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THE SECOND REAL JOHNSON-WILSON THEORY AND NONIMMERSIONS OF RP^n, PART II

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(*communicated by Donald M. Davis*)

Abstract

This paper is a continuation of the study begun in the previous paper with the same title. We analyze $ER(2)^{16*+8}(RP^{2n})$ and compute $ER(2)^*(RP^{16K+1})$, and use these to prove more nonimmersion theorems for RP^n, including many in fairly low dimensions. In particular, we get 12 new nonimmersion results for RP^n where $n < 192$, the range included in the tables Don Davis keeps. These complement the 10 already found in the first paper.

1. Introduction

This paper is a continuation of [**KW**], which we refer to as Part I. We make free use of the notation and results of Part I.

The main theorem of [**Dav84**] states that for

$$n = m + \alpha(m) - 1, \qquad k = 2m - \alpha(m);$$

there does not exist an axial map

$$RP^{2^K-2k-2} \times RP^{2n} \longrightarrow RP^{2^K-2n-2},$$

and so, by [**Jam63**], $RP^{2n} \not\subseteq \mathbb{R}^{2k}$ for these n and k.

This is proven using the equivalent of $E(2)^*(-)$ by showing that the $u^{2^{K-1}-n} = 0$ on the right would have to go to a nonzero element on the left. That prevents the existence of the axial map. In Part I we constructed a purely algebraic surjection

$$ER(2)^{16*}(RP^{2^K-2k-4}) \longrightarrow E(2)^{16*}(RP^{2^K-2k-2}),$$

that allowed us to show the axial map

$$RP^{2^K-2k-4} \times RP^{2n} \longrightarrow RP^{2^K-2n-2}$$

did not exist if we added the restrictions to k and n that $n \equiv 7$ or $0 \bmod 8$ and $-k-2 \equiv 1, 2, 5$ or $6 \bmod 8$. This improved some nonimmersion results by 2.

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In this paper we are able to include the $n \equiv 3$ and 4 mod 8 cases by analyzing $ER(2)^{16*+8}(RP^{2n})$ and constructing a similar algebraic map

$$ER(2)^{16*+8}(RP^{2^K-2k-4}) \longrightarrow E(2)^{16*+8}(RP^{2^K-2k-2})$$

with the previous restrictions on k.

In order to describe $ER(2)^{16*+8}(RP^{2n})$ properly we need to define and study an element $y \in ER(2)^8(RP^\infty)$. The simple version of our answer, similar to our understanding of $ER(2)^{16*}(RP^{2n})$ in Theorem 1.6 of Part I, is:

Theorem 1.1. *A 2-adic basis for $ER(2)^{16*+8}(RP^{2n})$ consists of the elements $\alpha_2\alpha^k u^j$, with $0 \leqslant k$ and $0 < j < n$, yu^j, with $0 \leqslant j \leqslant n-4$, and when*

$n \equiv 3$ or 4 modulo 8, no other elements, with $yu^{n-3} = 0$;

$n \equiv 2$ or 5 modulo 8, $\alpha^k yu^{n-3}$, with $yu^{n-2} = 0$;

$n \equiv 1$ or 6 modulo 8, $\alpha^k yu^{n-3}$, and yu^{n-2}, with $yu^{n-1} = 0$;

$n \equiv 7$ or 0 modulo 8, yu^{n-3}, yu^{n-2}, and yu^{n-1}, with $yu^n = 0$;

and no others.

In addition to this we need some information about $ER(2)^*(-)$ of products and then we are able to prove:

Theorem 1.2. *When the pair $(m, \alpha(m))$ is, modulo 8, $(0,3)$,$(5,6)$, $(4,7)$ or $(1,2)$, then*

$$RP^{2(m+\alpha(m))} \not\subseteq \mathbb{R}^{2(2m-\alpha(m)+1)}.$$

Don Davis points out that by combining this with Theorem 1.9 of Part I, we really have the result for $(m, \alpha(m))$ equal to $(0,3)$ and $(1,2)$ mod 4.

The most interesting pair to us is $(m, \alpha(m)) = (0,3)$. Let $m = 8 + 16 + 2^i$, then $2(m + \alpha(m)) = 54 + 2^{i+1}$ and $2(2m - \alpha(m) + 1) = 92 + 2^{i+2}$. With this we get

$$RP^{54+2^{i+1}} \not\subseteq \mathbb{R}^{92+2^{i+2}}.$$

The lowest dimensional cases are

$$RP^{118} \not\subseteq \mathbb{R}^{220}; \qquad RP^{182} \not\subseteq \mathbb{R}^{348}.$$

However, the importance to us is that it gets on Don Davis's tables, [**Dav**]. Notice also that for these cases there is now only a knowledge gap of 1 between best known nonimmersions and best known immersions.

Next we move on to compute $ER(2)^*(RP^{16K+1})$, analyze $ER(2)^{8*}(RP^{16K+1})$, and construct an algebraic map

$$ER(2)^{8*}(RP^{16K+1}) \longrightarrow E(2)^{8*}(RP^{16K+2})$$

that allows us to do similar things for nonimmersions when, in our axial maps, $-k-1 \equiv 1$ mod 8. The theory $E(2)^*(-)$ cannot make use of the odd spaces because the top cell is not connected algebraically, but for $ER(2)^*(-)$ the connection is strong for $16K+1$ and $16K+9$. We have not done the computation for $16K+9$ because, although there are surely more nonimmersions there, they are not of low enough dimension to inspire us to do the work, whereas the $16K+1$ case gives lots of nice new low dimensional results.

Theorem 1.3. *A 2-adic basis for $ER(2)^{16*}(RP^{16K+1})$ is given by the elements $\alpha^k u^j$, $0 \leqslant k$, $0 < j \leqslant 8K+1$, with $u^{8K+2} = 0$.*

A 2-adic basis for $ER(2)^{16*+8}(RP^{16K+1})$ is given by the elements

$$\alpha_2 \alpha^k u^j, \qquad 0 \leqslant k, \qquad 0 < j < 8K;$$
$$\alpha_2 \alpha^k u^{8K} = \alpha^{k+1} y u^{8K-3};$$
$$w \alpha^k u;$$
$$y u^j, \qquad 0 \leqslant j < 8K, \text{ with } y u^{8K} = 0.$$

Using this, we get:

Theorem 1.4. *For the mod 8 pairs $(m, \alpha(m)) = (6,6), (1,4), (2,6), (5,4)$ we have:*

$$RP^{2(m+\alpha(m)-1)} \not\subseteq \mathbb{R}^{2(2m-\alpha(m))+1}.$$

Let's look at the numbers. First, the pair $(1,4)$. The lowest possible nonimmersions we get from this are

$$RP^{56+2^{i+1}} \not\subseteq \mathbb{R}^{93+2^{i+2}},$$

which also implies another new result:

$$RP^{57+2^{i+1}} \not\subseteq \mathbb{R}^{93+2^{i+2}}.$$

The lowest dimensional examples are:

$$RP^{120} \not\subseteq \mathbb{R}^{221}; \qquad RP^{121} \not\subseteq \mathbb{R}^{221}; \qquad RP^{184} \not\subseteq \mathbb{R}^{349}; \qquad RP^{185} \not\subseteq \mathbb{R}^{349}.$$

The next pair to look at is $(5,4)$. From this we get

$$RP^{16+2^{i+1}+2^{j+1}} \not\subseteq \mathbb{R}^{13+2^{i+2}+2^{j+2}}.$$

When $i = 3$, this is:

$$RP^{32+2^{j+1}} \not\subseteq \mathbb{R}^{45+2^{j+2}}.$$

The lowest dimensional examples are:

$$RP^{96} \not\subseteq \mathbb{R}^{173}; \qquad RP^{160} \not\subseteq \mathbb{R}^{301}.$$

When $i = 4$, this is:

$$RP^{48+2^{j+1}} \not\subseteq \mathbb{R}^{77+2^{j+2}}.$$

which also implies that

$$RP^{49+2^{j+1}} \not\subseteq \mathbb{R}^{77+2^{j+2}}.$$

The lowest dimensional examples are:

$$RP^{112} \not\subseteq \mathbb{R}^{205}; \qquad RP^{113} \not\subseteq \mathbb{R}^{205}; \qquad RP^{176} \not\subseteq \mathbb{R}^{333}; \qquad RP^{177} \not\subseteq \mathbb{R}^{333}.$$

In the tables, [**Dav**], the best known results for nonimmersions for RP^n for $n < 192$ are listed. Of these, 95 are solved completely because it is known that RP^n immerses in the next higher dimension. Of the remaining 96 cases, we improve on 12 in this paper; $n =$ 96, 112, 113, 118, 120, 121, 160, 176, 177, 182, 184, and 185, making for a total of 22 when combined with 10 from Part I; $n = 48$, 62, 80, 94, 110, 126, 144, 158, 174, and 190.

The tables also list what is known for $n = d + 2^i$ ($d < 2^i$) for $0 \leqslant d < 64$. Of these 64 cases, 24 are known completely. Of the remaining 40, we improve on 10; 6 from this paper, d =32, 48, 49, 54, 56, and 57, and 4 from Part I; d =16, 30, 46, and 62.

We are fairly confident that $ER(2)^*(-)$ will not give any more results in these low dimensions. Before attacking the present cases in this paper, computer computations were made on all of the cases we believed we could approach below 192 and we have now proven all of the results that seemed to be there.

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2. Injections

$ER(2)^*(X)$ has already been computed for $X = RP^\infty$, RP^{2n}, and $RP^\infty \wedge RP^\infty$. However, we need a more detailed analysis.

Theorem 2.1. *The map* $ER(2)^{16*+8}(RP^\infty) \to E(2)^{16*+8}(RP^\infty)$ *is an injection with cokernel given by* $v_2^4 u^{\{1-3\}}$.

Proof. Recall that a 2-adic basis for $E(2)^*(RP^\infty)$ is given by

$$v_2^i \alpha^k u^j, \qquad 0 \leqslant i < 8, \qquad 0 \leqslant k, \qquad 1 \leqslant j.$$

The elements of degree 8 mod 16 are

$$v_2^4 \alpha^k u^j, \qquad 0 \leqslant k, \qquad 1 \leqslant j.$$

From Theorem 8.1 of Part 1 we can read off the elements of $ER(2)^{16*+8}(RP^\infty)$. From the x^1-torsion we have

$$\alpha_2 \alpha^k u^j, \qquad 0 \leqslant k, \qquad 1 \leqslant j.$$

From the x^3-torsion we have

$$w\alpha^k u, \quad k \geqslant 0, \text{ and } wu^j, \quad 1 < j.$$

Mapping these elements to $E(2)^*(RP^\infty)$ we have

$$\alpha_2 \alpha^k u^j \longrightarrow 2v_2^4 \alpha^k u^j \equiv v_2^4 \alpha^{k+1} u^{j+1}$$

plus higher filtration terms

$$w\alpha^k u \longrightarrow v_2^4 \alpha^{k+1} u$$

and

$$wu^j \longrightarrow v_2^4 \alpha u^j \equiv v_2^4 u^{j+2}, \qquad j > 1,$$

modulo higher terms. From this we can see the injection and that the only terms missed are $v_2^4 u^{\{1-3\}}$. □

Theorem 2.2. *The map* $ER(2)^{16*+8}(RP^\infty \wedge RP^\infty) \to E(2)^{16*+8}(RP^\infty \wedge RP^\infty)$ *is an injection with cokernel given by* $v_2^4 u_1^{\{1-3\}} u_2^{\{1-3\}}$.

Proof. We have both $E(2)^*(RP^\infty \wedge RP^\infty)$ and $ER(2)^*(RP^\infty \wedge RP^\infty)$ written down in Theorem 17.1 of Part I. A 2-adic basis for $E(2)^*(RP^\infty \wedge RP^\infty)$ is given by

$$v_2^s \alpha^k u_1^i u_2, \quad 0 \leqslant s < 8, \quad 0 \leqslant k, \quad 0 < i;$$
$$v_2^s u_1^i u_2^j, \quad 0 \leqslant s < 8, \quad 0 < i, \quad 1 < j.$$

The elements in degree $16*+8$ are those with $s = 4$. We can also write down $ER(2)^*(RP^\infty \wedge RP^\infty)$ and the map to $E(2)^*(RP^\infty \wedge RP^\infty)$ in degrees $16*+8$: From the x^1-torsion we get

$$\alpha_2 \alpha^k u_1^i u_2 \longrightarrow 2v_2^4 \alpha^k u_1^i u_2 \equiv v_2^4 \alpha^{k+1} u_1^{i+1} u_2;$$
$$\alpha_2 u_1^i u_2^j \longrightarrow 2v_2^4 u_1^i u_2^j \equiv v_2^4 u_1^{i+1} u_2^{j+2}, \qquad 0 < i, \quad 1 < j,$$

all modulo higher filtrations. From the x^3-torsion we get:

$$w\alpha^k u_1 u_2 \longrightarrow v_2^4 \alpha^{k+1} u_1 u_2, \qquad 0 \leqslant k;$$
$$w u_1^i u_2^{\{1,2,3\}} \longrightarrow v_2^4 u_1^{i+2} u_2^{\{1,2,3\}}, \qquad 1 < i;$$
$$w u_1 u_2^j \longrightarrow v_2^4 u_1 u_2^{j+2}, \qquad 1 < j.$$

From this we see the injection and that the only elements missed are those stated.

Note that there are no elements in degrees $16*+8$ divisible by x. □

Corollaries 8.3 and 17.3 of Part I give us an isomorphism for these spaces in degrees $16*$ so we get:

Corollary 2.3. *The map $ER(2)^{8*}(X) \to E(2)^{8*}(X)$ is an injection for $X = RP^\infty$ and $RP^\infty \wedge RP^\infty$.*

Not much more work is required to prove:

Proposition 2.4. *The map $ER(2)^{16*+i}(X) \to E(2)^{16*+i}(X)$ is an injection for $X = RP^\infty$ and $RP^\infty \wedge RP^\infty$ when $i = 0, 1, 2, 3, 4, 5$ and 8.*

From this we know that there are no elements divisible by x in any of these degrees.

3. y, a new element

We need to introduce a new element that we have good control over. We know we have an isomorphism of $ER(2)^{16*}(RP^\infty)$ and $E(2)^{16*}(RP^\infty)$ and that we have the same relation, $0 = 2u +_F \alpha u^2 +_F u^4$, in both. We can use this to solve for u^4 as

$$u^4 = -_F(2u) -_F (\alpha u^2) = 2ug + \alpha u^2 h,$$

where g and h are invertible power series. As it stands, g and h are not uniquely determined, but if we insist that none of the terms of h be divisible by 2 (we can move such terms to g) then we can make our choice of g and h unique.

Recall that in $E(2)^*(-)$ we have set $v_2^8 = 1$. We now multiply this relation, when viewed only as being in $E(2)^*(RP^\infty)$, by v_2^4, which is a unit in $E(2)^*(-)$, to get a relation

$$v_2^4u^4 = v_2^4(2ug + \alpha u^2h) = (2v_2^4)ug + (v_2^4\alpha)u^2h.$$

The image of the element α_2 from $ER(2)^*$ in $E(2)^*$ is $2v_2^4$ and the image of w is $v_2^4\alpha$ so, in this relation, all of the terms on the right hand side are in the image from $ER(2)^*(RP^\infty)$ and we can use them to define a new element

$$y = \alpha_2 ug + wu^2h$$

that reduces to $v_2^4u^4 \in E(2)^8(RP^\infty)$. (The lack of uniqueness of g and h would not affect anything here. It does not matter whether we convert $2v_2^4$ to α_2 or $v_2^4\alpha$ to w if we have a $v_2^4 2\alpha$ that could be factored either way because $\alpha_2\alpha = 2w \in ER(2)^*$, [**KW07**].)

Although we struggled with this element a great deal in our original computations and then managed to eliminate it for our work in Part I, it was only with the work of Bruner, Davis and Mahowald in [**BDM02, DM**] that we realized its importance for our work with nonimmersion theorems.

The element y has many interesting properties. We collect a few here.

Theorem 3.1. *There is an element $y \in ER(2)^8(RP^\infty)$ that maps to $v_2^4u^4 \in E(2)^8(RP^\infty)$. We have relations:*

$$y^2 = u^8, \qquad \alpha y = wu^4, \qquad wy = \alpha u^4, \qquad 2y = \alpha_2 u^4, \qquad \alpha_2 y = 2u^4,$$
$$\alpha_3 y = \alpha_1 u^4, \qquad \alpha_1 y = \alpha_3 u^4, \qquad xy = xwu^2h, \qquad x^3y = 0.$$

Proof. We have already constructed y with the property that it reduces to $v_2^4u^4$. The first five relations take place in degrees $8*$ where we have we can prove the relations by substituting $v_2^4u^4$ for y, $2v_2^4$ for α_2 and $v_2^4\alpha$ for w. They all follow quickly then. The next relation is in degree 4 modulo 16 and we have an injection here too as well from Proposition 2.4 so it also follows by replacing α_i with $2v_2^{2i}$. Only the next relation requires anything else. It is in degree -4 modulo 16 and we do not have an injection in this degree. We have to resort to the definition (which we could also have used for the other relations)

$$\alpha_1 y = \alpha_1(\alpha_2 ug + wu^2h) = (\alpha_1\alpha_2)ug + (\alpha_1 w)u^2h =$$
$$(2\alpha_3)ug + (\alpha\alpha_3)u^2h = \alpha_3(2ug + \alpha u^2h) = \alpha_3 u^4.$$

This uses the relations in the coefficient ring, $2\alpha_3 = \alpha_1\alpha_2$ and $\alpha\alpha_3 = \alpha_1 w$, from [**KW07**].

$$y = \alpha_2 ug + wu^2h,$$

so, since $x\alpha_2 = 0$, we get the next relation. Since $x^3w = 0$, the last one follows. □

We know $E(2)^*(-)$ and $ER(2)^*(-)$ for RP^∞ and $RP^\infty \wedge RP^\infty$. From Theorem 3.4 of Part I we know that we have a Künneth theorem for $RP^\infty \wedge RP^\infty$. The standard map $RP^\infty \times RP^\infty \to RP^\infty$ induces a coproduct that can be computed from the formal group law; i.e. $u \to u_1 +_F u_2$. However, things are much nicer than that:

Theorem 3.2. *The coproduct of u, up to a unit, is $u_1 - u_2$. The coproduct of y, up to a unit, is*

$$y_1 - 2\alpha_2 u_1^3u_2 + 3\alpha_2 u_1^2u_2^2 - 2\alpha_2 u_1u_2^3 + y_2.$$

Proof. Because we have injections in degrees $8*$, it is enough to prove this for the image in $E(2)^*(-)$. The first statement is well-known and comes from the fact that $0 = [2](X) = X +_F X$. This implies that $X +_F Y$ is divisible by $X - Y$ (because plugging in $Y = X$ gives zero) and so $X +_F Y = (X - Y)g$ where g is a power series in X and Y that is invertible, i.e. a unit. Since the coproduct is given by the formal group law, we have $u \to u_1 +_F u_2 = (u_1 - u_2)$ up to a unit.

To compute the coproduct of y up to a unit we can just compute for u^4. Up to a unit this is $u_1^4 - 4u_1^3u_2 + 6u_1^2u_2^2 - 4u_1u_2^3 + u_2^4$. Multiply this by v_2^4 and replace $v_2^4u_i^4$ with y_i and $2v_2^4$ with α_2. □

4. Rewriting $ER(2)^*(RP^\infty)$

We would like to rewrite our answer for $ER(2)^{16*+8}(RP^\infty)$ using our new element y. Recall, from Theorem 8.1 of Part I, our description of $ER(2)^*(RP^\infty)$.

The x^1-torsion generators are given by

$$\alpha_i\alpha^k u^j, \qquad 0 \leqslant i < 4, \qquad 0 \leqslant k \qquad 1 \leqslant j,$$

where $\alpha_0 = 2$.

The x^3-torsion generators are given by:

$$w^\epsilon\alpha^k u, \quad \epsilon + k > 0; \quad wu^j, \quad 1 < j; \ \text{ and } \ u^j, \quad 3 < j.$$

The only x^7-torsion generators are

$$u^{\{1-3\}}.$$

From $y = \alpha_2 ug + wu^2h$ it is easy to see that we can replace the x^3-torsion generators, wu^j, $1 < j$, using yu^{j-2}.

We would also like to replace some of the x^1-torsion generators,

$$\alpha_2\alpha^k u^j,$$

with $\alpha^{k+1}yu^{j-3}$. This last element is not x^1-torsion, though. When we are in $ER(2)^*(RP^{2n})$ and j is big enough, this can be x^1-torsion;

$$y = \alpha_2 ug + wu^2h,$$
$$\alpha_2 ug = y - wu^2h,$$
$$\alpha_2 u = yg^{-1} - wu^2hg^{-1},$$
$$\alpha_2 u^3 = yu^2g^{-1} - wu^4hg^{-1}.$$

We know that $wu^4 = \alpha y$, so this is

$$\alpha_2 u^3 = yu^2g^{-1} - \alpha yhg^{-1}.$$

The lead term (i.e. the term with lowest filtration) here is αy and the whole right hand side must be x^1-torsion even if the lead term isn't. When the higher filtration terms are all zero, we can replace $\alpha_2\alpha^k u^j$ with $\alpha^{k+1}yu^{j-3}$.

This is enough to give us what we want.

5. $ER(2)^{16*+8}(RP^{2n})$

We have, for $ER(2)^{16*+8}(-)$, a theorem similar to Theorem 13.4 of Part I for $ER(2)^{16*}(-)$.

Theorem 5.1. *For all $n > 3$ there is a short exact sequence*

$$0 \longleftarrow ER(2)^{16*+8}(RP^{2n-2}) \longleftarrow ER(2)^{16*+8}(RP^{2n}) \longleftarrow ER(2)^{16*+8}(RP^{2n}/RP^{2n-2}) \longleftarrow 0. \quad (5.2)$$

We have elements $\alpha_2\alpha^k u^j \in ER(2)^{16*+8}(RP^{2n})$, $0 \leqslant k$, $0 < j < n$. We also have elements yu^j for $0 \leqslant j \leqslant n-4$.

Depending on n modulo 8 there are other elements in $ER(2)^{16*+8}(RP^{2n})$.

For $n = 8K+4$ and $8K+3$ there are no other elements and $yu^{8K+1} = 0$.

For $n = 8K+2$ there is an x^5-torsion element, z_{16K-38}, that reduces to v_2u^{8K+2} in the Bockstein spectral sequence such that

$$x^2\alpha^k z_{16K-38} = \alpha^k yu^{8K-1}$$

with $yu^{8K} = 0$.

For $n = 8K+1$ there is an x^5-torsion element, z_{16K-22}, that reduces to v_2u^{8K+1} in the Bockstein spectral sequence such that

$$x^2\alpha^k z_{16K-22} = \alpha^k yu^{8K-2},$$

and an x^7-torsion element, z_{16K-4} that reduces to $v_2^6u^{8K+1}$ in the Bockstein spectral sequence such that

$$x^2uz_{16K-22} = x^4z_{16K-4} = yu^{8K-1}$$

with $yu^{8K} = 0$.

For $n = 8K$ there are x^7-torsion elements, z_{16K-20} and z_{16K-18}, that reduce to $v_2^6u^{8K-1}$ and $v_2^3u^{8K}$ respectively in the Bockstein spectral sequence such that

$$x^4z_{16K-20} = yu^{8K-3},$$
$$x^4uz_{16K-20} = yu^{8K-2}$$

and

$$x^4u^2z_{16K-20} = x^6z_{16K-18} = yu^{8K-1}$$

with $yu^{8K} = 0$.

For $n = 8K+7$ there are x^7-torsion elements, z_{16K-36} and z_{16K-34}, that reduce to $v_2^6u^{8K+6}$ and $v_2^3u^{8K+7}$ respectively in the Bockstein spectral sequence such that

$$x^4z_{16K-36} = yu^{8K+4},$$
$$x^4uz_{16K-36} = yu^{8K+5}$$

and

$$x^4u^2z_{16K-36} = x^6z_{16K-34} = yu^{8K+6}$$

with $yu^{8K+7} = 0$.

For $n = 8K + 6$ there is an x^5-torsion element, z_{16K-8}, that reduces to $v_2 u^{8K+6}$ in the Bockstein spectral sequence such that

$$x^2 \alpha^k z_{16K-8} = \alpha^k y u^{8K+3},$$

and an x^7-torsion element, z_{16K-36} that reduces to $v_2^6 u^{8K+6}$ in the Bockstein spectral sequence such that

$$x^2 u z_{16K-8} = x^4 z_{16K-36} = y u^{8K+4}$$

with $yu^{8K+5} = 0$.

For $n = 8K + 5$ there is an x^5-torsion element, z_{16K+10}, that reduces to $v_2 u^{8K+5}$ in the Bockstein spectral sequence such that

$$x^2 \alpha^k z_{16K+10} = \alpha^k y u^{8K+2},$$

with $yu^{8K+3} = 0$.

This gives us our Theorem 1.1.

Proof. We have computed the Bockstein spectral sequence for all of the spaces RP^{2n-2}, RP^{2n}, and RP^{2n}/RP^{2n-2}. From this we can just read off the elements in degree $16* + 8$. In every case, the x^1-torsion elements $\alpha_2 \alpha^k u^j$ for $j < n - 1$ correspond using the map induced by $RP^{2n-2} \to RP^{2n}$. Likewise for the elements $w\alpha^k u$, and yu^j, $0 \leqslant j \leqslant n - 5$ so we will ignore these elements. In the proof we are constantly using the fact that we already know all of the groups. We also make use of the fact that the map $ER(2)^*(RP^{2n}/RP^{2n-2}) \to ER(2)^*(RP^{2n})$ was computed explicitly in (13.1) of Part I.

First note that $\alpha_0 \alpha^k u^{n-1} = 2\alpha^k u^{n-1} = \alpha^{k+1} u^n$.

For $n \equiv 4 \mod 8$, there is nothing else in $ER(2)^{16*+8}(RP^{2n-2})$. All that is left of (5.2) is $\alpha^k z_{2n} \in ER(2)^{16*+8}(RP^{2n}/RP^{2n-2})$ and $\alpha_2 \alpha^k u^{n-1}$ and yu^{n-4} in $ER(2)^{16*+8}(RP^{2n})$. Since there are no elements of higher filtration, we can use Section 4 to replace $\alpha_2 \alpha^k u^{n-1}$ with $\alpha^{k+1} y u^{n-4}$. Note that $\alpha^k y u^{n-4}$ is represented by $v_2^4 \alpha^k u^n$ in the Bockstein spectral sequence. The long exact sequence forces $\alpha^k z_{2n} \to \alpha^k y u^{n-4}$, but so does our direct computation using (13.1) of Part I.

Because $uz_{2n} = 0$, we must have $yu^{n-3} = yu^{8K+1} = 0$. Because z_{2n} maps to yu^{8K}, this element goes to zero in $ER(2)^{16*+8}(RP^{2n})$ when $n \equiv 3$, 2, 1, and 0 modulo 8.

For $n \equiv 3 \mod 8$, $ER(2)^{16*+8}(RP^{2n}/RP^{2n-2}) = 0$. We must have $\alpha^k y u^{n-4} \to x^2 \alpha^k u^{n-1} v_2$. (Technically, we need to worry that perhaps yu^{n-4} goes to $x^2 \alpha^{3k} u^{n-1} v_2$ for some k. If this is the case, then the boundary homomorphism on $x^2 u^{n-1} v_2$ must be nontrivial but we can check that there is nowhere for it to go. Consequently we will ignore this kind of possibility in the rest of this proof.)

For $n \equiv 2 \mod 8$, things are a little more complicated. The only elements in $ER(2)^{16*+8}(RP^{2n}/RP^{2n-2})$ are $x^2 w \alpha^k z_{2n-18}$ and we can compute directly that they go to $x^2 \alpha^{k+1} u^n v_2$. The element $\alpha^k y u^{n-4}$ must go to $x^2 \alpha^k u^{n-1} v_2$. The only possibility left is for $x^2 u^n v_2$ to go to $x^4 u^{n-1} v_2^6$. Recall from above that this last element is yu^{n-3}.

For $n \equiv 1 \bmod 8$, we compute the map to $ER(2)^{16*+8}(RP^{2n})$ directly and we have

$$w\alpha^k z_{2n-18} \longrightarrow \alpha^{k+1} y u^{n-4};$$
$$x^2 w\alpha^k z_{2n} \longrightarrow x^2\alpha^{k+1} u^n v_2.$$

Keep in mind that this last represents $\alpha^{k+1} y u^{n-3}$. The element yu^{n-4} must map to $x^4 v_2^6 u^{n-2}$, $x^2 v_2 u^n = yu^{n-3}$ to $x^4 v_2^6 u^{n-1}$, and $x^4 v_2^6 u^n = yu^{n-2}$ to $x^6 v_2^3 u^{n-1}$.

For $n \equiv 0 \bmod 8$ we compute $x^6 z_{2n-18} \to x^6 u^n v_2^3 = yu^{n-1}$ and $w\alpha^k z_{2n} \to \alpha^{k+1} y u^{n-4}$. That leaves $yu^{n-4} \to x^4 u^{n-2} v_2^6$, $x^4 u^{n-1} v_2^6 = yu^{n-3} \to x^4 u^{n-1} v_2^6$, and $x^4 u^n v_2^2 = yu^{n-2} \to x^6 u^{n-1} v_2^3$.

Because yu^{n-1} is hit above, we must have $yu^{8K-1} = 0$ below.

For $n \equiv 7 \bmod 8$, we compute $x^4 z_{2n-18} \to x^4 u^n v_2^6 = yu^{n-2}$ and $x^6 z_{2n} \to x^6 u^n v_2^3 = yu^{n-1}$. That leaves $x^4 u^{n-1} v_2^2 = yu^{n-3} \to x^4 u^{n-1} v_2^6$ and $\alpha^k y u^{n-2} \to x^2 \alpha^k u^{n-1} v_2$.

Because yu^{n-2} and yu^{n-1} are hit above, we must have $yu^{8K+5} = 0$ below.

For $n \equiv 6 \bmod 8$, we compute $x^2\alpha^k z_{2n-18} \to x^2 \alpha^k u^n v_2 = \alpha^k y u^{n-3}$ and $x^4 z_{2n} \to x^4 u^n v_2^6 = yu^{n-2}$. All that is left is $\alpha^k y u^{n-4} \to x^2 \alpha^k u^{n-1} v_2$.

Because yu^{n-3} and yu^{n-2} are hit above, we must have $yu^{8K+3} = 0$ below.

The $n \equiv 5 \bmod 8$ case is simple again with $\alpha^k z_{2n-18} \to \alpha^k y u^{n-4}$ and $x^2 \alpha^k z_{2n} \to x^2 \alpha^k u^n v_2 = \alpha^k y u^{n-3}$. □

6. Algebraic maps

We can now see, from Theorem 5.1 and Theorem 1.6 of Part I, that we have purely algebraic maps, no topology used or implied, of

$$ER(2)^{8*}(RP^{2n}) \longrightarrow E(2)^{8*}(RP^{2n+2}), \qquad n \equiv 1, 2, 5, 6 \bmod 8.$$

These maps are neither injective nor surjective. However, they are close enough to surjective for our purposes since the only elements they miss are the $v_2^4 u^{\{1-3\}}$. These low powers of u are never involved with our nonimmersion results.

7. Last of the even spaces

The goal of this section is to prove Theorem 1.2.

We begin again with the main theorem of [**Dav84**]: for

$$n = m + \alpha(m) - 1, \qquad k = 2m - \alpha(m),$$

there does not exist an axial map

$$RP^{2n} \times RP^{2^K-2k-2} \longrightarrow RP^{2^K-2n-2},$$

and so $RP^{2n} \not\subseteq \mathbb{R}^{2k}$. This is proven by using the equivalent of $E(2)^*(-)$ and showing that the $u^{2^{K-1}-n} = 0$ on the right would have to go to a nonzero element on the left.

That same element would prevent the existence of an axial map,

$$RP^{2n+2} \times RP^{2^K-2k-2} \longrightarrow RP^{2^K-2n-2},$$

and likewise

$$RP^{2n+2} \times RP^{2^K-2k-2} \longrightarrow RP^{2^K-2n-4}.$$

Furthermore, if $u^{2^{K-1}-n}$ went to nonzero then we must also have $u^{2^{K-1}-n-1} = 0$ going to a nonzero element. If $n+1 \equiv 3 \mod 8$, then, from Theorem 1.1, we know $yu^{2^{K-1}-n-5} = 0$ in $ER(2)^*(RP^{2^K-2n-4})$ and if $-k-2 \equiv \{1,2,5,6\}$ mod 8, we have a purely algebraic surjection $ER(2)^*(RP^{2^K-2k-4}) \longrightarrow E(2)^*(RP^{2^K-2k-2})$. Our element $yu^{2^{K-1}-n-5}$ maps to $v_2^4 u^{2^{K-1}-n-1}$. We know v_2^4 is a unit, so the result of Davis shows that this element maps nontrivially to $E(2)^*(RP^{2n+2} \times RP^{2^K-2k-2})$. This obstruction can be written in terms of a 2-adic basis. We show that the same 2-adic basis exists in $ER(2)^*(RP^{2n+2} \times RP^{2^K-2k-4})$ and so our $ER(2)^*(-)$ obstruction improves the result.

When this is accomplished, we will have a proof of the following:

Proposition 7.1. *When $n = m + \alpha(m) - 1$ and $k = 2m - \alpha(m)$, with $n+1 \equiv 3$ mod 8 and $-k-2 \equiv \{1,2,5,6\}$ mod 8, there is no axial map*

$$RP^{2n+2} \times RP^{2^K-2k-4} \longrightarrow RP^{2^K-2n-4},$$

and so $RP^{2n+2} \not\subseteq \mathbb{R}^{2k+2}$.

To derive the proof of Theorem 1.2 from this we have to untangle some equations to get our $(m, \alpha(m))$ pairs. We have

$$n + 1 = m + \alpha(m) \equiv 3 \mod 8$$

and

$$-k - 2 = -2m + \alpha(m) - 2 \equiv \{1,2,5,6\} \mod 8.$$

The equation for k gives

$$2m - \alpha(m) \equiv \{5,4,1,0\} \mod 8.$$

The equation for n gives

$$m + \alpha(m) \equiv 3 \mod 8.$$

Adding, we have

$$3m \equiv \{0,7,4,3\} \mod 8.$$

Multiply by 3 to get

$$m \equiv \{0,5,4,1\} \mod 8.$$

Substituting into

$$\alpha(m) \equiv -m + 3 \mod 8,$$

we get

$$\alpha(m) = \{3,6,7,2\}$$

and this gives us $(m, \alpha(m))$ pairs $(0,3)$,$(5,6)$, $(4,7)$ and $(1,2)$ mod 8.

To complete our proof of Theorem 1.2, all we have to do is show that the 2-adic basis elements for $E(2)^*(RP^{2n+2} \times RP^{2^K-2k-2})$ that the obstruction can be written in terms of, i.e. those $u_1^i u_2^j$ with $i+j$ big, also form a 2-adic basis for $ER(2)^*(RP^{2n+2} \times RP^{2^K-2k-4})$ and that the same powers of u_1 and u_2 are zero in each.

The basis for $E(2)^*(RP^{2n+2} \times RP^{2^K-2k-2})$, in degrees $16*+8$, is given by $v_2^4\alpha^k u_1^i u_2$ with $i \leqslant n+1$ and $v_2^4 u_1^i u_2^j$ with $i \leqslant n+1$ and $1 < j \leqslant 2^{K-1}-k-1$.

We need to discuss the obstruction just a little in order to be careful with our comparison. Recall from Theorem 3.2 that the coproduct of $v_2^4 u^{2^{K-1}-n-1}$ is, up to a unit, $(u_1-u_2)^{2^{K-1}-n-1}$. The algorithm, Remark 15.3 of Part I, never lowers the powers of the u's, so the obstruction must be a linear combination of the 2-adic basis elements $v_2^4 u_1^i u_2^j$ with $i \leqslant n+1$, $1 < j \leqslant 2^{K-1}-k-1$, and $i+j \geqslant 2^{K-1}-n-1$. (The missing $v_2^4 u_1^{\{1,2,3\}} u_2^{\{1,2,3\}}$ never figure in here.)

We now proceed to show that the elements of the same name in $ER(2)^*(RP^{2n+2} \times RP^{2^K-2k-4})$ are part of its 2-adic basis. This is easy for most of these elements. Consider the elements $u_1^i u_2^j y_2$ with $3 < i \leqslant n+1$ and for $3 < j \leqslant 2^{K-1}-k-6$. These reduce nontrivially (and independently) to $v_2^4 u_1^i u_2^{j+4}$ in $E(2)^{8*}(-)$. That's not quite enough, though. We also need the elements $u_1^i u_2^{2^{K-1}-k-5} y_2$ for $3 < i \leqslant n+1$. For the submodules of our groups generated by the u_i, in the range of our obstruction where $u_1^i u_2^j$ has $i+j$ big, we have identical 2-adic bases. In addition, we need $u_1^{n+2} u_2^j y_2 = 0$ and $u_1^i u_2^{2^{K-1}-k-4} y_2 = 0$. The two groups are really isomorphic as $\mathbb{Z}_{(2)}[\alpha, u_1, u_2]$-modules.

The following result finishes off what we need.

Proposition 7.2. *When $q \equiv 1, 2, 5$ or 6 modulo 8 and $m \leqslant 8K$ and $8K+8 < q$, the element $u_1^m u_2^{q-3} y_2 \in ER(2)^*(RP^{2m} \wedge RP^{2q})$ is nonzero. When $i \geqslant 4$, we have $u_1^i u_2^{q-2} y_2 = 0$. When $m \equiv 3$ mod 8, $u_1^{m+1} u_2^j y_2 = 0$.*

Proof. The element yu^{q-3} is represented in the spectral sequence for $ER(2)^*(RP^{2q})$ by $x^2 v_2 u^q$ (from Theorem 5.1) so the element $u_1^m u_2^{q-3} y_2$ is represented by $x^2 u_1^m v_2 u_2^q$. Thus it is enough to show that the element $v_2 u_1^m u_2^q$ survives in the spectral sequence to E^3. In Theorem 19.2 of Part I, we have computed the entire E^2-term of the spectral sequence and this term is there with no restrictions on q. There can be no differentials on this element since it is the product of two honest elements (this does use the restriction on q).

All we need to do now is show that this element is not in the image of d^2. The differential d^2 has degree $35 \equiv -13$. Our element $u_1^m u_2^q v_2$ has degree $-16(m+q)-6$ so the source that would have to hit it would have to have degree $-16(m+q)-41$; in particular, it must be odd degree. From Theorem 19.2 of Part I, the odd degree elements in the E^2-term of our Bockstein spectral sequence are:

$$v_2^{2s}\alpha^k z_{-16q-17};$$
$$v_2^{2s} u_1^i z_{-16q-17}, \quad 0 < i < m;$$
$$\text{and} \quad v_2^{2s+1}\alpha^k u_1^{m-1} z_{-16q-17}.$$

The only elements with degree equal to −9 modulo 16 are

$$v_2^4\alpha^k z_{-16q-17}$$
$$\text{and} \quad v_2^4 u_1^i z_{-16q-17}, \quad 0 < i < m.$$

In the proof of Proposition 19.3 of Part I, we showed that d^2 must be trivial on $z_{-16q-17}$ when $m \leqslant 8K$. The differential d^2 also commutes with multiplication by α and u_1. We also know that since $d^2(v_2^4) = 0$, d^2 commutes with multiplication by v_2^4. Consequently, d^2 is trivial on all of the elements listed above when $m \leqslant 8K$.

For the next statement of the proof we use the fact that $u_2^{q-2}y_2$ is divisible by x^4 and that $x^3u_1^4 = 0$. For the final statement we use the fact that u_1^{m+1} is divisible by x^4 and that $x^3y_2 = 0$. □

This completes the proof of Theorem 1.2.

8. $ER(2)^*(RP^{16K+1})$

We begin our computation by setting up the Bockstein spectral sequence.

E^1 is just $E(2)^*(RP^{16K+1})$, which is nothing more than

$$E(2)^*(RP^{16K}) \oplus E(2)^*(S^{16K+1}).$$

So we have a 2-adic basis (it isn't really necessary to use this notation for the torsion free part and so it isn't necessary to go to the 2-adic completion of $ER(2)$; it is just convenient notation now), letting $2 = \alpha_0$,

E^1:

$$v_2^i\alpha^k u^j, \quad 0 \leqslant i < 8, \quad 0 \leqslant k, \quad 0 < j \leqslant 8K;$$
$$v_2^i\alpha_0^q\alpha^k\iota_{16K+1}, \quad 0 \leqslant i < 8, \quad 0 \leqslant q, \quad 0 \leqslant k.$$

All of the first part is even degree and all of the second part is odd degree. The differential d^1 is even degree, so it is induced by the maps

$$RP^{16K} \longrightarrow RP^{16K+1} \longrightarrow S^{16K+1}$$

where we already know it. Thus we can just read off our d^1 from Theorem 13.2, Part I, for the RP^{16K} part and Section 5 Part I for the S^{16K+1} part.

$$d^1(v_2^{2s-5}\alpha^k u^j) = 2v_2^{2s}\alpha^k u^j \equiv v_2^{2s}\alpha^{k+1}u^{j+1}, \qquad j < 8K$$

(modulo higher powers of u).

$$d^1(\alpha_0^q v_2^{2s+1}\alpha^k\iota_{16K+1}) = \alpha_0^{q+1}v_2^{2s-2}\alpha^k\iota_{16K+1}.$$

E^2 is given by

$$v_2^{2s}\alpha^k u, \;\; 0 \leqslant k; \qquad v_2^{2s}u^j, \;\; 1 < j \leqslant 8K; \qquad v_2^{2s+1}\alpha^k u^{8K}, \;\; 0 \leqslant k;$$
$$v_2^{2s}\alpha^k\iota_{16K+1}.$$

We confront a new problem now. The differential d^2 has degree 35 and we have both odd and even degree elements so it could be nonzero. If so, by naturality it must have its source in the RP^{16K} part and its target in the S^{16K+1} part. Furthermore, the source cannot be something in the image from the E^2 for $ER(2)^*(RP^\infty)$ because we know that d^2 is zero on all of those elements. All we are left with for possible sources is

$$v_2^{2s+1}\alpha^k u^{8K}, \qquad 0 \leqslant k.$$

The differential d^2 is trivial on v_2^2 and α so it commutes with multiplication by these elements. Since v_2^2 is a unit, if there is a d^2 then it must be nonzero on $v_2 u^{8K}$, which has degree $-6-16(8K) \equiv 16K-6$. The degree of the target must be this plus 35, or $16K+29 \equiv 16K-19$. The possible targets have degrees $-12s-32k+16K+1$. Working modulo 16, we need $-3 \equiv -12s+1$, so we see that $s=3$ and we have $16K-19 \equiv -36-32k+16K+1$, which gives $-19 \equiv -36-32k+1$ modulo 48. This is $16 \equiv -32k$, which suggests the solution of $k=1$ (other alternatives are $k=3q+1$).

If there is a d^2, we would conjecture that it starts with

$$d^2(v_2 u^{8K}) = v_2^6 \alpha \iota_{16K+1},$$

and this would lead to

$$d^2(v_2^{2s+1}\alpha^k u^{8K}) = v_2^{6+2s}\alpha^{k+1}\iota_{16K+1}.$$

We now know what to look for. If we can show that the element $\alpha\iota_{16K+1}$ is in the image from S^{16K+1} and that x^2 times it must be zero, then our conjectured d^2 is correct.

If we look carefully at $ER(2)^*(S^{16K+1})$, we see that there are a number of known 2-torsion free elements, namely all of $w^\epsilon \alpha^k \iota_{16K+1}$, $\alpha_{\{1,3\}}\alpha^k\iota_{16K+1}$ and $\alpha_2\iota_{16K+1}$. If we look at

$$RP^{16K} \longrightarrow RP^{16K+1} \longrightarrow S^{16K+1},$$

we know that $ER(2)^*(RP^{16K})$ is all torsion, so our torsion free elements must all inject into $ER(2)^*(RP^{16K+1})$. From this we know that the element $\alpha\iota_{16K+1}$ is in $ER(2)^*(RP^{16K+1})$, but we don't yet know if x^2 kills it.

For that we need the diagram:

$$\begin{array}{ccccc}
RP^{16K} & \xrightarrow{=} & RP^{16K} & & \\
\downarrow & & \downarrow & & \\
RP^{16K+1} & \longrightarrow & RP^{16K+2} & \longrightarrow & S^{16K+2} \\
\downarrow & & \downarrow & & \downarrow = \\
S^{16K+1} & \longrightarrow & RP^{16K+2}/RP^{16K} & \longrightarrow & S^{16K+2}.
\end{array} \tag{8.1}$$

Each row and column is a cofibration giving rise to a long exact sequence. Our goal is to show that $x^2\alpha\iota_{16K+1}$ in $ER(2)^*(RP^{16K+1})$ is zero. It is the image of the same named element in $ER(2)^*(S^{16K+1})$. From Corollary 9.3 of Part I we know that the element

$$z_{16K-16} \in ER(2)^*(RP^{16K+2}/RP^{16K})$$

maps to $x\iota_{16K+1}$ in $ER(2)^*(S^{16K+1})$. Consequently, $x\alpha z_{16K-16}$ must map to our element of interest, $x^2\alpha\iota_{16K+1}$.

Rather than go through S^{16K+1} on our way to RP^{16K+1}, we can now go through RP^{16K+2}. The element z_{16K-16} maps to u^{8K+1} ((13.1) Part I) and so $x\alpha z_{16K-16}$ maps to $x\alpha u^{8K+1}$. Since $2x = 0$, we can use the relation ((1.3) Part I) on αu^2 and we will get xu^{8K+3} plus even higher terms, but we know that u^{8K+3} is zero in $ER(2)^*(RP^{16K+2})$ (Theorem 1.6, Part I). So it follows that we have $x^2\alpha\iota_{16K+1} = 0$ in $ER(2)^*(RP^{16K+1})$ and we can compute our d^2 as we conjectured.

Although uz_{16K-16} maps to u^{8K+2}, it comes from S^{16K+2} and so goes to zero in $ER(2)^*(RP^{16K+1})$.

We get more information out of that computation. It shows us that u^{8K+1} is represented in the spectral sequence by $x\iota_{16K+1}$ because both come from z_{16K-16}. Multiply u^{8K} by α_2, and in $E(2)^*(RP^{16K+2})$ this is $v_2^4\alpha u^{8K+1}$. Consequently, $\alpha_2\alpha^k u^{8K}$ is represented by $v_2^4\alpha^{k+1}u^{8K+1}$ and in RP^{16K+1} this is represented by $v_2^4\alpha^{k+1}x\iota_{16K+1}$. From our discussion in Section 4 we can replace this $\alpha_2\alpha^k u^{8K}$ with $\alpha^{k+1}yu^{8K-3}$.

We now have our
E^3:

$$v_2^{2s}\alpha^k u,\ \ 0 \leqslant k; \qquad v_2^{2s}u^j,\ \ 1 < j \leqslant 8K; \qquad v_2^{2s}\iota_{16K+1}.$$

The differential d^3 is even degree again, so the even and odd degree parts don't mix. In S^{16K+1}, d^3 takes v_2^2 to αv_2^4 but this element is not there, so there is no d^3 on the odd part. On the even part we already know the d^3 differentials:

$$d^3(v_2^{\{6,2\}}\alpha^k u) = v_2^{\{0,4\}}\alpha^{k+1}u;$$
$$d^3(v_2^{\{6,2\}}u^j) = v_2^{\{0,4\}}\alpha u^j = v_2^{\{0,4\}}u^{j+2}, \quad 1 < j \leqslant 8K-2.$$

We get
E^4:

$$v_2^{\{0,4\}}u^{\{1-3\}}; \qquad v_2^{\{6,2\}}u^{\{8K-1,8K\}}; \qquad v_2^{2s}\iota_{16K+1}.$$

Our d^4 is odd degree again and so must go from the RP^{16K} part to the S^{16K+1} part if at all. The differential d^4 has degree 21 and must be zero on things in the image from RP^∞, so a nonzero differential must start out on

$$v_2^{\{6,2\}}u^{8K-1}$$

and hit one of

$$v_2^{2s}\iota_{16K+1}.$$

The source degrees are $-36 - 16(8K-1) = 16K - 20$ and $16K + 4$. Adding 21 to see what degree our target would have to be, we get $16K + 1$ and $16K + 25$. Since v_2^4 commutes with d^4, if we have a d^4 it must be

$$d^4(v_2^6u^{8K-1}) = \iota_{16K+1}; \qquad d^4(v_2^2u^{8K-1}) = v_2^4\iota_{16K+1}.$$

In our discussion of d^2 we showed that ι_{16K+1} lives in $ER(2)^*(RP^{16K+1})$. So, because it comes from $ER(2)^*(S^{16K+1})$, it has no differential on it. The only question is what differential hits it. We have already computed d^1, d^2, and d^3, so we know that $x^3\iota_{16K+1} \neq 0$ in $ER(2)^*(RP^{16K+1})$. If $x^4\iota_{16K+1} = 0$, then ι_{16K+1} must be hit by d^4 and our differential must be as above.

The argument here is the same as before using the diagram (8.1). The element z_{8K-16} maps to $x\iota_{16K+1}$ in S^{16K+1} so we want to study $x^3 z_{8K-16}$ and we will again get to RP^{16K+1} by way of RP^{16K+2} where $x^3 z_{16K-16}$ maps to $x^3u^{8K+1} = 0$ (because $x^3u^4 = 0$).

This computes our d^4 and we have left for our

E^5:

$$v_2^{\{0,4\}}u^{\{1-3\}}; \qquad v_2^{\{6,2\}}u^{8K}; \qquad v_2^{\{2,6\}}\iota_{16K+1}.$$

d^5 is once again even degree and the dimensions don't work for the odd part so it is zero there and RP^{8K} determines it is zero on the even part.

We have to consider the possibility of a d^6 which is of degree 7. Again, it must go from even to odd by naturality. It would commute with v_2^4, so if d^6 is nonzero on $v_2^2u^{8K}$, it will be nonzero on the other element. The degree here is $-12 + 16K$. Add 7 to look for the target to get $-5 + 16K$. Elements that are in odd degrees are in degrees $16K + 1 - 12$ and $16K + 1 - 36$, so there can be no d^6.

All we have left is d^7, and we know, for starters, that $d^7(v_2^4u^{1-3}) = u^{1-3}$. We can also read off from RP^{8K} that $d^7(v_2^2u^{8K}) = v_2^6u^{8K}$.

The only issue remaining is how d^7 works on

$$v_2^{\{2,6\}}\iota_{16K+1}.$$

The element yu^{8K-1} is represented by $x^4v_2^6u^{8K+1}$ in the Bockstein spectral sequence for $ER(2)^*(RP^{16K+2})$ and by $x^6v_2^3u^{8K}$ in the Bockstein spectral sequence for $ER(2)^*(RP^{16K})$ (Theorem 5.1).

It must pass through $ER(2)^*(RP^{16K+1})$ nontrivially and it must be divisible by x^4 in here. The only even degree candidates for such an element in the Bockstein spectral sequence for $ER(2)^*(RP^{16K+1})$ are $x^{\{4,6\}}v_2^6u^{8K}$ and $x^5v_2^{\{2,6\}}\iota_{16K+1}$. The degree of yu^{8K-1} is $16K \pm 24$, which is 8 mod 16. Checking $x^{\{4,6\}}v_2^6u^{8K}$ and $x^5v_2^{\{2,6\}}\iota_{16K+1}$, we see that, modulo 16, their degrees are $8, -10, 0$, and 8 respectively so the only possibilities are $x^4v_2^6u^{8K}$ and $x^5v_2^6\iota_{16K+1}$. However, modulo 48, these are $-20 - 36 + 16K$ and $-36 - 36 + 16K$, or, $16K - 8$ and $16K - 24$, so it must be represented by $x^5v_2^6\iota_{16K+1}$ and so our last undecided differential must be $d^7(v_2^2\iota_{16K+1}) = v_2^6\iota_{16K+1}$.

From Theorem 5.1 we already have $yu^{8K} = 0$ in $ER(2)^*(RP^{16K+2})$ and so it is also zero in $ER(2)^*(RP^{16K+1})$.

This concludes our computation of $ER(2)^*(RP^{16K+1})$ using the Bockstein spectral sequence and we collect our results here.

Theorem 8.2. *The Bockstein spectral sequence for $ER(2)^*(RP^{16K+1})$ is as follows.*

E^1*:*

$$v_2^i\alpha^k u^j, \quad 0 \leqslant i < 8, \quad 0 \leqslant k, \quad 0 < j \leqslant 8K;$$
$$v_2^i\alpha_0^q\alpha^k\iota_{16K+1}, \quad 0 \leqslant i < 8, \quad 0 \leqslant q, \quad 0 \leqslant k.$$

$$d^1(v_2^{2s-5}\alpha^k u^j) = 2v_2^{2s}\alpha^k u^j \equiv v_2^{2s}\alpha^{k+1}u^{j+1}, \qquad j < 8K$$

(modulo higher powers of u).

$$d^1(\alpha_0^q v_2^{2s+1}\alpha^k \iota_{16K+1}) = \alpha_0^{q+1} v_2^{2s-2}\alpha^k \iota_{16K+1}.$$

E^2:

$$v_2^{2s}\alpha^k u,\ \ 0 \leqslant k; \qquad v_2^{2s}u^j,\ \ 1 < j \leqslant 8K; \qquad v_2^{2s+1}\alpha^k u^{8K},\ \ 0 \leqslant k\ ;$$
$$v_2^{2s}\alpha^k \iota_{16K+1}.$$

$$d^2(v_2^{2s+1}\alpha^k u^{8K}) = v_2^{6+2s}\alpha^{k+1}\iota_{16K+1}.$$

E^3:

$$v_2^{2s}\alpha^k u,\ \ 0 \leqslant k; \qquad v_2^{2s}u^j,\ \ 1 < j \leqslant 8K; \qquad v_2^{2s}\iota_{16K+1}.$$

$$d^3(v_2^{\{6,2\}}\alpha^k u) = v_2^{\{0,4\}}\alpha^{k+1}u;$$
$$d^3(v_2^{\{6,2\}}u^j) = v_2^{\{0,4\}}\alpha u^j = v_2^{\{0,4\}}u^{j+2}, \quad 1 < j \leqslant 8K-2.$$

E^4:

$$v_2^{\{0,4\}}u^{\{1-3\}}; \qquad v_2^{\{6,2\}}u^{\{8K-1,8K\}}; \qquad v_2^{2s}\iota_{16K+1}.$$

$$d^4(v_2^{\{6,2\}}u^{8K-1}) = v_2^{\{0,4\}}\iota_{16K+1}.$$

$E^5 = E^6 = E^7$:

$$v_2^{\{0,4\}}u^{\{1-3\}}; \qquad v_2^{\{6,2\}}u^{8K}; \qquad v_2^{\{2,6\}}\iota_{16K+1}.$$

$$d^7(v_2^4 u^{\{1-3\}}) = u^{\{1-3\}};$$
$$d^7(v_2^2 u^{8K}) = v_2^6 u^{8K}.$$

$$d^7(v_2^2\iota_{16K+1}) = v_2^6\iota_{16K+1}.$$

We identify all of the elements in degree $8*$. This completes the proof of Theorem 1.3.

Theorem 8.3. *A 2-adic basis for the elements in $ER(2)^{8*}(RP^{16K+1})$ is given by the following.*

From the x^1-torsion elements we have $\alpha^k u^j$, $0 < k$, $1 < j \leqslant 8K$, representing elements with the same name. Also, $\alpha_2\alpha^k u^j$, $0 \leqslant k$, $0 < j < 8K$, is represented by $2v_2^4\alpha^k u^j \equiv v_2^4\alpha^{k+1}u^{j+1}$ modulo higher powers of u.

From the x^2-torsion we have $\alpha^k u^{8K+1}$, $k > 0$, is represented by $\alpha^k x\iota_{16K+1}$ and $\alpha_2\alpha^k u^{8K} = \alpha^{k+1}yu^{8K-3}$ is represented by $v_2^4\alpha^{k+1}x\iota_{16K+1}$.

From the x^3-torsion we have $\alpha^{k+1}u$ represents the element with the same name and $v_2^4\alpha^{k+1}u$ represents $w\alpha^k u$.

The elements u^j, $3 < j \leqslant 8K$, represent the elements with the same name and $v_2^4 u^j$, $3 < j \leqslant 8K$, represents yu^{j-4}.

From the x^4-torsion we have $x\iota_{16K+1}$ represents u^{8K+1} and $v_2^4 x\iota_{16K+1}$ represents yu^{8K-3}.

From the x^7-torsion we have $u^{\{1-3\}}$ represents elements of the same name and $x^4 v_2^6 u^{8K}$ represents yu^{8K-2}. Finally, $x^5 v_2^6\iota_{16K+1}$ represents yu^{8K-1}.

We also have $u^{8K+2} = 0 = yu^{8K}$.

Proof. We can find all of the elements in degrees $8*$ by looking at the Bockstein spectral sequence. If we check the elements that are x^1-torsion, i.e. the image of d^1, the elements in degrees $8*$ are just $2v_2^{\{0,4\}}\alpha^k u^j$, and these are, modulo higher filtrations, $v_2^{\{0,4\}}\alpha^{k+1}u^{j+1}$, $0 < j < 8K$. These can be written as $\alpha_i\alpha^k u^j$ with $i = 0$ and 2, $0 < j < 8K$.

Elements in degree $8*$ coming from the x^2-torsion are represented by x times $v_2^{\{0,4\}}\alpha^{k+1}\iota_{16K+1}$. In our computation of d^2 in the Bockstein spectral sequence, we showed that z_{16K-16} mapped to $x\iota_{16K+1}$. We also showed it mapped to u^{8K+1}. This was only x^2-torsion if we multiplied by α^{k+1}. This gives us our $\alpha^{k+1}u^{8K+1}$, which is $2\alpha^k u^{8K}$.

From the proof of Theorem 5.1, we know that $w\alpha^k z_{16K-16}$ maps to $\alpha^{k+1}yu^{8K-3}$ and is represented by $v_2^4\alpha^{k+1}u^{8K+1}$, or, x times $v_2^4\alpha^{k+1}\iota_{16K+1}$. This identifies our remaining $8*$ degree elements coming from the x^2-torsion.

The only elements that come from the x^3-torsion are again standard elements, $w^\epsilon\alpha^k u$ with $\epsilon + k > 0$, u^j with $3 < j \leqslant 8K$, and yu^j with $0 \leqslant j \leqslant 8K - 4$.

From the x^4-torsion, we get $xv_2^{\{0,4\}}\iota_{16K+1}$ in the expected degrees. We have already identified each of these as u^{8K+1} and yu^{8K-3} respectively.

Of course our x^7-torsion $u^{\{1-3\}}$ is standard.

We have only the x^7-torsion elements $v_2^6u^{8K}$ and $v_2^6\iota_{16K+1}$ remaining to consider. The only elements in the appropriate degrees are x^4 times the first and x^5 times the second. We have already identified the last one as yu^{8K-1}. We have not identified the necessary element yu^{8K-2} which we now see must be $x^4v_2^6u^{8K}$.

We have already shown $u^{8K+2} = 0 = yu^{8K}$. □

We have a corollary:

Corollary 8.4. *There is a purely algebraic map*

$$ER(2)^{8*}(RP^{16K+1}) \longrightarrow E(2)^{8*}(RP^{16K+2})$$

which only misses the elements $v_2^4u^{\{1-3\}}$.

9. Axial maps and odd spaces

Recall that Don Davis uses $E(2)^*(-)$ to show that the axial map

$$RP^{2^K-2k-2} \times RP^{2n} \longrightarrow RP^{2^K-2n-2}$$

does not exist when $n = 2(m + \alpha(m) - 1)$ and $k = 2(2m - \alpha(m))$, giving him

$$RP^{2(m+\alpha(m)-1)} \not\subseteq \mathbb{R}^{2(2m-\alpha(m))}.$$

From the previous section, we know that there is an algebraic map, which for our purposes is surjective enough when $-2k-2 \equiv 2 \bmod 16$:

$$ER(2)^*(RP^{2^K-2k-3}) \longrightarrow ER(2)^*(RP^{2^K-2k-2}).$$

We will be able to use our standard tricks to show that

$$RP^{2(m+\alpha(m)-1)} \not\subseteq \mathbb{R}^{2(2m-\alpha(m))+1}$$

when

$$-k-1 \equiv 1 \equiv -2m+\alpha(m)-1 \quad \bmod 8$$

and

$$m+\alpha(m)-1 = n = \{3,4,7,0\}.$$

Presumably one could compute $ER(2)^*(RP^{16K+9})$ and prove a similar theorem with $-k-1=5$. Although the results are probably new they are not of sufficiently low dimensions to interest us.

Before we proceed, let's check out the numbers here. We have

$$2m-\alpha(m) \equiv -2 \quad \bmod 8;$$
$$m+\alpha(m) \equiv \{4,5,0,1\}.$$

Adding, we get

$$3m \equiv \{2,3,6,7\}.$$

Multiply by 3 (always mod 8);

$$m \equiv \{6,1,2,5\}.$$

Then

$$\alpha(m) \equiv \{6,4,6,4\}.$$

This is Theorem 1.4 in the introduction.

The rest of the paper is dedicated to the proof that there is no axial map

$$RP^{2^K-2k-3} \times RP^{2n} \longrightarrow RP^{2^K-2n-2}$$

so that the derivation of Theorem 1.4 in this section holds. This proof breaks up into two separate pieces. The case for $n \equiv 7$ or 0 is done in the next two sections and $n \equiv 3$ or 4 is done in the last section.

10. The 16∗ cases

By now our arguments should seem fairly standard. Don Davis has computed the obstruction to the axial map

$$RP^{2^K-2k-2} \times RP^{2n} \longrightarrow RP^{2^K-2n-2}$$

in $E(2)^*(RP^{2^K-2k-2} \times RP^{2n})$. We will show that the same 2-adic basis that the obstruction lives in is also in $ER(2)^*(RP^{2^K-2k-3} \times RP^{2n})$ and that the same powers of u_1 and u_2 are zero.

We assume throughout that 2^K-2k-3 is equal to 1 mod 16.

In the case of $n \equiv 7$ or 0 mod 8, we have isomorphisms

$$ER(2)^{16*}(RP^{2n}) \longrightarrow E(2)^{16*}(RP^{2n}).$$

We use the fact that $u^{2^{K-1}-n}$ is zero in $ER(2)^*(RP^{2^K-2n-2})$.

All we have to do now is show that the elements corresponding to a 2-adic basis where Davis's obstruction lives also exist in

$$ER(2)^{16*}(RP^{2n} \wedge RP^{2^K-2k-3}).$$

As in previous cases, because the algorithm always increases the number of u's and because the coproduct of u can be computed as $u_1 - u_2$ up to a unit, we just need the elements $u_1^i u_2^j$ to be nonzero when $i+j$ is big. (We don't really have to worry about the $\alpha u_1^i u_2$ terms because they don't have enough u's in them.)

Most of the $u_1^i u_2^j$ are obviously nonzero and independent because they reduce to $E(2)^*(-)$. The only elements this doesn't work for are taken care of by the following theorem.

Theorem 10.1. *When $n \leqslant 8M < 8M+8 < 8K$, in*

$$ER(2)^{16*}(RP^{2n} \wedge RP^{16K+1}),$$

the element $u_1^n u_2^{8K+1}$ is nonzero.

If this element is nonzero, since the elements $u_1^i u_2^{8K+1}$ are defined and u_1^{n-i} times them is nonzero, they too are all nonzero.

We already know that $u_1^{n+1} = 0 = u_2^{8K+2}$, so the 2-adic basis for big products of u_1 and u_2 are the same for both cohomology theories and their respective spaces ($n \equiv 7$ or 0 modulo 8).

This result, proven in the next section, will complete the proof of the nonexistence of the axial map mentioned at the end of the last section for the $n = 7, 0$ cases.

11. Products with an odd space

We study the Bockstein spectral sequence for

$$ER(2)^*(RP^{2n} \wedge RP^{16K+1})$$

where $2n < 16K+1$. The E^1-term is, as usual, just

$$E(2)^*(RP^{2n} \wedge RP^{16K+1}).$$

This has a few more pieces than we are used to because

$$E(2)^*(RP^{16K+1}) \simeq E(2)^*(RP^{16K}) \oplus E(2)^*(S^{16K+1}).$$

Since $E(2)^*(S^{16K+1})$ is free, it doesn't affect the Tor term, only the tensor product term. So our E^1 is, from Theorem 14.3 of Part I,

$$E(2)^*(RP^{2n}) \otimes E(2)^*(RP^{16K}) \oplus E(2)^*(RP^{2n}) \otimes E(2)^*(S^{16K+1})$$
$$\oplus \Sigma^{-16(8K)-1} E(2)^*(RP^{2n}).$$

Keep in mind that

$$E(2)^*(RP^{2n}) \otimes E(2)^*(S^{16K+1}) \simeq \Sigma^{16K+1} E(2)^*(RP^{2n}).$$

The $-16(8K)-1$ looks silly and can be replaced with $16K-1$ since we are working modulo 48.

We need our 2-adic basis for our E^1-term:

$$v_2^s \alpha^k u_1^i u_2, \quad 0 \leqslant k, \quad 0 < i \leqslant n, \quad s < 8;$$
$$v_2^s u_1^i u_2^j, \quad 0 < i \leqslant m, \quad 1 < j \leqslant 8K, \quad s < 8;$$

and

$$v_2^s \alpha^k u_1^i \iota_{16K+1}, \quad 0 \leqslant k, \quad 0 < i \leqslant n, \quad s < 8;$$
$$v_2^s \alpha^k u_1^i z_{16K-17}, \quad 0 \leqslant k, \quad 0 \leqslant i < n, \quad s < 8.$$

We know that $x\iota_{16K+1}$ represents u_2^{8K+1}, so $xu_1^n\iota_{16K+1}$ represents $u_1^n u_2^{8K+1}$. There can be no differential on $u_1^n\iota_{16K+1}$ because it is a product of elements. All we have to do is show that it is not in the image of d^1. Since d^1 is even degree, we only have to worry about it on the odd degree elements, since $u_1^n\iota_{16K+1}$ is odd degree.

d^1 has degree 18, so if d^1 is to hit $u_1^n\iota_{16K+1}$, it must start on some $\alpha^k u_1^i z_{16K-17}$ because they are the only elements in the appropriate degree mod 16. Since d^1 commutes with α, we would have to have $u_1^i z_{16K-17}$ hitting $u_1^n\iota_{16K+1}$ and since d^1 commutes with multiplication by u_1, we would have to have d^1 be nontrivial on z_{16K-17}.

In the Bockstein spectral sequence for $ER(2)^*(RP^{16M+16} \times RP^{16K+2})$, with $8M+8 < 8K$, we have, from Theorem 19.2 of Part I, that $d^1(z_{16K-33}) = 0$. From Theorem 1.2 of [**GW**], z_{16K-33} maps to $u_1 z_{16K-17}$ in the spectral sequence for $RP^{16M+16} \times RP^{16K}$. Since this passes through the spectral sequence for $RP^{16M+16} \times RP^{16K+1}$, z_{16K-33} maps to $u_1 z_{16K-17}$ here as well, so $d^1(u_1 z_{16K-17}) = 0$; i.e. u_1 multiplied times $d^1(z_{16K-17})$ is zero. All elements killed by multiplication by u_1 go to zero under the map to $RP^{8M} \times RP^{16K+1}$, and so in here, our $d^1(z_{16K-17}) = 0$ and our result follows by naturality.

This concludes our proof for the $n = 7, 0$ cases.

12. The 16∗ + 8 cases

We continue to assume that $2^K - 2k - 3$ is equal to 1 mod 16.

We now switch to the cases of $n \equiv 3, 4 \bmod 8$. We now have $yu^{2^{K-1}-n-4} = 0$ in $ER(2)^*(RP^{2^K-2n-2})$ and this maps to $v_2^4 u^{2^{K-1}-n} = 0$ in $E(2)^*(RP^{2^K-2n-2})$. We know from [**Dav84**] that this goes to nonzero in $E(2)^*(RP^{2n} \times RP^{2^K-2k-2})$. We will, as usual, show that the 2-adic basis elements of whose terms this obstruction can be written also live in $ER(2)^*(RP^{2n} \times RP^{2^K-2k-3})$.

The discussion now is nearly identical to that for the even products we studied first in this paper. The end result that we need to complete the work is:

Theorem 12.1. *When $m \leqslant 8M$ and $8M+8 < 8K$, the element*

$$u_1^m u_2^{8K-3} y_2 \in ER(2)^*(RP^{2m} \wedge RP^{16K+1})$$

is nonzero. When $i \geqslant 4$, $u_1^i u_2^{8K-2} y_2 = 0$. When $n \equiv 3$ or 4 modulo 8, $u_1^{n+1} u_2^j y_2 = 0$.

Proof. We have already written down the E^1-term for this. The element that represents $u_1^m u_2^{8K-3} y_2$ is $v_2^4 x u_1^m \iota_{16K+1}$. All we have to do is show that $v_2^4 u_1^m \iota_{16K+1}$ is not the target of a d^1. If there is such a differential for $8M+8$, just as in the last case, $d^1(z_{-16n-17})$ must be nonzero and u_1 times the target must be zero. All such target elements go to zero when we map down to $m \leqslant 8M < 8M+8$ and so the differential is trivial there.

For the last statements, we note that $u_2^{8K-2}y_2$ is divisible by x^4 and x^3 kills u_1^4 and that u_1^{n+1} is divisible by x^4 and x^3 kills y_2. □

This concludes the proof of the final cases.

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