

REALIZATION THEOREMS FOR END OBSTRUCTIONS

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(communicated by Daniel Grayson)

Abstract

A stratified space is a filtered space with manifolds as its strata. Connolly and Vajiac proved an end theorem for stratified spaces, generalizing earlier results of Siebenmann and Quinn. Their main result states that there is a single K -theoretical obstruction to completing a tame-ended stratified space. A necessary condition to completeness is to find an exhaustion of the stratified space, i.e. an increasing sequence of stratified spaces with bicollared boundaries, whose union is the original space. In this paper we give an example of a stratified space that is not exhaustible. We also prove that the Connolly-Vajiac end obstructions can be realized.

1. Introduction

The results of this paper are closely related to the problem of characterizing those topological manifolds which can be the interior of a compact manifold with boundary. After the starting result due to Browder, Levine and Livesay in [5], L.C. Siebenmann showed that a topological manifold of dimension $n \geq 6$, X , which is tame at infinity, is the interior of a compact manifold with boundary if and only if a single obstruction $\sigma(X) \in \tilde{K}_0(\mathbb{Z}\pi)$ vanishes. Here π denotes the fundamental group of the end of X .

An important contribution in this direction is due to F. Quinn ([29, 30, 32]). He defines an obstruction, $q_0(X, A) \in \tilde{K}_0^{lf}(A, p_X)$, which vanishes if A has a mapping cylinder neighborhood in X . Quinn's results generalize Siebenmann's theorem; in particular, a mapping cylinder neighborhood can be described as a "controlled" completion of $X - A$. We will review the definitions and the statement of Quinn's theorem in Section 3 below.

In [12] Connolly and Vajiac proved the following end theorem for stratified spaces:

Theorem. *For any tame ended stratified space X one can define a "controlled Quinn-type" obstruction $\gamma_*(X)$. Assume it vanishes; i.e., $\gamma_*(X) = 0$. Assume that the 5-dimensional skeleton, X^5 , already has a completion; i.e., it is the interior of a compact stratified space. Then X itself is the interior of a compact stratified space \bar{X} .*

Received May 7, 2007, revised April 16, 2008; published on July 14, 2008.

2000 Mathematics Subject Classification: 57Q10; 57Q20, 57Q40, 57N40, 57N80.

Key words and phrases: stratified spaces, homology, homotopy.

This article is available at <http://intlpress.com/HHA/v10/n2/a1>

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The obstruction $\gamma_*(X)$ lies in a localization of the Quinn's group mentioned above, and it is a direct sum of obstructions (one for each stratum):

$$\gamma_*(X) = \bigoplus_m \gamma_m(X) \in \bigoplus_m K_0^{lf}((\hat{X}^{m-1}, p_{\hat{X}^m})_{(\infty)}).$$

See [12] for more details on the definition of γ and of the obstruction groups.

When a completion is impossible, one can ask at least to find an *exhaustion* of X . By this we mean an increasing sequence of compact stratified subspaces of X , with bicollared boundaries in X , whose union is X . Note that a stratified space that admits a completion is, in particular, exhaustible. Suppose now that X is a stratified space with n -dimensional top stratum. Since the obstruction $\gamma_*(X)$ is a direct sum of obstructions, one for each stratum, we can concentrate our discussion on the top stratum of X . That is to say, assume that X^{n-1} , its $(n-1)$ -dimensional skeleton, has a completion Y . Let $B = \partial Y$. Choose a collar neighborhood for ∂Y in Y , say, $B \times (0, \infty] \subset Y$. Then the *open cone* of B , OB , can be identified to $B \times (0, \infty] / B \times \{\infty\}$. The open cone becomes a neighborhood of $\{\infty\}$ in \hat{X}^{n-1} , the one-point compactification of X^{n-1} . Let $p_{\hat{X}}|_{OB}$ be the projection map $\text{Holink}(\hat{X}, \hat{X}^{n-1})|_{OB} \xrightarrow{p_{\hat{X}}|_{OB}} OB$ and denote by p_B the restriction of $p_{\hat{X}}|_{OB}$ to $B \times \{1\}$, where we identify B with $B \times \{1\} \subset B \times (0, \infty) \subset X^{n-1}$. The Holink is discussed in detail in Section 2.

In this particular case Theorem 1.12 in [12] can be stated as follows:

Theorem (Exhaustibility Theorem from [12]). *Let X be a tame ended n -dimensional stratified space, $n \geq 6$, so that X^{n-1} has a completion Y . Let $B = \partial Y$ be as above. Assume that X^5 admits an exhaustion and that all the fundamental groups of the holink fibers $\text{Holink}(X^n, X^{n-1}) \rightarrow X^{n-1}$ are good. Then there is an element $\partial\gamma_n(X) \in K_{-1}(B, p_B)$ so that $\partial\gamma_n(X) = 0$ if and only if X admits an exhaustion.*

Remark 1.1. A group G is good if $K_i(\mathbb{Z}[G]) = 0$ for $i \leq -2$. No example of a bad group is known.

The main results of this paper are Theorems 4.1 and 4.2. The first one provides a simple example of a stratified space which is not exhaustible.

Theorem (Theorem 4.1). *Let $n \geq 3$. There exist manifolds \widetilde{M}^{2n+1} , homeomorphic to $\mathbb{R}^{2n+1} - \{0\}$, with a properly discontinuous cocompact semifree action of $C_6 \times \mathbb{Z}$ so that:*

- (i) \widetilde{M}^{C_6} is isomorphic to $\mathbb{R} \amalg \mathbb{R}$ and it is locally flat embedded in \widetilde{M} .
- (ii) (\widetilde{M}, C_6) has two ends; neither end is exhaustible (and therefore \widetilde{M} is not exhaustible).
- (iii) $(\widetilde{M}, C_6 \times \mathbb{Z})$ is equivariantly h -cobordant to $(\mathbb{R}^{2n+1} - \{0\}, C_6 \times \mathbb{Z})$. The action of C_6 is linear on \mathbb{R}^{2n+1} , and \mathbb{Z} acts (linear also) by multiplication by powers of two.

The second theorem states that any element in the group $K_{-1}(B, p_B)$ above can be realized in the form $\partial\gamma_n(W)$, for some stratified space W . Here is the precise statement:

Theorem (Theorem 4.2). *Let Y be a compact n -dimensional stratified space. Denote by σY its $(n-1)$ -skeleton. Let $p_Y: \text{Holink}(Y, \sigma Y) \rightarrow Y$ be the projection map. Denote by \mathbf{a} the forget-control assembly map ([35]):*

$$\tilde{K}_{-1}(\sigma Y, p_Y) \xrightarrow{\mathbf{a}} \tilde{K}_{-1}(\mathbb{Z}\pi_1(Y - \sigma Y)).$$

Let $x \in \text{Ker}(\mathbf{a})$ be an arbitrary element. Then there exists a stratified space W , containing Y , so that $\sigma W \approx \sigma Y \times \mathbb{R}$, and the obstruction to exhaust W is $\partial\gamma_n(W) = i_(x)$. The map i_* is induced by inclusion and gives an isomorphism between*

$$K_{-1}(\sigma Y, p_Y) = \tilde{K}_0^{lf}(\sigma Y \times \mathbb{R}, p_Y \times 1_{\mathbb{R}}) \quad \text{and} \quad \tilde{K}_0^{lf}(\sigma W, p_W).$$

The structure of this paper is the following: in Section 2 we review the theory of geometric modules and continuously controlled K -theory. More details are provided in [41]. Section 3 recalls the definition of the finiteness obstruction as introduced by F. Quinn in [30]. We will use a more algebraic approach, as suggested by F. Quinn in [33] and Ranicki and Yamasaki in [35]. We provide a quick account, for the reader's convenience. More details are provided in [41]. We also discuss the relation between the torsion of an h -cobordism and the mapping cylinder obstructions.

The last section, Section 4, contains the proofs of the main theorems, Theorems 4.1 and 4.2.

The author wants to thank his advisor, Professor Frank Connolly, for his guidance and support during the process of completing this work.

2. Geometric modules and K -theory

We will briefly recall the theory of geometric modules and the definition of the K -theory groups. These concepts were first introduced by Connell and Hollingsworth in [11]. We follow an approach similar to the ones of Quinn ([29, 30]) and Ranicki ([34, 35]). The full details are provided in [40] and [41].

Let X be a topological space and let $\Pi(X)$ be the fundamental groupoid of X . This is a category with objects being the points of X and morphisms from x_1 to x_0 being the homotopy classes of paths from x_0 to x_1 .

The *homotopy category of finite geometric modules on X* is:

$$\mathcal{G}(X) = \text{Ad } \mathbb{Z}[\Pi(X)].$$

Here the categorical constructions $\text{Ad } \mathbb{Z}[\Pi(X)]$ mean the abelian category generated by $\Pi(X)$. See [41] for a detailed description.

The continuously controlled K -theory of a pair (X, B) was introduced in [2]. Here is a very short explanation.

An *abelian category over a space X* , is an abelian category where every morphism is assigned a compact subset of X , called its support. The support of a morphism satisfies the following properties:

- $\text{supp}(\sigma \circ \tau) \subset \text{supp}(\sigma) \cup \text{supp}(\tau)$.
- $\text{supp}(\sigma + \tau) \subset \text{supp}(\sigma) \cup \text{supp}(\tau)$.
- If $\sigma = \sum_{i=1}^k n_i \sigma_i$, then $\text{supp}(\sigma)$ is the union of the supports of those σ_i for which $n_i \neq 0$.

Definition 2.1. Let B be a subspace of X . A collection, $\{S_\lambda : \lambda \in \Lambda\}$, of compact subsets of $X - B$, is *continuously controlled*, or *cc* over (X, B) , if:

- (a) For each X -neighborhood U of B , $\{\lambda \in \Lambda : S_\lambda \not\subset U\}$ is a finite subset of Λ .
- (b) For each point $p \in B$ and each X -neighborhood, $U(p)$, there is an X -neighborhood $V(p)$ so that any set S_λ meeting V must lie in U .

Using the continuously controlled notion, we define the homotopy category of geometric modules of a pair $\mathcal{G}(X, B)$ in a similar way. Objects in this category are pairs (S, j) , where S is a set, and $j : S \rightarrow |\mathbb{C}|$ is a function for which the indexed collection $\{\text{supp}(j(s)) : s \in S\}$ is a *cc*-collection over (X, B) . For a morphism $f = (f_t^s)_{(s,t) \in S \times T}$, we require that $\{\text{supp}(f_t^s) : (s, t) \in S \times T\}$ is a *cc*-collection over (X, B) .

In particular, the *locally finite* geometric modules over X are the geometric modules of the pair (\hat{X}, ∞) . Here \hat{X} is a one point compactification of X .

Definition 2.2. Let B be a subset of the topological space X . The groups $K_i(X, B) = K_i(\mathcal{G}(X, B))$, $i \leq 1$ are defined in the sense of Quillen (see [28] and [39]).

Similarly, $K_i^{lf}(X, B) = K_i(\mathcal{G}(\hat{X}, \hat{B}))$, $K_i^{lf}(X) = K_i(\mathcal{G}(\hat{X}, \infty)) = K_i(\mathcal{G}^{lf}(X))$.

The reduced K_0 -groups for these categories are defined as the cokernel of the natural map, $K_0(\mathcal{G}(X, B)) \rightarrow K_0(\mathcal{G}^{\text{id}}(X, B))$, where the $\mathcal{G}^{\text{id}}(X, B)$ denotes the idempotent completion (see [34] for more details).

The *Whitehead group*, $\text{Wh}(X, B)$ is defined as

$$\text{Wh}(X, B) = K_1(X, B)/H, \text{Wh}(X) = \text{Wh}(X, \emptyset) = K_1(\mathcal{GM}(X))/H.$$

Here H is the subgroup of $K_1(X, B)$ generated by basis change matrices. Note that $\text{Wh}(X) \approx \text{Wh}(\pi_1(X, x_0))$ if X is path connected.

The *Whitehead torsion* of an isomorphism in $\mathcal{G}(X, B)$ is defined as usual:

$$\tau(f) = [b \circ f] \in K_1(X, B)/H = \text{Wh}(X, B);$$

here b is a base change isomorphism.

By using the ideas from [35] one can think of $\text{Wh}(X, B)$ as the equivalence classes of contractible continuously controlled chain complexes over $\mathcal{G}(X, B)$. Hence an element in $\text{Wh}(X, B)$ will be one of those complexes.

Since we will use the following notions, we recall a few definitions from [32], for the reader's convenience.

Definitions 2.3. Let A be a subspace of a topological space X .

1. **Holink.**

$$\text{Holink}(X, A) = \{\sigma \in \text{Map}([0, 1], X) \mid \sigma^{-1}(A) = \emptyset\}.$$

(It is a topological space, using the compact-open topology.) There are two evaluation maps:

$$p_X : \text{Holink}(X, A) \rightarrow A; p_X(\sigma) = \sigma(0)$$

and

$$j_X : \text{Holink}(X, A) \rightarrow (X - A); j_X(\sigma) = \sigma(1).$$

2. **Forward tame.** If A is closed, then A is said to be forward tame in X if there exists a neighborhood U of A in X a map $F: U \times [0, \infty] \rightarrow X$ such that $F^{-1}(A) = A \times [0, \infty] \cup U \times \{\infty\}$, and $F(a, t) = a, \forall (a, t) \in A \times [0, \infty]$.
3. **Reverse tame.** If A is closed, then A is said to be reverse tame in X if there is a map $R: (X - A) \times [0, \infty] \rightarrow X - A$ so that:
 - (a) for each $t \in [0, \infty) \text{Cl}_X R((X - A) \times [0, t]) \subset X - A$.
 - (b) each point $x \in X - A$ has a neighborhood U and a number $t_x \in [0, \infty)$ so that $R_t|_U = \text{identity}|_U$ for all $t \geq t_x$. Here $R_t: X - A \rightarrow X - A$ is defined by $R_t(x) = R(x, t)$.
 - (c) R extends continuously to a map $(X - A) \times [0, \infty] \cup (A \times \{\infty\}) \rightarrow X$ by setting $R(a, \infty) = a, \forall a \in A$.
4. **Tame.** A closed subset of X is tame if it is both forward and reverse tame.
5. **Stratified space.** A *stratified space* is a finitely filtered, locally compact Hausdorff space $(X, \{X_i\}_0^n)$ such that:
 - (a) Each stratum X_k is a k -dimensional topological manifold (possibly with boundary).
 - (b) For each $i < j, X_i$ is tame in $X_i \cup X_j$.
 - (c) $\text{Holink}(X_i \cup X_j, X_i) \xrightarrow{p} X_i$ is a fibration, and the inclusion

$$\text{Holink}(\partial X_i \cup \partial X_j, \partial X_i) \rightarrow \text{Holink}(X_i \cup X_j, X_i)|_{\partial X_i}$$

is a fiber homotopy equivalence over ∂X_i .

The k -th skeleton of a stratified space is defined as $X^k = X_1 \cup X_2 \cup \dots \cup X_k$.

3. Finiteness obstructions

The controlled \tilde{K}_0 -groups of the controlled finiteness obstruction were first introduced by Quinn in [30]. We will recall their definitions, using the algebraic approach of [33] and [35]. See also [40] and [41] for more details.

Definition 3.1. $\tilde{K}_0^{lf}(B, p)$ is defined as $\lim_{\leftarrow \delta} \tilde{K}_0^{lf}(B, p; \delta)$, as δ ranges over the con-

tinuous functions $\delta: B \rightarrow (0, \infty)$. (If B is compact, then ϵ and δ can be chosen to be positive real numbers; in this case the K_0 groups will not have the locally finite superscript.)

The negative controlled K groups of (B, p) are defined (B compact) as

$$\tilde{K}_{-i}(B, p) = \tilde{K}_0^{lf}(B \times \mathbb{R}^i, p \times 1_{\mathbb{R}^i}).$$

A few explanations are necessary about the above notation.

$p: E \rightarrow B$ is a continuous map from a topological space E to a locally compact metric space B , and $\delta: B \rightarrow (0, \infty)$ is a continuous map. A subset $S \subset E$ is a δ -subset if $p(S)$ lies in the $\delta(p(x))$ ball around $p(x)$, for each $x \in S$. A morphism is a δ -morphism if $\text{supp}(f_t^s)$ is a δ -subset for each (s, t) . Following the ideas in the previous section, we can define the ‘‘category of δ geometric modules’’. A δ -projective module is a pair (M, e) , where $e: M \rightarrow M$ has the property that e^2 is δ -homotopic to e . The group $\tilde{K}_0^{lf}(B, p; n, \delta)$ is defined as the set of equivalence classes of δ -projective

chain complexes of geometric modules. We suppress quite a few details. See [35] for a full account.

In the above notation, we will define now Quinn's mapping cylinder obstruction ([30]). Let C be an n -dimensional locally finite δ -chain complex over E which is δ -dominated by a δ -chain complex over (B, p) . In [35, Section 3.1], it is shown that $(C, 1)$ is ϵ -chain equivalent to an n -dimensional ϵ -projective chain complex, (D, e) , over (B, p) . The class

$$q_\delta(C) = \chi([D, e]) \in K_0^{lf}(B, p; 9\delta)$$

depends only on C ([35, 41]) and is called the δ -controlled finiteness obstruction of C over B .

Definition 3.2 (Quinn's mapping cylinder obstruction). Let $f: M \rightarrow B$ be a map from a manifold M with compact boundary to a compact metric space B . Assume that $\dim(M) \geq 6$, and (M, f) is tame ([29, Section 1.1]). Quinn defines the obstruction $q_0(M, f)$ as follows.

Let $M = M(0) \supset M(1) \supset M(2) \supset \dots$ be an infinite sequence of submanifolds with compact boundaries and relatively compact complements in M . Assume that $\bigcap_{k=1}^{\infty} M(k) = \emptyset$. By using the tameness, one can choose this sequence so that $M(k)$ deforms, rel $\partial M(k)$, into the set $N(k) = M(k) - \text{Int}(M(k+1))$, and the diameter of the deformations are $\langle \delta(k), \text{ where } \delta(1) > \delta(2) \rangle \dots$, and $\lim_{k \rightarrow \infty} \delta(k) = 0$. All measurements are made in B . Set $f(k) = f|_{M(k)}: M(k) \rightarrow B$.

Choose a handle decomposition of $(N(k), \partial M(k))$ and an infinite handle decomposition of $(M(k), \partial M(k))$ of diameter much less than $\delta(k)$. The chain complex $C(M(k), \partial M(k))$ is $\delta(k)$ -finitely dominated by the finite complex $C(N(k), \partial M(k))$, so $q_{\epsilon(k)}(C(k))$ is defined as above. It turns out that the sequence

$$q_0(M, f) = \{q_{\epsilon(k)}(C(k))\}_{k=1}^{\infty}$$

is an element of the group: $\lim_k \tilde{K}_0(B, f(k), 9\epsilon(k)) \approx \tilde{K}_0(B, p)$. Recall that p is the holink projection. The element $q_0(M, f)$ is called the controlled end obstruction of (M, f) .

Remark 3.3. In [29, Chapter 7] and [30], Quinn proves that $q_0(M, f) = 0$ if and only if M is the interior of a compact manifold \bar{M} such that f extends to a map $\bar{f}: \bar{M} \rightarrow B$.

According to [1] (see also [41] for details), the following sequence

$$\text{Wh}(X, B) \xrightarrow{\Delta} \tilde{K}_0(B, p) \xrightarrow{\alpha} \tilde{K}_0(\mathbb{Z}\pi_1(X - B))$$

is exact at $\tilde{K}_0(B, p)$.

A *relative manifold* is a compact Hausdorff pair (X, A) , for which $X - A$ is a paracompact manifold of dimension $\dim(X - A) \geq 5$, possibly with boundary, with A tame in X .

A *cobordism of relative manifolds* between (X, A) and (Y, A) is a relative manifold (W, A) such that $\partial W - A$ is the union of the two open sets $X - A$ and $Y - A$.

An *h-cobordism of relative manifolds* is a cobordism (W, A) between (X, A) and (Y, A) for which there are strict maps $r_i^t: (W, A) \rightarrow (W, A)$, $0 \leq t \leq 1$, $i = 1, 2$ deforming (W, A) into (X, A) , respectively, (Y, A) , by a deformation which fixes X , respectively Y . The set of equivalence classes of *h-cobordisms* on (X, A) is parametrized by $\text{Wh}(X, A)$ ([41]).

We will denote Quinn's mapping cylinder obstruction of A in X by $q_0(X, A)$. (Recall that there is a map $X - A \rightarrow A$, given by the Holink.)

Theorem 3.4 ([41]). *Let (X, B) be a relative manifold. Let W be an *h-cobordism of relative manifolds*, from (X, B) to (Y, B) . Let $\tau \in \text{Wh}(X, B)$ be the torsion of (W, B) . Then*

$$\Delta(\tau) = r_*q_0(W, B) - q_0(X, B). \quad (*)$$

The map Δ is explicitly constructed in [41] and $r_*: \tilde{K}_0(B, p_W) \rightarrow K_0(B, p_X)$ is the isomorphism induced by the retraction r from W to X .

4. A stratified space which is not exhaustible

This section contains the two main results. They are closely related to Theorem 1.12 in [12], as was already mentioned in the introduction.

Recall that, for a stratified space X , an *exhaustion* means an increasing sequence of compact stratified subspaces of X , with bicollared boundaries in X , whose union is X .

Suppose now that X^{n-1} has a completion Y . Let $B = \partial Y$. As explained in the introduction, Theorem 1.12 in [12] identifies an element $\partial\gamma_n(X) \in K_{-1}(B, p_B)$ so that $\partial\gamma_n(X) = 0$ if and only if X admits an exhaustion.

The goal of the present section is to describe a simple example of a tame ended stratified space which is not exhaustible. We then go on to prove that any element in the kernel of the assembly map can be represented, in a sense explained below, in the form $\partial\gamma_n(X')$, for some X' . This can be viewed as a realization theorem for elements in $K_{-1}(B, p)$.

Let C_6 be the cyclic group of order 6. Let $\mathbb{Z}C_6$ denote its group ring, equipped with the standard involution. It is known that $\tilde{K}_0(\mathbb{Z}C_6) = 0$ and $K_{-1}(\mathbb{Z}C_6) \approx \mathbb{Z}$. Moreover, the standard involution acts trivially on $K_{-1}(\mathbb{Z}C_6)$. Hence every non-zero element $x \in K_{-1}(\mathbb{Z}C_6)$ satisfies the equation $x + \bar{x} \neq 0$, where \bar{x} stands for the involution applied to x .

The proof of the fact that $\tilde{K}_0(\mathbb{Z}C_6) = 0$ is number-theoretic in nature. It was first done (in a more general setting) by Bass and Murthy [4]. The method involves including $\mathbb{Z}C_n$ in the \mathbb{Z} -maximal order in $\mathbb{Q}C_n$ and carefully analyzing the kernel. See also [14] for a detailed exposition. The computation of $K_{-1}(\mathbb{Z}G)$, for G finite abelian, is done in the book of H. Bass [3]. An easier approach that is specific to the group C_6 can be found in [36]. For complete details, refer to [3] or [36].

Construction of the counterexample

Theorem 4.1. *Let $n \geq 3$. There exist manifolds \tilde{M}^{2n+1} , homeomorphic to $\mathbb{R}^{2n+1} - \{0\}$, with a properly discontinuous cocompact semifree action of $C_6 \times \mathbb{Z}$ so that:*

- (i) \widetilde{M}^{C_6} is isomorphic to $\mathbb{R} \amalg \mathbb{R}$ and it is locally flat embedded in \widetilde{M} .
- (ii) (\widetilde{M}, C_6) has two ends; neither end is exhaustible (and therefore \widetilde{M} is not exhaustible).
- (iii) $(\widetilde{M}, C_6 \times \mathbb{Z})$ is equivariantly h -cobordant to $(\mathbb{R}^{2n+1} - \{0\}, C_6 \times \mathbb{Z})$. The action of C_6 is linear on \mathbb{R}^{2n+1} , and \mathbb{Z} acts (linear also) by multiplication by powers of two.

Idea of proof. Let V be a $(2n+1)$ -dimensional $\mathbb{R}C_6$ module so that C_6 acts semi-freely, and V^{C_6} is 1-dimensional. Let \mathbb{Z} act on $V - \{0\}$ multiplying by powers of two. Set $M(V) = (V - \{0\})/\mathbb{Z}$. $M(V)$ is a compact manifold which is (non-equivariantly) homeomorphic to $S^1 \times S^{2n}$. The C_6 -fixed point set, $\sigma M(V)^{C_6}$, consists of two copies of S^1 . The construction of \widetilde{M} goes as follows: we will first alter $M(V)$ by an h -cobordism which will be a product cobordism on the singular set. At the other end of the h -cobordism we will get a C_6 -manifold, say M , whose universal cover \widetilde{M} will satisfy the conclusion of the theorem. The proof involves an analysis of the relationship between the Whitehead group parametrizing the h -cobordisms in question, and the mapping cylinder neighborhood obstruction of the singular set in the ambient manifold. \square

The proof relies on the following facts:

1. The C_6 -equivariant h -cobordisms on $M(V)$, which are product cobordisms on $M(V)^{C_6}$, are parametrized by $\text{Wh}(M(V)/C_6, M(V)^{C_6}/C_6)$. We will further denote the stratified space $M(V)/C_6$ by X in order to simplify the notation. It follows that $\sigma X = X - \text{top stratum} = M(V)^{C_6}/C_6 \approx S^1 \amalg S^1$.
2. There is an exact sequence

$$\text{Wh}(X, \sigma X) \xrightarrow{\partial} \widetilde{K}_0(\sigma X, p) \xrightarrow{j} \widetilde{K}_0(\mathbb{Z}\pi_1(X - \sigma X)).$$

See the previous section for references.

In our case it is easy to see that

$$\widetilde{K}_0(\mathbb{Z}\pi_1(X - \sigma X)) \approx \widetilde{K}_0(C_6 \times \mathbb{Z}) \approx \widetilde{K}_0(\mathbb{Z}C_6) \oplus K_{-1}(\mathbb{Z}C_6) \approx \mathbb{Z}.$$

This follows from the Bass-Heller-Swan formula and the previous observations. Also, $\widetilde{K}_0(S^1, p) \approx \mathbb{Z}$. This can be accomplished either by an E^2 -term computation, using Quinn's spectral sequence (see [30]), or invoking the controlled version of the Bass-Heller-Swan formula (see [35, Chapter 7]). Note that, by the homogeneity of the above construction, the map

$$p: \text{Holink}(X, \sigma X) \rightarrow \sigma X$$

is fiber homotopy equivalent to $p_{pt} \times 1_{S^1}$. It follows that the map j above can be identified with

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j} \mathbb{Z},$$

$$j(1, 0) = j(0, 1) = 1.$$

3. The boundary map in the exact sequence above takes the torsion, τ , of the homotopy equivalence between the two ends of an h -cobordism to the difference

of the mapping cylinder obstructions of the singular sets for the two ends of the h -cobordism. More precisely: Let W be a stratified h -cobordism from X to Y , so that σW is homeomorphic to $\sigma X \times I$. Let $\tau = \tau(W, X) \in \text{Wh}(X, \sigma X)$. Then

$$\partial(\tau + \bar{\tau}) = q_0(X, \sigma X) - r_* q_0(Y, \sigma Y). \quad (*)$$

This formula follows from Theorem 3.4.

The proof is quite formal from now on.

Note that

$$\text{Ker}(j) = \{y \oplus (-y) \mid y \in \mathbb{Z}\}.$$

Let $0 \neq x \in K_{-1}(\mathbb{Z}C_6)$. Then $(x \oplus (-x)) = 0$; hence there exist $\tau \in \text{Wh}(X, \sigma X)$ so that $\partial\tau = x \oplus (-x)$. By the above cc - h -cobordism theorem, τ is the torsion of an C_6 -equivariant h -cobordism from $(M(V), C_6)$ to a topological C_6 -manifold, say M . By formula (*) we have:

$$q_0(M/C_6, M^{C_6}/C_6) = -\partial(\tau + \bar{\tau}) \neq 0.$$

One can easily conclude now that neither end of (\tilde{M}, C_6) has any stratified space neighborhoods. Recall that a stratified space neighborhood is a stratified space with bicollared boundary, required to be a neighborhood of only one end. Suppose that there exists a stratified space neighborhood N of one of the ends of \tilde{M} , say ε . One can assume without loss of generality that

$$(\sigma N, \sigma\partial N) \approx ([0, \infty), 0).$$

(If this is not the case, then one can pass to a smaller subneighborhood.) Consider the non-compact manifold $(\partial N - \sigma\partial N)/C_6$; because $\sigma\partial N$ is a point, it follows that the Siebenmann obstructions for the ends of this manifold, in $\tilde{K}_0(\mathbb{Z}C_6)$, are zero. So by [37], $\sigma\partial N$ has a mapping cylinder neighborhood W in ∂N . By Quinn's collar extension theorem ([32, pp. 492–494]), the C_6 -collar $[0, \infty)$ for $\sigma\partial N$ in σN extends to a C_6 -collar of ∂N in N . Similarly, if $N' = \tilde{M} - \text{Int}_{\tilde{M}} N$, $\partial N' = \partial N$, then $(\sigma N', \sigma\partial N') \approx ((-\infty, 0], 0)$. This is a collar for $\sigma\partial N'$ in $\sigma N'$, and it extends by the same argument to a collar for $\partial N'$ in N' . Therefore we have a neighborhood U of $\sigma\tilde{M} \cup \partial N$ of the form $U = \mathbb{R} \times \partial N$ so that $\sigma U = \mathbb{R} \times \sigma\partial N$. $\mathbb{R} \times W$ is then a mapping cylinder neighborhood of $\sigma\tilde{M}$ in \tilde{M} . But this is absurd, by the above calculation.

Realization theorem for elements of $K_{-1}(\sigma Y, p_Y)$

Let Y be a compact n -dimensional stratified space. Denote by σY be its $(n-1)$ -skeleton. Let $p_Y: \text{Holink}(Y, \sigma Y) \rightarrow Y$ be the projection map. Denote by \mathfrak{a} the forget-control assembly map (as defined above or see [35]):

$$\tilde{K}_{-1}(\sigma Y, p_Y) \xrightarrow{\mathfrak{a}} \tilde{K}_{-1}(\mathbb{Z}\pi_1(Y - \sigma Y)).$$

Here is the main result:

Theorem 4.2. *In the above notation, let $x \in \text{Ker}(\mathfrak{a})$ be an arbitrary element. Then there exists a stratified space W , so that $\sigma W \approx \sigma Y \times \mathbb{R}$, and the obstruction to exhaust W , as explained before, is represented by x . This means that $\partial\gamma_n(X) = i_*(x)$. The*

map i_* provides an (inclusion induced) isomorphism between $K_{-1}(\sigma Y, p_Y)$ and the group $\tilde{K}_0^{lf}(\sigma W, p_W)$. It will be explained below.

Proof. The argument is similar to the one used in the construction of the counterexample above. Here is the outline. First form $Y \times S^1$. It is clear that $\sigma Y \times S^1$ has a mapping cylinder neighborhood in $Y \times S^1$. This follows from an argument due to S. Ferry [17]. See also Corollary 1.15 in [12]. By [35], we have the isomorphism:

$$\tilde{K}_0(\sigma Y \times S^1, p_{(Y \times S^1)}) \approx \tilde{K}_0(\sigma Y, p_Y) \oplus \tilde{K}_{-1}(\sigma Y, p_Y).$$

Look now at the exact sequence:

$$\begin{aligned} \text{Wh}(Y \times S^1, \sigma Y \times S^1) &\xrightarrow{\Delta} \tilde{K}_0(\sigma Y \times S^1, p_{Y \times S^1}) \\ &\xrightarrow{\mathbf{a}} \tilde{K}_0(\mathbb{Z}\pi_1(Y \times S^1 - (\sigma Y \times S^1))) \longrightarrow \dots \end{aligned}$$

Since $x \in \text{Ker}(\mathbf{a})$, it follows that there exists an element

$$\tau \in \text{Wh}(Y \times S^1, \sigma Y \times S^1),$$

so that $\Delta(\tau) = x$. Construct an h -cobordism W' on $(Y \times S^1, \sigma Y \times S^1)$, so that, according to formula (*) mentioned above, we have:

$$q_0(W', \sigma W') = x.$$

Take the infinite cyclic cover and get a stratified space W , with $\sigma W \approx \sigma Y \times \mathbb{R}$, so that the mapping cylinder obstruction of σW in W is exactly as required. To be more precise, we should identify $K_{-1}(\sigma Y, p_Y)$ and $\tilde{K}_0^{lf}(\sigma W, p_W) \approx K_{-1}(\sigma Y, p_W|_{\sigma Y})$. But the construction provides a fiber homotopy equivalence between the corresponding holinks, and hence an induced isomorphism i_* for the K -groups. \square

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