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# QUOTIENTS OF THE MULTIPLIHEDRON AS CATEGORIFIED ASSOCIAHEDRA

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(*communicated by Ronald Brown*)

*Abstract*

We describe a new sequence of polytopes which characterize $A_\infty$-maps from a topological monoid to an $A_\infty$-space. Therefore each of these polytopes is a quotient of the corresponding multiplihedron. Our sequence of polytopes is demonstrated not to be combinatorially equivalent to the associahedra, as was previously assumed in both topological and categorical literature. They are given the new collective name composihedra. We point out how these polytopes are used to parametrize compositions in the formulation of the theories of enriched bicategories and pseudomonoids in a monoidal bicategory. We also present a simple algorithm for determining the extremal points in Euclidean space whose convex hull is the $n^{th}$ polytope in the sequence of composihedra, that is, the $n^{th}$ composihedron $\mathcal{CK}(n)$.

## 1. Introduction

Categorification, as described in [**1**], refers to the process of creating a new mathematical theory by:

1) choosing a demonstrably useful concept that is well understood,
2) replacing some of the sets and functions in its definition with categories and functors, and
3) replacing some of the axiomatic equalities with morphisms.

The associahedra are a sequence of polytopes, invented by Stasheff in [**34**], whose face poset is defined to correspond to bracketings of a given list of elements from a set. If instead we take lists of elements from a category, and define a sequence of polytopes whose faces correspond to either bracketings or to certain maps in that category, then we may naively describe our new definition as a categorified version of the associahedra. Of course there may be many interesting ways to make this idea precise. Here we focus on just one, which arises naturally in the study of both topological monoids and category theory.

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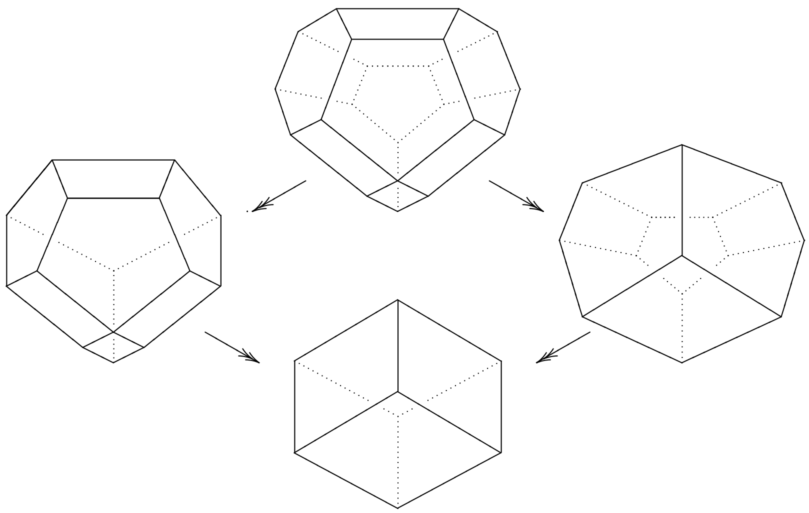


Figure 1: The cast of characters. Left to right: $\mathcal{CK}(4)$, $\mathcal{J}(4)$, 3-d cube, and $\mathcal{K}(5)$.

Categorification of the concept of category itself can be achieved by replacing the Hom-sets of arrows (and/or the set of objects) by categories. This implies considering enriched (or internal) categories of **Cat**, the category of categories. There is room for the composition laws in enriched or internal categories in **Cat** to be weakened. To achieve this in a coherent way, the familiar definitions of bi- and tricategory utilize the Stasheff polytopes, or associahedra, as the underlying shapes of axiomatic commuting diagrams.

A parallel categorification of enriched categories creates a definition allowing the hom-objects to be located in a monoidal bicategory $\boldsymbol{\mathcal{W}}$. This is known as the theory of enriched bicategories. Rather than the associahedra, it is a new sequence of polytopes which arises in the corresponding coherence axioms of enriched bicategories. This new sequence comprises the categorified version of the associahedra which we will be studying. Since they appear in the composition laws of enriched categories, we have chosen to refer to them collectively as the composihedra, or singularly as the $n^{th}$ composihedron, denoted $\mathcal{CK}(n)$. The capital "$\mathcal{C}$" stands for "compose", or for "categorify," or for "cone"– the last since the $n^{th}$ composihedron can be seen as being a subdivision of the topological cone of the $n^{th}$ associahedron. Indeed the polytope $\mathcal{CK}(n)$ is of dimension $n-1$, while the polytope $\mathcal{K}(n)$ is of dimension $n-2$. Decompositions of the boundaries of the earliest terms in our new sequence of polytopes have been seen before, in the classic sources on enriched categories such as [**17**], the definition of enriched bicategories in [**6**], and in the single-object version of the latter: pseudomonoids as defined in [**31**].

The first half of our title refers to the fact that the polytopes we study here also arise in the study of $A_n$ and $A_\infty$-maps. The multiplihedra were invented by Stasheff [**34**], described by Iwase and Mimura [**16**], and generalized by Boardman and Vogt [**5**]. They represent the fundamental structure of a weak map between weak structures, such as weak $n$-categories or $A_n$-spaces. They form a bimodule over the associahedra, and collapse under a quotient map to become the associahedra in the special case of a strictly associative monoid as range. In the case of a strictly associative monoid as domain, the multiplihedra collapse to form our new family of polytopes. This is pictured above in Figure 1, in dimension 3. The multiplihedron is at the top, composihedron on the left, associahedron on the right, and the cube which results in the case of both strict domain and range structures at the bottom.

In Section 2 we briefly review the appearance of polytope sequences in topology and category theory. In Section 3 we provide a complete recursive definition of our new polytopes, as well as a description of them as quotients of the multiplihedra. In Section 4 we go over some basic combinatorial results about the composihedra. In Section 5 we present an alternative definition of the composihedra as a convex hull, based upon the convex hull realization of the multiplihedra in [**10**] and which reflects the quotienting process. In Section 6 we prove that the convex hull defined in Section 5 is indeed combinatorially equivalent to the complex defined in Section 3.

A word of introduction is appropriate in regard to the convex hull algorithm in Sections 5 and 6. In the paper on the multiplihedra, [**10**], we described how to represent Boardman and Vogt's spaces of painted trees with $n$ leaves as convex polytopes, which are combinatorially equivalent to the CW-complexes described by Iwase and Mimura. Our algorithm for the vertices of the polytopes is flexible in that it allows an initial choice of a constant $q$ between zero and one. In the limit as $q \to 1$, the convex hull approaches that of Loday's convex hull representation of the associahedra as described in [**23**]. The limit as $q \to 1$ corresponds to the case for which the mapping is a homomorphism in that it strictly respects the multiplication. The limit as $q \to 0$ represents the case for which multiplication in the domain of the morphism in question is strictly associative. In the limit as $q \to 0$, the convex hulls of the multiplihedra approach our newly discovered sequence of polytopes, the composihedra.

There are two projects for the future that are supported by this work. One is to make rigorous the implication that enriched bicategories may be exemplified by certain maps of topological monoids. It could be hoped that if this endeavor is successful, then $A_\infty$ categories and their maps might also be amenable to the same approach, yielding more interesting examples of enriched bicategories. The other project already underway is to extend the concept of quotient multiplihedra described here to the graph associahedra introduced by Carr and Devadoss in [**7**]. Initial results are found in [**9**].

There is also a philosophical conclusion to be argued from the results of this work. Historically, weak 2- and 3-categories were defined using the associahedra, which form an operad of topological spaces. Then the operad structure was taken as fundamental in many of the functioning definitions of weak $n$-category, as described in [**20, 21**] and [**22**]. Again historically, weak maps of bi- and tricategories were defined using the multiplihedra, which form a 2-sided operad bimodule over the associahedra. More recently, the operad bimodule structure has been used to define weak maps of weak $n$-categories, in [**4**] and [**15**]. Thus the facts that the composihedra are used for defining enriched categories and bicategories, and that they form an operad bimodule (left-module over the associahedra and right-module over the associative operad $t(n) = *$), lead us to propose that enriching over a weak $n$-category should in general be accomplished by use of operad bimodules as well. The philosophy here is that the structure of a bimodule will take into account the weakness of the base of enrichment, (where a weakly associative product is used to form the domain for composition) as well as providing for the weakness of the enriched composition itself.

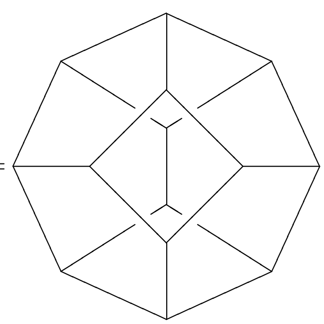
Thanks to Jim Stasheff, Mike Reed, and Jerome Lecointre for suggestions, inspiration, and taking the time to discuss the material in this work.

## 2. Polytopes in topology and category theory

Here we review the appearance of fundamental families of polytopes in the axioms of higher-dimensional category theory. In both topology and in category theory, the use of these polytopes has proven to be a source of important clues rather than the final solution. The algebraic structure of the polytope sequence is more important than its combinatorial structure, although certainly one depends on the other. Thus the operad structure on the associahedra can be seen as foreshadowing the intuition for the use of actions of the operad of little $n$-cubes to recognize loop spaces, as well as the use of $n$-operad actions in Batanin's definition of $n$-category [**4**].

### 2.1. Associahedron

The associahedra are the famous sequence of polytopes denoted $\mathcal{K}(n)$ from [**34**], which characterize the structure of weakly associative products: $\mathcal{K}(1) = \mathcal{K}(2) =$ a single point, $\mathcal{K}(3)$ is the line segment, $\mathcal{K}(4)$ is the pentagon, and $\mathcal{K}(5)$ is the following 3d shape:

$\mathcal{K}(5) =$ 

The original examples of weakly associative product structure are the $A_n$-spaces, topological $H$-spaces with weakly associative multiplication of points. Here "weak" should be understood as "up to homotopy." That is, there is a path in the space from $(ab)c$ to $a(bc)$. An $A_\infty$-space $X$ is characterized by its admission of an action

$$\mathcal{K}(n) \times X^n \to X$$

for all $n$.

Categorical examples begin with the monoidal categories as defined in [**25**], where there is a weakly associative tensor product of objects. Here "weak" officially means "naturally isomorphic." There is a natural isomorphism

$$\alpha\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W).$$

Recall that a *monoidal category* is a category $\mathcal{V}$, together with a functor $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, such that $\otimes$ is associative up to the coherent natural isomorphism $\alpha$.

The coherence axiom is given by a commuting pentagon, which is a copy of $\mathcal{K}(4)$.

$$\begin{tikzcd}
& ((U\otimes V)\otimes W)\otimes X \arrow[r, "\alpha_{UVW}\otimes 1_X"] \arrow[dl, "\alpha_{(U\otimes V)WX}"] & (U\otimes (V\otimes W))\otimes X \arrow[dr, "\alpha_{U(V\otimes W)X}"] & \\
(U\otimes V)\otimes (W\otimes X) \arrow[drr, "\alpha_{UV(W\otimes X)}"'] & & & U\otimes ((V\otimes W)\otimes X) \arrow[dl, "1_U\otimes \alpha_{VWX}"] \\
& & U\otimes (V\otimes (W\otimes X)) &
\end{tikzcd}$$

Monoidal categories can be viewed as single object bicategories. In a bicategory, each $\hom(a, b)$ is located in **Cat** rather than *Set*. The composition of morphisms is not exactly associative: there is a 2-cell $\sigma\colon (f\circ g)\circ h \to f\circ (g\circ h)$. This associator obeys the same pentagonal commuting diagram as for monoidal categories, as seen in [**19**].

Another iteration of categorification results in the theory of tricategories. These obey an axiom in which a commuting pasting diagram has the underlying form of the associahedron $\mathcal{K}(5)$, as noticed by the authors of [**13**]. The term "cocycle condition" for this axiom was popularized by Ross Street, and its connections to homology are described in many of his papers, including [**36**]. The pattern continues in Trimble's unpublished definition of tetracategories, where $\mathcal{K}(6)$ is found as the underlying structure of the corresponding cocycle condition. The associahedra are also seen as the classifying spaces of certain categories of trees, as illustrated in [**20**], and as the foundation for the free strict $\omega$-categories defined in [**36**].

The associahedra are also the starting point for defining the one-dimensional analogs of the full $n$-categorical comparison of delooping and enrichment. One-dimensional weakened versions of enriched categories have been well-studied in the field of differential graded algebras and $A_\infty$-categories, the many object generalizations of Stasheff's $A_\infty$-algebras [**34**]. An $A_\infty$-category is basically a category "weakly" enriched over chain complexes of modules, where the weakening in this case is accomplished by summing the composition chain maps to zero (rather than by requiring commuting diagrams). It is also easily described as an algebra over a certain operad.

### 2.2. Multiplihedron

The complexes now known as the multiplihedra $\mathcal{J}(n)$ were first pictured by Stasheff, for $n \leqslant 4$ in [**35**]. They were introduced in order to approach a full description of the category of $A_\infty$-spaces by providing the underlying structure for morphisms which preserved the structure of the domain space "up to homotopy" in the range. Recall that an $A_\infty$-space itself is a monoid only "up to homotopy," and is recognized by a continuous action of the associahedra as described in [**34**]. Thus the multiplihedra are used to characterize the $A_\infty$-maps. A map $f\colon X\to Y$ between $A_\infty$-spaces is an $A_\infty$-map if there exists an action

$$\mathcal{J}(n)\times X^n \to Y$$

for all $n$, which is equal to the action of $f$ for $n = 1$, and obeying associativity constraints as described in [**35**]. Stasheff described how to construct the 1-skeleton of these complexes in [**35**], but stopped short of a full combinatorial description. Iwase and Mimura in [**16**] give the first detailed definition of the sequence of complexes $\mathcal{J}(n)$ now known as the multiplihedra, and describe their combinatorial properties.

Spaces of painted trees were first introduced by Boardman and Vogt in [**5**] to help describe multiplication in (and morphisms of) topological monoids that are not strictly associative (and whose morphisms do not strictly respect that multiplication.) The $n^{th}$ multiplihedron is a CW-complex whose vertices correspond to the unambiguous ways of multiplying and applying an $A_\infty$-map to $n$ ordered elements of an $A_\infty$-space. Thus the vertices correspond to the binary painted trees with $n$ leaves. In [**10**] a new realization of the multiplihedra, based upon a map from these binary painted trees to Euclidean space, is used to unite the approach to $A_n$-maps of Stasheff, Iwase and Mimura to that of Boardman and Vogt.

The overall structure of the associahedra is that of a topological operad, with the composition given by inclusion. The multiplihedra together form a bimodule (left and right module) over this operad, with the action again given by inclusion. That is, there exist inclusions:

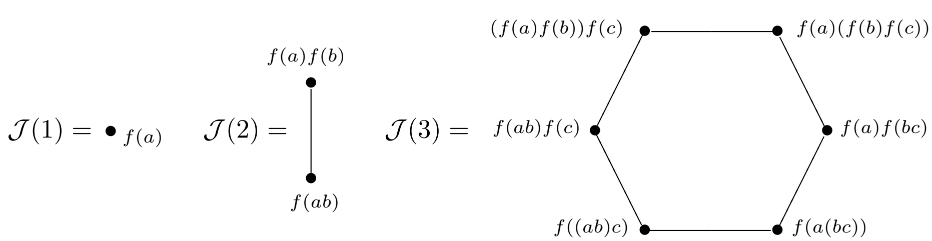
$$\mathcal{K}(k) \times (\mathcal{J}(j_1) \times \cdots \times \mathcal{J}(j_k)) \hookrightarrow \mathcal{J}(n),$$

where $n$ is the sum of the $j_i$. This is the left module structure. The right module structure is from existence of inclusions:

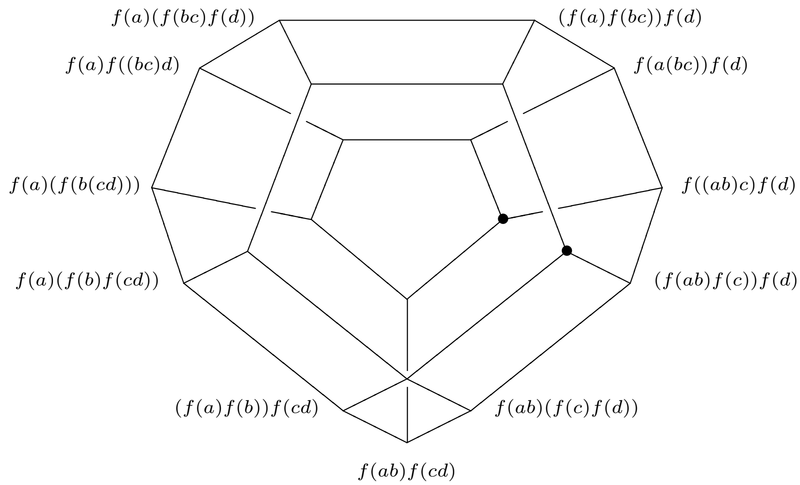
$$\mathcal{J}(k) \times (\mathcal{K}(j_1) \times \cdots \times \mathcal{K}(j_k)) \hookrightarrow \mathcal{J}(n),$$

where $n$ is the sum of the $j_i$. This structure mirrors the fact that the spaces of painted trees form a bimodule over the operad of spaces of trees, where the compositions and actions are given by the grafting of trees, root to leaf. The definition of operad bimodule we use here is that stated in some detail in [**27**] and briefly in [**15**]. A related definition of operad bimodules can be found in [**29**, p. 138].

Here are the first few low-dimensional multiplihedra. The vertices are labeled, all but some of those in the last picture. There the bold vertex in the large pentagonal facet has label $((f(a)f(b))f(c))f(d)$ and the bold vertex in the small pentagonal facet has label $f(((ab)c)d)$. The others can be easily determined based on the fact that those two pentagons are copies of the associahedron $\mathcal{K}(4)$, that is to say all their edges are associations.



$\mathcal{J}(4) =$




The multiplihedra also appear in higher category theory. The definitions of bicategory and tricategory homomorphisms each include commuting pasting diagrams as seen in [**19**] and [**13**] respectively. The two halves of the axiom for a bicategory homomorphism together form the boundary of the multiplihedra $\mathcal{J}(3)$, and the two halves of the axiom for a tricategory homomorphism together form the boundary of $\mathcal{J}(4)$. Since weak $n$-categories can be understood as being the algebras of higher operads, these facts can be seen as the motivation for defining morphisms of operad (and $n$-operad) algebras in terms of their bimodules. This definition was mentioned in [**4**] and is developed in detail in [**15**]. In the latter paper it is pointed out that the bimodules in question must be co-rings, which have a co-multiplication with respect to the bimodule product over the operad.

The study of the $A_\infty$-spaces and their maps is still in progress. There is an open question about the correct way of defining composition of these maps in order to form a category. On page 100 in [**5**], an obvious sort of composition is described as not being associative in the obvious way. A diagonal map is constructed for the multiplihedra in [**32**], extrapolated from an analogous diagonal on the associahedra. These maps allow a functorial monoidal structure for certain categories of $A_\infty$-algebras and $A_\infty$-categories. Different, conjecturally equivalent, versions of diagonals on the associahedra are presented in [**24**] and [**28**]. Eventually it needs to be understood whether any of the possible diagonals make the multiplihedra into a co-ring as defined in [**15**], as well as how such a structure relates to the canonical compositions defined in [**5**].

The multiplihedra have also appeared in several areas related to deformation theory and $A_\infty$ category theory. The 3-dimensional version of the multiplihedron is given the name "Chinese lantern diagram" in [**38**], and is used to describe deformation of functors. There is a forthcoming paper by Woodward and Mau in which a new realization of the multiplihedra as moduli spaces of disks with additional structure is presented [**30**]. This realization allows the authors to define $A_n$-functors as well as morphisms of cohomological field theories. There are also interesting questions about the extension of $A_n$-maps, as in [**14**], and about the transfer of $A_\infty$ structure through these maps, as in [**26**].

### 2.3. Quotients of the multiplihedron

The special multiplihedra in the case for which multiplication in the range is strictly associative were found by Stasheff in [**35**] to be precisely the associahedra. Specifically, the quotient of $\mathcal{J}(n)$ under the equivalence generated by $(f(a)f(b))f(c) = f(a)(f(b)f(c))$ is combinatorially equivalent to $\mathcal{K}(n+1)$. This projection is pictured on the upper right-hand side of Figure 1, in dimension 3. Recall that the edges of the multiplihedra correspond to either an association $(ab)c \to a(bc)$ or to a preservation $f(a)f(b) \to f(ab)$. The associations can either be in the range: $(f(a)f(b))f(c) \to f(a)(f(b)f(c))$, or the image of a domain association: $f((ab)c) \to f(a(bc))$.

It was long assumed that the case for which the domain was associative would likewise yield the associahedra, but we will demonstrate otherwise. The $n^{th}$ composihedron may be described as the quotient of the $n^{th}$ multiplihedron under the equivalence generated by $f((ab)c) = f(a(bc))$. Of course this is implied by associativity in the domain, where $(ab)c = a(bc)$. We will take this latter view throughout, but we note that there may be interesting functions for which $f((ab)c) = f(a(bc))$ even if the domain is not strictly associative.

Below are the first few composihedra with vertices labeled as in the multiplihedra, but with the assumption that the domain is associative. (For these pictures, the label actually appears over the vertex.) Notice how the $n$-dimensional composihedron is a subdivided topological cone on the $(n-1)$-dimensional associahedron. The Schlegel diagram is shown for $\mathcal{CK}(4)$, viewed with a copy of $\mathcal{K}(4)$ as the perimeter.

Any confusion can probably be traced to the fact that the two sequences of polytopes are identical for the first few terms. They diverge first in dimension three, at which dimension the associahedron has nine facets and 14 vertices, while the composihedron has ten facets and 15 vertices. Another similarity at this dimension is that both polytopes have exactly six pentagonal facets; the difference is in the number of quadrilateral facets. The difference is also clear from the fact that the associahedra are simple polytopes, whereas starting at dimension three the composihedra are not.

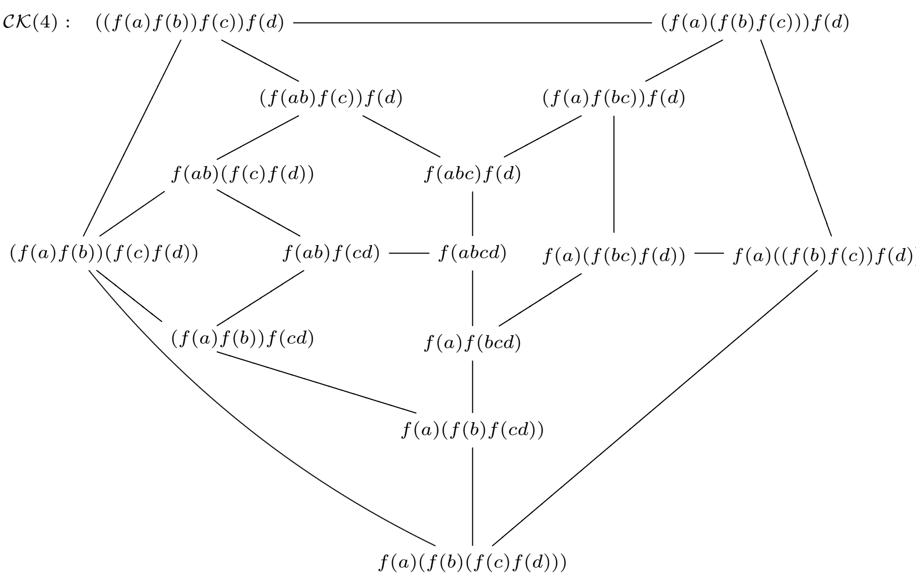
$$\mathcal{CK}(1): \quad f(a)$$

$$\mathcal{CK}(2): \quad f(a)f(b) \text{ ——— } f(ab)$$

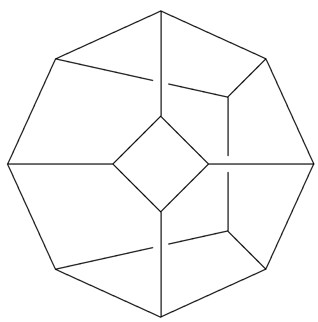
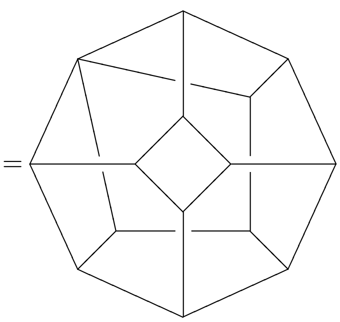
$$\mathcal{CK}(3):$$

$$\begin{tikzcd}
& (f(a)f(b))f(c) \arrow[rr, dash] & & f(a)(f(b)f(c)) \arrow[dr, dash] & \\
f(ab)f(c) \arrow[ur, dash] \arrow[drr, dash] & & & & f(a)f(bc) \arrow[dll, dash] \\
& & f(abc) & &
\end{tikzcd}$$

$\mathcal{CK}(4):$




Here are easily compared pictures of the associahedron and composihedron in dimension 3.

$\mathcal{K}(5) =$  $\mathcal{CK}(4) =$ 

J. Stasheff points out that we can obtain the complex $\mathcal{K}(5)$ by deleting a single edge of $\mathcal{CK}(4)$.

### 2.4. Enriched bicategories and the composihedron

Remarkably, the same minor error of recognition between the associahedron and composihedron may have been made by category theorists who wrote down the coherence axioms of enriched bicategory theory. The first few polytopes in our new sequence correspond to cocyle coherence conditions in the definition of enriched bicategories, as in [**6**]. It is incorrectly implied there that the final cocycle condition has the combinatorial form of the associahedron $\mathcal{K}(5)$. The axiom actually consists of two pasting diagrams, which, when glued along their boundary, are seen to form the composihedron $\mathcal{CK}(4)$ instead. Also note that the first few composihedra correspond as well to the axioms for pseudomonoids in a monoidal bicategory, as seen in [**31**]. This fact is to be expected, since pseudomonoids are just single object enriched bicategories.

Little has been published about enriched bicategories, although the theory is used in recent research papers such as [**11**] and [**33**]. The full definition of enriched bicategory is worked out in [**6**]. It is repeated with the simplification of a strict monoidal $\boldsymbol{\mathcal{W}}$ in [**18**], in which case the commuting diagrams have the form of cubes. (Recall that when both the range and domain are strictly associative that the multiplihedra collapse to become the cubes, as shown in [**5**].) An earlier definition of a (lax) enriched bicategory as a lax functor of certain tricategories is found in [**13**] and is also reviewed in [**18**]; in retrospect this formulation is to be expected since the composihedra are special cases of multiplihedra.

Here for reference is the definition of enriched bicategories, closely following [**6**]. Let $(\boldsymbol{\mathcal{W}}, \otimes, \alpha, \pi, I)$ be a monoidal bicategory, as defined in [**6**] or in [**13**], or comparably in [**2**], with $I$ a strict unit (but we will include the identity cells in our diagrams). Note that item (5) in the following definition corrects an obvious typo in the corresponding item of [**6**], and that the 2-cells in item (6) have been somewhat rearranged from that source, for easier comparison to the polytope $\mathcal{CK}(4)$.

**Definition 2.1.** An enriched bicategory $\boldsymbol{\mathcal{A}}$ over $\boldsymbol{\mathcal{W}}$ is:

1. a collection of objects $\mathrm{Ob}\boldsymbol{\mathcal{A}}$,
2. hom-objects $\boldsymbol{\mathcal{A}}(A,B) \in \mathrm{Ob}\boldsymbol{\mathcal{W}}$ for each $A, B \in \mathrm{Ob}\boldsymbol{\mathcal{A}}$,
3. a composition 1-cell
$$\boldsymbol{\mathcal{M}}_{ABC} \colon \boldsymbol{\mathcal{A}}(B,C) \otimes \boldsymbol{\mathcal{A}}(A,B) \to \boldsymbol{\mathcal{A}}(A,C)$$
in $\boldsymbol{\mathcal{W}}$ for each $A, B, C \in \mathrm{Ob}\boldsymbol{\mathcal{A}}$,
4. an identity 1-cell $\mathcal{J}_A \colon I \to \boldsymbol{\mathcal{A}}(A,A)$ for each object $A$,
5. 2-cells $\boldsymbol{\mathcal{M}}_2$ in $\boldsymbol{\mathcal{W}}$ for each $A, B, C, D \in \mathrm{Ob}\mathcal{A}$:
$$\begin{tikzcd}
(\boldsymbol{\mathcal{A}}(C,D) \otimes \boldsymbol{\mathcal{A}}(B,C)) \otimes \boldsymbol{\mathcal{A}}(A,B) \arrow[rr, "\alpha"] \arrow[d, "\boldsymbol{\mathcal{M}}\otimes 1"] & & \boldsymbol{\mathcal{A}}(C,D) \otimes (\boldsymbol{\mathcal{A}}(B,C) \otimes \boldsymbol{\mathcal{A}}(A,B)) \arrow[d, "1\otimes\boldsymbol{\mathcal{M}}"'] \\
\boldsymbol{\mathcal{A}}(B,D) \otimes \boldsymbol{\mathcal{A}}(A,B) \arrow[rr, Rightarrow, "\boldsymbol{\mathcal{M}}_2"] \arrow[dr, "\boldsymbol{\mathcal{M}}"'] & & \boldsymbol{\mathcal{A}}(C,D) \otimes \boldsymbol{\mathcal{A}}(A,C) \arrow[dl, "\boldsymbol{\mathcal{M}}"] \\
& \boldsymbol{\mathcal{A}}(A,D) &
\end{tikzcd}$$
6. which obey a cocycle condition for each $A, B, C, D, E$. This is shown in Figure 2, where we abbreviate the hom-objects and composition by $\boldsymbol{\mathcal{M}} \colon BC, AB \to AC$.

7. Unit 2-cells:

$$\begin{tikzcd}
I \otimes \boldsymbol{\mathcal{A}}(A,B) \arrow[drr, "="] \arrow[dd, "\mathcal{J}_B \otimes 1"'] & & & & \boldsymbol{\mathcal{A}}(A,B) \otimes I \arrow[dll, "="'] \arrow[dd, "1 \otimes \mathcal{J}_A"] \\
& & \boldsymbol{\mathcal{A}}(A,B) \arrow[l, phantom, "\lambda \Uparrow"] \arrow[r, phantom, "\rho \Uparrow"] & & \\
\boldsymbol{\mathcal{A}}(B,B) \otimes \boldsymbol{\mathcal{A}}(A,B) \arrow[urr, "\boldsymbol{\mathcal{M}}_{ABB}"'] & & & & \boldsymbol{\mathcal{A}}(A,B) \otimes \boldsymbol{\mathcal{A}}(A,A), \arrow[ull, "\boldsymbol{\mathcal{M}}_{AAB}"]
\end{tikzcd}$$

8. which obey a pasting condition of their own, which we will omit for brevity.

If, instead of the existence of 2-cells postulated in (5) and (7), we had required that the diagrams commute, we would recover the definition of an enriched category. Note that the pentagonal axiom for a monoidal category at the beginning of this section has the form of the associahedron $\mathcal{K}(4)$ but the commuting pentagon for enriched categories (here the domain and range of $\boldsymbol{\mathcal{M}}_2$), is actually better described as having the form of $\mathcal{CK}(3)$.

Here is the cocycle condition (6), pasted together and shown as a Schlegel diagram for comparison to the similarly displayed picture above of $\mathcal{CK}(4)$. To save space, "$\bullet\ \bullet \to \bullet$" will represent $\boldsymbol{\mathcal{M}}_1 \colon \boldsymbol{\mathcal{A}}(B,C) \otimes \boldsymbol{\mathcal{A}}(A,B) \to \boldsymbol{\mathcal{A}}(A,C)$.

$$\begin{tikzcd}
& ((\bullet\ \bullet)\bullet)\bullet \arrow[rrrr] \arrow[dr] \arrow[dddl] & & & & (\bullet(\bullet\ \bullet))\bullet \arrow[dl] \arrow[dddr] & \\
& & (\bullet\ \bullet)\bullet \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\overset{\boldsymbol{\mathcal{M}}_2}{\Rrightarrow}"] & & (\bullet\ \bullet)\bullet \arrow[dl] \arrow[ddr] & & \\
& \bullet(\bullet\ \bullet) \arrow[dr] \arrow[rr, phantom, "\boldsymbol{\mathcal{M}}_2 \Downarrow"] & & \bullet\ \bullet \arrow[d] \arrow[dr, phantom, "\Downarrow \boldsymbol{\mathcal{M}}_2"] & & & \\
(\bullet\ \bullet)(\bullet\ \bullet) \arrow[ur] \arrow[dr] \arrow[dddrrr] & & \bullet\ \bullet \arrow[r] & \bullet & & \bullet(\bullet\ \bullet) \arrow[dll] & \bullet((\bullet\ \bullet)\bullet) \arrow[l] \arrow[dddlll] \\
& (\bullet\ \bullet)\bullet \arrow[ur] \arrow[drr] \arrow[rr, phantom, "\boldsymbol{\mathcal{M}}_2 \Downarrow"] & & \bullet\ \bullet \arrow[u] \arrow[dr, phantom, "\Downarrow \boldsymbol{\mathcal{M}}_2"] & & & \\
& & & \bullet(\bullet\ \bullet) \arrow[u] & & & \\
& & & \bullet(\bullet(\bullet\ \bullet)) \arrow[u] & & &
\end{tikzcd}$$

From this perspective it is easier to visualize how, in the case that $\boldsymbol{\mathcal{W}} = \mathbf{Cat}$, an enriched bicategory is just a bicategory. In that case $\boldsymbol{\mathcal{M}}_2$ is renamed $\sigma$, and the enriched cocycle axiom becomes the ordinary bicategory cocycle axiom with the underlying form of $\mathcal{K}(4)$. Thus the $n^{th}$ composihedron can be seen to contain in its structure two copies of the $n^{th}$ associahedron: one copy as a particular upper facet

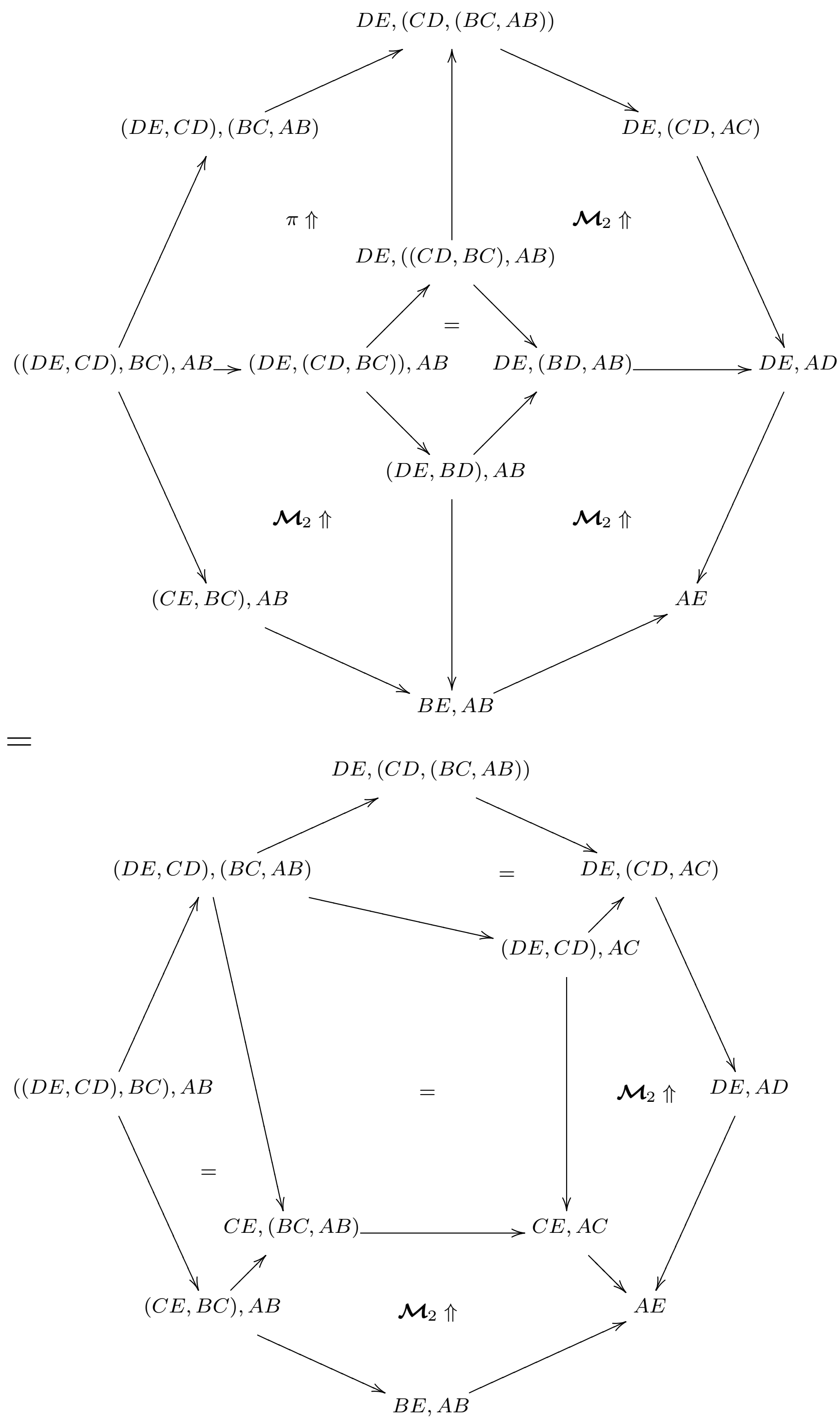


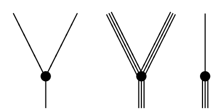

Figure 2: Cocycle condition for enriched bicategories.

and another as a certain collection of facets that can function as labels for the facets of the corresponding associahedron. Of course the second copy is only seen upon decategorification!

## 3. Recursive definition

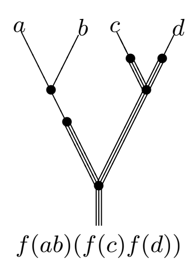
In [**16**] the authors give a geometrically defined CW-complex definition of the multiplihedra, and then demonstrate the recursive combinatorial structure. Here we describe how to collapse that structure for the case of a strictly associative domain, and achieve a recursive definition of the composihedra.

Pictures in the form of *painted binary trees* can be drawn to represent the multiplication of several objects in a monoid, before or after their passage to the image of that monoid under a homomorphism. We use the term "painted" rather than "colored" to distinguish our trees with two edge colorings, "painted" and "unpainted," from the other meaning of colored, as in colored operad or multicategory. We will refer to the exterior vertices of the tree as the root and the leaves , and to the interior vertices as nodes. This will be handy since then we can reserve the term "vertices" for reference to polytopes. A painted binary tree is painted beginning at the root edge (the leaf edges are unpainted), and always painted in such a way that there are only three types of nodes. They are:

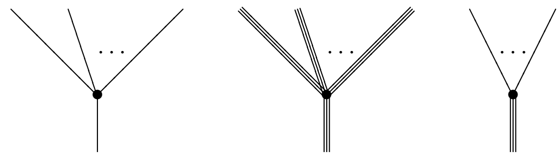


(1) (2) (3)

This limitation on nodes implies that painted regions must be connected, that painting must never end precisely at a trivalent node, and that painting must proceed up both branches of a trivalent node. To see the promised representation, we let the left-hand type (1) trivalent node above stand for multiplication in the domain; the middle, painted, type (2) trivalent node above stand for multiplication in the range; and the right-hand type (3) bivalent node stand for the action of the mapping. For instance, given $a, b, c, d$ elements of a monoid, and $f$ a monoid morphism, the following diagram represents the operation resulting in the product $f(ab)(f(c)f(d))$:

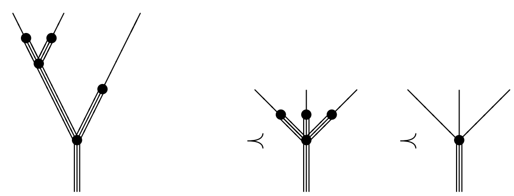



To define the face poset structures of the multiplihedra and composihedra we need painted trees that are no longer binary. Here are the three new types of node allowed in a general painted tree: They correspond to the node types (1), (2) and (3) in that they are painted in similar fashion. They generalize types (1), (2), and (3) in that each has greater or equal valence than the corresponding earlier node type.



(4) (5) (6)

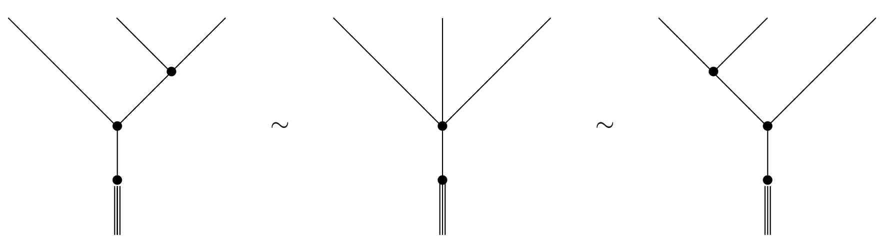
**Definition 3.1.** By *refinement* of painted trees we refer to the relationship: $t$ refines $t'$ means that $t'$ results from the collapse of some of the internal edges of $t$. This is a partial order on $n$-leaved painted trees, and we write $t \prec t'$. Thus the binary painted trees are refinements of the trees having nodes of type (4)–(6). *Minimal refinement* refers to the covering relation in this poset: $t$ minimally refines $t''$ means that $t$ refines $t''$ and also that there is no $t'$ such that both $t$ refines $t'$ and $t'$ refines $t''$. The poset of painted trees with $n$ leaves is precisely the face poset of the $n^{th}$ multiplihedron. Here is an example of a chain in the poset of 3-leaved painted trees:



**Definition 3.2.** Two painted trees are said to be *domain equivalent* if two requirements are satisfied:

1) they both refine the same tree, and
2) the collapses involved in both refinements are of edges whose two nodes are of type (1) or type (4). That is, these collapses will be of internal unpainted edges with no adjacent painted edges.

Locally the equivalences will appear as follows:



That is, the domain equivalence is generated by the two moves illustrated above. Therefore we often choose to represent a domain equivalence class of trees by the unique member with least refinement, that is with its unpainted subtrees all corollas.

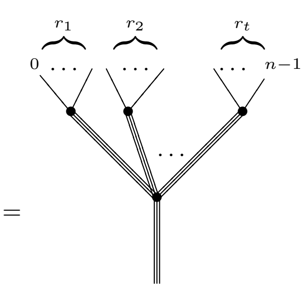
**Definition 3.3.** A *painted corolla* is a painted tree with only one node, of type (6).

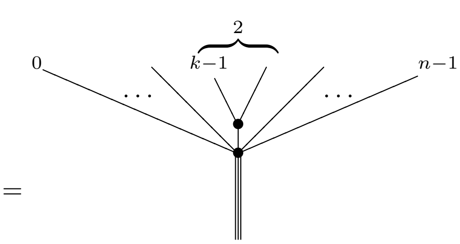
The $n^{th}$ composihedron may be described as the quotient of the $n^{th}$ multiplihedron under domain equivalence. This quotient can be performed by applying the equivalence either to the metric trees, which define the multiplihedra in [**5**] or to the combinatorial definition of the multiplihedra, i.e. to the face poset of the multiplihedra. Here we will follow the latter scheme to unpack the definition of the composihedra into a recursive description utilizing facet inclusions.

The facets of the $n^{th}$ multiplihedron are of two types: upper facets whose vertices correspond to sets of related ways of multiplying in the range, and lower facets whose vertices correspond to sets of related ways of multiplying in the domain. A lower facet is denoted $\mathcal{J}_k(r,s)$ and is a combinatorial copy of the complex $\mathcal{J}(r)\times\mathcal{K}(s)$. Here $r+s-1=n$. A vertex of a lower facet represents a way in which $s$ of the points are multiplied in the domain, and then how the images of their product and of the other $r-1$ points are multiplied in the range. In the case for which the domain is strictly associative, the copy of $\mathcal{K}(s)$ here should be replaced by a single point, denoted $\{*\}$. Thus in the composihedra the lower faces will have reduced dimension. In fact only certain of them will still be facets.

The recursive definition of the new sequence of polytopes is as follows:

**Definition 3.4.** The first composihedron denoted $\mathcal{CK}(1)$ is defined to be the single point $\{*\}$. It is associated to the painted tree with one leaf, and thus one type (3) internal node. Assume that the $\mathcal{CK}(k)$ have been defined for $k=1,\dots,n-1$. To $\mathcal{CK}(k)$ we associate the $k$-leaved painted corolla. We define an $(n-2)$-dimensional CW-complex $\partial\mathcal{CK}(n)$ as follows, and then define $\mathcal{CK}(n)$ to be the cone on $\partial\mathcal{CK}(n)$. Now the top-dimensional cells of $\partial\mathcal{CK}(n)$ (facets of $\mathcal{CK}(n)$) are in bijection with the set of painted trees of two types:

*upper* trees $u(t;r_1,\dots,r_t)=$ 


and *minimal lower* trees $l(k,2)=$ 

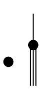

There are $2^{n-1}-1$ upper trees, as counted in [**10**]. The index $t$ can range from $2,\dots,n$, and $r_i\geqslant 1$. The upper facet, which we call $\mathcal{CK}(t;,r_1,\dots,r_t)$ associated with the upper tree of identical indexing, is a copy of $\mathcal{K}(t) \times \mathcal{CK}(r_1) \times \cdots \times \mathcal{CK}(r_t)$. Here the sum of the $r_i$ is $n$.

There are $n-1$ minimal lower trees. The lower facet, which we call $\mathcal{CK}_k(n-1,2)$ associated with the lower tree of identical indexing, is a copy of $\mathcal{CK}(n-1) \times \mathcal{K}(2)$, that is, a copy of $\mathcal{CK}(n-1)$. Here $k$ ranges from 1 to $n-1$. $B(n)$ is the union of all these facets, with intersections described as follows:

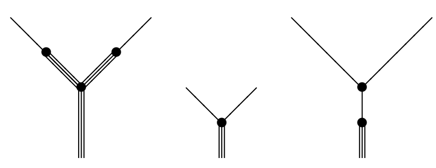
Since the facets are product polytopes, their sub-facets in turn are products of faces (of smaller associahedra and composihedra) whose dimensions sum to $n-3$. Each of these sub-facets thus comes (inductively) with a list of associated trees. There will always be a unique way of grafting these trees to construct a painted tree that is a minimal refinement of the upper or minimal lower tree associated to the facet in question. For the sub-facets of an upper facet the recipe is to paint entirely the $t$-leaved tree associated to a face of $\mathcal{K}(t)$ and to graft to each of its branches in turn the trees associated to the appropriate faces of $\mathcal{CK}(r_1)$ through $\mathcal{CK}(r_t)$ respectively. A sub-facet of the lower facet $\mathcal{CK}_k(n-1,2)$ inductively comes with an $(n-1)$-leaved associated upper or minimal lower tree $T$. The recipe for assigning our sub-facet an $n$-leaved minimal refinement of the $n$-leaved minimal lower tree $l(k,2)$ is to graft an unpainted 2-leaved tree to the $k^{th}$ leaf of $T$. See the following Example 3.5 for this pictured in low dimensions.

The intersection of two facets in $\partial\mathcal{CK}(n)$ consists of the sub-facets of each which have associated trees that are domain equivalent. Then $\mathcal{CK}(n)$ is defined to be the cone on $\partial\mathcal{CK}(n)$. To $\mathcal{CK}(n)$ we assign the painted corolla of $n$ leaves.

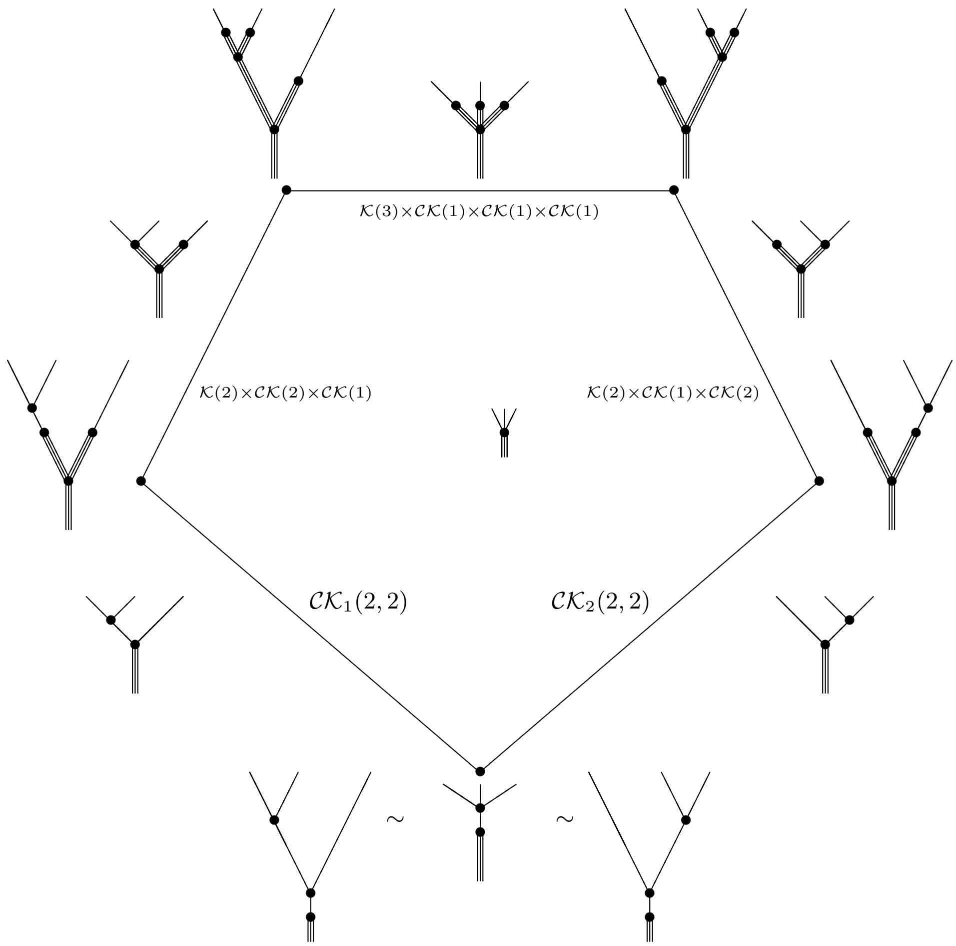
*Example 3.5.*

$\mathcal{CK}(1) = \bullet$ 

Here is $\mathcal{CK}(2)$ with the upper facet $\mathcal{K}(2) \times \mathcal{CK}(1) \times \mathcal{CK}(1)$ on the left and the lower facet $\mathcal{CK}(1) \times \mathcal{K}(2)$ on the right.



And here is the complex $\mathcal{CK}(3)$. The product structure of the upper facets is listed, and the two lower facets are named. Note that both lower facets are copies of $\mathcal{CK}(2) \times \mathcal{K}(2)$. Notice also how the sub-facets (vertices) are labeled. For instance, the upper right vertex is labeled by a tree that could be constructed by grafting three copies of the single leaf painted corolla onto a completely painted binary tree with three leaves, or by grafting a single leaf painted corolla and a 2-leaf painted binary tree onto the leaves of a 2-leaf completely painted binary tree.




*Remark 3.6.* The total number of facets of the $n^{th}$ composihedron is $2^{n-1}+n-2$; therefore the number of facets of $\mathcal{CK}(n+1)$ is

$$2^n + n - 1 = 0, 2, 5, 10, 19, 36, \dots .$$

This is sequence A052944 in the OEIS. This sequence also gives the number of vertices of an $n$-dimensional cube with one truncated corner. Now the relationship between the polytopes in Figure 1 can be algebraically stated as an equation:

$$(2^n + n - 1) + \left(\frac{(n+1)(n+2)}{2} - 1\right) - \left(\frac{n(n+1)}{2} + 2^n - 1\right) = 2n.$$

That is:

$$|\{\text{facets of } \mathcal{CK}(n+1)\}| + |\{\text{facets of } \mathcal{K}(n+2)\}| - |\{\text{facets of } \mathcal{J}(n+1)\}| \\ = |\{\text{facets of } C(n)\}|,$$

where $C(n)$ is the $n$-dimensional cube.

*Remark 3.7.* Consider the existence of facet inclusions:

$$\mathcal{K}(k) \times (\mathcal{CK}(j_1) \times \cdots \times \mathcal{CK}(j_k)) \to \mathcal{CK}(n),$$

where $n$ is the sum of the $j_i$. This is the left module structure: the composihedra together form a left operad module over the operad of spaces formed by the associahedra. Notice that the description of the assignation of trees to sub-facets (via grafting) in the definition above guarantees the module axioms.

*Remark 3.8.* By definition, the $n^{th}$ composihedron is seen to be the quotient of the $n^{th}$ multiplihedron under domain equivalence. Recall that the lower facets of $\mathcal{J}(n)$ are equivalent to copies of $\mathcal{J}(r) \times \mathcal{K}(s)$. The actual quotient is achieved by identifying any two points $(a, b) \sim (a, c)$ in a lower facet, where $a$ is a point of $\mathcal{J}(r)$ and $b, c$ are points in $\mathcal{K}(s)$. The associated quotient map from $\mathcal{J}(n)$ to $\mathcal{CK}(n)$ is in fact a cellular projection. In other words, $\mathcal{CK}(n)$ is the polytope achieved by taking the $n^{th}$ multiplihedron $\mathcal{J}(n)$ and sequentially collapsing certain faces. For a picture of this, see the upper left projection of Figure 1, where the small pentagon and two rectangles on the back of $\mathcal{J}(4)$ collapse to a vertex and two edges respectively.

*Remark 3.9.* In [**32**], a projection $\pi$ from the $n^{th}$ permutohedron $\mathcal{P}(n)$ to $\mathcal{J}(n)$ is described; thus there follows a composite projection from $\mathcal{P}(n) \to \mathcal{CK}(n)$. For comparison sake, recall that in the case of a strictly associative range the multiplihedra become the associahedra, in fact the quotient under range equivalence of $\mathcal{J}(n)$ is $\mathcal{K}(n+1)$. The implied projection of this quotient composed with $\pi$ yields a new projection from $\mathcal{P}(n) \to \mathcal{K}(n+1)$. Saneblidze and Umble describe a very different projection from $\mathcal{J}(n)$ to $\mathcal{K}(n+1)$. When composed with $\pi$ this yields a (different) projection from $\mathcal{P}(n)$ to $\mathcal{K}(n+1)$ that is shown in [**32**] to be precisely the projection $\theta$ described in [**37**].

*Remark 3.10.* There is a bijection between the 0-cells, or vertices, of $\mathcal{CK}(n)$ and the domain equivalence classes of $n$-leaved painted binary trees. This follows from the recursive construction, since the 0-cells must be associated to completely refined painted trees. However, the domain equivalent trees must be assigned to the same vertex since domain equivalence determines intersection of sub-facets. Recall that we can also label the vertices of $\mathcal{CK}(n)$ by applications of an $A_\infty$ function $f$ to $n$-fold products in a topological monoid.

*Remark 3.11.* As defined in [**35**], an $A_n$-map between $A_n$-spaces $f\colon X \to Y$ is described by an action $\mathcal{J}(n) \times X^n \to Y$. If the space $X$ is a topological monoid, then we may equivalently describe $f\colon X \to Y$ by simply replacing the action of $\mathcal{J}(n)$ with an action of $\mathcal{CK}(n)$ in the definition of $A_n$-map.

## 4. Vertex combinatorics

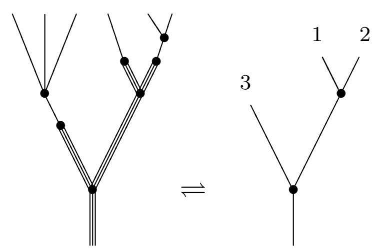
We now mention results regarding the counting of the vertices of the composihedra. Recall that vertices correspond bijectively to domain equivalence classes of painted binary trees.

**Theorem 4.1.** *The number of vertices $a_n$ of the $n^{th}$ composihedron is given recursively by:*

$$a_n = 1 + \sum_{i=1}^{n-1} a_i a_{n-i},$$

*where* $a_0 = 0$.

*Proof.* By a weighted tree we refer to a tree with a real number assigned to each of its leaves. The total weight of the tree is the sum of the weights of its leaves. The set of domain equivalence classes of painted binary trees with $n$-leaves is in bijection with the set of binary weighted trees of total weight $n$, where leaves have positive integer weights. This is evident from the representation of a domain equivalence class by its least refined member. Here is a picture that illustrates the general case of the bijection.




There is one weighted 1-leaved tree with total weight $n$. Now we count the binary trees with total weight $n$ that have at least one trivalent node. Each of these consists of a choice of two binary subtrees whose root is the initial trivalent node, and whose weights must sum to $n$. Thus we sum over the ways that $n$ can be split into two natural numbers. □

*Remark 4.2.* This formula gives the sequence which begins:

$$0, 1, 2, 5, 15, 51, 188, 731, 2950, 12235 \dots \ .$$

It is sequence A007317 of the *On-line Encyclopedia* of integer sequences. This formula for the sequence was originally stated by Benoit Cloitre [**8**].

The recursive formula above yields the equation

$$A(x) = \frac{x}{1-x} + (A(x))^2,$$

where $A(x)$ is the ordinary generating function of the sequence $a_n$ above. Thus by use of the quadratic formula we have

$$A(x) = \frac{x}{1-x} c(\frac{x}{1-x}),$$

where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of the Catalan numbers. This formula was originally derived by Emeric Deutsch from a comment of Michael Somos [**8**].

Now if we want to generate the sequence $\{a_{n+1}\}_{n=0}^{\infty} = 1, 2, 5, 15, 51, \dots$, then we must divide our generating function by $x$, to get $\frac{1}{1-x} c(\frac{x}{1-x})$. This we recognize as the generating function of the binary transform of the Catalan numbers, from the definition of binary transform in [**3**]. Therefore a direct formula for the sequence is given by

$$a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} C(k),$$

where $C(n)$ are the Catalan numbers.

*Remark 4.3.* This sequence also counts the non-commutative non-associative partitions of $n$. Two other combinatorial items that this sequence enumerates are: the number of Schroeder paths (i.e. consisting of steps $U = (1,1), D = (1,-1), H = (2,0)$ and never going below the x-axis) from $(0,0)$ to $(2n-2,0)$, with no peaks at even level, and the number of tree-like polyhexes (including the non-planar *helicenic* polyhexes) with $n$ or fewer cells [**8**].

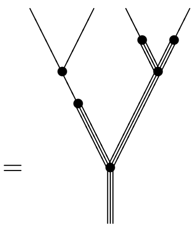
## 5. Convex hull realizations

In [**23**] Loday gives an algorithm for taking the binary trees with $n$ leaves and finding for each an extremal point in $\mathbf{R}^{n-1}$, together whose convex hull is $\mathcal{K}(n)$, the $(n-2)$-dimensional associahedron. Note that Loday writes formulas with the convention that the number of leaves is $n+1$, where we instead always use $n$ to refer to the number of leaves. Given a (non-painted) binary $n$-leaved tree $t$, Loday arrives at a point $M(t)$ in $\mathbf{R}^{n-1}$ by calculating a coordinate from each trivalent node. These are ordered left to right based upon the ordering of the leaves from left to right. Following Loday we number the leaves $0, 1, \ldots, n-1$ and the nodes $1, 2, \ldots, n-1$. The $i^{th}$ node is "between" leaf $i-1$ and leaf $i$, where "between" might be described to mean that a raindrop falling between those leaves would be caught at that node. Each trivalent node has a left and right branch, which each support a subtree. To find the Loday coordinate for the $i^{th}$ node we take the product of the number of leaves of the left subtree ($l_i$) and the number of leaves of the right subtree ($r_i$) for that node. Thus $M(t) = (x_1, \ldots, x_{n-1})$, where $x_i = l_i r_i$. Loday proves that the convex hull of the points thus calculated for all $n$-leaved binary trees is the $n^{th}$ associahedron. He also shows that the points thus calculated all lie in the $n-2$-dimensional affine hyperplane $H$ given by the equation $x_1 + \cdots + x_{n-1} = S(n-1) = \frac{1}{2}n(n-1)$.

In [**10**] we adjust Loday's algorithm to apply to painted binary trees as described above, with only nodes of type (1), (2), and (3), by choosing a number $q \in (0,1)$. Then given a painted binary tree $t$ with $n$ leaves we calculate a point $M_q(t)$ in $\mathbf{R}^{n-1}$ as follows: we begin by finding the coordinate for each trivalent node from left to right given by Loday's algorithm, but if the node is of type (1) (unpainted, or colored by the domain), then its new coordinate is found by further multiplying its Loday coordinate by $q$. Thus

$$M_q(t) = (x_1, \ldots, x_{n-1}), \text{ where } x_i = \begin{cases} q l_i r_i, & \text{if node } i \text{ is type (1)} \\ l_i r_i, & \text{if node } i \text{ is type (2).} \end{cases}$$

Note that whenever we speak of the numbered nodes ($1, \ldots, n-1$ from left to right) of a binary tree, we are referring only to the trivalent nodes, of type (1) or (2). For an example, let us calculate the point in $\mathbf{R}^3$ which corresponds to the 4-leaved tree:

$t =$ 

Then $M_q(t) = (q, 4, 1)$.

Now we turn to consider the case when in the algorithm described above we let $q = 0$.

$$M_0(t) = (x_1, \ldots, x_{n-1}), \text{ where } x_i = \begin{cases} 0, & \text{if node } i \text{ is type (1)} \\ l_i r_i, & \text{if node } i \text{ is type (2).} \end{cases}$$

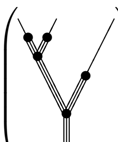
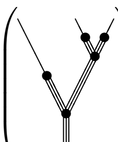
For the same tree $t$ in the above example we have $M_0(t) = (0, 4, 1)$.

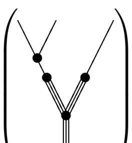
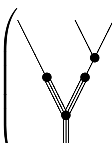
**Lemma 5.1.** *If (painted binary) tree $t$ is domain equivalent to $t'$, then $M_0(t) = M_0(t')$. That is, each domain equivalence class of binary painted trees contributes exactly one point.*

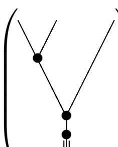
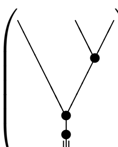
*Proof.* This is clear from the fact that all the unpainted nodes of a tree contribute a 0 coordinate. □

**Theorem 5.2.** *The convex hull of all the resulting points $M_0(t)$ for $t$ in the set of $n$-leaved binary painted trees is the $n^{th}$ composihedron. That is, our convex hull is combinatorially equivalent to the CW-complex $\mathcal{CK}(n)$.*

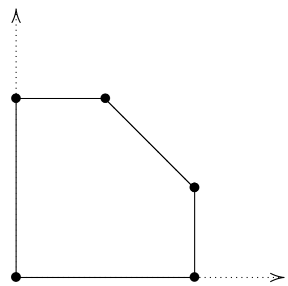
The proof will follow in Section 6. Here are all the painted binary trees with three leaves, together with their points $M_0(t) \in \mathbf{R}^2$.

$M_0$ (  ) $= (1, 2)$, $M_0$ (  ) $= (2, 1)$

$M_0$ (  ) $= (0, 2)$ $M_0$ (  ) $= (2, 0)$

$M_0$ (  ) $= (0, 0)$, $M_0$ (  ) $= (0, 0)$

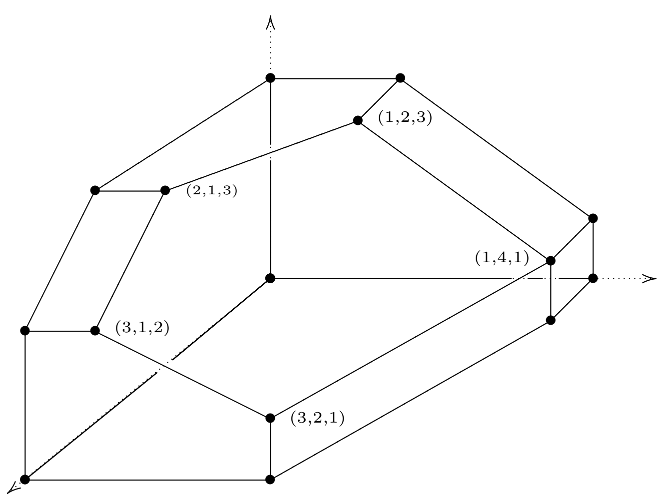
Note that the bottom two points are both the origin. The convex hull of the five total distinct points appears as follows:



The (redundant) list of vertices for $\mathcal{CK}(4)$ based on painted binary trees with four leaves is:

| | | | | |
|---|---|---|---|---|
| (1, 2, 3) | (0, 2, 3) | (0, 0, 3) | (0, 0, 0) | |
| (2, 1, 3) | (2, 0, 3) | (0, 0, 3) | (0, 0, 0) | |
| (3, 1, 2) | (3, 0, 2) | (3, 0, 0) | (0, 0, 0) | |
| (3, 2, 1) | (3, 2, 0) | (3, 0, 0) | (0, 0, 0) | |
| (1, 4, 1) | (0, 4, 1) | (1, 4, 0) | (0, 4, 0) | (0, 0, 0) . |

These are suggestively listed as a table where the first column is made up of the coordinates calculated by Loday for $\mathcal{K}(4)$, which here correspond to trees with every trivalent node entirely painted. The rows may be found by applying the factor 0 to each coordinate in turn, in order of increasing size of those coordinates. Here is the convex hull of the 15 total distinct points, where we see that each row of the table corresponds to shortest paths from the big pentagon to the origin. Of course, sometimes there are multiple such paths.



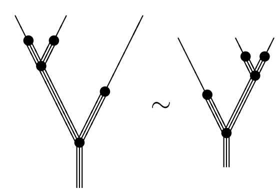

To see the picture of $\mathcal{CK}(4)$, that is in Figure 1 of this paper, just rotate this view of the convex hull by 90 degrees clockwise. To compare to other pictures of $\mathcal{CK}(4)$ in this paper note the smallest quadrilateral facet in this picture containing the unique vertex with four incident edges.

To see a rotatable version of the convex hull, which is the fourth composihedron, enter the following homogeneous coordinates into the Web Demo of polymake [**12**] (with option visual). Indeed polymake was instrumental in the experimental phase of this research. This table may be pasted into polymake as input, but with each point listed on a new line.

POINTS

| | | |
|---|---|---|
| 1 1 2 3 | 1 0 2 3 | 1 0 0 3 |
| 1 0 0 0 | 1 2 1 3 | 1 2 0 3 |
| 1 3 1 2 | 1 3 0 2 | 1 3 0 0 |
| 1 3 2 1 | 1 3 2 0 | 1 1 4 1 |
| 1 0 4 1 | 1 1 4 0 | 1 0 4 0 |

*Remark 5.3.* It is also fairly simple to devise a mapping from $n$-leaved painted binary trees to Euclidean space which reflects the quotient of the multiplihedron by range equivalence, where $f(a)(f(b)f(c)) = (f(a)f(b))f(c)$. It may be done by reflecting such equivalences as:



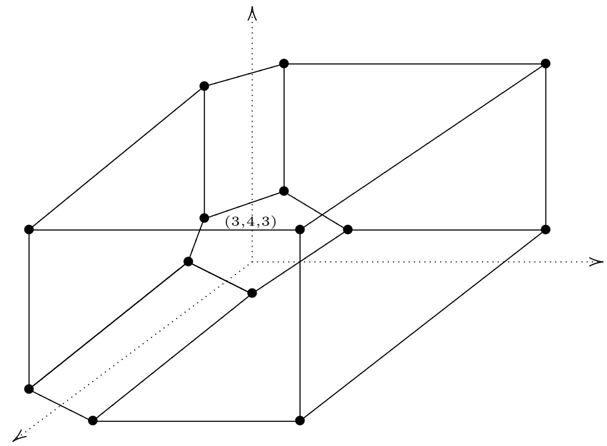
by mapping them both to a single vertex in $\mathbb{R}^2$. One possible map is

$$M'(t) = (x_1, \ldots, x_{n-1}), \text{ where } x_i = \begin{cases} ql_ir_i, & \text{if node } i \text{ is type (1)} \\ i(n-i), & \text{if node } i \text{ is type (2).} \end{cases}$$

This will yield a new realization of $\mathcal{K}(n+1)$ in $\mathbb{R}^{(n-1)}$, with vertices corresponding to range equivalence classes of $n$-leaved painted trees. Here is the (redundant) list of vertices for $\mathcal{K}(5)$ based on painted binary trees with four leaves: The list of vertices for $\mathcal{J}(4)$ based on painted binary trees with four leaves, for $q = \frac{1}{2}$, is:

| | | | | |
|---|---|---|---|---|
| (3, 4, 3) | (1/2, 4, 3) | (1/2, 2/2, 3) | (1/2, 2/2, 3/2) | |
| (3, 4, 3) | (3, 1/2, 3) | (2/2, 1/2, 3) | (2/2, 1/2, 3/2) | |
| (3, 4, 3) | (3, 1/2, 3) | (3, 1/2, 2/2) | (3/2, 1/2, 2/2) | |
| (3, 4, 3) | (3, 4, 1/2) | (3, 2/2, 1/2) | (3/2, 2/2, 1/2) | |
| (3, 4, 3) | (1/2, 4, 3) | (3, 4, 1/2) | (1/2, 4, 1/2) | (1/2, 4/2, 1/2) . |

These are suggestively listed as a table where the first column is the single image of all the trees with every trivalent node entirely painted. Indeed there are 14 total different vertices. Here is the convex hull of those vertices.




## 6. Proof of Theorem 5.2

Our algorithm for generating extremal points whose convex hull gives the composihedra can be described as the $q \to 0$ limit of the algorithm for the multiplihedra, which was given in [**10**]. What we need to demonstrate is that the limiting process does indeed deliver the combinatorial equivalent of the CW-complex $\mathcal{CK}(n)$.

To demonstrate that our convex hulls are each combinatorially equivalent to the corresponding CW-complexes of Definition 3.4, we need only check that they both have the same vertex-facet incidence. We will show that for each $n$ there is an isomorphism $f$ between the vertex sets (0-cells) of our convex hull and $\mathcal{CK}(n)$ which preserves the sets of vertices corresponding to facets; i.e. if $S$ is the set of vertices of a facet of our convex hull then $f(S)$ is a vertex set of a facet of $\mathcal{CK}(n)$.

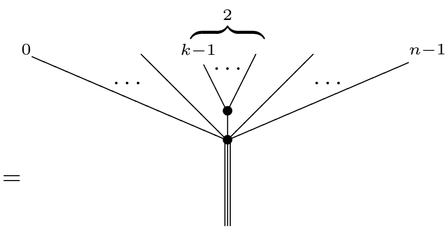
To demonstrate the existence of the isomorphism, noting that the vertices of $\mathcal{CK}(n)$ correspond to the domain equivalence classes of binary painted trees, we only need to check that the points we calculate from those classes are indeed the vertices of their convex hull. Recall that the calculation of a point from a class is well defined by Lemma 5.1. The isomorphism $f$ is the one that takes the vertex calculated from a certain class to the 0-cell associated to the same class. Now a given facet of $\mathcal{CK}(n)$ corresponds to a tree $T$ which is one of the two sorts of trees pictured in Definition 3.4. To show that our implied isomorphism of vertices preserves vertex sets of facets we need to show that that our facet is the convex hull of the points corresponding to the classes of binary trees represented by *refinements* of $T$. By refinement of painted trees we refer to the relationship: $t$ refines $t'$ if $t'$ results from the collapse of some of the internal edges of $t$.

The proofs of both key points will proceed in tandem, and will be inductive. That is, we will show that for each facet tree $T$, the points $M_0(t)$ for $t < T$ are precisely those points that lie in a a bounding hyperplane of our convex hull. Then we will check that those points $M_0(t)$ are the extremal points of their convex hull, which is indeed an $(n-2)$-dimensional polytope of the type decreed in Definition 3.4.

We will use the fact that if $P(q)$ and $R(q)$ are two polytopes with some vertices parametrized continuously by $q$ that

$$\lim_{q\to a}(P\times Q)\equiv\lim_{q\to a}P\times\lim_{q\to a}Q.$$

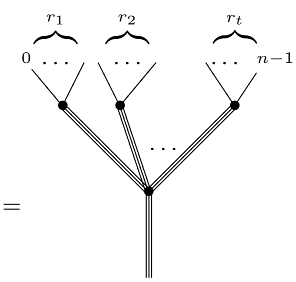
**Definition 6.1.** The lower facets $\mathcal{CK}_k(n-1,2)$ correspond to minimal lower trees such as:

$l(k,2) =$ 


These are assigned a hyperplane $H_0(l(k,2))$ determined by the equation $x_k = 0$.

Recall that $n-1$ is the number of branches extending from the lowest node; thus $1\leqslant k\leqslant n-1$. Notice that if $q$ were not zero, as in Definition 5.1 of [**10**], then the equation would appear: $x_k = q$. We conjecture that these latter hyperplanes would function as replacements for the ones given by $x_k = 0$ in that after the replacement the resulting polytope would still be the composihedron.

**Definition 6.2.** The upper facets $\mathcal{CK}(t; r_1,\ldots,r_t)$ correspond to upper trees such as:

$u(t;r_1,\ldots,r_t) =$ 


These are assigned a hyperplane $H_0(u(t;r_1,\ldots,r_t))$ determined by the equation

$$x_{r_1}+x_{(r_1+r_2)}+\cdots+x_{(r_1+r_2+\cdots+r_{t-1})}=\frac{1}{2}\left(n(n-1)-\sum_{i=1}^{t}r_i(r_i-1)\right),$$

or equivalently:

$$x_{r_1}+x_{(r_1+r_2)}+\cdots+x_{(r_1+r_2+\cdots+r_{t-1})}=\sum_{1\leqslant i<j\leqslant t}r_ir_j.$$

Note that if $t=n$ (so $r_i=1$ for all $i$), then this becomes the hyperplane given by

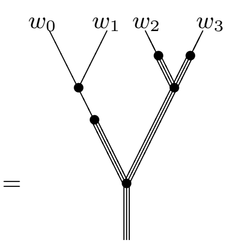
$$x_1+\cdots+x_{n-1}=\frac{1}{2}n(n-1)=S(n-1).$$

Therefore the points $M_0(t)$ for $t$, a binary tree with only nodes type (2) and (3), will lie in the hyperplane $H$ by Lemma 2.5 of [**23**] (using notation $S(n)$ and $H$ as in that source). Also note that these upper hyperplanes are exactly the same as those defined for $\mathcal{J}(n)$ in Definition 5.2 of [**10**].

In order to prove Theorem 5.2 it turns out to be expedient to prove a more general result. This consists of an even more flexible version of the algorithm for assigning points to binary trees in order to achieve a convex hull of those points which is the composihedron. To assign points in $\mathbf{R}^{n-1}$ to the domain equivalence classes of binary painted trees with $n$ leaves we choose an ordered $n$-tuple of positive integers $w_0, \dots, w_{n-1}$. Now given a tree $t$ we calculate a point $M_0^w(t)$ in $\mathbf{R}^{n-1}$ as follows: we begin by assigning the weight $w_i$ to the $i^{th}$ leaf. We refer to the result as a weighted tree. Then we modify Loday's algorithm for finding the coordinate for each trivalent node by replacing the number of leaves of the left and right subtrees with the sums of the weights of the leaves of those subtrees. Thus we let $L_i = \sum w_k$, where the sum is over the leaves of the subtree supported by the left branch of the $i^{th}$ node. Similarly, we let $R_i = \sum w_k$, where $k$ ranges over the leaves of the subtree supported by the right branch. Then

$$M_0^w(t) = (x_1, \dots x_{n-1}), \text{ where } x_i = \begin{cases} 0, & \text{if node } i \text{ is type (1)} \\ L_i R_i, & \text{if node } i \text{ is type (2).} \end{cases}$$

Note that the original points $M_0(t)$ are recovered if $w_i = 1$ for $i = 0, \dots, n-1$. Thus proving that the convex hull of the points $M_0^w(t)$, where $t$ ranges over the binary painted trees with $n$ leaves is the $n^{th}$ composihedron, will imply the main theorem. For an example, let us calculate the point in $\mathbf{R}^3$ which corresponds to the 4-leaved tree:

$t =$ 


Now $M_0^w(t) = (0, (w_0 + w_1)(w_2 + w_3), w_2 w_3)$. To motivate this new weighted version of our algorithm we mention that the weights $w_0, \dots, w_{n-1}$ are to be thought of as the sizes of various trees to be grafted to the respective leaves. This weighting is therefore necessary to make the induction go through, since the induction is itself based upon the grafting of trees.

Since we are proving that the points $M_0^w(t)$ are the vertices of the composihedron, we need to define hyperplanes $H_0^w(t)$ for this weighted version, which we will show to be the bounding hyperplanes when $t$ is a facet tree.

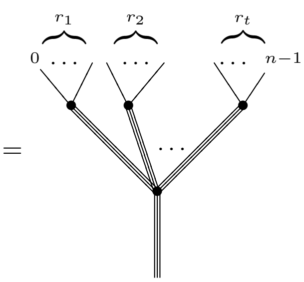
**Definition 6.3.** Recall that the lower facets $\mathcal{CK}_k(n-1, 2)$ correspond to minimal lower trees. These are assigned a hyperplane $H_0^w(l(k,2))$ determined by the equation

$$x_k = 0.$$

**Lemma 6.4.** *For any painted binary tree $t$, the point $M_0^w(t)$ lies in the hyperplane $H_0^w(l(k,2))$ if and only if $t$ is domain equivalent to a refinement of $l(k,2)$. Also, the* *hyperplane* $H_0^w(l(k,2))$ *bounds below the points* $M_0^w(t)$ *for* $t$ *any binary representative of a domain class of painted trees.*

*Proof.* The $k^{th}$ node of any binary tree that is a refinement of $l(k,2)$ will be unpainted, and will have left and right subtrees with only one leaf each. Thus the $k^{th}$ node of a binary tree that is domain equivalent to a refinement of $l(k,2)$ will be unpainted. Furthermore, any binary tree with its $k^{th}$ node unpainted is domain equivalent to a refinement of $l(k,2)$, by a series of domain equivalence moves. (Recall that no branches may be painted above an unpainted node.) Thus any binary tree $t$ which is domain equivalent to a refinement of the lower tree $l(k,2)$ will yield a point $M_0^w(t)$ for which $x_k = 0$ since the $k^{th}$ node will be unpainted in any such tree. Also, if a binary tree $t$ is not equivalent to a refinement of a lower tree $l(k,2)$ then the point $M_0^w(t)$ will have the property that $x_k > 0$, since the $k^{th}$ node will be painted. □

**Definition 6.5.** Recall that the upper facets $\mathcal{CK}(t; r_1, \ldots, r_t)$ correspond to upper trees such as:

$u(t; r_1, \ldots, r_t) =$ 


These are assigned a hyperplane $H_0^w(u(t; r_1, \ldots, r_t))$ determined by the equation

$$x_{r_1} + x_{(r_1+r_2)} + \cdots + x_{(r_1+r_2+\cdots+r_{t-1})} = \sum_{1 \leqslant i < j \leqslant t} R_i R_j,$$

where $R_i = \sum w_j$, where the sum is over the leaves of the $i^{th}$ subtree (from left to right) with root the type (5) node; the index $j$ goes from $(r_1 + r_2 + \cdots + r_{i-1})$ to $(r_1 + r_2 + \cdots + r_i - 1)$ (where $r_0 = 0$.) Note that if $t = n$ (so $r_i = 1$ for all $i$) that this becomes the hyperplane given by

$$x_1 + \cdots + x_{n-1} = \sum_{1 \leqslant i < j \leqslant n-1} w_i w_j.$$

**Lemma 6.6.** *For any painted binary tree* $t$*, the point* $M_0^w(t)$ *lies in the hyperplane* $H_0^w(u(t; r_1, \ldots, r_t))$ *if and only if* $t$ *is (domain equivalent to) a refinement of* $u(t; r_1, \ldots, r_t)$*. Also, the hyperplane* $H_0^w(u(t; r_1, \ldots, r_t))$ *bounds above the points* $M_0^w(t)$ *for* $t$ *any binary painted tree.*

*Proof.* These hyperplanes are precisely those defined as $H_q^w(u(t; r_1, \ldots, r_t))$ in Definition 5.7 of [**10**]. Thus the proof of Lemma 5.8 of [**10**] holds here as well. Letting $q = 0$ in the proof of that lemma does not change the argument, since in neither definition does the value of $q$ actually appear. In fact the only use of $q$ in the proof relies on the fact that $q < 1$, and when $q = 0$ this is certainly also true. □

*Proof of Theorem 5.2.* Now we may proceed with our inductive argument. The base case of $n = 2$ leaves is trivial to check. The points in $\mathbf{R}^1$ are $w_0 w_1$ and 0 Their convex hull is a line segment, combinatorially equivalent to $\mathcal{CK}(2)$. Now we assume that for all $i < n$ and for positive integer weights $w_0, \ldots, w_{i-1}$, that the convex hull of the points $\{M_0^w(t) \mid t \text{ is a painted binary tree with } i \text{ leaves}\}$ in $\mathbf{R}^{i-1}$ is combinatorially equivalent to the complex $\mathcal{CK}(i)$, and that the points $M_0^w(t)$ are the vertices of the convex hull. Now for $i = n$ we need to show that the equivalence still holds. Recall that the two items we plan to demonstrate are that the points $M_0^w(t)$ are the vertices of their convex hull and that the facet of the convex hull corresponding to a given lower or upper tree $T$ is the convex hull of just the points corresponding to the binary trees that are refinements of $T$. The first item will be seen in the process of checking the second.

Given an $n$-leaved lower tree $l(k, 2)$ we have from Lemma 6.4 that the points corresponding to binary refinements of $l(k, 2)$ lie in an $(n-2)$-dimensional hyperplane $H_0^w(l(k, 2))$ which bounds the entire convex hull. To see that this hyperplane does indeed contain a facet of the entire convex hull we use the induction hypothesis to show that the dimension of the convex hull of just the points in $H_0^w(l(k, 2))$ is $n - 2$. For $q \in (0, 1)$ we know from [**10**] (proof of Theorem 3.1) that the points $M_q^w(t)$ corresponding to the binary trees $t < l(k, 2)$ are the vertices of a polytope combinatorially equivalent to $\mathcal{J}(n-1) \times \mathcal{K}(2) = \mathcal{J}(n-1)$. Taking the limit as $q \to 0$ piecewise allows us to use the induction hypothesis to show that the points $M_0^w(t)$ corresponding to the binary trees $t < l(k, 2)$ are the vertices of a polytope combinatorially equivalent to $\mathcal{CK}(n-1)$, which is the required type of polytope.

Given an $n$-leaved upper tree $u(t, r_1, \ldots, r_t)$ we have from Lemma 6.6 that the points corresponding to binary refinements of $u(t, r_1, \ldots, r_t)$ lie in an $n-2$-dimensional hyperplane $H_0^w(u(t, r_1, \ldots, r_t))$ which bounds the entire convex hull. To see that this hyperplane does indeed contain a facet of the entire convex hull we use the induction hypothesis to show that the dimension of the convex hull of just the points in $H_0^w(u(t, r_1, \ldots, r_t))$ is $n - 2$.

For $q \in (0, 1)$ we know from [**10**] (proof of Theorem 3.1) that the points $M_q^w(t)$ corresponding to the binary trees $t < u(t, r_1, \ldots, r_t)$ are the vertices of a polytope combinatorially equivalent to $\mathcal{K}(t) \times \mathcal{J}(r_1) \times \cdots \times \mathcal{J}(r_t)$. Here the vertices of the associahedron do not depend on the factor $q$. Taking the limit as $q \to 0$ piecewise allows us to use the induction hypothesis to show that the points $M_0^w(t)$ corresponding to the binary trees $t < u(t, r_1, \ldots, r_t)$ are the vertices of a polytope combinatorially equivalent to $\mathcal{K}(t) \times \mathcal{CK}(r_1) \times \cdots \times \mathcal{CK}(r_t)$. This is the required type of polytope.

Since each $n$-leaved binary painted tree is a refinement of some upper and or lower trees, then the point associated to that tree is found as a vertex of some of the facets of the entire convex hull, and thus is a vertex of the convex hull. This completes the proof. □

## References


[**1**] J.C. Baez and J. Dolan, Categorification, in *Higher category theory* (E. Getzler and M. Kapranov, Eds.), Contemp. Math. **230**, 1–36, Amer. Math. Soc., Providence, RI, 1998.

[**2**] J.C. Baez and M. Neuchl, Higher-dimensional algebra I: Braided monoidal 2-categories, *Adv. Math.* **121**, no. 2 (1996), 196–244.

[**3**] P. Barry, A Catalan transform and related transformations on integer sequences, *J. Integer Sequences* **8** (2005), no. 4, 1–24.

[**4**] M. Batanin, Monoidal globular categories as a natural environment for the theory of weak $n$-categories, *Adv. Math.* **136** (1998), no. 1, 39–103.

[**5**] J.M. Boardman and R.M. Vogt, Homotopy invariant algebraic structures on topological spaces, *Lecture Notes in Mathematics* **347**, Springer-Verlag, New York, 1973.

[**6**] S.M. Carmody, Cobordism categories, Ph.D. thesis, University of Cambridge, 1995.

[**7**] M. Carr and S.L. Devadoss, Coxeter complexes and graph-associahedra, *Topology Appl.* **153** (2006), no. 12, 2155–2168.

[**8**] E. Deutsch and M. Somos, `http://research.att.com/~njas/sequences/A007317`, 2005.

[**9**] S.L. Devadoss and S. Forcey, Marked tubes and the graph multiplihedron, to appear in *Algebr. Geom. Topol.*, 2008, `http://arxiv.org/abs/0807.4159`.

[**10**] S. Forcey, Convex hull realizations of the multiplihedra, to appear in *Topology Appl.*, 2008, `http://arxiv.org/abs/0706.3226`.

[**11**] R. Garner and N. Gurski, The low-dimensional structures that tricategories form, `http://arxiv.org/abs/0711.1761`, 2007.

[**12**] E. Gawrilow and M. Joswig, polymake: a framework for analyzing convex polytopes, in *Polytopes—combinatorics and computation* (Oberwolfach, 1997), DMV Sem. **29**, 43–73, Birkhäuser, Basel, 2000. Software available at `http://www.math.tu-berlin.de/polymake/index.html#apps/polytope`.

[**13**] R. Gordon, A.J. Power, and R. Street, *Coherence for tricategories*, *Mem. Amer. Math. Soc.* **117**, no. 558, 159–165, AMS, Providence, RI (1995).

[**14**] Y. Hemmi, Retractions of $H$-spaces, *Hiroshima Math. J.* **35** (2005), no. 1, 159–165.

[**15**] K. Hess, P.-E. Parent, and J. Scott, Co-rings over operads characterize morphisms, `math.AT/0505559`.

[**16**] N. Iwase and M. Mimura, Higher homotopy associativity, in *Algebraic topology* (Arcada, CA, 1986), Lecture Notes in Mathematics **1370**, 193–220, Springer-Verlag, New York, 1989.

[**17**] G.M. Kelly, *Basic concepts of enriched category theory*, London Math. Society Lecture Note Series **64**, Cambridge Univ. Press, Cambridge, 1982.

[**18**] S.G. Lack, The algebra of distributive and extensive categories, Ph.D. thesis, University of Cambridge, 1995.

[**19**] T. Leinster, Basic bicategories, `http://arxiv.org/abs/math.CT/9810017`.

[**20**] T. Leinster, *Higher operads, higher categories,* London Math. Society Lecture Note Series **298**, Cambridge Univ. Press, Cambridge, 2004.

[**21**] T. Leinster, Operads in higher-dimensional category theory, *Theory Appl. Categ.* **12** (2004), no. 3, 73–194.

**[22]** T. Leinster, A survey of definitions of $n$-category, *Theory Appl. Categ.* **10** (2002), no. 1, 1–70.

**[23]** J.L. Loday, Realization of the Stasheff polytope, *Arch. Math. (Basel)* **83** (2004), no. 3, 267–278.

**[24]** J.L. Loday, The diagonal of the Stasheff polytope, `http://arxiv.org/abs/0710.0572`, 2007.

**[25]** S. Mac Lane, *Categories for the working mathematician*, second edition, Graduate Texts in Mathematics **5**, Springer-Verlag, New York, 1998.

**[26]** M. Markl, Transferring $A_\infty$ (strongly homotopy associative) structures, *Ren. Circ. Mat. Palermo (2)*, Suppl. No. **79** (2006), 139–151.

**[27]** M. Markl, Simplex, associahedron, and cyclohedron, in *Higher homotopy structures in topology and mathematical physics* (Poughkeepsie, NY, 1996), Contemp. Math. **227**, 235–265, Amer. Math. Soc., Providence, RI, 1999.

**[28]** M. Markl and S. Shnider, Associahedra, cellular $W$-construction and products of $A_\infty$-algebras, *Trans. Amer. Math. Soc.* **358** (2006), no. 6, 2353–2373.

**[29]** M. Markl, S. Shnider, and J. Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monograph **96**, Amer. Math. Soc., Providence, RI, 2002.

**[30]** S. Mauo and C.T. Woodward, Geometric realizations of the multiplihedron and its complexification, `http://arxiv.org/abs/0802.2120`.

**[31]** P. McCrudden, Balanced coalgebroids, *Theory Appl. Categ.* **7** (2000), no. 6, 71–147.

**[32]** S. Saneblidze and R. Umble, Diagonals on the permutahedra, multiplihedra and associahedra, homology, *Homology Homotopy Appl.* **6** (2004), no. 1, 363–411.

**[33]** M. Shulman, Homotopy limits and colimits and enriched homotopy theory, `http://arxiv.org/abs/math/0610194`.

**[34]** J.D. Stasheff, Homotopy associativity of $H$-spaces I, *Trans. Amer. Math. Soc.* **108** (1963), 275–292.

**[35]** J.D. Stasheff, $H$-spaces from a homotopy point of view, *Lecture Notes in Mathematics* **161**, Springer-Verlag, New York (1970).

**[36]** R. Street, The algebra of oriented simplexes, *J. Pure Appl. Algebra* **49** (1987), no. 3, 283–335.

**[37]** A. Tonks, Relating the associahedron and the permutohedron, in *Operads: Proc. of Renaissance conferences* (Hartford CT/Luminy France 1995), Contemp. Math. **202**, 33–36, AMS, Providence, RI (1997).

**[38]** D. Yetter, Functorial knot theory: Categories of tangles, coherence, categorical deformations and topological tnvariants, *Series on knots and everything* **26**, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

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