} f_j^Y\}_{j\geqslant 0}$$

  is the spectrum $f^\natural(Y)$.
- The square

  $$\begin{array}{ccc} \mathcal{E}(Y)_j & \longrightarrow & \mathcal{E}(X)_j \\ \downarrow & & \downarrow \\ B\Delta_Y & \longrightarrow & B\Delta_X \end{array}$$

  is homotopy cartesian.

Set $\mathcal{E} := \mathcal{E}(X)$. Then $\mathcal{E}$ is a fibered spectrum over X, and it is a straightforward consequence of the above properties that there is a natural weak equivalence $f^\natural(Y) \simeq H^\bullet(Y;\mathcal{E})$. This completes the proof of Theorem A.4. □

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